TESTING a FUNCTION for DIFFERENTIABILITY

Here are some important facts:

* **DEF:** $f(x,y)$ is differentiable at (a,b) if:
  (i) $f_x(a,b)$ and $f_y(a,b)$ BOTH exist;
  (ii) $$\lim_{(x,y)\to(a,b)} \frac{R_{1,(a,b)}(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0.$$

* **THM 1:** $\boxed{f \text{ diff. at } (a,b) \Rightarrow f \text{ cont. at } (a,b)}$

  OR, EQUIVALENTLY,

  $\boxed{f \text{ NOT cont. at } (a,b) \Rightarrow f \text{ NOT diff. at } (a,b)}$

* **THM 2:** $(f_x \text{ and } f_y \text{ cont. at } (a,b)) \Rightarrow (f \text{ diff. at } (a,b))$.

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NOTE: The second THM is very useful when testing differentiability of f a general points in its domain, BUT I suggest using the definition at special points.

E.g.: $$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$D(f) = \mathbb{R}^2.$

* $\underline{(x,y) \neq (0,0)}$: $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ (which is continuous for all $(x,y)\neq(0,0)$)

$$\rightsquigarrow f_x = \frac{y^2(x^2+y^2) - xy^2(2x)}{(x^2+y^2)^2}$$

$$f_y = \frac{2xy(x^2+y^2) - xy^2(2y)}{(x^2+y^2)^2}$$

continuous $\forall (x,y)\neq(0,0)$

$\Longrightarrow$ by THM 2, f is diff. $\forall (x,y) \neq (0,0)$

* $\underline{(x,y) = (0,0)}$: this is a special point since the definition of f changes there.

$\Longrightarrow$ USE DEFINITION.

$$f_x(0,0) = \lim_{h\to 0} \frac{\left(h\cdot 0/h^2+0\right) - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{\left(0\cdot h^2/0+h^2\right) - 0}{h} = 0$$

$\Longrightarrow L_{(0,0)}(x,y) \equiv 0$ and $R_{1,(0,0)}(x,y) = f(x,y)$.

THUS,

$$\lim_{(x,y)\to(0,0)} \frac{R_{1,(0,0)}(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\dfrac{xy^2}{x^2+y^2}}{\sqrt{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{(x^2+y^2)^{3/2}} \neq 0$$ since along the path $y=x$,

$$\lim_{\substack{x\to 0\\ y=x}} \frac{x\cdot x^2}{(x^2+x^2)^{3/2}} = \lim_{x\to 0} \frac{1}{\sqrt{2}} \cdot \frac{x^3}{|x^3|} \quad \text{DNE}.$$

$\Longrightarrow$ f is NOT differentiable at $(0,0)$.

HENCE, f is diff. $\forall (x,y) \neq (0,0)$, but NOT diff. at $(0,0)$.

NOTE: In the above example, f is continuous at $(0,0)$ since $\left|\frac{xy^2}{x^2+y^2} - 0\right| = \frac{|x|\cdot y^2}{x^2+y^2} \leq |x| \longrightarrow 0$.

Therefore, we could NOT use THM 1 to show that it is not differentiable at $(0,0)$.

In general, I suggest the following steps for testing differentiability:

HOW TO TEST DIFFERENTIABILTY of $f(x,y)$ at (a,b).

(A) If (a,b) is a general point of $D(f)$:

(i) Find f_x and f_y.

(ii) IF f_x and f_y are continuous at (a,b), THEN f is differentiable at (a,b).

If steps (i) or (ii) fail, go to (B).

(B) If (a,b) is a special point of $D(f)$: use the definition of differentiability.

(i) Do $f_x(a,b)$ and $f_y(a,b)$ exist?

NO $\Rightarrow$ f is NOT diff. at (a,b)

YES: keep testing.

(ii) Is $\displaystyle\lim_{(x,y)\to(a,b)} \frac{R_{1,(a,b)}(x,y)}{\sqrt{(x-a)^2+(y-b)^2}} = 0$?

YES $\Rightarrow$ f is diff. at (a,b).

NO $\Rightarrow$ f is NOT diff. at (a,b).

NOTE: (i) If you already know that f is NOT cont. at (a,b), then f is NOT diff. at (a,b) by THM 1, so there is no need to test the point (a,b).

(ii) If you are having a hard time computing the limit in (B)(ii), then think of using THM 1 and check whether f is even continuous at (a,b).

Ex: 1) $f(x,y) = x^2 \cos(3y - e^x) \rightsquigarrow D(f) = \mathbb{R}^2$.
Where is f differentiable?

Note that f is continuous for all $(x,y) \in \mathbb{R}^2$.
Moreover,

$$f_x = 2x\cos(3y - e^x) + x^2 \cdot (-\sin(3y - e^x)) \cdot (-e^x)$$

$$f_y = x^2 \cdot (-\sin(3y - e^x)) \cdot (3),$$

for all $(x,y) \in \mathbb{R}^2$. Since f, f_x, f_y can each be expressed by a single formula in $D(f) = \mathbb{R}^2$, there are no "special" points in $D(f)$. We therefore proceed as in (A).

Now, since f_x and f_y are BOTH continuous for ALL $(x,y) \in D(f)$, f is diff. for ALL $(x,y) \in \mathbb{R}^2$.

2) $f(x,y) = x y^{1/3} \rightsquigarrow D(f) = \mathbb{R}^2$.
Where is f differentiable?

Again, note that f is continuous and given by a single expression for all $(x,y) \in \mathbb{R}^2 = D(f)$
HOWEVER,

$$f_x = y^{1/3} \rightsquigarrow \text{defined for all } (x,y) \in \mathbb{R}^2$$

$$f_y = \tfrac{1}{3} x y^{-2/3} \rightsquigarrow \text{ONLY defined for } y \neq 0.$$

$\Rightarrow$ The "general" points in $D(f)$ are $\{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ and for those points, since $f_x = y^{1/3}$ and $f_y = \frac{1}{3} x\, y^{-2/3}$ are cont., f is differentiable.

$\rightsquigarrow$ f is diff. $\forall (x,y) \in \mathbb{R}^2$ with $y \neq 0$.

NOW, let's consider the "special" points in $D(f)$: $(a,0)$, $a \in \mathbb{R}$.

$\rightarrow$ $\underline{(a,0) = (0,0)}$:
$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{h \cdot 0 - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0 \cdot h^{1/3} - 0}{h} = 0 .$$

$\Rightarrow$ $L_{(0,0)}(x,y) \equiv 0$ and

$$R_{1,(0,0)}(x,y) = f(x,y) = x y^{1/3}$$

AND since
$$\lim_{(x,y) \to (x,y)} \frac{x y^{1/3}}{\sqrt{x^2+y^2}} = 0,$$
because $\left| \frac{x y^{1/3}}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|x| \cdot |y|^{1/3}}{\sqrt{x^2+y^2}} \leq |y|^{1/3} \to 0$ as $(x,y) \to (0,0)$,

f is differentiable at $(0,0)$.

→ $(a,0)$ with $a \neq 0$:

$$f_x(a,0) = \lim_{h\to 0} \frac{f(a+h,0) - f(a,0)}{h}$$

$$= \lim_{h\to 0} \frac{(a+a)\cdot 0 - 0}{h} = 0$$

$$f_y(a,0) = \lim_{h\to 0} \frac{f(a,h) - f(a,0)}{h}$$

$$= \lim_{h\to 0} \frac{a\cdot h^{1/3} - 0}{h}$$

$$= \lim_{h\to 0} \frac{a}{h^{2/3}} \quad \text{DNE since } a \neq 0.$$

$\Rightarrow$ $f_y(a,0)$ DNE if $a \neq 0$

$\Rightarrow$ f is NOT differentiable at $(a,0)$ with $a \neq 0$.

THUS, f is diff. $\forall (x,y)$ with $y \neq 0$ AND at $(0,0)$, BUT, f is NOT diff. at $(x,y) = (a,0)$ with $a \neq 0$.

3) $f(x,y) = |x|\, y^2 \leadsto D(f) = \mathbb{R}^2$.
Where is f differentiable?

Note that since $|x|$ is a piece-wise-defined function, then so is f:

$$f(x,y) = \begin{cases} xy^2 & \text{if } x > 0 \\ -xy^2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The "special" points in $D(f)$ are along $x=0$, and so those points should be treated as in (B).

* (x,y) with $x>0$: $f(x,y) = xy^2$ which is continuous for all (x,y) with $x>0$.
ALSO, $f_x = y^2$ and $f_y = 2xy$ which are defined and continuous for all (x,y) with $x>0$. $\Longrightarrow$ f is differentiable for all (x,y) with $x>0$.

* (x,y) with $x<0$: $f(x,y) = -xy^2$, which is continuous and has continuous first partials $f_x = -y^2$ and $f_y = -2xy$, for all (x,y) with $x<0$ $\Longrightarrow$ f is diff. for all (x,y) with $x<0$.

* $(x,y) = (0,b)$:

$\rightarrow$ $(x,y) = (0,0)$: $f_x(0,0) = f_y(0,0) = 0$, so that

$L_{(0,0)}(x,y) \equiv 0$ and

$$\lim_{(x,y)\to(0,0)} \frac{R_{1,(0,0)}(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}} = 0$$

(since $\left| \frac{xy^2}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|x|\, y^2}{\sqrt{x^2+y^2}} \le y^2 \to 0$ as $(x,y)\to(0,0)$)

$\Rightarrow$ f is differentiable at $(0,0)$.

$\rightarrow$ $(x,y) = (0,b)$, $b \neq 0$:

$$f_x(0,b) = \lim_{h\to 0} \frac{f(h,b) - f(0,b)}{h}$$

$$= \lim_{h\to 0} \frac{|h|\cdot b^2 - 0}{h}$$

$$= \lim_{h\to 0} b^2 \cdot \left(\frac{|h|}{h}\right) \text{ DNE since } b \neq 0.$$

$\Rightarrow$ f is NOT diff. at $(0,b)$, $b \neq 0$.

So, f is diff. $\forall (x,y)$ except $(x,y) = (0,b)$, $b \neq 0$.

4) $f(x,y) = |x^3 y| = \begin{cases} x^3 y & \text{if } x,y>0 \text{ or } x,y<0 \\ -x^3 y & \text{if } x>0, y<0 \text{ or } x<0, y>0 \\ 0 & \text{if } x=0 \text{ or } y=0 \end{cases}$ (10)

$\Downarrow$

diff. $\forall (x,y)$ with $x,y \neq 0$ or at $(0,0)$.

$\rightsquigarrow$ $D(f) = \mathbb{R}^2$ and the "special" points in the domain lie on the lines $x=0$ and $y=0$.

* For $x,y \neq 0$, one proves as in example 3) that f is diff. $\forall (x,y)$ with $x,y \neq 0$.

* For $(x,y) = (0,0)$, one shows as in examples 2) and 3) that f is diff. at $(0,0)$

* For $(x,y) = (a,0)$, $a \neq 0$, one shows as in ex. 2) that $f_y(a,0)$ DNE $\Rightarrow$ f is NOT diff. at $(a,0)$, $a \neq 0$.

* For $(x,y) = (0,b)$, $b \neq 0$, one shows as in ex. 3) that $f_x(0,b)$ DNE $\Rightarrow$ f is NOT diff. at $(0,b)$, $b \neq 0$.

5)
$$f(x,y) = \begin{cases} \dfrac{2x^3+y^3}{|x|+|y|} + 2, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0) \end{cases}$$

$D(f) = \mathbb{R}^2$ and the "special" point in the domain is $(0,0)$.

* $\underline{(x,y) \neq (0,0)}$: $$f(x,y) = \begin{cases} \dfrac{2x^3+y^3}{x+y} + 2, & x,y > 0 \\ \dfrac{2x^3+y^3}{-x+y} + 2, & x<0,\ y>0 \\ \dfrac{2x^3+y^2}{x-y} + 2, & x>0,\ y<0 \\ \dfrac{2x^3+y^2}{-x-y} + 2, & x,y<0. \end{cases}$$

If $x,y>0$,
$$f_x = \frac{6x^2(x+y) - (2x^3+y^3)\cdot 1}{(x+y)^2}$$
and
$$f_y = \frac{3y^2(x+y) - (2x^3+y^3)\cdot 1}{(x+y)^2},$$
which are both defined and continuous $\forall (x,y)$ with $x,y>0$, since they quotients of polynomials and the denominator is never zero.

Similarly, one proves that if $\{x<0, y>0\}$, or $\{x>0, y<0\}$, or $\{x, y<0\}$, then f_x and f_y are defined and continuous at all those points.

$\Rightarrow$ f_x and f_y are cont. $\forall (x,y) \neq (0,0)$

$\Rightarrow$ by THM 2, f is diff. $\forall (x,y) \neq (0,0)$.

* $\underline{(x,y)=(0,0)}$: since $(0,0)$ is a special point, use the definition.

(i) $$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\left(\frac{2h^3}{|h|}+2\right)-2}{h}$$

$$= \lim_{h\to 0} \frac{2h^2}{|h|} = \lim_{h\to 0} \frac{2|h|\cdot|h|}{|h|} = \lim_{h\to 0} 2|h| = 0.$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{\left(\frac{h^3}{|h|}+2\right)-2}{h} = \lim_{h\to 0} |h| = 0.$$

$\Rightarrow$ $f_x(0,0) = f_y(0,0) = 0$

and $L_{(0,0)}(x,y) = 0\cdot(x-0) + 0\cdot(y-0) + 2 = 2$

$$\Rightarrow R_{1,(0,0)}(x,y) = \begin{cases} \dfrac{2x^3+y^3}{|x|+|y|}, & (x,y)\neq(0,0) \\[2ex] 0, & (x,y)=(0,0) \end{cases}$$

(ii) $$\lim_{(x,y)\to(0,0)} \frac{R_{1,(0,0)}(x,y)}{\sqrt{x^2+y^2}} = \frac{2x^3+y^3}{(|x|+|y|)\sqrt{x^2+y^2}} \overset{?}{=} 0$$

Note that the numerator is a polynomial of degree 3, and the denominator

$$(|x|+|y|)\sqrt{x^2+y^2} \simeq (|x|+|y|)(|x|+|y|) = \begin{pmatrix}\text{expression of degree} \\ 2 \text{ in } x \text{ and } y\end{pmatrix}$$

$\Rightarrow$ the numerator converges faster to 0 than the denominator, and so the limit likely exists and is equal to 0.

$\rightsquigarrow$ USE SQUEEZE with $L=0$.

$$\left|\frac{2x^3+y^3}{(|x|+|y|)\sqrt{x^2+y^2}} - 0\right| = \frac{|2x^3+y^3|}{(|x|+|y|)\sqrt{x^2+y^2}}$$

$$\underset{\text{TRIANGLE INEQUALITY}}{\leq} \frac{2|x|^3+|y|^3}{(|x|+|y|)\sqrt{x^2+y^2}} = \frac{2|x|\cdot|x|\cdot\sqrt{x^2}}{(|x|+|y|)\sqrt{x^2+y^2}} + \frac{|y|\cdot|y|\cdot\sqrt{y^2}}{(|x|+|y|)\sqrt{x^2+y^2}}$$

$$\leq \frac{2|x|\cdot\cancel{(|x|+|y|)}\cdot\cancel{\sqrt{x^2+y^2}}}{\cancel{(|x|+|y|)}\cancel{\sqrt{x^2+y^2}}} + \frac{|y|\cdot\cancel{(|x|+|y|)}\cdot\cancel{\sqrt{x^2+y^2}}}{\cancel{(|x|+|y|)}\cdot\cancel{\sqrt{x^2+y^2}}}$$

$$= 2|x|+|y| \longrightarrow 0 \quad \text{as } (x,y)\to(0,0)$$

$\Rightarrow$ (ii) holds and f is differentiable at $(0,0)$

$\Rightarrow$ f is differentiable on ALL $\mathbb{R}^2$.