Directional derivatives.

2-var. Let f be a function of 2 variables and $(a,b) \in D(f)$.

DEF.: Let $\vec{u} = (u_1, u_2)$ be a unit vector, i.e., $\|\vec{u}\| = 1$.

$$D_{\vec{u}} f(a,b) := \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

= directional derivative of f at (a,b) in the direction $\vec{u}$.

NOTE: 1) $f_x(a,b) = D_{(1,0)} f(a,b)$

and $f_y(a,b) = D_{(0,1)} f(a,b)$.

2) $D_{\vec{u}} f(a,b) = \left(\begin{array}{c} \text{rate of change of } f \\ \text{at } (a,b) \text{ in the direction} \\ \vec{u} \text{ (at speed 1)} \end{array} \right)$

IMPORTANT: $\|\vec{u}\| = 1$ in the definition of $D_{\vec{u}} f(a,b)$!

3) $\left(\begin{array}{c} \text{rate of change of } f \\ \text{at } (a,b) \text{ in the direction} \\ \vec{u} \text{ at speed } k \end{array} \right) = k D_{\vec{u}} f(a,b), \; k > 0.$

CONVENTION: By "rate of change of f at (a,b) in the direction $\vec{u}$", we mean $D_{\vec{u}} f(a,b)$. So, if the speed at which one travels in direction $\vec{u}$ is not mentioned, it is assumed to be 1.

THM: If f is differentiable at (a,b), then

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}. \quad (*)$$

IMPORTANT! Formula $(*)$ only makes sense if f is differentiable at (a,b).

e.g.: Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Then, f is NOT diff. at $(0,0)$ since it's not continuous at $(0,0)$. [Indeed, $\lim_{(x,y)\to(0,0)} xy/(x^2+y^2) \neq 0 = f(0,0)$ because along the path $y=x$

$$\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \neq 0.]$$

Nonetheless, the partials of f exist at $(0,0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and similarly

$$f_y(0,0) = 0.$$

Consider $D_{\vec{u}} f(0,0)$ with $\vec{u} = (1/\sqrt{2}, 1/\sqrt{2})$.

Then, since f is not differentiable at $(0,0)$, we have to use the limit definition to compute $D_{\vec{u}} f(0,0)$. Note that $\|\vec{u}\| = 1$. Then,

$$D_{\vec{u}} f(0,0) = \lim_{h\to 0} \frac{f(0 + h/\sqrt{2}, 0 + h/\sqrt{2}) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(h/\sqrt{2})(h/\sqrt{2})}{(h/\sqrt{2})^2+(h/\sqrt{2})^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^2/2}{h\left(h^2/2 + h^2/2\right)} = \lim_{h\to 0} \frac{1}{2h} \text{ DNE.}$$

$\Rightarrow D_{\vec{u}} f(0,0)$ <u>does not exist</u>.

HOWEVER, the right-hand side of formula $(*)$ gives us:

$$\nabla f(0,0) \cdot \vec{u} = (0,0) \cdot (1/\sqrt{2}, 1/\sqrt{2}) = 0.$$

$\rightsquigarrow$ we see that

$$D_{\vec{u}} f(0,0) \neq \nabla f(0,0) \cdot \vec{u}$$

in this case, and this is due to the fact that f is <u>NOT</u> differentiable at $(0,0)$.

RECAP:

1) To compute the directional derivative of f at (a,b) in the direction $\vec{u}$:

* **NEED** $\|\vec{u}\| = 1$ $\rightsquigarrow$ if the direction is given by a non-unit vector $\vec{v}$, normalise! I.e., use $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$.

* If f is differentiable at (a,b), use the formula:
$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}.$$

* If f is not differentiable at (a,b) OR (a,b) is a "special" point in the domain of f, use the definition:
$$D_{\vec{u}} f(a,b) := \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}.$$

2) To compute the rate of change of f at (a,b) in the direction $\vec{u}$, $\|\vec{u}\| = 1$, at speed k, $k > 0$:
$$\begin{pmatrix} \text{rate of change} \\ \text{in direction} \\ \vec{u} \text{ at speed } k \end{pmatrix} = k D_{\vec{u}} f(a,b)$$
(but **ONLY** if $\|\vec{u}\| = 1$!).

Ex. 1) Find the directional derivative of $f(x,y)=x^2+3y^2$ in the direction $\vec{u}=(1/\sqrt{2}, 1/\sqrt{2})$ at $(-1,3)$.

First note that $\|\vec{u}\|=1$ and that f is diff. at $(-1,3)$ since it is a polynomial. Thus,

$$D_{\vec{u}} f(-1,3) = \nabla f(-1,3) \cdot \vec{u}.$$

Since $\nabla f = (2x, 6y)$, we get:

$$D_{\vec{u}} f(-1,3) = (-2,18)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 16/\sqrt{2} = \boxed{8\sqrt{2}}.$$

2) What is the rate of change of $f(x,y)=e^x \cos y$ at $(0,\frac{\pi}{4})$ in the direction $\vec{v}=(1,3)$.

* $\|\vec{v}\|=\sqrt{10}\neq 1 \Rightarrow$ use $\vec{u}=\dfrac{\vec{v}}{\|\vec{v}\|}=\dfrac{1}{\sqrt{10}}(1,3)$.

* Also, $f_x = e^x \cos y$ and $f_y = e^x(-\sin y)$ are cont. $\forall (x,y)$, so f is diff. $\forall(x,y)$ and

$$\begin{aligned} D_{\vec{u}} f\left(0,\tfrac{\pi}{4}\right) &= \nabla f\left(0,\tfrac{\pi}{4}\right)\cdot \vec{u} \\ &= \left(e^0\cos(\pi/4),\, e^0(-\sin(\pi/4))\right)\cdot \frac{1}{\sqrt{10}}(1,3) \\ &= \left(1/\sqrt{2}, -1/\sqrt{2}\right)\cdot\frac{1}{\sqrt{10}}(1,3) = -2/\sqrt{20} = -1/\sqrt{5}. \end{aligned}$$

3) The temperature of a metal sheet as a function of position (x,y) is given by $T(x,y)=100+10e^{-x}\sin y$. If a bug at $(0,\pi/6)$ moves west at a speed of 6 units/sec, what rate of change does it experience?

Since $\vec{u} = (-1,0)$ is a unit vector pointing west, and the bug moves at speed 6, the rate of change of temperature it experiences at $(0,\frac{\pi}{6})$ is:

$$6\, D_{\vec{u}} f(0,\tfrac{\pi}{6}).$$

Now, $f_x = -10e^{-x}\sin y$ and $f_y = 10e^{-x}\cos y$ are both cont. at $(0,\pi/6)$, f is diff. at $(0,\pi/6)$ and

$$\begin{pmatrix}\text{rate of temperature} \\ \text{change at} \\ \text{speed 6}\end{pmatrix} = 6\,\nabla f(0,\pi/6)\cdot\vec{u}$$

$$= 6\left(-10\cdot\left(\tfrac{1}{2}\right), 10\left(\tfrac{\sqrt{3}}{2}\right)\right)\cdot(-1,0)$$

$$= \boxed{30 \text{ deg/sec}}.$$

4) What is the rate of change of $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$ at $(0,0)$ in the direction $\vec{v} = (-1,\sqrt{3})$?

* $\|\vec{v}\| = 2 \Rightarrow$ use $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = (-1/2, \sqrt{3}/2)$.

* since $(0,0)$ is a "special" of f, use limit definition:

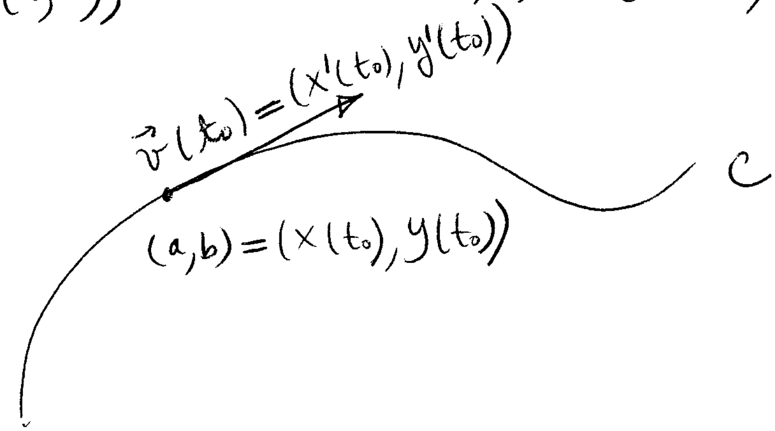
$$D_{\vec{u}} f(0,0) = \lim_{h\to 0} \frac{f(0+h(-1/2), 0+h(\sqrt{3}/2)) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(-h/2)^3}{(-h/2)^2+(\sqrt{3}h/2)^2} - 0}{h} = \lim_{h\to 0} \frac{\dfrac{h^3/8}{h^2/4+3h^2/4}}{h}$$

$$= \lim_{h\to 0} \frac{h}{8} = \boxed{0}.$$

Rate of change along a path.

Let $C : (x(t), y(t))$ be a parametric curve passing through (a,b), so that $(a,b) = (x(t_0), y(t_0))$ for some t_0.



Then,
$$\begin{pmatrix} \text{rate of change of} \\ f \text{ at } (a,b) \\ \text{along the path } C \end{pmatrix} = g'(t_0),$$

where $g(t) = f(x(t), y(t)) = (\text{restriction of } f \text{ to } C)$.

If f is differentiable at (a,b), then by the Chain Rule,

$$\begin{pmatrix} \text{rate of change} \\ \text{of } f \text{ at } (a,b) \\ \text{along } C \end{pmatrix} = \nabla f(a,b) \cdot (x'(t_0), y'(t_0))$$

$$= \|\vec{v}(t_0)\| \cdot D_{\frac{\vec{v}(t_0)}{\|\vec{v}(t_0)\|}} f(a,b),$$

where $\vec{v}(t_0) := (x'(t_0), y'(t_0)) = (\text{tgt vector to } C \text{ at } (a,b))$
$= \text{velocity at } t_0$.

NOTE: The rate of change of f at (a,b) along C ONLY depends on the velocity $\vec{v}(t_0)$ at t_0, and is equal to the rate of change of f at (a,b) in the direction $\vec{v}(t_0)$ at speed $\|\vec{v}(t_0)\|$.

E.g.: We have seen in example 3 on page 5 that the temperature of a metal sheet is given at position (x,y) by

$$T(x,y) = 100 + 10e^{-x}\sin y.$$

If a bug at $(0, \pi/6)$ moves along the path $C: (x(t), y(t)) = \left(\ln(t^2+1), \frac{\pi}{6}e^{3t}\right)$, what rate of temperature change does the bug experience?

HERE, $\vec{v}(t) = \left(\frac{2t}{t^2+1}, \frac{\pi}{2}e^{3t}\right)$ and since $(0, \pi/6) = (x(0), y(0))$, we are working at $t_0 = 0$.

Then,

$$\begin{pmatrix} \text{rate of temperature} \\ \text{change of bug} \\ \text{at } (0, \pi/6) \end{pmatrix} = \nabla T(0, \pi/6) \cdot \vec{v}(0)$$

we have seen that f is diff. at $(0, \pi/6)$

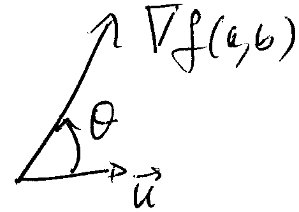
$$= \left(-10\sin(\pi/6), 10\cos(\pi/6)\right) \cdot \left(0, \frac{\pi}{2}\right)$$

$$= \left(-10 \cdot (1/2), 10 \cdot (\sqrt{3}/2)\right) \cdot \left(0, \frac{\pi}{2}\right) = \frac{5\pi\sqrt{3}}{2}$$

Geometric interpretation of the gradient.

Suppose that f is diff. at (a,b) and that $\nabla f(a,b) \neq (0,0)$. Let $\vec{u} = (u_1, u_2)$ be a unit vector. Then,

$$D_{\vec{u}} f(a,b) = \| \nabla f(a,b) \| \cos\theta$$



where $\theta = \text{angle}\left(\nabla f(a,b), \vec{u} \right)$.

ALSO: $\begin{pmatrix} \text{rate of change} \\ \text{of } f \text{ at } (a,b) \\ \text{at speed } k \end{pmatrix} = k \| \nabla f(a,b) \| \cos\theta, \quad k > 0.$

So:

* $\begin{pmatrix} \text{maximal rate of} \\ \text{change at speed } k \end{pmatrix} = k \| \nabla f(a,b) \|$ $\rightsquigarrow$ occurs in the direction $\nabla f(a,b)$ (when $\theta = 0$).

* $\begin{pmatrix} \text{minimal rate of} \\ \text{change at speed } k \end{pmatrix} = - k \| \nabla f(a,b) \|$ $\rightsquigarrow$ occurs in the direction $-\nabla f(a,b)$ (when $\theta = \pi$).

* $\begin{pmatrix} \text{rate of change} \\ \text{at speed } k \end{pmatrix} = C \iff \theta_0 = \arccos\left(\dfrac{C}{k \| \nabla f(a,b) \|} \right).$

OR for $\vec{u} = (u_1, u_2)$ such that

$$\begin{cases} \nabla f(a,b) \cdot \vec{u} = C/k \\ \| \vec{u} \| = 1. \end{cases}$$

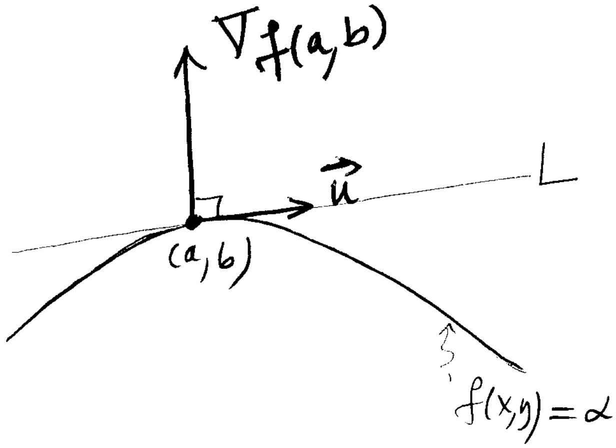
$*$ $\begin{pmatrix}\text{rate of change}\\ \text{at speed } k\end{pmatrix} = 0 \iff \theta_0 = \pi/2$

$\iff \nabla f(a,b) \perp \vec{u}$

$\iff$ $\nabla f(a,b)$ is perpendicular to the level curve of f passing through (a,b):

$$f(x,y) = \alpha$$

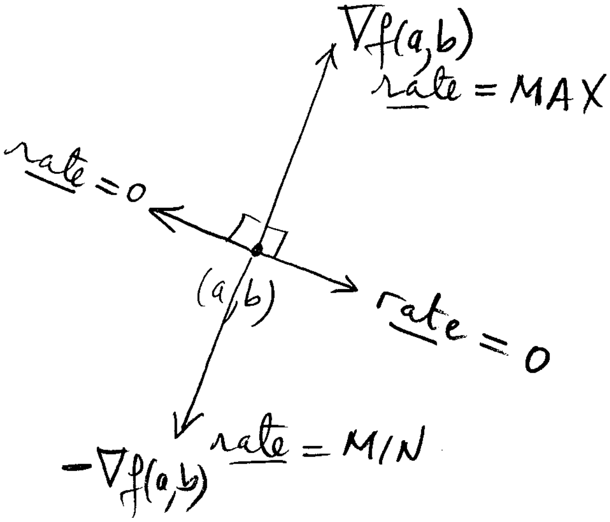
with $\alpha = f(a,b)$.

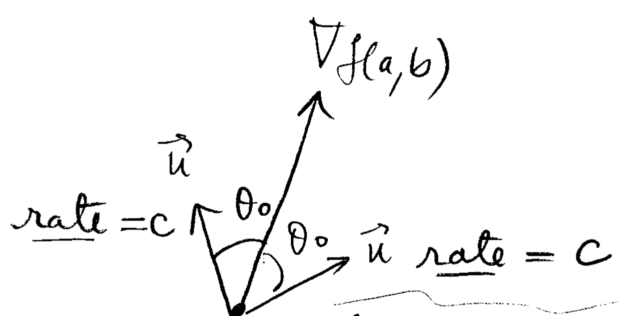


<u>CONSEQUENCE</u>: The tangent line L to the level curve $f(x,y) = \alpha$ passing through (a,b) has equation:

$$\boxed{L: \quad \nabla f(a,b) \cdot (x-a,\, y-b) = 0}.$$

In general, we have the following picture:





GET <u>two</u> distinct directions $\vec{u}$ where rate $= c$ if $\theta_0 \neq 0, \pi$

Ex. 1) Consider $f(x,y) = xy^2$ at $(2,1)$.

Note that f is differentiable at $(2,1)$ since it is a polynomial. Moreover, $\nabla f = (y^2, 2xy)$ and

$$\nabla f(2,1) = (1,4) \neq (0,0)$$

(i) Maximal rate of change: $\|\nabla f(2,1)\| = \sqrt{17}$,

in the unit direction $\vec{u} = \dfrac{\nabla f(2,1)}{\|\nabla f(2,1)\|} = \dfrac{1}{\sqrt{17}}(1,4)$.

(ii) Minimal rate of change: $-\|\nabla f(2,1)\| = -\sqrt{17}$,

in the unit direction $\vec{u} = -\dfrac{\nabla f(2,1)}{\|\nabla f(2,1)\|} = \dfrac{-1}{\sqrt{17}}(1,4)$.

(iii) No rate of change: $\vec{u} \perp \nabla f(2,1)$ and $\|\vec{u}\| = 1$

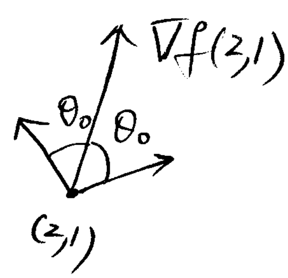
$\rightsquigarrow \vec{u} = \pm\left(4/\sqrt{17}, -1/\sqrt{17}\right) \Rightarrow$ get 2 directions.

(iv) In what directions is the rate of change equal to 5? NONE since $5 > \sqrt{17} =$ (max. rate of change).

(V) In what directions is the rate of change equal to 2? Since MIN. $= -\sqrt{17} < 2 < \sqrt{17} =$ MAX., the rate of change can be 2. It occurs at unit directions $\vec{u}$ at an angle

$$\theta_0 = \arccos\left(\frac{2}{\|\nabla f(2,1)\|}\right) = \arccos\left(2/\sqrt{17}\right) \simeq \frac{\pi}{3}$$

from $\nabla f(2,1)$, so in 2 directions.



We can also find $\vec{u}$ explicitly by solving:

$$\begin{Bmatrix} \nabla f(2,1)\cdot\vec{u} = 2 \\ \|\vec{u}\| = 1 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} (1,4)\cdot(u_1,u_2) = 2 \\ u_1^2+u_2^2 = 1 \end{Bmatrix}$$

$$\Leftrightarrow \begin{cases} u_1 + 4u_2 = 2 & \Rightarrow \boxed{u_1 = 2-4u_2} \rightsquigarrow \text{sub. into } (*): \\ u_1^2+u_2^2 = 1 \ (*) & \rightsquigarrow (2-4u_2)^2 + u_2^2 = 1 \end{cases}$$

$$\Leftrightarrow 17u_2^2 - 16u_2 + 3 = 0$$

$$\Leftrightarrow u_2 = \frac{16 \pm \sqrt{256-204}}{34} = \frac{8 \pm \sqrt{13}}{17}$$

$$\Rightarrow \vec{u} = (u_1,u_2) = \left(\frac{2-4\sqrt{13}}{17}, \frac{8+\sqrt{13}}{17}\right)$$

OR

$$= \left(\frac{2+4\sqrt{13}}{17}, \frac{8-\sqrt{13}}{17}\right)$$

because $u_1 = 2-4u_2$.

$\rightsquigarrow$ So we again get 2 distinct <u>unit</u> directions in which the rate of change of f is equal to 2.

<u>NOTE</u>: In (i) and (ii), it is not necessary to give <u>unit</u> directions. In fact, $\nabla f(2,1)$ and $-\nabla f(2,1)$ are also correct answers. BUT, in (v) is easier to give unit directions, since we are trying to find <u>ALL</u> directions, and although there are only 2 distinct unit directions, there are infinitly many non-unit ones.

2) We have seen in example 3 on page 5 that the temperature of a metal sheet is given at position (x,y) by

$$T(x,y) = 100 + 10e^{-x}\sin y.$$

In what directions should a bug move from the point $(0, \pi/6)$ at speed 3 in order to experience a rate of temperature change of $20\ C°/\text{sec}$?

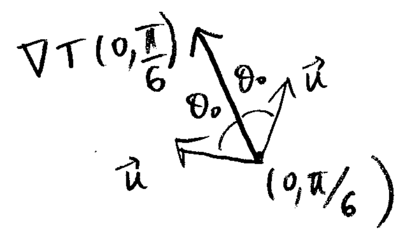
Let $\vec{u}$ be a unit vector and $\theta_0 = \text{angle}\left(\nabla T(0, \frac{\pi}{6}), \vec{u}\right)$.

Then,
$$\begin{pmatrix} \text{rate of change at} \\ (0,\pi/6) \text{ in direction} \\ \vec{u} \text{ at speed } 3 \end{pmatrix} = 20\ C°/\text{sec}$$

$$\Leftrightarrow \quad \theta_0 = \arccos\left(\frac{20}{3\,\|\nabla T(0,\pi/6)\|}\right) = \arccos\left(\frac{2}{3}\right).$$

since $\nabla T(0, \pi/6) = 10(-1/2, \sqrt{3}/2)$.

$\rightsquigarrow$ get 2 distinct unit directions:



3) Find the tangent line to the curve $C: 5x^2y = 6y - 3$ at $(-1, 3)$.

Note that C can be thought of as the level set $f(x,y) = -3$ of the function $f(x,y) = 5x^2y - 6y$.
The tangent line L to C at $(-1,3)$ is therefore:

$$L: \nabla f(-1,3) \cdot (x-(-1), y-3) = 0$$

since $\nabla f = (10xy, 5x^2 - 6)$ $\rightsquigarrow$ $\Leftrightarrow (-30, -1)\cdot(x+1, y-3) = 0 \Leftrightarrow \boxed{30(x+1) + (y-3) = 0}$

3-var. Let f be a function of 3 variable and $(a,b,c) \in D(f)$.

DEF. Let $\vec{u} = (u_1, u_2, u_3)$ be a unit vector, i.e., $\|\vec{u}\| = 1$.

$$D_{\vec{u}} f(a,b,c) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$$

= directional derivative of f at (a,b,c) in the direction $\vec{u}$.

The directional derivative for 3 variables has similar properties to 2 variables. We highlight some of them.

1)
$$k D_{\vec{u}} f(a,b,c) = \begin{pmatrix} \text{rate of change of } f \\ \text{at } (a,b,c) \text{ in the direction} \\ \vec{u} \text{ at speed } k \end{pmatrix}$$

for all $k > 0$.

2) **THM:** Suppose that f is differentiable at (a,b,c). Then,

$$D_{\vec{u}} f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u},$$

AND, if $\nabla f(a,b,c) \neq (0,0,0)$,

$$D_{\vec{u}} f(a,b,c) = \|\nabla f(a,b,c)\| \cos\theta,$$

where $\theta = \text{angle}(\nabla f(a,b,c), \vec{u})$.

3) Geometric interpretation of the gradient.

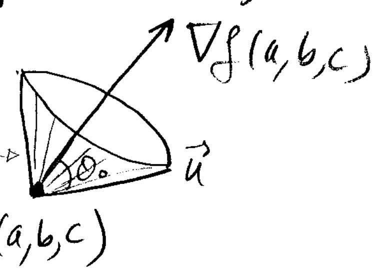
Suppose that f is differentiable at (a,b,c) and that $\nabla f(a,b,c) \neq (0,0,0)$. Then:

* $\begin{pmatrix} \text{maximal rate of} \\ \text{change at speed } k \end{pmatrix} = k\,\|\nabla f(a,b,c)\| \rightsquigarrow$ occurs in the direction $\nabla f(a,b,c)$.

* $\begin{pmatrix} \text{minimal rate of} \\ \text{change at speed } k \end{pmatrix} = -k\,\|\nabla f(a,b,c)\| \rightsquigarrow$ occurs in the direction $-\nabla f(a,b,c)$.

* $\begin{pmatrix} \text{rate of change} \\ \text{at speed } k \end{pmatrix} = C \rightsquigarrow$ occurs in unit directions $\vec{u}$ such that $\theta_0 = \arccos\left(\dfrac{C}{k\,\|\nabla f(a,b,c)\|}\right)$, where $\theta_0 = \text{angle}(\nabla f(a,b,c), \vec{u})$.

$\vec{u}$'s form a cone when $\theta_0 \neq 0, \pi$ $\rightsquigarrow$



NOTE: In this case, if $\theta_0 \neq 0, \pi$ we get an infinite number of unit directions where the rate of change is C: we get a cone of directions.

* $\nabla f(a,b,c)$ is perpendicular to the level set $f(x,y,z) = \alpha$ passing through (a,b,c) so that the equation to the tangent plane to

$$f(x,y,z) = \alpha \quad \text{with} \quad \alpha = f(a,b,c)$$

at (a,b,c) is:

$$\boxed{\nabla f(a,b,c) \cdot (x-a,\, y-b,\, z-c) = 0}$$

Ex. 1) Let $V(x,y,z) = x^2 + 4y^2 + 2z^2$ be the function giving the electrical potential at the point (x,y,z) in space.

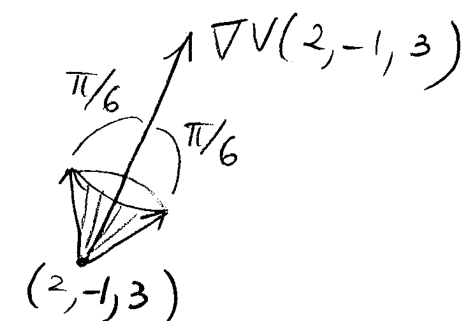
(i) In what directions is the rate of change of the potential equal to $6\sqrt{2}$ at the point $(2,-1,3)$?

Note that since no speed is specified, it is assumed to be 1. So, if $\vec{u}$ is a unit vector and $\theta_0 = \text{angle}(\nabla V(2,-1,3), \vec{u})$, we have

$$\theta_0 = \arccos\left(\frac{6\sqrt{2}}{1 \cdot \|\nabla V(2,-1,3)\|}\right),$$

where $\nabla V(2,-1,3) = (4,-8,4)$ since $\nabla V = (2x, 8y, 4z)$.

So,
$$\theta_0 = \arccos\left(\frac{6\sqrt{2}}{\sqrt{96}}\right) = \arccos\left(\frac{\sqrt{3}}{2}\right) = \pi/6.$$



$\leadsto$ get the cone of (unit) directions at angle of $\pi/6$ from $\nabla V(2,-1,3)$, so an infinite number of unit directions (since we are in $\mathbb{R}^3$).

(ii) What is the maximal rate of change of the potential if one move away from $(2,-1,3)$ at speed 5 units/sec?

$$\begin{pmatrix} \text{max. rate of change} \\ \text{at } (2,-1,3) \text{ at speed } 5 \end{pmatrix} = 5\,\|\nabla V(2,-1,3)\|$$
$$= 5\sqrt{96} = \boxed{20\sqrt{6}}$$

2) Find the tangent plane to the surface

$$S: \; xz^2 - 3yx + 2z = 0$$

at the point $(1,1,1)$.

The surface S is the level set $f(x,y,z) = 0$ of the function $f(x,y,z) = xz^2 - 3yx + 2z$. The tangent plane to S at $(1,1,1)$ is therefore:

$$\nabla f(1,1,1) \cdot (x-1,\, y-1,\, z-1) = 0.$$

Since $\nabla f = (z^2 - 3y,\, -3x,\, 2xz + 2)$, we get:

$$(-2,-3,4)\cdot(x-1,\, y-1,\, z-1) = 0$$
$$\Longleftrightarrow \boxed{-2(x-1) - 3(y-1) + 4(z-1) = 0}.$$

3) Find the tangent plane at $(1, \pi/2, 0)$ to the surface

$$S': \; x\cos y - y^2 z = 2e^z - 3.$$

Since S' is the level set $f(x,y,z) = -3$ of the function $f(x,y,z) = x\cos y - y^2 z - 2e^z$ and $\nabla f = (\cos y,\, -x\sin y - 2yz,\, -y^2 - 2e^z)$, the tangent plane is:

$$\nabla f\left(1, \tfrac{\pi}{2}, 0\right)\cdot\left(x-1,\, y-\tfrac{\pi}{2},\, z-0\right) = 0 \Longleftrightarrow \boxed{-\left(y - \tfrac{\pi}{2}\right) + \left(-\tfrac{\pi^2}{4} - 2\right) z = 0.}$$