Math 237: Suggested problems for the final – Fall 2010.

Graphing.

- Assignment 1, # 1; also describe/sketch/visualise the surface $z = f(x, y)$ in 3-space.
- Problem Set 1: A1 (a), (b).
- To set up the triple integrals, you should be comfortable sketching/visualising basic surfaces such as planes, paraboloids, cylinders, parabolic cylinders, cones, and spheres in 3-space. For example, sketch/visualise the following surfaces.

  (a) $x - y + 2z = 4$
  (b) $x = 2y$
  (c) $3y + 2z = 1$
  (d) $z = 3 - x^2 - y^2$
  (e) $z = \sqrt{x^2 + y^2}$
  (f) $x^2 + y^2 + z^2 = 5$
  (g) $x^2 + y^2 = 4$
  (h) $z = x^2$
  (i) $z = -2\sqrt{x^2 + y^2}$

- Sketch the following polar curves.

  (a) $r = 3\sin\theta$
  (b) $r = 1 + \cos\theta$
  (c) $r = 2 - \sin\theta$
  (d) $r = 1 + 2\cos\theta$

Limits and continuity.

- State the definition of the limit of a two variable function.
- State the definition of continuity of a two variable function.
- Assignment 1, #2.
- Problem Set 1: A2, A3, B2 (i), (ii), (iii), (iv), (v), (vii).
- Problem Set 1: C1.

Partials, linear approximations, and differentiability.

- State the definition of the first and higher order partials, and of the linear approximation of a function f of two or three variables.
- State the definition of differentiability of a function f of two variables.
- Problem Set 2: A3, A4 (iii).

  Also, use the linear approximation found in A3 (b) to approximate $f(3.02, -0.99)$, where f is the function in A3 (i). Similarly, use the linear approximation found in A3 (b) to approximate $f(0.99, 1.01, -1.02)$, where f is the function in A3 (iii).

- For each of the following functions f, determine all the points where f is continuous and all the points where f is differentiable.

  (a) $f(x, y) = x^2 + (3x - 1)e^{y^2}$

  (b) $f(x, y) = y\cos(x^2y - 3x)$

- Assignment 2, #4; Assignment 3, ##2, 3; Assignment 4, #1 (a).

  Also, in questions #2 (a), (b), and #3 on Assignment 3, determine all the points where the function f is continuous and and all the points where f is differentiable.

- Problem Set 2: A1, A2, B3 (i), (ii).

- Is the following statement true: "A function of two variables $f(x, y)$ is differentiable at a point (a, b) if and only if its first order partials $f_x(a, b)$ and $f_y(a, b)$ both exist at (a, b)"? Justify your answer.

  Hint: See question B3 (ii) on Problem Set 2.

The Chain Rule.

- Problem Set 3: A1, A2, A3, A4, A6, A17, A19.

Directional derivatives and tangent planes.

- State the definition of the directional derivative of a function f of two or three variables.

- State the definition of the tangent plane to the graph of a function of two variables f at the point $(a, b, f(a, b))$.

- What is the equation of the tangent plane to the level surface $f(x, y, z) = k$ of a function of three variables f at the point (a, b, c)?

- Problem Set 3: A8, A9, A10, A14, A16, B1.

- Consider the function $f(x, y) = xy^2 - 3e^{x-y}$.

  (a) What are the maximal and minimal rates of change of f at the point (1,1) and in what directions do they occur?

  (b) Is there a direction in which the rate of change of f at the point (1,1) is 7? If your answer is yes, find all possible directions.

  (c) Is there a direction in which the rate of change of f at the point (1,1) is 0? If your answer is yes, find all possible directions.

- The path of a space-craft is given by $(x, y, z) = (e^{2t}\cos t, e^{2t}\sin t, 2t + 1)$ where t denotes time. The temperature at position (x, y, z) is given by a function $u : \mathbb{R}^3 \to \mathbb{R}$, and the temperature gradient at $(1, 0, 1)$ is $\nabla u(1, 0, 1) = (\frac{1}{5}, -\frac{1}{3}, -\frac{1}{4})$.

  (a) Find the velocity of the spacecraft at time t.

  (b) At $t = 0$, find the rate of change of temperature experienced by the spacecraft with respect to time.

  (c) Find the rate of change of temperature per units of distance experienced by the spacecraft if it moves in the direction $(4, -1, 2)$ at time $t = 0$.

Taylor polynomials and Taylor's Theorem.

- State the definition of second degree Taylor polynomial; also state Taylor's Theorem.

- Problem Set 4: A2, B2, B4.
- Let $f(x, y) = \sin(x+2y)$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(0,0)}(x, y)$ is at most $3(x^2 + y^2)$.
- Let $f(x, y) = \ln(3x-2y)$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(1,1)}(x, y)$ is at most $\frac{15}{2}3^2[(x-1)^2 + (y-1)^2]$ if $1 \leq x \leq 3, 1 \leq y \leq \frac{4}{3}$.
- Let $f(x, y) = e^{xy}$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(0,0)}(x, y)$ is at most $\frac{7}{2}(x^2 + y^2)$ if $-1 \leq x \leq 0, 0 \leq y \leq 2$.

Critical points and optimisation problems.

- State the Second Derivative Test.
- State the Extreme Value Theorem for a function of one variable on an interval.
- State the Extreme Value Theorem for a function of two variables on a subset of $\mathbb{R}^2$.
- State the Lagrange Multiplier Method for functions of two variables and for functions of three variables.
- Assignment 7, ##1, 2, 4.
- Problem Set 5: A1 (i), (iii), A3, A4, A5, A8.

Cylindrical and spherical coordinates.

- Change the following equations to cylindrical coordinates:
  (a) $x^2 + y^2 = 4$
  (b) $z = x^2 + y^2$
  (c) $z = -\sqrt{3}\sqrt{x^2 + y^2}$
  (d) $z = -\sqrt{1 - x^2 - y^2}$
  (e) $x^2 + y^2 + z^2 = 2z$
- Change the following equations to spherical coordinates:
  (a) $x^2 + y^2 + z^2 = 4$
  (b) $x^2 + y^2 = 4$
  (c) $z = -\sqrt{3}\sqrt{x^2 + y^2}$
  (d) $x^2 + y^2 + z^2 = 2z$
  (e) $z = 5$

Mappings.

- State the definition of a derivative matrix and of the Jacobian of a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$.
- State the definition of the inverse of a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$.
- State the Inverse Function Theorem.
- Assignment 8, ##2, 3, 4.
- Problem Set 7: A3, A5, A6, B1, B2.

- Find the inverse of the following mappings:

  (a) $F(x, y) = (2x - 3y, x + 2y) = (u, v)$
  (b) $F(x, y) = (y + \ln x, 3y - 2\ln x) = (u, v)$
  (c) $F(x, y) = (xy + x, 2xy - x) = (u, v)$

  Moreover, for each of the above mappings, find the image under F of the rectangle $R = \{1 \leq x \leq 2,\ 2 \leq y \leq 4\}$.

Double and triple integrals.

- Assignment 10, ##2, 3.
- Assignment 11, ##2, 3, 8.
- Problem Set 8: A1, A3, A4, A6, A10, A11, A12.
- Problem Set 9: A1, A5, A6, A8, B1, B4.
- A city occupies the region D of the xy-plane bounded by $y = x^2$ and $y = 1$. The population density in the city (measured as people/unit area) depends on position (x, y), and is given by the function $p(x, y) = y$. How many people live in the city?
- Express the volume of the following regions as triple integrals in rectangular coordinates.

  (a) R is the region bounded by $z = y$, $z = 2 - y^2$, $z = 0$, $x = 0$, and $x = 2$.
  (b) R is the region in the first octant bounded by $z = x^2$, $z = y^2$, and $z = 1$.
  (c) R is the region bounded by $z = x$, $2x + z = 2$, $y = 0$, $y = 3$, and $z = 0$.
  (d) R is the region bounded by $x + z = 1$, $z = 2y$, $y = x$, and $z = 0$.

- Use cylindrical coordinates to solve the following questions.

  (a) Suppose a beehive is shaped like the region R below $z = 4 - x^2 - y^2$ and inside $x^2 + y^2 + z^2 = 4z$, and the number of bees per unit of volume is $f(x, y, x) = 3$ in R. Determine the number of bees in the hive.
  (b) Compute the volume of the region R inside $x^2 + y^2 + z^2 = 4z$, outside $z = x^2 + y^2$, and below $z = 2$.
  (c) Compute the volume of the region R inside $x^2 + y^2 = 9$, outside $z = 3\sqrt{x^2 + y^2}$, below $z = 4$, and above the xy-plane.

- Use spherical coordinates to solve the following problems.

  (a) Consider a solid that has the shape of the three-dimensional region R lying inside $x^2 + y^2 + z^2 = 2$, outside $x^2 + y^2 + z^2 = 1$, below $z = \sqrt{3x^2 + 3y^2}$, above $z = 0$, and with $x \geq 0$. Suppose that the solid has density $f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ per unit of volume at every point in R. Find the total mass of the solid.
  (b) Compute the volume of the "ice cream cone" bounded by $x^2 + y^2 + z^2 = 2z$ and $z = \sqrt{x^2 + y^2}$.
  (c) Evaluate $\int\int\int_R z dV$, where$R$ is the region inside $x^2 + y^2 + z^2 = 1$ and outside $x^2 + y^2 + z^2 = 2z$.