TO DRAW the graph of a surface $z = f(x,y)$:

1) Find $D(f)$, $R(f)$.

2) Draw level curves: $f(x,y) = k$, $k \in R(f)$.

NOTE: If $R(f)$ is a bounded interval, always treat the level curves of the bounds separately.

E.g: If $R(f) = [2, +\infty)$, consider
* $k = 2$, and
* $k > 2$.

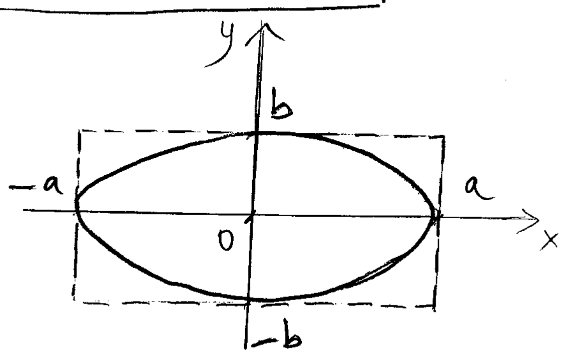
3) Take some vertical cross-sections to get the "vertical shape" of the graph.

IMPORTANT:
* Don't take too many cross-sections.
* Use vertical planes that intersect as many level curves as possible, and follow the symmetry of the level curves.
* KEEP IT SIMPLE!!

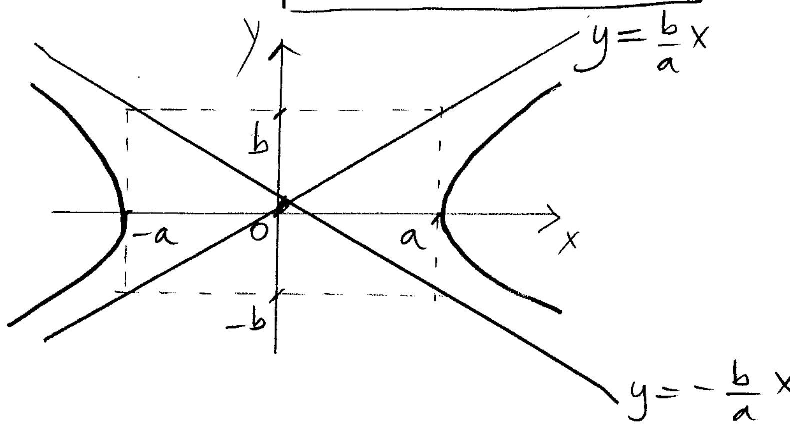
E.g. start with simple planes such as $x=0$, $y=0$, and $y = \pm x$.

BEFORE looking at examples, here are some families of curves that often appear as level curves:

1) Ellipses: $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$ $\rightsquigarrow$ centered at $(0,0)$.



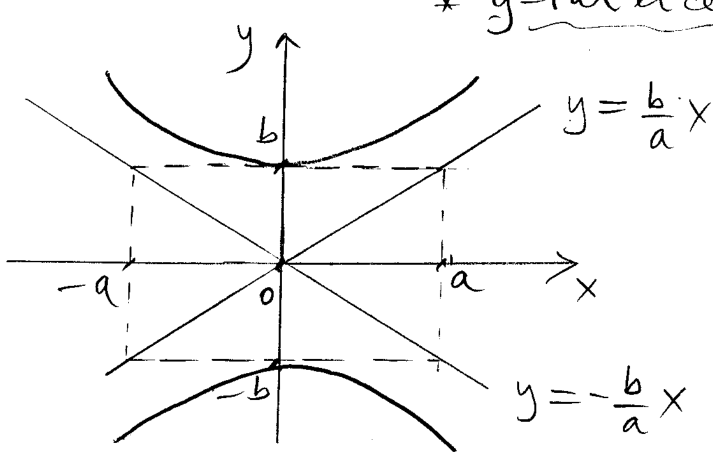
2) Hyperbolas: $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ $\rightsquigarrow$ * asymptotes $y = \pm \dfrac{b}{a} x$



* x-intercepts: $x = \pm a$
(No y-int.)

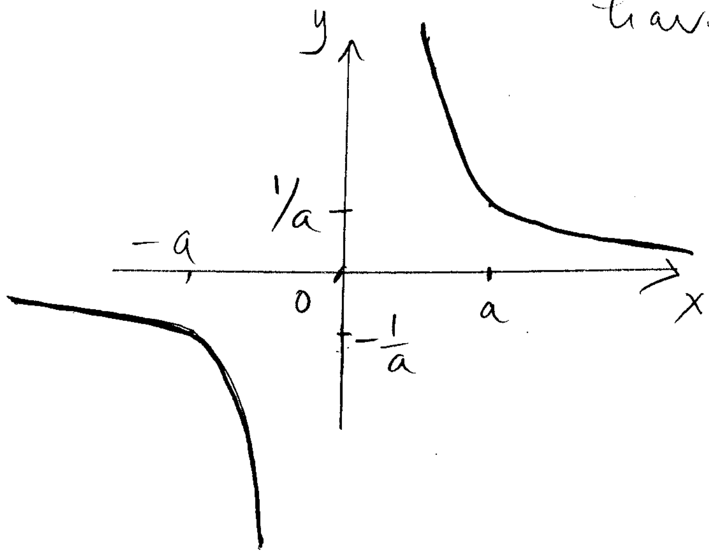
$\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1}$ $\rightsquigarrow$ * asymptotes $y = \pm \dfrac{b}{a} x$

* y-intercepts: $y = \pm b$ (No x-int.)



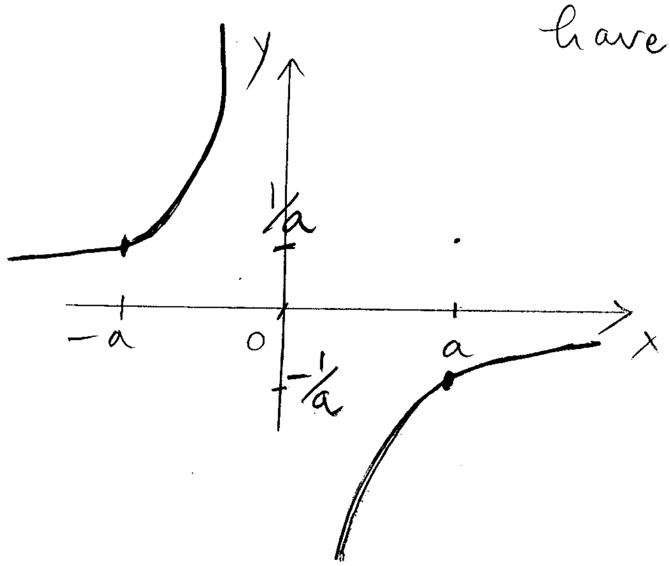
$\boxed{xy = a, \quad a > 0}$ ⇝ * asymptotes $x = 0$ and $y = 0$

* branches lie in the 1^{st} and 3^{rd} quadrants since $xy = a > 0$, so that x & y have the **same** sign.



$\boxed{xy = -a, \quad a > 0}$ ⇝ * asymptotes $x = 0$ and $y = 0$

* branches lie in the 2^{nd} and 4^{th} quadrants since $xy = -a < 0$, so that x & y have **opposite** signs.

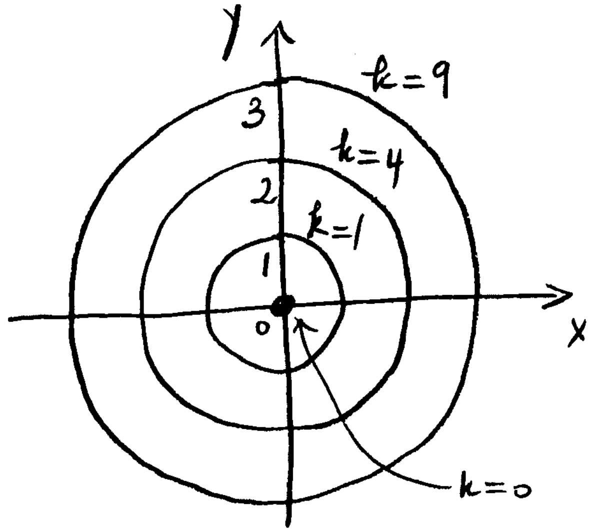


E.g. 1) $f(x,y) = x^2 + y^2 \rightsquigarrow D(f) = \mathbb{R}^2,\ R(f) = [0,\infty)$.

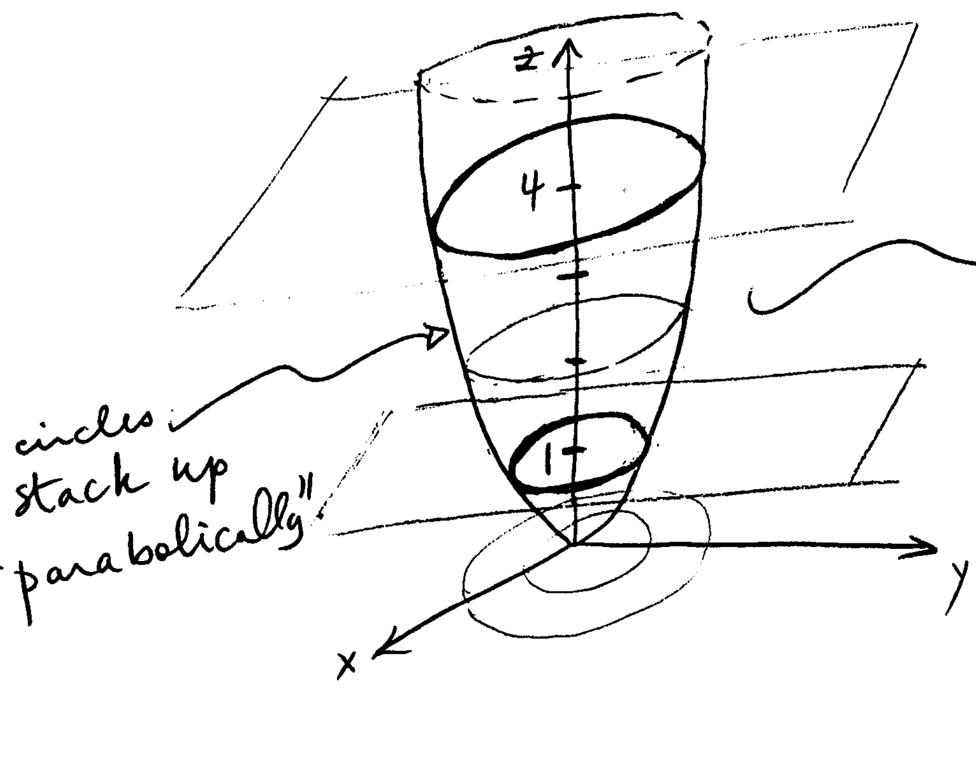
Level sets: $f(x,y) = k,\ k \in R(f)$

$$\iff x^2 + y^2 = k,\ k \geqslant 0.$$

* $\underline{k=0}$: $x^2+y^2=0 \iff (x,y)=(0,0) \rightsquigarrow$ point.
* $\underline{k=1}$: $x^2+y^2=1$: circle centered at $(0,0)$ of radius 1.
* In general, $\underline{k\geq 0}$: $x^2+y^2=k$: circle centered at $(0,0)$ of radius $\sqrt{k}$.



$\Longrightarrow$ get concentric circles centered at $(0,0)$.

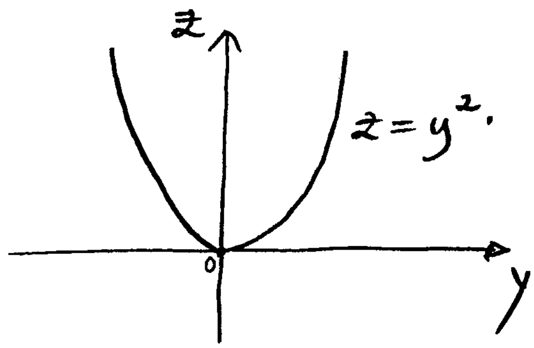


$z = x^2+y^2$: quadric surface called (circular) $\underline{\text{paraboloid}}$.

Why is the surface $z = x^2 + y^2$ called a paraboloid? Take vertical cross-sections:

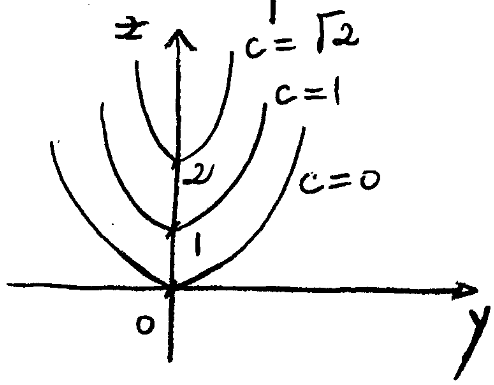
e.g.: * The intersection of $z = x^2 + y^2$ with the yz-plane is obtained by setting $x = 0$:

$$z = y^2.$$

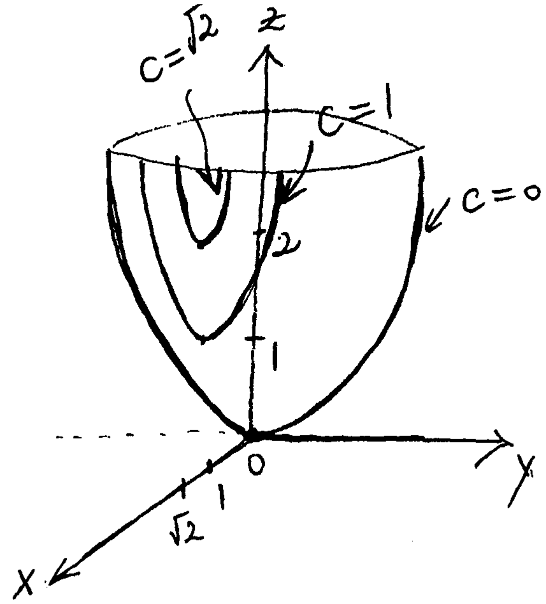


I.e., the trace of the surface in the yz-plane is the parabola $z = y^2$.

* More generally, if we intersect $z = x^2 + y^2$ with the vertical plane $x = c$, we get the vertical cross-sections: $z = c^2 + y^2$, which are parabolas.

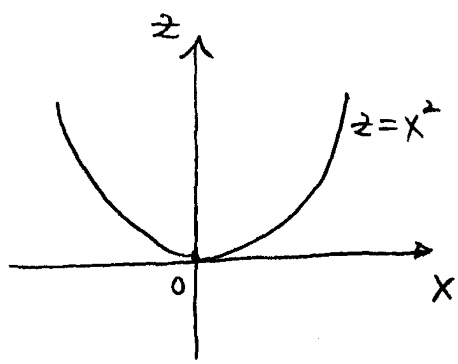


By intersecting the graph $z = x^2 + y^2$ with the planes $x = 0$, $x = 1$, $x = \sqrt{2}$, we see that we indeed get parabolas that are shifting up a c increases:



* One can also take vertical cross-sections of the graph $z = x^2+y^2$ by intersecting it with vertical planes of the form $y = c$.

 ex. $y=0$: $z=x^2$, which is the intersection of the graph with the xy-plane.



NOTE: The traces of $z=x^2+y^2$ show that the radii of the circles making up the graph increase parabolically, hence the name paraboloid.

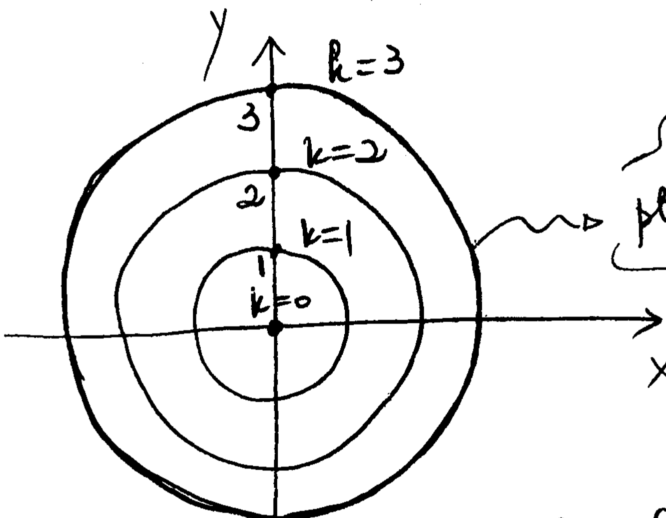
RMK: To get the basic vertical shape of the graph of a simple function, it's often enough to just intersect with the yz- and xz-planes.

2) $f(x,y)=\sqrt{x^2+y^2}$, $D(f)=\mathbb{R}^2$, $R(f)=[0,+\infty)$.

Level sets: $\sqrt{x^2+y^2}=k \iff x^2+y^2=k^2$.

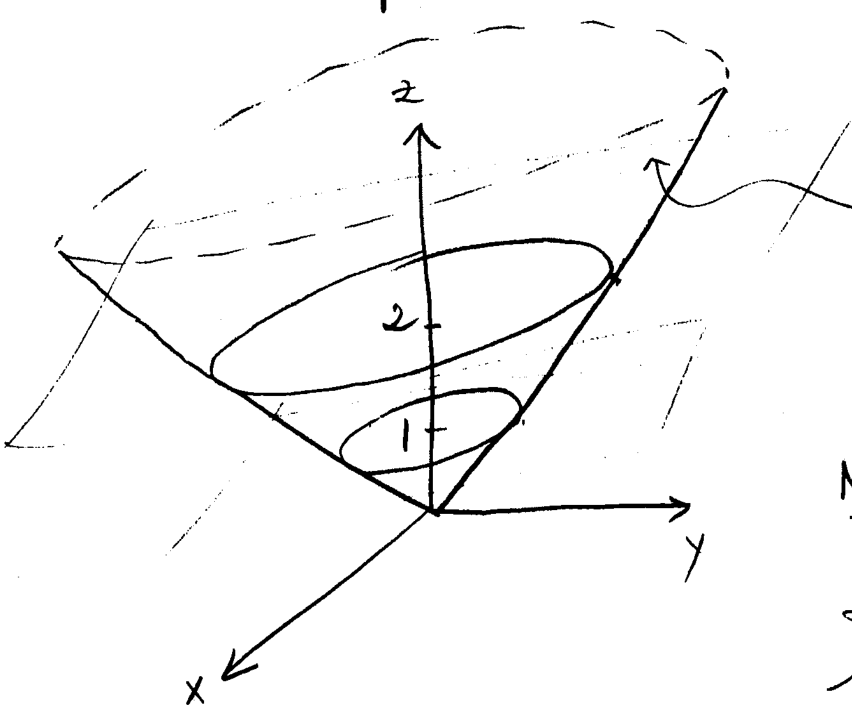
$k=0$: $(0,0)$.

$k>0$: $x^2+y^2=k^2 \rightsquigarrow$ circle centered at $(0,0)$ of radius k.



$\rightsquigarrow$ shape of set of pts on the graph at height $k=3$

Like a topographical map.

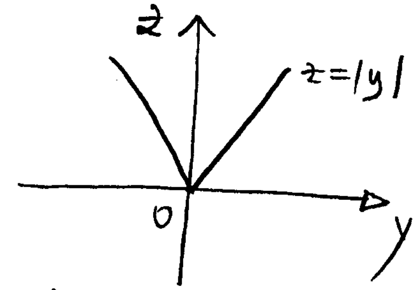


graph $z=\sqrt{x^2+y^2}$: (circular) cone.

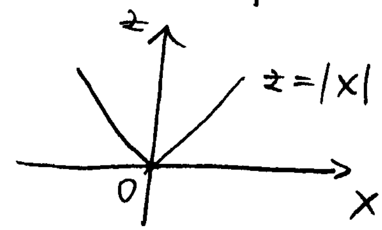
NOTE: In this case, the radius of the circles increases linearly with $z=k$ $\rightsquigarrow$ the circles stack up linearly.

To verify the vertical shape of the graph, let's take two vertical cross-sections:

e.g.: $x=0$: $z = \sqrt{y^2} = |y|$



$y=0$: $z = \sqrt{x^2} = |x|$

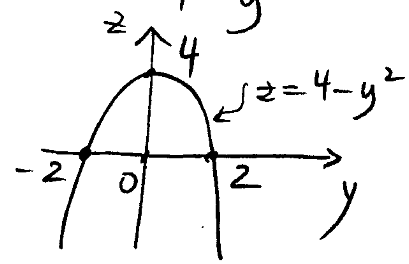


3) $f(x,y) = 4 - x^2 - y^2$, $D(f) = \mathbb{R}^2$, $R(f) = (-\infty, 4]$.

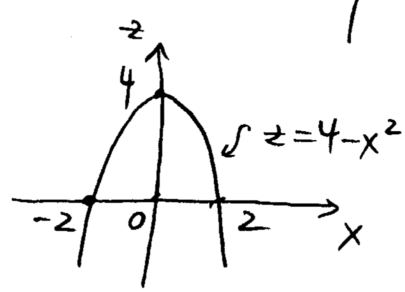
Level sets: $x^2 + y^2 = 4 - k$, $k \le 4$.

* $k=4$: $x^2+y^2 = 0 \rightsquigarrow (0,0)$ point
* $k=0$: $x^2+y^2 = 4 \rightsquigarrow$ (intersection with (x,y)-plane) = (circle of radius 2, centered at 0)
* $k<0$: $x^2+y^2 = 4-k \rightsquigarrow$ circle centered at $(0,0)$ of radius $\sqrt{4-k}$.

Vertical cross-sections: • $x=0$: $z = 4 - y^2$



• $y=0$: $z = 4 - x^2$



3) $f(x,y) = xy \rightsquigarrow D(f) = \mathbb{R}^2$

$R(f) = \mathbb{R}$ (since, $\forall t \in \mathbb{R}$, $t = f(t,1)$.)

Level curves: $xy = k$, $k \in \mathbb{R}$.

* $k=0$: $xy = 0 \iff x = 0$ and $y = 0$.

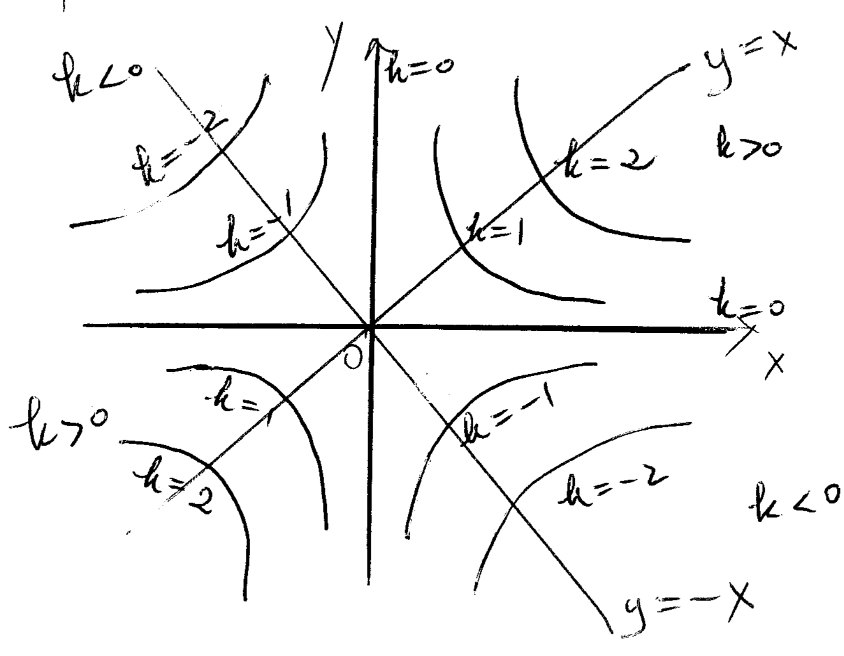
* $k=1$: $xy = 1 \iff$ hyperbola with asymptotes $x=0$ and $y=0$ with branches in the 1st and 3rd quadrants (since $xy = 1 > 0 \rightsquigarrow$ x and y have the same sign).

* $k=-1$: $xy = -1 \iff$ hyperbola with asymptotes $x=0$ and $y=0$ with branches in the 2nd and 4th quadrants (since $xy = -1 < 0 \rightsquigarrow$ x and y have opposite signs).

In general, for $xy = k$, with $k \neq 0$, get a hyperbola with asymptotes $x=0$ and $y=0$.

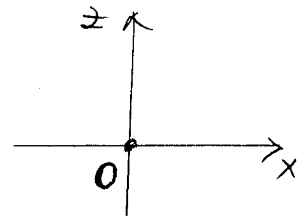
AND if $k > 0 \rightsquigarrow$ branches are in 1st & 3rd quadrants

if $k < 0 \rightsquigarrow$ ——————— 2nd & 4th ———.



Vertical cross-sections:

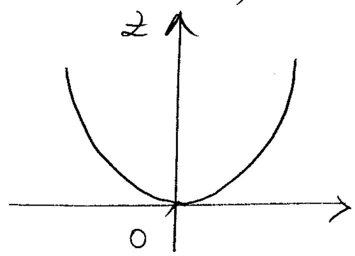
* Let's first try intersecting the graph with the plane $y=0$: $z = f(x,0) = 0$ ⇝ get the single point $(0,0)$ in the xy-plane.



This is not so surprising since $y=0$ only intersects one level curve, namely $x=0$.

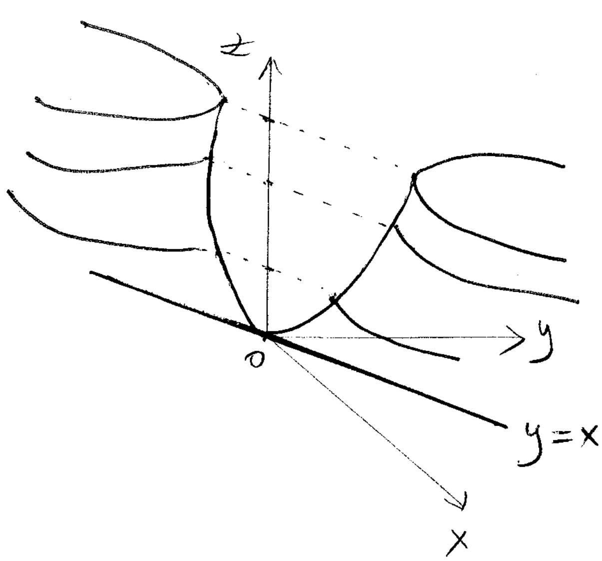
⇒ Try using vertical planes that intersect as many level curves as possible. From the sketch of the level curves, the planes $y=x$ and $y=-x$ are good choices.

* $y=x$: $z = f(x,x) = x^2$ ⇝ upward parabola.

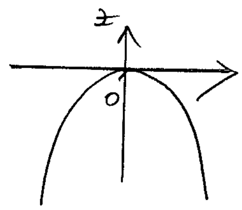


NOTE: $y=x$ intersects the hyperbolas
$$xy = k$$
with $k>0$.

⇒ The family of hyperbolas $xy=k$, $k>0$, stacks up vertically along an upward parabola.

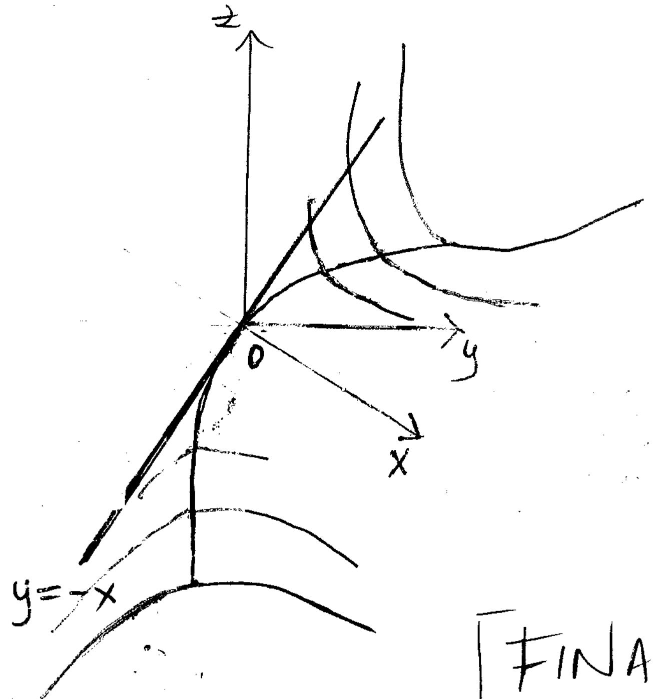


* $y=-x$: $z = f(x,-x) = -x^2 \rightsquigarrow$ downward parabola.



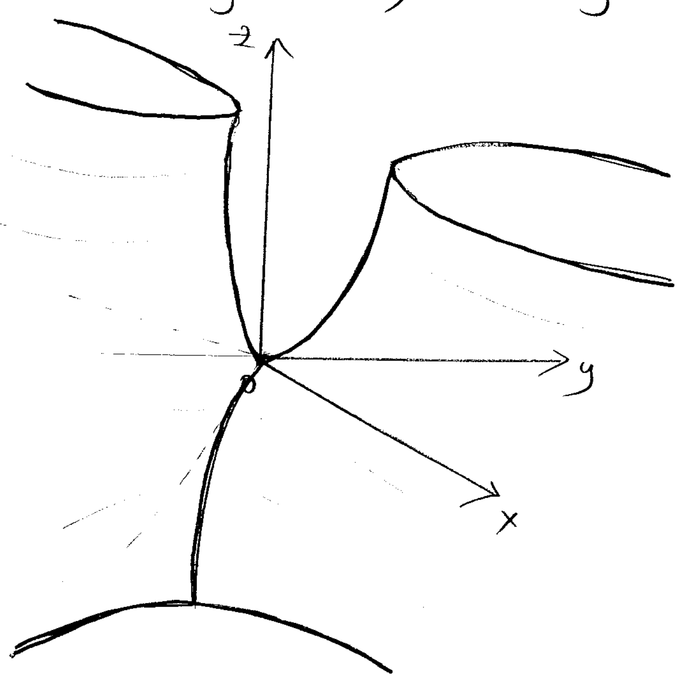
NOTE: $y=-x$ intersects the hyperbolas $xy=k$ with $k<0$

$\Rightarrow$ The family of hyperbolas $xy=k$, $k<0$, stacks up vertically along a downward parabola.



FINALLY, note that the parabolas $z=x^2$ and $z=-x^2$ lie in perpendicular planes: $y=x$ & $y=-x$.

Putting it all together, we get:



4) $f(x,y) = 4 - x^2 - 9y^2 \rightsquigarrow D(f) = \mathbb{R}^2$

This tells us that the graph lies below the plane $z = 4$

$$\begin{cases} R(f) = (-\infty, 4] \\ (\text{since } 4 - (x^2 + 9y^2) \leq 4 \text{ because } \\ x^2 + 9y^2 \geqslant 0, \text{ and } \forall t \leq 4, \\ t = f(\sqrt{4-t}, 0).) \end{cases}$$

Level curves: $4 - x^2 - 9y^2, \quad k \leq 4$

$$\Longleftrightarrow \boxed{x^2 + 9y^2 = 4 - k, \quad k \leq 4}$$

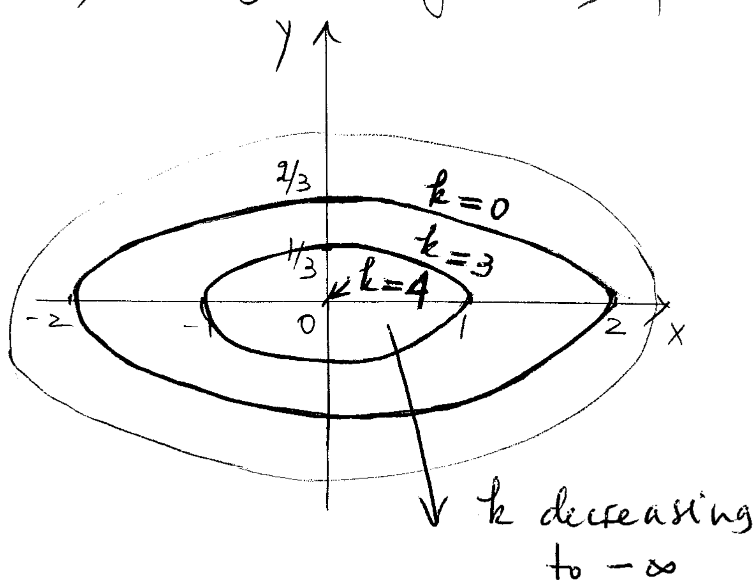
* $k = 4$: $x^2 + 9y^2 = 0 \Longleftrightarrow (x,y) = (0,0)$ point
* $k = 3$: $x^2 + 9y^2 = 1 \Longleftrightarrow x^2 + \dfrac{y^2}{(1/9)} = 1$

  $\rightsquigarrow$ ellipse centered at $(0,0)$ with $a = 1$ and $b = 1/3$
* $k = 0$: $x^2 + 9y^2 = 4 \Longleftrightarrow \dfrac{x^2}{4} + \dfrac{y^2}{(4/9)} = 1$

  $\rightsquigarrow$ ellipse centered at $(0,0)$ with $a = 2$ and $b = 2/3$.

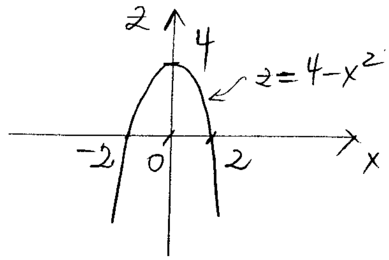
etc...

In general, we get a family of ellipses:

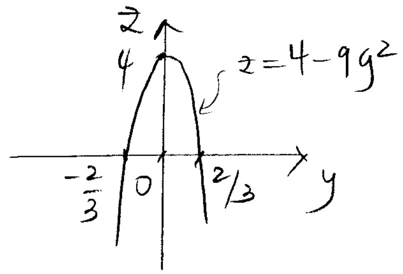


Vertical cross-sections: by the symmetry of the level curves, let's pick $y=0$ and $x=0$.

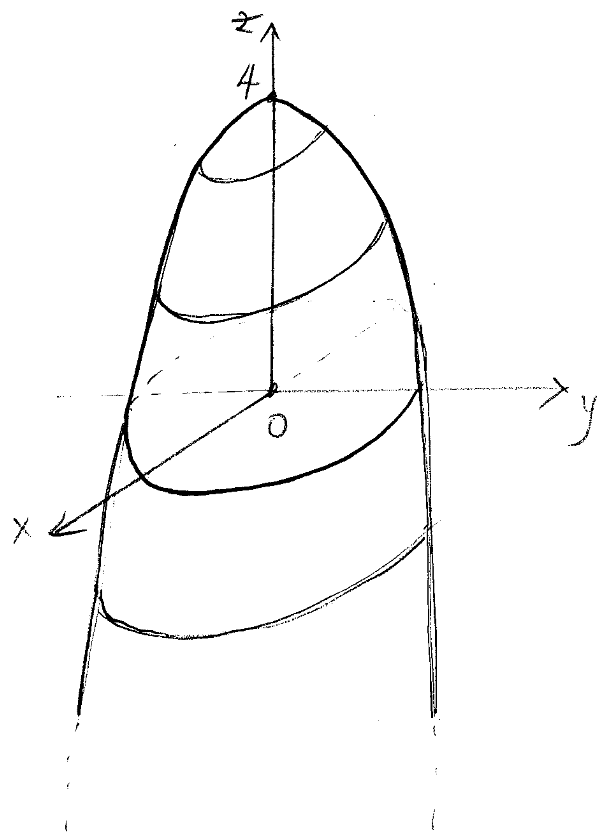
* $y=0$: $z=f(x,0)=4-x^2$ ⇝ downward parabola with x-intercepts $x=\pm 2$



* $x=0$: $z=f(0,y)=4-9y^2$ ⇝ downward parabola with y-intercepts $y=\pm 2/3$



Putting it all together, we obtain an elliptic (downward) paraboloid!



5) $f(x,y) = \ln(x^2+y^2-4)$.

We have seen that $D(f) = \{x^2+y^2>4\}$ and

$$R(f) = \mathbb{R}.$$

Level curves: $\ln(x^2+y^2-4) = k,\ k\in\mathbb{R}$

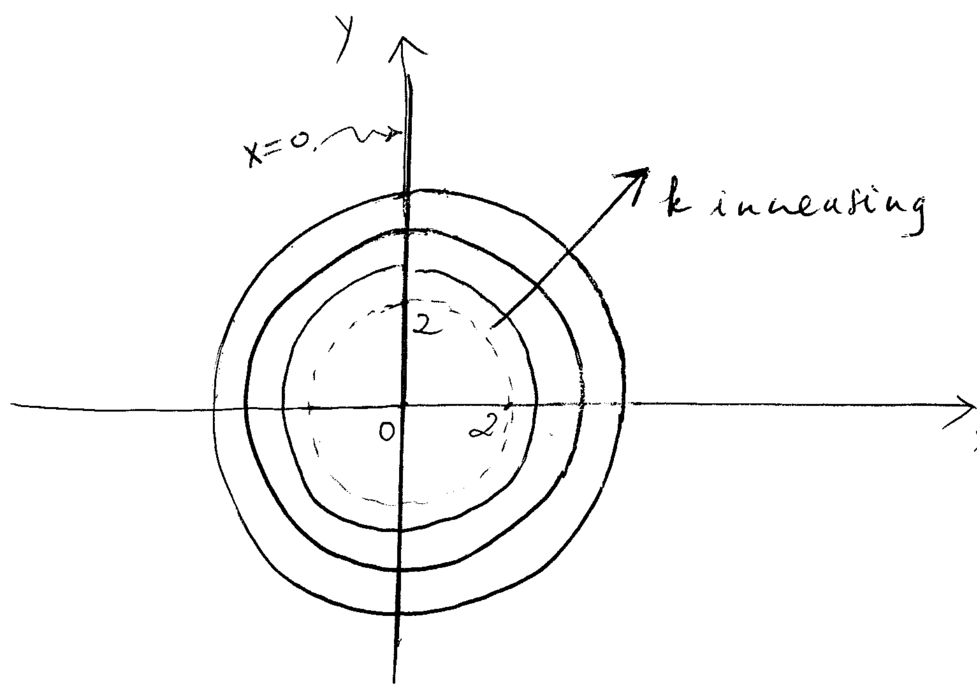
$$\Leftrightarrow \boxed{x^2+y^2 = 4+e^k,\ k\in\mathbb{R}}$$

$\rightsquigarrow$ circles centered at $(0,0)$ of radius $r=\sqrt{4+e^k}$.

NOTE: • Since $e^k>0$ for all $k\in\mathbb{R}$, the radii of these circles are >2.

• Also, $\lim_{k\to-\infty} r = 2$, $\lim_{k\to+\infty} r = +\infty$, and

$$r=\sqrt{5} \text{ at } k=0.$$



Vertical cross-sections: because of the symmetry of the level curves, it is enough to check $x=0$: get

$$\boxed{z = \ln(y^2-4)}.$$

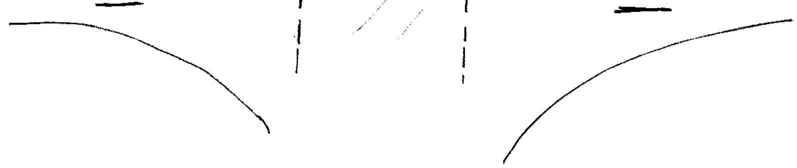
Let $g(y) = \ln(y^2-4)$. Then, $g(y)$ is only defined for $y^2-4>0 \iff y<-2$ and $y>2$. Also,

$$g'(y) = \frac{2y}{(y^2-4)}$$

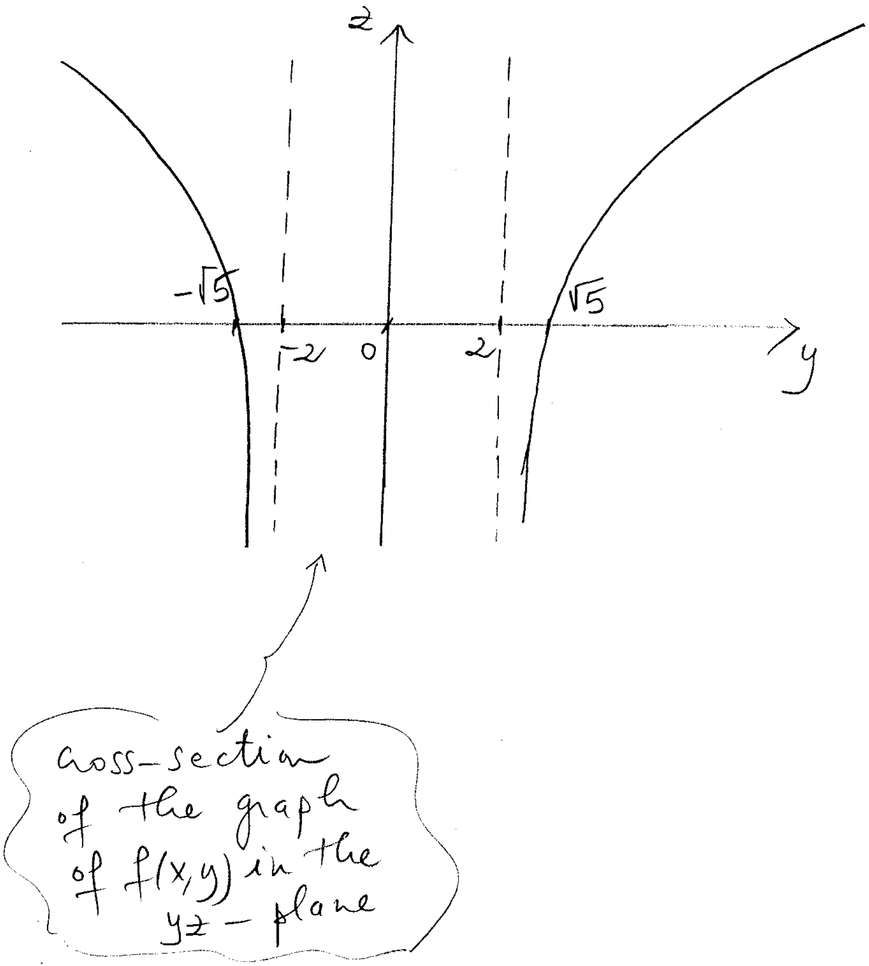
and

$$g''(y) = -\frac{2(y^2+4)}{(y^2-4)^2}.$$

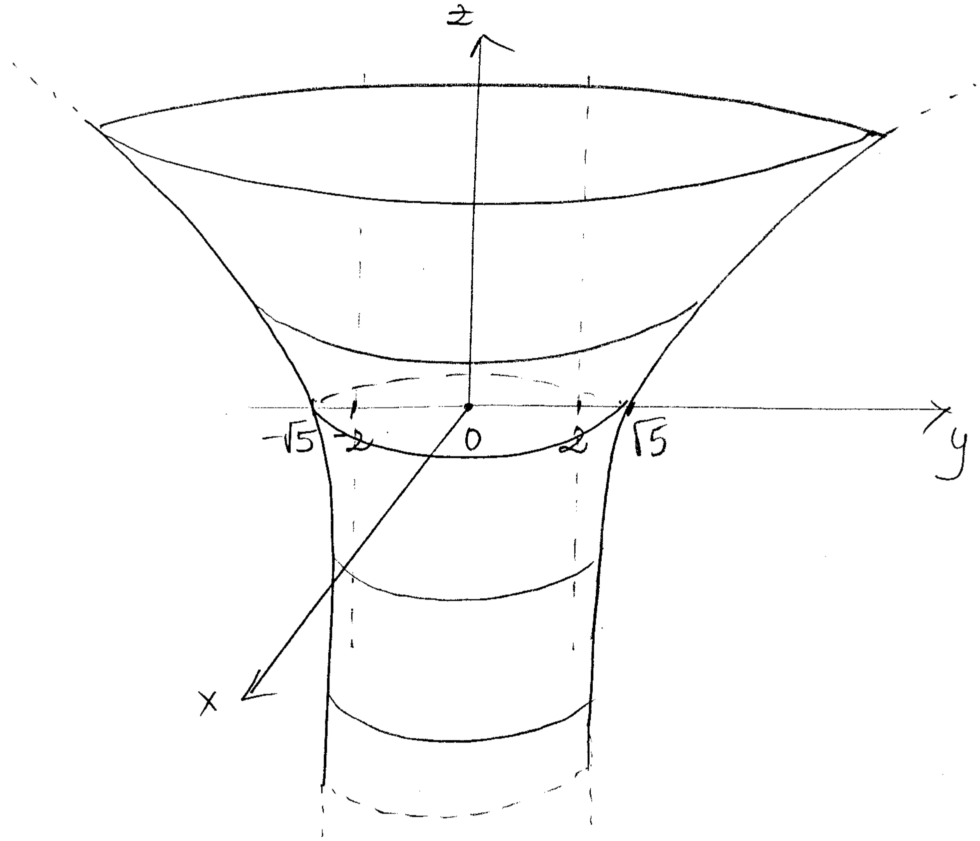
y	$-\infty$	-2	2	
$g'(y)$	$-$			$+$
$g''(y)$	$-$			$-$



The graph $z = y^2-4$ is then



Putting it all together, we find:



NOTE: The intersection of the surface $z = f(x,y)$ with the xy-plane is the circle

$$\boxed{x^2 + y^2 = 5}.$$

6) $f(x,y) = \sqrt{|x^2-y^2|}$. $\leadsto$ $D(f) = \mathbb{R}^2$

$R(f) = [0, +\infty)$

(since, $\forall t \geq 0$, $t = f(t,0)$.)

Level sets: $\sqrt{|x^2-y^2|} = k$, $k \geq 0$

$\iff |x^2-y^2| = k^2$, $k \geq 0$

$\iff \boxed{x^2-y^2 = \pm k^2, \ k \geq 0}$.

* $k=0$: $x^2-y^2=0 \iff y=x$ and $y=-x$

* $k=1$: $x^2-y^2=1$ and $x^2-y^2=-1$.

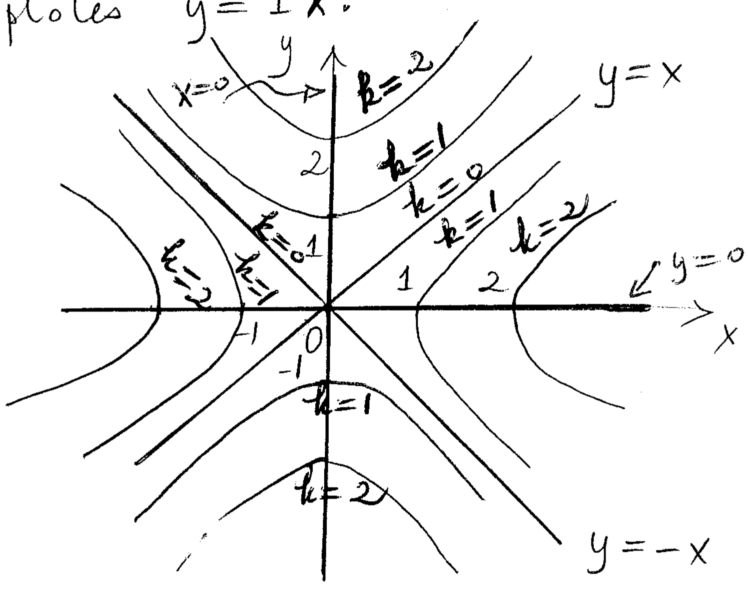
$\leadsto$ both are hyperbolas with asymptotes $y=\pm x$.

BUT, $x^2-y^2=1$ has x-intercepts: $x=\pm 1$ and $x^2-y^2=-1$ has y-intercepts: $y=\pm 1$.

* $k=2$: $x^2-y^2=\pm 2$ $\leadsto$ hyperbolas with asymptotes $y=\pm x$ and x-int. / y-int. at $x=\pm\sqrt{2}$ / $y=\pm\sqrt{2}$.

etc...

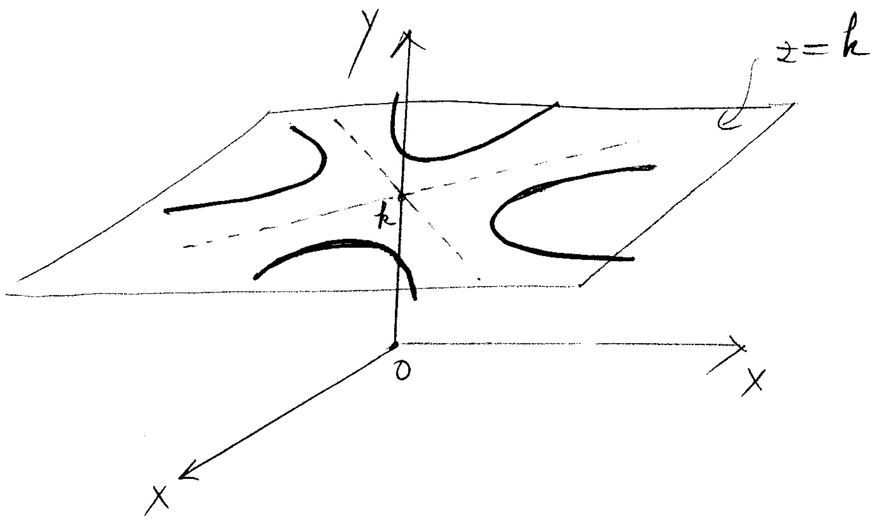
In general, we get a family of hyperbolas, each with asymptotes $y=\pm X$.



NOTE: Although we again have a family of hyperbolas, like in example 3), there are two important differences:

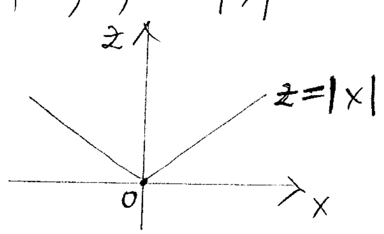
1) In this example, $k \geq 0$, which means that the graph lies above the xy-plane.

2) For each $k \geq 0$, there correspond 2 hyperbolas, with 2 branches each, meaning that there is a hyperbola branch in each quadrant for every value of k.

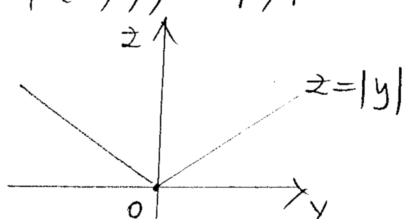


Vertical cross-sections: we want to pick vertical planes that intersect as many level curves as possible. A good choice is: $y=0$ and $x=0$.

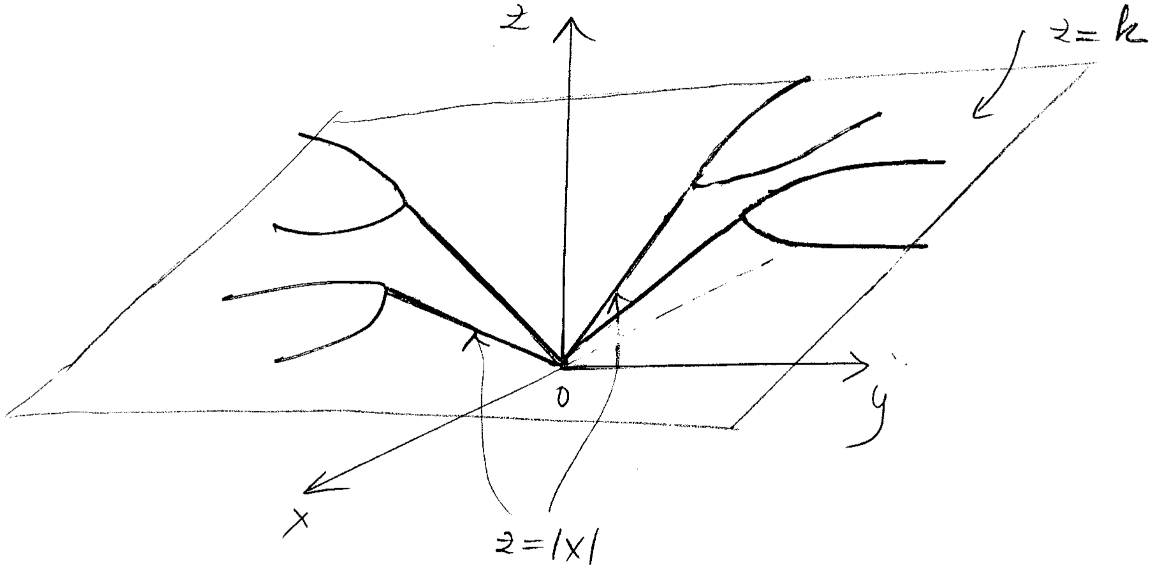
* $y=0$: $z = f(x,0) = |x|$



* $x=0$: $z = f(0,y) = |y|$



<u>NOTE</u>: The vertical cross-sections are on perpendicular planes, and each intersects one of the hyperbolas at a given height $z = k$.



Putting it all together, we get a graph of the following shape: