Recall that

f is diff. at (a,b) if:

(i) $f_x(a,b)$, $f_y(a,b)$ both exist;

(ii) $$\lim_{(x,y)\to(a,b)} \frac{R_{1,(a,b)}(x,y)}{\|(x,y)-(a,b)\|} = 0,$$

where $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

and
$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

THM: f_x and f_y are cont. at (a,b)
$\Rightarrow f$ is diff. at (a,b).

Before proving the theorem, let's look at an example.

ex. $f(x,y) = \sin(x^2 y)$. Let's show that f is diff. at $(0,0)$!

* $f_x = 2xy\cos(x^2y) \Rightarrow f_x(0,0) = 0$
  $f_y = x^2\cos(x^2y) \Rightarrow f_y(0,0) = 0$.
  $f(0,0) = 0 \Rightarrow L_{(0,0)}(x,y) = 0$ and
  $R_{1,(0,0)}(x,y) = \sin(x^2y)$.

* $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{\sqrt{x^2+y^2}} = 0.$$

Here, since $\sin(x^2y)$ is not a polynomial, it's difficult to find an upper bound by inspection.
HOWEVER, recall the MVT:

MVT: If $g(t)$ is diff. on $[a,b]$, then
$$g(b) - g(a) = g'(c)(b-a)$$
for some $c \in (a,b)$.

If we set $g(t) = \sin t$ and $[a,b] = [0,t]$, then
$$\sin t - \sin 0 = \sin'(c)(t-0)$$
for some $c \in (0,t)$. So,

$$\sin t = \cos(c)\, t \Rightarrow |\sin t| = \underbrace{|\cos(c)|}_{\leq 1} \cdot |t|$$

$$\Rightarrow \boxed{|\sin t| \leq |t|, \ \forall t.}$$

$$\Rightarrow \left| \frac{\sin(x^2 y)}{\sqrt{x^2+y^2}} - 0 \right| \leq \frac{x^2 |y|}{\sqrt{x^2+y^2}} \leq x^2 \to 0$$

$|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$

as $(x,y) \to (0,0)$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{\sqrt{x^2+y^2}} = 0$$ and f is diff. at $(0,0)$.

<u>Pf of the theorem:</u>

By MVT,

$$f(x,y) - f(a,y) = f_x(\bar{x}, y)(x-a), \quad \bar{x} \text{ between } a \,\&\, x$$

y fixed

and

$$f(a,y) - f(a,b) = f_y(a, \bar{y})(y-b), \quad \bar{y} \text{ between } b \,\&\, y.$$

$$\Rightarrow f(x,y) - f(a,b) = [f_x(\bar{x}, y)(x-a) + f_y(a,\bar{y})(y-b)]$$

and putting into f^ula of $R_{1,(a,b)}(x,y)$ we get

$$R_1 = R_{1,(a,b)}(x,y) = [f_x(\bar{x},y) - f_x(a,b)](x-a) + [f_y(a,\bar{y}) - f_y(a,b)](y-b).$$

So,

$$\frac{|R_1|}{\|(x,y)-(a,b)\|} \le \frac{|f_x(\bar{x},y)-f_x(a,b)|\,|x-a| + |f_y(a,\bar{y})-f_y(a,b)|\,|y-b|}{\|(x,y)-(a,b)\|}$$

$$\le \underbrace{|f_x(\bar{x},y)-f(a,b)|}_{\to 0} + \underbrace{|f_y(a,\bar{y})-f_y(a,b)|}_{\to 0}$$

since $|x-a| \le \|(x,y)-(a,b)\|$ and $|y-b| \le \|(x,y)-(a,b)\|$

0 since f_x cont. at (a,b) & as $(x,y)\to(a,b)$, $(\bar{x},y)\to(a,b)$

0 since f_y cont. at (a,b) & $(a,\bar{y})\to(a,b)$ as $(x,y)\to(a,b)$.

$$\Rightarrow \lim_{(x,y)\to(a,b)} \frac{R_1}{\|(x,y)-(a,b)\|} = 0.$$

$\Rightarrow$ f is diff. at (a,b). $\boxed{\Sigma}$

ex: 1) polynomials a diff. on $\mathbb{R}^2$.

2) $f(x,y) = \sin(x^2y)$ is diff. on $\mathbb{R}^2$ since

$$\left.\begin{aligned} f_x &= 2xy\cos(x^2y) \\ f_y &= x^2\cos(x^2y) \end{aligned}\right\} \text{ cont. } \forall (x,y) \in \mathbb{R}^2.$$

3) We have seen that $f(x,y) = \sqrt{x^2+y^2}$ is NOT diff. at $(0,0)$. BUT, $\forall (x,y) \neq (0,0)$:

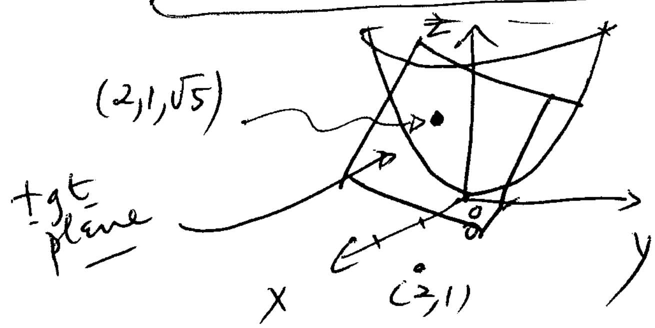
$$\left.\begin{aligned} f_x &= \frac{x}{\sqrt{x^2+y^2}} \\ f_y &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}\right\} \text{ cont. at all } (x,y) \neq (0,0)$$

$\Longrightarrow f(x,y) = \sqrt{x^2+y^2}$ is diff. $\forall (x,y) \neq (0,0)$

$\leadsto \underbrace{z = \sqrt{x^2+y^2}}_{\text{cone}}$ has a tgt plane for all $(x,y) \neq (0,0)$

E.g. At $(a,b) = (2,1)$, $f(2,1) = \sqrt{5}$, $f_x(2,1) = \frac{2}{\sqrt{5}}$, $f_y(2,1) = \frac{1}{\sqrt{5}}$
and the tgt plane is:

$$\boxed{z = L_{(2,1)}(x,y) = \sqrt{5} + \frac{2}{\sqrt{5}}(x-2) + \frac{1}{\sqrt{5}}(y-1)}$$



RECAP: how to test differentiability of f at (a,b).

1) Is f cont. @ (a,b)? NO $\Rightarrow$ f is <u>not</u> diff. at (a,b).
YES, keep testing.

2) Do $f_x(a,b)$ & $f_y(a,b)$ exist?
NO $\Rightarrow$ f is <u>not</u> diff. at (a,b)
YES $\Rightarrow$ keep testing.

3) Are f_x, f_y cont. at (a,b)?
YES $\Rightarrow$ f <u>is</u> diff. at (a,b).

NO <u>or</u> difficult to show that f_x & f_y are cont.

$\Rightarrow$ use definition.

4) Use definition: determine whether or not

$$\lim_{(x,y)\to(a,b)} \frac{R_{1,(a,b)}(x,y)}{\|(x,y)-(a,b)\|} = 0.$$

<u>Ex.</u> 1) $f(x,y) = ye^{x^2}$, $D(f) = \mathbb{R}^2$.
$\to$ f cont. $\forall (x,y)$
$\to$ $f_x = 2xye^{x^2}$, $f_y = e^{x^2}$ cont. $\forall (x,y)$
$\Rightarrow$ f diff. $\forall (x,y)$.

2) $f(x,y) = x^{1/3} y^{1/3}$, $D(f) = \mathbb{R}^2$.

$\rightarrow$ f cont. $\forall (x,y)$.

$\rightarrow$ $f_x = \frac{1}{3} x^{-2/3} y^{1/3}$ for $x \neq 0$

and

$f_y = \frac{1}{3} x^{1/3} y^{-2/3}$ for $y \neq 0$.

$\Longrightarrow$ f_x & f_y are cont. $\forall (x,y)$ such that $x, y \neq 0$.

$\rightsquigarrow$ $\boxed{f \text{ diff. } \forall (x,y) \text{ with } x,y \neq 0.}$

$\rightarrow$ $\underbrace{(x,y) = (0,b)}$, $b \in \mathbb{R}$:

$$f_x(0,b) = \lim_{h \to 0} \frac{f(h,b) - f(0,b)}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3} b^{1/3} - 0}{h} = \begin{cases} 0 & \text{if } b = 0 \\ \text{DNE} & \text{if } b \neq 0. \end{cases}$$

$\Longrightarrow$ f is $\underline{\underline{\text{not}}}$ diff. for $(x,y) = (0,b)$, $b \neq 0$.

At $(0,0)$, we have:

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0 \cdot h^{1/3} - 0}{h} = 0.$$

$\Longrightarrow$ $f(0,0) = f_x(0,0) = f_y(0,0) = 0$ and

$L_{(0,0)}(x,y) = 0$.

$\rightsquigarrow$ $R_{1,(0,0)}(x,y) = x^{1/3} y^{1/3}$ and

$$\lim_{(x,y) \to (0,0)} \frac{R_{1,(0,0)}(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y) \to (0,0)} \frac{x^{1/3} y^{1/3}}{\sqrt{x^2+y^2}} \quad \text{DNE}$$

$\rightarrow$ order $2/3 < 1$

$\rightarrow$ order $\frac{2}{2} = 1$

since along path $y=x$:

$$\lim_{x\to 0} \frac{x^{1/3}x^{1/3}}{\sqrt{x^2+x^2}} = \lim_{x\to 0} \frac{x^{2/3}}{\sqrt{2}\,|x|} = \lim_{x\to 0} \frac{1}{\sqrt{2}\,|x|^{1/3}} \text{ DNE.}$$

$\Longrightarrow f$ is <u>not</u> diff. at $(0,0)$.

$\rightarrow$ <u>$(x,y) = (a,0)$, $a \in \mathbb{R}\setminus 0$</u>: (only need to check for $a \neq 0$ since $(0,0)$ already done).

$$f_y(a,0) = \lim_{h\to 0} \frac{f(a,h)-f(a,0)}{h}$$

$$= \lim_{h\to 0} \frac{a^{1/3}\cdot h^{1/3} - 0}{h} = \lim_{h\to 0} \frac{a^{1/3}}{h^{2/3}} \quad \text{DNE since } a \neq 0.$$

$\Longrightarrow f$ is <u>not</u> diff. at $(0,0)$.

So, f is <u>only</u> diff. $\forall (x,y)$ with $x,y \neq 0$.

3) $f(x,y) = |x-1| \cdot y^2$, $D(f) = \mathbb{R}^2$. (8)

$\to$ f cont. $\forall (x,y)$.

$$\to f(x,y) = \left\{ \begin{array}{ll} (x-1)\,y^2 & \text{if } x \geq 1 \\ -(x-1)\,y^2 & \text{if } x < 1 \end{array} \right\} \text{ piece-wise defined.}$$

$\rightsquigarrow$ def. changes on the line $x = 1$

$\Rightarrow$ will need to use the limit definitions at those points.

* $\forall (x,y)$ with $x > 1$ or $x < 1$, $f(x,y)$ is a polyn.

$\Rightarrow$ $\boxed{f \text{ diff. } \forall (x,y) \text{ with } x \neq 1.}$

* $\underline{(x,y) = (1,b),\ b \in \mathbb{R}:}$

$$f_x(1,b) = \lim_{h \to 0} \frac{f(1+h,b) - f(1,b)}{h}$$

$$= \lim_{h \to 0} \frac{|h| \cdot b^2 - 0}{h} = \left\{ \begin{array}{ll} 0 & \text{if } b = 0 \\ \text{DNE} & \text{if } b \neq 0. \end{array} \right.$$

$\Rightarrow$ f is $\underline{\underline{\text{not}}}$ diff. at $(1,b)$, $b \neq 0$.

At $(1,0)$, we have:

$$f_y(1,0) = \lim_{h \to 0} \frac{f(1,h) - f(1,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 \cdot h^2 - 0}{h} = 0.$$

$\Rightarrow$ $f(1,0) = f_x(1,0) = f_y(1,0) = 0$.

and $L_{(1,0)}(x,y) = 0$.

So, $R_{1,(1,0)}(x,y) = |x-1| \cdot y^2$ and

$$\lim_{(x,y)\to(1,0)} \frac{R_{1,(1,0)}(x,y)}{\sqrt{(x-1)^2+y^2}} = \lim_{(x,y)\to(1,0)} \frac{\overbrace{|x-1|}^{=\sqrt{(x-1)^2}}\, y^2}{\sqrt{(x-1)^2+y^2}} = 0$$

since

$$\frac{|x-1|\cdot y^2}{\sqrt{(x-1)^2+y^2}} \leq \frac{\sqrt{(x-1)^2+y^2}}{\sqrt{(x-1)^2+y^2}} \cdot y^2 = y^2 \longrightarrow 0$$

as $(x,y)\to(1,0)$.

$\Longrightarrow f(x,y)$ is diff. at $(1,0)$.

So: $f(x,y)$ is diff. $\forall (x,y)$ with $x \neq 1$, and also at $(1,0)$, BUT, f is NOT diff. at $(1,y)$ with $y \neq 0$.