Let $f(x,y)$ be a function of 2 variables.

DEF.: First order partials.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

NOTATION: If $z = f(x,y)$, we write:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z_x = f_1$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = z_y = f_2$$

NOTE: "∂": partial (pronounced "del")
"d": ordinary.

HIGHER ORDER PARTIALS:

- 2nd order partials: there are four of them.

Two pure:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx} = f_{11}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy} = f_{22}$$

Two mixed:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx} = f_{21}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy} = f_{12}$$

(+) $$Hf(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \text{Hessian matrix of } f$$

• 3rd order partials: there are eight of them.

$f_{xxx}, f_{xxy}, f_{xyx}, f_{yxx}, f_{xyy}, f_{yxy}, f_{yyx}, f_{yyy}$.

ETC.....

ex. $f(x,y) = x^4y^2 - 2xy^5$ ⇝ find the 2nd partials, the Hessian matrix, and f_{xyy}, f_{yyy}.

$f_x = 4x^3y^2 - 2y^5$

$f_y = x^4\cdot(2y) - 2x\cdot(5y^4) = 2x^4y - 10xy^4$.

$$\begin{cases} f_{xx} = (f_x)_x = 12x^2y^2 - 0 = 12x^2y^2 \\ f_{xy} = (f_x)_y = 4x^3\cdot(2y) - 10y^4 = 8x^3y - 10y^4 \\ f_{yx} = (f_y)_x = 8x^3y - 10y^4 \\ f_{yy} = (f_y)_y = 2x^4 - 10x(4y^3) = 2x^4 - 40xy^3. \end{cases}$$

NOTE: $f_{xy} = 8x^3y - 10y^4 = f_{yx}$.

$$Hf(x,y) = \begin{pmatrix} 12x^2y^2 & 8x^3y - 10y^4 \\ 8x^3y - 10y^4 & 2x^4 - 40xy^3 \end{pmatrix}$$

$f_{xyy} = (f_{xy})_y = 8x^3 - 40y^3$

$f_{yyy} = (f_{yy})_y = -120xy^2$

etc.....

In the previous example, $f_{xy} = f_{yx}$, which meant that the order of differentiation doesn't matter. This will be the case for MANY functions, as stated by the next theorem:

THEOREM: Suppose that $f(x,y)$ is such that f, f_x, f_y, f_{xy} and f_{yx} exist in a neighbourhood of (a,b) AND f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

In this case, the Hessian matrix at (a,b)

$$Hf(a,b) = \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{pmatrix}$$

is symmetric.

HOWEVER, there are functions $f(x,y)$ for which the mixed partials are NOT equal at every point.

E.g. Question #3 from Assignment 2:

$$f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$\rightsquigarrow$ Here, $f_{xy}(0,0) = -1 \neq 1 = f_{yx}(0,0)$.

NOTE: If $f_{xy}(a,b) \neq f_{yx}(a,b)$, this means that f_{xy} OR f_{yx} is NOT continuous at (a,b) (otherwise, the theorem would force the mixed partials to be equal).

We finish with an example of how to compute the 2^{nd} order partials of a piece-wise defined function at one of the points where the definition changes.

ex:

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find $f_{xy}(0,0)$.

Since the definition of changes at $(0,0)$, we need to use the limit definition:

$$f_{xy}(0,0) = (f_x)_y(0,0)$$
$$= \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}.$$

To compute this limit, we NEED:

$f_x(0,h)$, $h \neq 0$, and $f_x(0,0)$.

* $f_x(0,0)$: since $(0,0)$ is the point where the definition of f changes, we use the limit definition.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3+0}{h^2+0} - 0}{h}$$

$$= \lim_{h \to} 1 = 1 \Rightarrow \boxed{f_x(0,0) = 1}$$

* $f_x(0,y)$, $y \neq 0$: since $(0,y) \neq (0,0)$, $\forall y \neq 0$, there are 2 ways of computing this.

→ using the limit definition:

$$f_x(0,y) = \lim_{h \to 0} \frac{f(h,y) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3+y^3}{h^2+y^2} - \frac{0+y^3}{0+y^2}}{h}$$

$$= \lim_{h \to 0} \frac{(h^3+\cancel{y^3}) - y(h^2+\cancel{y^2})}{(h^2+y^2)\,h}$$

$$= \lim_{h \to 0} \frac{h^{\cancel{2}}(h+y)}{\cancel{h}(h^2+y^2)} = \lim_{h \to 0} \frac{h(h+y)}{h^2+y^2}$$

$$= \frac{0}{y^2} = 0, \quad \forall y \neq 0.$$

$$\Rightarrow \boxed{f_x(0,y) = 0, \quad \forall y \neq 0}$$

OR

→ using ordinary differentiation with respect to x: if $y \neq 0$, then $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$, so that

$$f_x(x,y) = \frac{3x^2(x^2+y^2) - (x^3+y^3)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2+y^2)^2}$$

$$\Longrightarrow f_x(0,y) = 0, \quad \forall y \neq 0.$$

[IMPORTANT: $f_x(0,y)$ means $f_x(x,y)$ evaluated at $(0,y)$. So, to compute $f_x(0,y)$ you need to first compute $f_x(x,y)$.

~~ALSO, $f_x(0,y)$ is NOT EQUAL to the partial of $f(0,y)$ with respect to x! So, to compute $f_x(0,y)$, you CANNOT first compute $f(0,y)$ and then take the partial with respect to x!~~]

So, we have $f_x(0,0) = 1$

and

$$f_x(0,y) = 0, \quad \forall y \neq 0.$$

Therefore,

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 1}{h} = \lim_{h \to 0} \frac{-1}{h} \text{ DNE.}$$

$$\longrightarrow \boxed{f_{xy}(0,0) \text{ DNE.}}$$

NOTE: In the previous example,

1) To compute $f_x(0,0)$, we had **NO** choice but to use the limit definition of the partial derivative since the definition changes at $(0,0)$.

2) To compute $f_x(0,y)$, $y \neq 0$, we could either use the limit definition or ordinary differentiation with respect to x because we have an expression for f that is valid for **all** $y \neq 0$: $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$, $\forall y \neq 0$.

HOWEVER, since we were computing f_x at specific points $(0,y)$, $y \neq 0$, using the limit definition was not that much more complicated.

§4.2. Linear approximations.

We would like to find a good definition of differentiability for functions of 2 var.

1-var: f is diff. at $x=a$ $\Leftrightarrow$ $f'(a)$ exists.

AND

f is diff. at $x=a$ $\Rightarrow$ f is cont. at $x=a$.

[although the converse is not true:
e.g. $f(x)=|x|$ is cont. at $x=0$, but not diff. there since $\lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{|h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h} \text{ DNE.}$$]

2-var.: Is it enough to have $f_x(a,b)$ and $f_y(a,b)$ exist? We would need to make sure that our definition of differentiability imply that

$$\boxed{f(x,y) \text{ diff. at } (a,b) \Rightarrow f(x,y) \text{ cont. at } (a,b).}$$

Answer: NO. There are functions such that $f_x(a,b)$ and $f_y(a,b)$ BOTH exist, BUT $f(x,y)$ is NOT continuous at (a,b).

E.g.: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

* f is NOT cont. at $(0,0)$: because

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \neq 0$$

since along $y=x$: $\displaystyle\lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2} \neq 0,$

* $$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h\cdot 0}{h^2+0} - 0}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

and by symmetry, $f_y(0,0) = 0$,

$\Rightarrow$ $f_x(0,0) = f_y(0,0) = 0$ ~~us~~ BOTH <u>exist</u>.

$\Rightarrow$ NEED a better definition of differentiability.

DEF: $f(x,y)$ is differentiable at (a,b) if

(i) $f_x(a,b)$ and $f_y(a,b)$ BOTH exist;

(ii) $$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L_{(a,b)}(x,y)}{\|(x,y)-(a,b)\|} = 0,$$

where $L_{(a,b)}(x,y) := f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

= linear approximation of f at (a,b).