4) Approximate $\sqrt{(0.1)^2+(0.97)^2}$. (10)

$\rightsquigarrow$ use the linear approximation of $f(x,y)=\sqrt{x^2+y^2}$ at $(0,1)$.

We have seen that f is diff. $\forall (x,y)\neq(0,0)$, which means that $L_{(0,1)}(x,y)$ is a good approximation of $f(x,y)$ near $(0,1)$:

$$f(\underbrace{0.1,\ 0.97}_{\text{close to } (0,1)}) \simeq L_{(0,1)}(0.1,\ 0.97).$$

$$\left.\begin{array}{ll} f_x = \dfrac{x}{\sqrt{x^2+y^2}} & \Rightarrow f_x(0,1)=0 \\ f_y = \dfrac{y}{\sqrt{x^2+y^2}} & \Rightarrow f_y(0,1)=1 \\ & f(0,1)=1 \end{array}\right\} \Rightarrow L_{(0,1)}(x,y) = 1+0\cdot(x-0)+1\cdot(y-1) = y.$$

and

$$\sqrt{(0.1)^2+(0.97)^2} = f(0.1,0.97) \simeq L_{(0,1)}(0.1,0.97) = \boxed{0.97}.$$

Note that the calculator value of $f(0.1,0.97)$ is $0.9751\ldots$, giving us an error of

$$\text{error} = f(0.1,0.97) - L_{(0,1)}(0.1,0.97) = \boxed{0.0051\ldots}$$

We'll see that 2^{nd} order Taylor polynomials often give better approximations.

5) Approximate $\sqrt{2(2.2)^2 + e^{-0.1}}$.

$\rightsquigarrow$ use linear approximation $f(x,y) = \sqrt{2x^2 + e^y}$ at $(2,0)$.

$$\left.\begin{aligned} f_x &= \frac{2x}{\sqrt{2x^2+e^y}} \\ f_y &= \frac{e^y}{2\sqrt{2x^2+e^y}} \end{aligned}\right\} \begin{aligned} &\Rightarrow \text{both cont. at } (2,0) \\ &\Rightarrow f \text{ is diff. at } (2,0). \end{aligned}$$

$\rightsquigarrow$ since $f_x(2,0) = \frac{4}{3}$, $f_y(2,0) = 1/6$, $f(2,0) = 3$, we have

$$L_{(2,0)}(x,y) = 3 + \frac{4}{3}(x-2) + \frac{1}{6}(y-0)$$

and

$$\sqrt{2(2.2)^2 + e^{-0.1}} = f(2.2, 0.2)$$

$$\simeq L_{(2,0)}(2.2, 0.2) = 3 + \frac{4}{3}(2.2-2) + \frac{1}{6}(-0.1-0)$$

$$= 3 + \frac{4}{3}(0.2) - \frac{0.1}{6}$$

$$= \frac{19.5}{6} = 3.25 .$$

$$\Longrightarrow \sqrt{2(2.2)^2 + e^{-0.1}} \simeq 3.25.$$

Chapter 6. Chain Rule.

1-var. Chain Rule.

Let $f = f(x)$ be a diff. fct of x, and $x = x(t)$ be a diff. fct of t. Then $u(t) = f(x(t))$ is a diff. fct of t with

$$u'(t) = f'(x(t))\, x'(t).$$

2-var. Chain Rule.

Let $f = f(x,y)$ be a diff. fct of x and y. And suppose that $x = x(t)$ and $y = y(t)$ are diff. fcts of t. Then, $u(t) = f(x(t), y(t))$ is a diff. fct of t and

$$u'(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

$$\left[= f_1(x(t), y(t))\, x'(t) + f_2(x(t), y(t))\, y'(t) \right.$$
$$\left. \text{where } f_x = f_1 \text{ and } f_y = f_2. \right]$$

OR

$$\frac{du}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

intermediate variable

$$\begin{array}{ccccl} & & u & & \text{dependent} \\ & \swarrow & & \searrow & \\ x & & & & y \quad \text{intermediate} \\ | & & & & | \\ t & & & & t \quad \text{independent} \end{array}$$

$\rightsquigarrow \quad u = f(x(t), y(t))$.

---

ex. 1) $f(x,y) = \sin(x^2 y)$, $(x,y) \in \mathbb{R}^2$.

$x = \sqrt{t}$

$y = t^2$ for $t > 0$.

$\rightarrow u(t) = f(\sqrt{t}, t^2) = \sin((\sqrt{t})^2 \cdot t^2) = \sin(t^3)$

$\Rightarrow u'(t) = 3t^2 \cos(t^3)$.

$\rightarrow$ Using the Chain Rule:

$$\left.\begin{aligned} f_x &= 2xy\cos(x^2y) \\ f_y &= x^2\cos(x^2y) \end{aligned}\right\} \quad f_x \text{ and } f_y \text{ are cont.} \Rightarrow f \text{ is diff. } \forall (x,y).$$

ALSO:

$$\left.\begin{aligned} x'(t) &= \frac{1}{2\sqrt{t}} \\ y'(t) &= 2t \end{aligned}\right\} \text{ both diff. for } t > 0.$$

$\Rightarrow$ by Chain Rule, if $u(t) = f(x(t), y(t))$,

$$\begin{aligned} u'(t) &= f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t) \\ &= f_x(\sqrt{t}, t^2) \cdot \frac{1}{2\sqrt{t}} + f_y(\sqrt{t}, t^2) \cdot 2t \\ &= \left[\cancel{2\sqrt{t}} \cdot t^2 \cos((\sqrt{t})^2 t^2)\right] \cdot \frac{1}{\cancel{2\sqrt{t}}} + \left[(\sqrt{t})^2 \cos((\sqrt{t})^2 t^2)\right] \cdot 2t \\ &= 3t^2 \cos(t^3), \text{ as expected.} \end{aligned}$$

2) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a fct of x and y, and consider

$$u(t) = f(\cos t, \sin t).$$

Suppose that f is diff. at $(1,0)$ and $f_x(1,0) = 8$, $f_y(1,0) = -4$. Find $u'(0)$.

NOTE: $u(t) = f(x(t), y(t))$ with $x(t) = \cos t$, $y(t) = \sin t$

also $(x(0), y(0)) = (\cos 0, \sin 0) = (1,0)$.

Since f is diff. at $(1,0) = (x(0), y(0))$

and $x'(t) = -\sin t$, $y'(t) = \cos t$ $\Rightarrow$ x & y are both diff. at $t=0$.

by the Chain Rule

$$u'(0) = f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0)) \cdot y'(0)$$
$$= f_x(1,0) \cdot (-\sin 0) + f_y(1,0) \cdot (\cos 0)$$
$$= 8 \cdot 0 + (-4) \cdot 1 = -4.$$

IMPORTANT: need f diff. in order for the Chain Rule to work. NOT enough for f_x and f_y to exist.

e.g.: Consider $f(x,y) = x^{1/3} y^{1/3}$ and $x(t) = t^2$, $y(t) = t$. Let $u(t) = f(t^2, t)$. Find $u'(0)$.

We have seen that f is not diff. at $(0,0) = (x(0), y(0))$, even though partials exist at $(0,0)$:

$$f_x(0,0) = f_y(0,0) = 0.$$

→ <u>DIRECT COMPUTATION</u>: $u(t) = (t^2)^{1/3} t^{1/3} = t$

$\Rightarrow u'(t) = 1 \Rightarrow \boxed{u'(0) = 1}$.

→ <u>Chain Rule f^ula</u>: here $x'(2t) = 2t$, $y'(t) = 1$, so

$$f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0)) \cdot y'(0)$$
$$= \underbrace{\cancel{f_x(0,0)}}_{0} \cdot 2(0) + \underbrace{\cancel{f_y(0,0)}}_{0} \cdot 1 = 0 \neq 1 = u'(0).$$

<u>Sketch of proof of the Chain Rule</u>: $t = t_0$.

Assume that $(a,b) = (x(t_0), y(t_0))$. Then,

$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$
$$u(t_0) = f(a,b)$$
$$u(t_0+h) = f(x(t_0+h), y(t_0+h))$$
$$= L_{(a,b)}(x(t_0+h), y(t_0+h)) + R_{1,(a,b)}(x(t_0+h), y(t_0+h))$$

So:

$$\frac{u(t_0+h) - u(t_0)}{h} = f_x(a,b)\left(\frac{x(t_0+h) - x(t_0)}{h}\right) + f_y(a,b)\left(\frac{y(t_0+h) - y(t_0)}{h}\right)$$
$$+ \frac{R_{1,(a,b)}(x(t_0+h), y(t_0+h))}{h}.$$

taking the limit as $h \to 0$ on both sides:

$$u'(t_0) = f_x(a,b)\, x'(t_0) + f_y(a,b)\, y'(t_0) + 0.$$ ☒

since f is diff. at (a,b)

GRADIENT NOTATION.

DEF: Given a fct of 2-var. $f(x,y)$, we define

$$\nabla f(a,b) := (f_x(a,b), f_y(a,b)) = \left(\underline{\text{gradient of } f \text{ at } (a,b)}\right),$$

at any point (a,b) where f_x and f_y exist.

e.g.: $f(x,y) = 2xy^2 + e^x$

$$\nabla f = (f_x, f_y) = (2y^2 + e^x, 4xy).$$

THEN: using the gradient notation, we have the following compact formulation of the Chain Rule:

$$\frac{d}{dt}\big(f(x(t), y(t))\big) = \nabla f(x(t), y(t)) \cdot (x'(t), y'(t)).$$

dot product

E.g.: Suppose that $g(t) = f(t^3, 4t - e^{t^2-1})$ with f diff. at $(1,3)$ and $\nabla f(1,3) = (-2,1)$. Find $g'(1)$.

Here, $x(t) = t^3$ and $y(t) = 4t - e^{t^2-1}$, so that $(x'(t), y'(t)) = (3t^2, 4 - 2t^2 e^{t^2-1})$ and, by Chain Rule,

$$g'(1) = \nabla f(x(1), y(1)) \cdot (x'(1), y'(1)) = \nabla f(1,3) \cdot (3, 2)$$
$$= (-2,1) \cdot (3,2) = (-2)(3) + (1)(2) = -4.$$

More than one variable: e.g. 3 variables.

For a fct $f(x,y,z)$ of 3 variable

- limits
- continuity
- partials $f_x, f_y, f_z, f_{xx}, f_{yx}, f_{zz}$, etc.....
- differentiability
- $\nabla f := (f_x, f_y, f_z)$ = gradient

are defined as in the 2 variable case.

ALSO, we have:

THM: f_x, f_y, f_z cont. at $(a,b,c) \Rightarrow f$ diff. at (a,b,c)

and

THM: (Chain Rule) Suppose that $f(x,y,z)$ and $x=x(t)$, $y=y(t)$, $z=z(t)$ are diff. fcts. If $u(t)=f(x(t),y(t),z(t))$, then

$$u'(t) = \nabla f(x(t),y(t),z(t)) \underset{\text{dot product}}{\cdot} (x'(t),y'(t),z'(t))$$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

E.g: Let $g(t) = f(\underbrace{t^2}_{x}, \underbrace{2t-1}_{y}, \underbrace{4\cos t}_{z})$, with $f(x,y,z)$ diff. at $(0,-1,4)$ and $\nabla f(0,-1,4) = (2,3,-1)$. Find $g'(0)$.

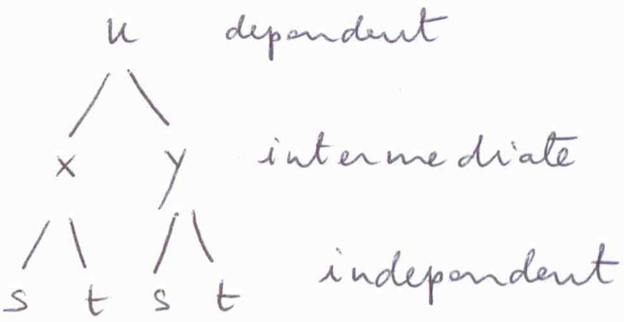
HERE: $x(t)=t^2$, $y(t)=2t-1$, $z(t)=4\cos t$ are all diff. and $x'(t)=2t$, $y'(t)=2$, $z'(t)=-4\sin t$. Then,

$$g'(0) = \nabla f(x(0),y(0),z(0)) \cdot (x'(0),y'(0),z'(0))$$
$$= \nabla f(0,-1,4\cos(0)) \cdot (0,2,-4\sin 0)$$
$$= (2,3,-1)\cdot(0,2,0) = (2)(0)+(3)(2)+(-1)(0) = 6$$

MORE GENERAL VERSION of CHAIN RULE:

$u = f(x,y)$, $x = x(s,t)$, $y = y(s,t)$.

THEN: $u = u(s,t)$



If f is diff., and x & y have partials that exist, then

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

OR

$$u_s = f_x\, x_s + f_y\, y_s = f_1 \cdot x_s + f_2 \cdot y_s$$

$$u_t = f_x\, x_t + f_y\, y_t = f_1 \cdot x_t + f_2 \cdot y_t.$$

Ex. 1) $u = f(st^2, s^2+t)$ ⇝ find $\frac{\partial u}{\partial s}$, $\frac{\partial u}{\partial t}$:

HERE: If $f = f(x,y)$, then $x(s,t) = st^2$, $y(s,t) = s^2+t$.

So,

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= f_x(st^2, s^2+t)\cdot t^2 + f_y(st^2, s^2+t)\cdot 2s$$

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$
$$= f_x(st^2, s^2+t)\cdot(2st) + f_y(st^2, s^2+t)\cdot 1.$$

2) $u = f(y^2x, y/x)$ ⇝ Find u_x, u_y, u_{xx}.

In this case, we don't specify the variables of f, and just denote the partials of f by f_1, f_2, f_{11}, f_{12}, etc....

$$u_x = f_1(y^2x, y/x)\cdot\frac{\partial}{\partial x}(y^2x) + f_2(y^2x, y/x)\cdot\frac{\partial}{\partial x}(y/x)$$
$$= f_1(y^2x, y/x)\cdot y^2 + f_2(y^2x, y/x)\cdot(-y/x^2)$$

$$u_y = f_1(y^2x, y/x)\cdot\frac{\partial}{\partial y}(y^2x) + f_2(y^2x, y/x)\cdot\frac{\partial}{\partial y}(y/x)$$
$$= f_1(y^2x, y/x)\cdot 2yx + f_2(y^2x, y/x)\cdot\left(\frac{1}{x}\right)$$

$$u_{xx} = (u_x)_x = \frac{\partial}{\partial x}\left[f_1(y^2x, y/x)\cdot y^2 + f_2(y^2x, y/x)\cdot(-y/x^2)\right]$$

first use the product rule.

$$= \frac{\partial}{\partial x}\left(f_1(y^2x, y/x)\right)\cdot y^2 + f_1(y^2x, y/x)\frac{\partial}{\partial x}(y^2)$$
$$+ \frac{\partial}{\partial x}\left(f_2(y^2x, y/x)\right)\cdot(-y/x^2) + f_2(y^2x, y/x)\cdot\frac{\partial}{\partial x}(-y/x^2)$$
$$= \left[(f_1)_1(y^2x, y/x)\cdot y^2 + (f_1)_2(y^2x, y/x)(-y/x^2)\right]\cdot y^2$$
$$+ f_1(y^2x, y/x)\cdot 0 + \left[(f_2)_1(y^2x, y/x)\cdot y^2 + (f_2)_2(y^2x, y/x)\cdot(-y/x^2)\right]\cdot\left(\frac{-y}{x^2}\right)$$
$$+ f_2(y^2x, y/x)\cdot(-2y/x^3).$$

So,

$$u_{xx} = f_{11}(y^2x, y/x)\cdot y^4 + f_{12}(y^2x, y/x)(-y^3/x^2) + f_{21}(y^2x, y/x)\cdot(-y^3/x^2)$$
$$+ f_{22}(y^2x, y/x)\cdot(y^2/x^4) + f_2(y^2x, y/x)\cdot(-2y/x^3).$$

3) Suppose that $g(x,y) = x f(2xy,\, x^2-y^2)$, with $f(2,0) = -1$, f diff. at $(2,0)$, and $\nabla f(2,0) = (2,3)$. Calculate $\frac{\partial g}{\partial x}(1,1)$. 9

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(x\, f(2xy,\, x^2-y^2) \right)$$

fist use product rule

$$= 1 \cdot f(2xy,\, x^2-y^2) + x \frac{\partial}{\partial x}\left(f(2xy,\, x^2-y^2) \right)$$

chain Rule

$$= f(2xy,\, x^2-y^2) + x\left[f_1(2xy, x^2-y^2)\, 2y + f_2(2xy, x^2-y^2) \cdot 2x \right]$$

$$\rightarrow \frac{\partial g}{\partial x}(1,1) = f(2,0) + 1 \cdot \left[f_1(2,0) \cdot 2(1) + f_2(2,0) \cdot 2(1) \right]$$

$$= -1 + 2 f_1(2,0) + 2 f_2(2,0).$$

Since $(f_1(2,0),\, f_2(2,0)) = \nabla f(2,0) = (2,3)$, we get:

$$\frac{\partial}{\partial x}(1,1) = -1 + 2(2) + 2(3) = 9.$$

4) Suppose that $g(t)$ has continuous 2^{nd} derivative, and f is defined by $f(x,y) = g(x^2 y)$. Calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ and verify that they are equal.

$$\frac{\partial f}{\partial x} = \underbrace{2xy\, g'(x^2y)}_{\text{product}} \Rightarrow \frac{\partial^2 f}{\partial y \partial x} = 2x \cdot g'(x^2y) + 2x^3 y\, g''(x^2y)$$

$$\frac{\partial f}{\partial y} = \underbrace{x^2 g'(x^2y)}_{\text{product}} \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 2x \cdot g'(x^2y) + x^2 \cdot (2xy)\, g''(x^2y)$$

$\boxed{\text{NOTE: } g', g'' \text{ exist by assumption}}$.

5) A function $f(x,y)$ is said to harmonic if it satisfies the Laplace equation

$$f_{xx} + f_{yy} = 0 \quad (\text{OR } f_{11} + f_{22} = 0).$$

Suppose that f, f_x, f_y are differentiable, and that f is harmonic. Show that

$$g(x,y) = f(ax+by,\ bx-ay)$$

is also harmonic.

We need to show that $g_{xx} + g_{yy} = 0$.

$$g_x = f_1 \cdot a + f_2 \cdot b$$

$$g_y = f_1 \cdot b + f_2 \cdot (-a).$$

$$g_{xx} = (f_{11} \cdot a + f_{12} \cdot b) \cdot a + (f_{21} \cdot a + f_{22} \cdot b) \cdot b$$

$$= a^2 f_{11} + ab(f_{12} + f_{21}) + b^2 f_{22}$$

$$g_{yy} = (f_{11} \cdot b + f_{12} \cdot (-a))\, b + (f_{21} \cdot b + f_{22} \cdot (-a))(-a)$$

$$= b^2 f_{11} - ab(f_{12} + f_{21}) + a^2 f_{22}$$

$$\Rightarrow g_{xx} + g_{yy} = (a^2+b^2)(\underbrace{f_{11} + f_{22}}_{=0}) = 0.$$

6) $w = F(\underbrace{t^4+1}_{x}, \underbrace{s^2-2t}_{y}, \underbrace{e^{3ts}}_{z})$.

Find $\frac{\partial w}{\partial t}$, $\frac{\partial w}{\partial s}$, $\frac{\partial w}{\partial s \partial t}$.

$$* \ \frac{\partial w}{\partial t} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial t}$$

$$= F_x \cdot (4t^3) + F_y \cdot (-2) + F_z \cdot (3se^{3ts}).$$

$$* \ \frac{\partial w}{\partial s} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial s}.$$

$$= \cancel{F_x \cdot 0} + F_y \cdot (2s) + F_z \cdot (3te^{3ts}).$$

$$* \ \frac{\partial w}{\partial s \partial t} = \frac{\partial}{\partial s}\left[4t^3 F_x - 2F_y + 3se^{3ts} F_z\right].$$

$$= 4t^3 \frac{\partial}{\partial s}(F_x) - 2\frac{\partial}{\partial s}(F_y) + (3e^{3ts} + 9ste^{3ts})F_z + 3se^{3ts}\frac{\partial}{\partial s}(F_z).$$

product Rule

$$= 4t^3\left[\cancel{F_{xx} \cdot 0} + F_{xy} \cdot (2s) + F_{xz} \cdot (3te^{3ts})\right]$$
$$- 2\left[\cancel{F_{yx} \cdot 0} + F_{yy} \cdot (2s) + F_{yz} \cdot (3te^{3ts})\right]$$
$$+ (3e^{3ts} + 9ste^{3ts})F_z + 3se^{3ts}\left[F_{zx} \cdot 0 + F_{zy} \cdot (2s) + F_{zz} \cdot (3te^{3ts})\right].$$

$$= 4t^3\left[2sF_{xy} + 3te^{3ts}F_{xz}\right] - 2\left[2sF_{yy} + 3te^{3ts}F_{yz}\right]$$
$$+ (3e^{3ts} + 9ste^{3ts})F_z + 3se^{3ts}\left[2sF_{zy} + 3te^{3ts}F_{zz}\right].$$

7) If $F(x,y,z) = f\left(\frac{y-z}{x}, \frac{z-x}{y}, \frac{x-y}{z}\right)$, with f diff., show that F satisfies:

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = 0.$$

$$\frac{\partial F}{\partial x} = f_1 \cdot \left(\frac{-(y-z)}{x^2}\right) + f_2 \cdot \left(-\frac{1}{y}\right) + f_3 \cdot \left(\frac{1}{z}\right)$$

$$\frac{\partial F}{\partial y} = f_1 \cdot \left(\frac{1}{x}\right) + f_2 \cdot \left(\frac{-(z-x)}{y^2}\right) + f_3 \cdot \left(-\frac{1}{z}\right)$$

$$\frac{\partial F}{\partial z} = f_1 \cdot \left(-\frac{1}{x}\right) + f_2 \cdot \left(\frac{1}{y}\right) + f_3 \cdot \left(-\frac{(x-y)}{z^2}\right)$$

$$\Rightarrow x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} =$$

$$\left[-\frac{(y-z)}{x} + \frac{y}{x} - \frac{z}{x}\right] \cdot f_1 + \left[-\frac{x}{y} - \frac{(z-x)}{y} + \frac{z}{y}\right] \cdot f_2$$
$$+ \left[\frac{x}{z} - \frac{y}{z} - \frac{(x-y)}{z}\right] \cdot f_3$$

$$\Rightarrow x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = 0 \quad \blacksquare$$