Ex.: 6) Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2|y|^{3/2}}{x^4+y^2} = 0.$

Let's use the Squeeze Theorem with $L=0$:

$$\left| \frac{x^2|y|^{3/2}}{x^4+y^2} - 0 \right| = \frac{x^2|y|^{3/2}}{x^4+y^2}$$

We need to find an upper bound $B(x,y)$ that's a multiple of (x^4+y^2) AND goes to 0 as $(x,y)\to(0,0)$.

* Try: $$\frac{x^2|y|^{3/2}}{x^4+y^2} = \frac{\sqrt{x^4}\,|y|^{3/2}}{x^4+y^2}$$
$$\leq \frac{\sqrt{x^4+y^2}\,|y|^{3/2}}{x^4+y^2} = \frac{|y|^{3/2}}{\sqrt{x^4+y^2}}.$$

Unfortunately, we STILL have a denominator that goes to 0 as $(x,y)\to(0,0)$.

* Let's try instead to manipulate the y's:

$$\frac{x^2|y|^{3/2}}{x^4+y^2} = \frac{x^2\left(\sqrt{y^2}\right)^{3/2}}{x^4+y^2}$$
$$\leq \frac{x^2\left(\sqrt{x^4+y^2}\right)^{3/2}}{x^4+y^2} = \frac{x^2(x^4+y^2)^{3/4}}{x^4+y^2}$$

$$= \frac{x^2}{(x^4+y^2)^{1/4}}$$

STILL have a denominator that goes to 0 as $(x,y) \to (0,0)$.

* BETTER: manipulate x & y simultaneously:

$$\frac{x^2|y|^{3/2}}{x^4+y^2} = \frac{\sqrt{x^4}\cdot\left(\sqrt{y^2}\right)^{3/2}}{x^4+y^2}$$

$$\leq \frac{\sqrt{x^4+y^2}\cdot\left(\sqrt{x^4+y^2}\right)^{3/2}}{x^4+y^2}$$

$$= \frac{(x^4+y^2)^{1/2+3/4}}{(x^2+y^2)} = (x^4+y^2)^{1/4}$$

$$= B(x,y) \to 0$$

as $(x,y) \to (0,0)$.

So, by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{x^2|y|^{3/2}}{x^4+y^2} = 0.$$

7) Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{(4x^2-y^2(x-2))}{2x^2+y^2}$.

Let's first SIMPLIFY the expression:

$$\frac{4x^2-y^2(x-2)}{2x^2+y^2} = 2 - \frac{y^2x}{2x^2+y^2}$$

* Pick path to find potential limit:

$$x=0: \quad \lim_{\substack{y\to 0\\ x=0}} \left(2-\frac{0}{0+y^2}\right) = \lim_{y\to 0} 2 = 2.$$

The potential limit is therefore $L=2$.

* Use Squeeze Theorem with $L=2$:

$$\left|\left(2-\frac{y^2x}{2x^2+y^2}\right)-2\right| = \frac{y^2|x|}{2x^2+y^2} \leq \frac{(2x^2+y^2)|x|}{2x^2+y^2}$$

$$= |x| \longrightarrow 0$$

as $(x,y)\to(0,0)$

$\Rightarrow$ By Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \left(\frac{4x^2-y^2(x-2)}{2x^2+y^2}\right) = 2.$$