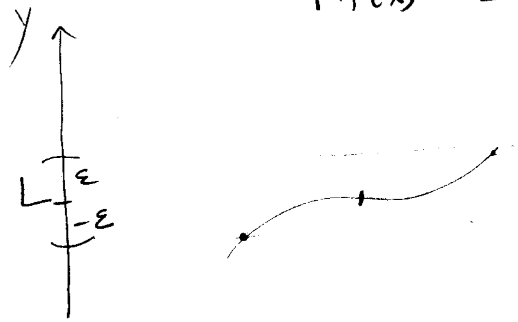
Chapter 2: Limits

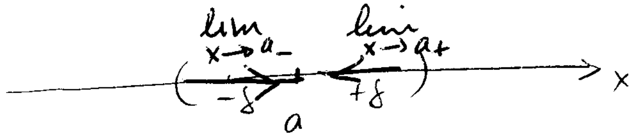
1-var.: given a fct of 1-var. f, we say that

$\lim_{x \to a} f(x)$ exists and is equal to L

if: $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$|f(x) - L| < \varepsilon, \ \forall \underbrace{|x-a|}_{\text{dist}(x,a)} < \delta.$$

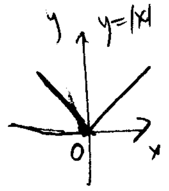




↳ in particular, limit exists if get value L for f no matter how one approaches a, i.e., $\underset{\text{from left}}{x \to a_-}$ OR $\underset{\text{from right}}{x \to a_+}$

Thus, same as

$$\underbrace{\lim_{x \to a_-} f(x) = \lim_{x \to a_+} f(x)}_{\text{BOTH exist and are equal}}$$



Ex.: • $\lim_{x \to 0} |x| = \lim_{x \to 0_-} -x = \lim_{x \to 0_+} x = 0$ exists

• $\lim_{x \to 0} \frac{|x|}{x}$ does not exist since $\lim_{x \to 0_+} \frac{|x|}{x} = \lim_{x \to 0_+} \frac{x}{x} = 1$ X

but $\lim_{x \to 0_-} \frac{|x|}{x} = \lim_{x \to 0_-} \frac{-x}{x} = -1$

Prop: If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then:

1) $\lim\limits_{x\to a} (f+g)(x) = L + M$

2) $\lim\limits_{x\to a} f(x)\,g(x) = LM$

3) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$.

4) $\lim\limits_{x\to a} H(f(x)) = H(L)$ if H is a 1-var. fct that is continuous at L.

⊕ **Squeeze THM:** if

$$m(x) \leq f(x) \leq M(x),$$

with $\lim\limits_{x\to a} m(x) = \lim\limits_{x\to a} M(x) = L$, then

$$\lim_{x\to a} f(x) = L.$$

OR, equivalently!

$0 \leq |f(x) - L| \leq B(x) \to 0$,

then $(f(x) - L) \xrightarrow{x\to a} 0$

$\Longleftrightarrow f(x) \xrightarrow[x\to a]{} L$

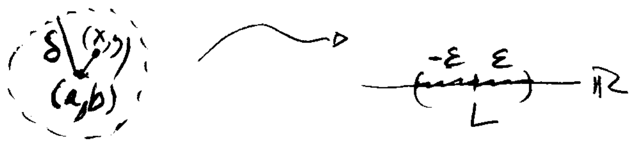
2-var.: All of the above applies to 2 variables, but now:

Def: $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to L

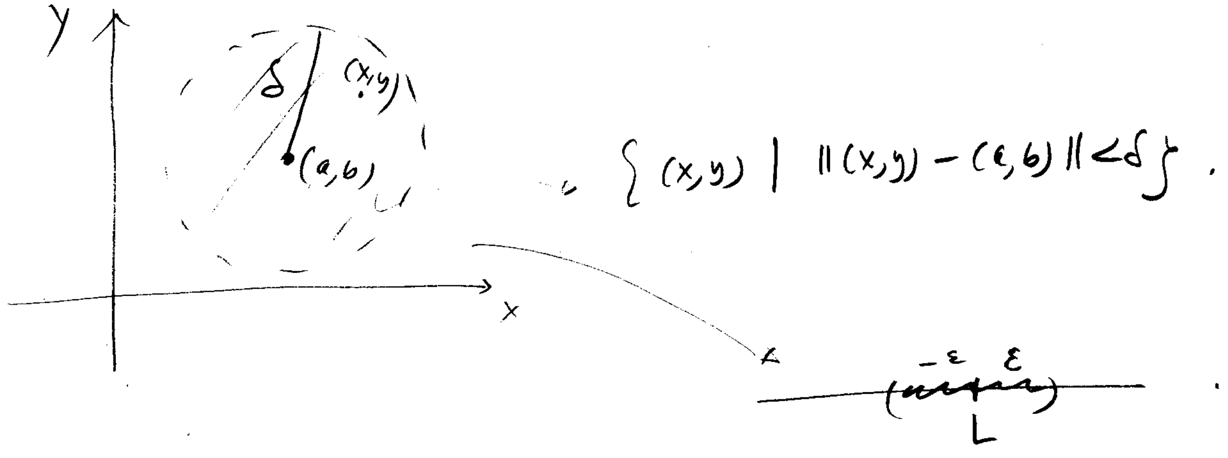
IF, $\underset{\mathbb{R}^2}{}$ $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x,y) - L| < \varepsilon$

for all (x,y) such that

$$\text{dist}((x,y),(a,b)) = \underbrace{\|(x,y) - (a,b)\|}_{\sqrt{(x-a)^2+(y-b)^2}} < \delta$$



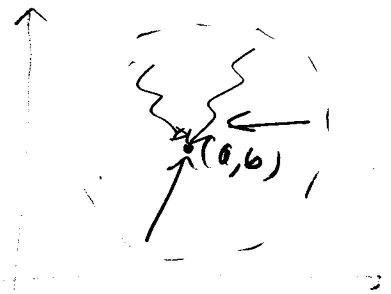
I.e., for *any* (x,y) in the ball of radius δ centered at $(0,0)$, $f(x,y)$ is very close to L.



$\{ (x,y) \mid \|(x,y) - (a,b)\| < \delta \}$.

In particular, if one restricts oneself to any path in the disc passing through (a,b), should get same limit:

[I.e, limits are <u>UNIQUE</u>]



$\Longrightarrow$ if $\exists$ 2 distinct paths in disc giving $\neq$ limits, then limit does not exist. OR if $\exists$ path along which limit DNE, then DNE

E.g.: 1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ does not exist.

* $x=0$: $\lim\limits_{(0,y)\to(0,0)} \dfrac{0}{0+y^2} = 0$

$\#$ $\Longrightarrow$ limit does not exist.

* $y=0$: $\lim\limits_{(x,0)\to(0,0)} \dfrac{x^2}{x^2+0} = 1$

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist: $x=0 \longrightarrow 0$ $\#$ $x=y \longrightarrow 1/2$

[$\Longrightarrow$ PICK simple paths: $x=0$, $y=0$, $y=x$

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{x^2+y^2}$ DNE.

* $\underline{y=0}$: $\lim\limits_{x\to 0} \dfrac{x}{x^2} = \lim\limits_{x\to 0} \dfrac{1}{x}$ DNE

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2-y^2)}{x^2+y^2}$ DNE

* $\underline{y=0}$: $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x^2} = 1$ $\rightsquigarrow$ (may use l'Hôpital $\lim\limits_{x\to 0} \dfrac{\sin(x^2)}{x^2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{2x\cos(x^2)}{2x} = 1$

OR $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$)

* $\underline{y=x}$: $\lim\limits_{y\to 0} \dfrac{\sin 0}{2x^2} = 0 \neq 1.$

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ DNE.

* $\underline{x=0}$: $\lim\limits_{y\to 0} \dfrac{0}{y^2} = 0$

* $\underline{y=0}$: $\lim\limits_{x\to 0} \dfrac{0}{x^2} = 0$

* $\underline{y=x}$: $\lim\limits_{x\to 0} \dfrac{x^3}{x^4+x^2} = \lim\limits_{x\to 0} \dfrac{x}{x^2+1} = \dfrac{0}{1} = 0$

} <u>all</u> give 0.

$\downarrow$

think of using other type of path, e.g., $y=x^2$, $x=y^2$, $y=x^3$, ...

* $\underline{y=x^2}$: $\lim\limits_{x\to 0} \dfrac{x^2(x^2)}{x^4+(x^2)^2} = \dfrac{1}{2} \neq 0.$

$\Rightarrow$ limit DNE.

NOTE: 1) • <u>Always</u> start with simple paths such as $y=0$, $x=0$, $y=x$.

• If those lines don't work try $y=x^2$, $x=y^2$, etc... $y^m=x^n$

But, KEEP IT SIMPLE!!!

2) In book, use test $y = mx$ to get family of lines. BUT, inefficient + avoids $x=0$, and may need more such as parabola....

Limit Theorems:

Let f & g be two 2-var. fcts and suppose that $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ and $\lim_{(x,y)\to(a,b)} g(x,y) = L_2$ (exist!).

Then,

1) $\lim_{(x,y)\to(a,b)} f(x,y) + g(x,y) = L_1 + L_2$

2) $\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = L_1 L_2$

3) $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$, if $L_2 \neq 0$

and

4) If $H(t)$ is continuous at $t = L_1$, then

$$\lim_{(x,y)\to(a,b)} H(f(x,y)) = H(L_1).$$

Proofs as in 1-var. case:

E.g.: Proof of 1): Since $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ and $\lim_{(x,y)\to(a,b)} f(x,y) = L_2$, then $\exists\ \delta_1$ & δ_2 such that if $\|(x,y)-(a,b)\| < \delta_1$, then $|f(x,y) - L_1| < \frac{\varepsilon}{2}$ and if $\|(x,y)-(a,b)\| < \delta_2$, then $|g(x,y) - L_2| < \frac{\varepsilon}{2}$. Set $\delta = \min\{\delta_1, \delta_2\}$. Then, if $\|(x,y)-(a,b)\| < \delta$,

$$|(f+g) - (L_1+L_2)| \leq |f - L_1| + |f - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

SQUEEZE THM:

If $0 \leq |f(x,y) - L| \leq B(x,y)$ for all $(x,y) \neq (a,b)$ near (a,b)

and $\displaystyle\lim_{(x,y)\to(a,b)} B(x,y) = 0$,

then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$.

Pf: Since $\displaystyle\lim_{(x,y)\to(a,b)} B(x,y) = 0$, then $\exists \delta > 0$ such that

if $\|(x,y)-(a,b)\| < \delta$, then $|B(x,y) - 0| < \varepsilon$.

$\Rightarrow$ if $\|(x,y)-(a,b)\| < \delta$, then

$$|f(x,y) - L| \leq B(x,y) = |B(x,y) - 0| < \varepsilon.$$

$$\Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = L.$$

$\square$

RMK: If limit exist, it is UNIQUE!

Pf: Suppose that $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ and

$\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L_2$,

then
$$0 = \lim_{(x,y)\to(a,b)} [f(x,y) - f(x,y)] = (\lim f) - (\lim f) = L_1 - L_2$$

$$\Rightarrow L_1 = L_2.$$

Examples:

1) $\lim_{(x,y)\to(1,1)} 2x^2+y = 3.$

2) $\lim_{(x,y)\to(2,1)} x^3y = 8.$

In general, if $f(x,y)=$ polyn., $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$

3) $\lim_{(x,y)\to(\pi,-1)} 3y^2\sin\left(\frac{x}{2}\right) = 3.$

4) $\lim_{(x,y)\to(2,1)} x e^{x^2+y} = 2e^5.$

5) $\lim_{(x,y)\to(1,0)} \sqrt{9-x^2+3y^4} = \sqrt{8} = 2\sqrt{2}.$

Situation gets more complicated if one has quotients:

6) $\lim_{(x,y)\to(-1,0)} \frac{\overset{\nearrow 1}{3y^2-x}}{\underset{\searrow -2\neq 0}{2x+y}} = \frac{1}{-2} = -\frac{1}{2}$ exists.

7) $\lim_{(x,y)\to(0,0)} \frac{\overset{\to 0}{x^2-y^4}}{\underset{\searrow 0}{x^2+y^4}} \rightsquigarrow \frac{0}{0}$ indeterminate case
$\Rightarrow$ need to look more closely.

What to do?

Pick a few paths:
- if get $\neq$ limits, DNE
- if always get limit L, use Squeeze to show that $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

HERE: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^4}{x^2+y^4}$ ~ same orde

$$\left.\begin{array}{ll} * \ \underline{y=0}: & \displaystyle\lim_{x\to 0}\frac{x^2}{x^2}=1 \quad X \\ * \ \underline{x=0}: & \displaystyle\lim_{y\to 0}\frac{-y^4}{y^4}=-1 \end{array}\right] \Rightarrow \text{limit DNE}$$

8) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\overset{\text{order } 2}{\boxed{y^2}}}{\underset{\text{order } 1}{\boxed{\sqrt{x^2+y^2}}}}$ ⇝ along lines, parabolas get 0

e.g. $\underline{y=0}$: $\displaystyle\lim_{y\to 0}\frac{0}{\sqrt{x^2}}=0.$

$\Rightarrow$ limit probably exists and is zero.

$$\left|\frac{y^2}{\sqrt{x^2+y^2}}-0\right| = \frac{y^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)}\frac{y^2}{\sqrt{x^2+y^2}} = 0.$$

RMK: If $f(x,y)=\dfrac{p(x,y)}{q(x,y)}$ is quotient of fcts which involve powers of x & y, then:

- if order$(p)\leq$ order(q), then the limit probably does not exist: FIND paths that give ≠ limits.
- if order$(p)>$ order(q), then the limit probably exists: use path to find L, then use squeeze to prove that $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)=L$.

[HERE: ord(q) & ord(p) = smallest power in expression.]

9) $$\lim_{(x,y)\to(0,0)} \left(\underbrace{\frac{4x^2 - y^2(x-2)}{2x^2+y^2}}_{\shortparallel} \right) = 2.$$

Simplify! $$\left(2 - \frac{y^2 x}{2x^2+y^2} \right)$$

$\Rightarrow$ need to find $B(x,y)$ such that

$$\lim_{(x,y)\to(0,0)} B(x,y) = 0$$

AND

$$|f(x,y) - 2| \leq B(x,y), \quad \forall (x,y) \text{ near } (0,0).$$

HERE,

$$|f(x,y) - 2| = \frac{y^2|x|}{2x^2+y^2} \leq \frac{(y^2+2x^2)}{(2x^2+y^2)}|x| = |x| \xrightarrow[(x,y)\to(0,0)]{} 0 \text{ as}$$

10) $$\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^2+y^4} = 0.$$

$$\left|\frac{x^3y^2}{x^2+y^4}\right| = \frac{|x|\, x^2 \overset{=\sqrt{y^4}}{\boxed{y^2}}}{x^2+y^4} \leq \frac{|x|\,\cancel{(x^2+y^4)}\sqrt{y^4+x^2}}{\cancel{x^2+y^4}}$$

$$= |x|\sqrt{y^4+x^2}$$

$$\longrightarrow 0 \text{ as } (x,y)\to(0,0).$$

RECAP:

A) To show that limit DNE, find 2 ≠ paths that give ≠ limits

E.g., try $y=0$, $x=0$, $y=x$, $y=mx$, $y=x^2$, ...

OR $\begin{cases} y=mx \\ x=0 \end{cases}$ → ALL lines

& if fails, then try $y=x^2$, $x=y^2$, ...

B) To show that $\lim_{(x,y)\to(a,b)} f(x,y)$ exist.

ONLY if NECESSARY $\boxed{1}$) Pick path to find potential limit L.

2) Use Squeeze Thm: find $B(x,y)$ such that

$$|f(x,y) - L| \le B(x,y) \quad \forall (x,y) \text{ near } (a,b)$$

$\underline{\text{and}}$

$$B(x,y) \longrightarrow L \quad \text{as} \quad (x,y) \to (0,0)$$