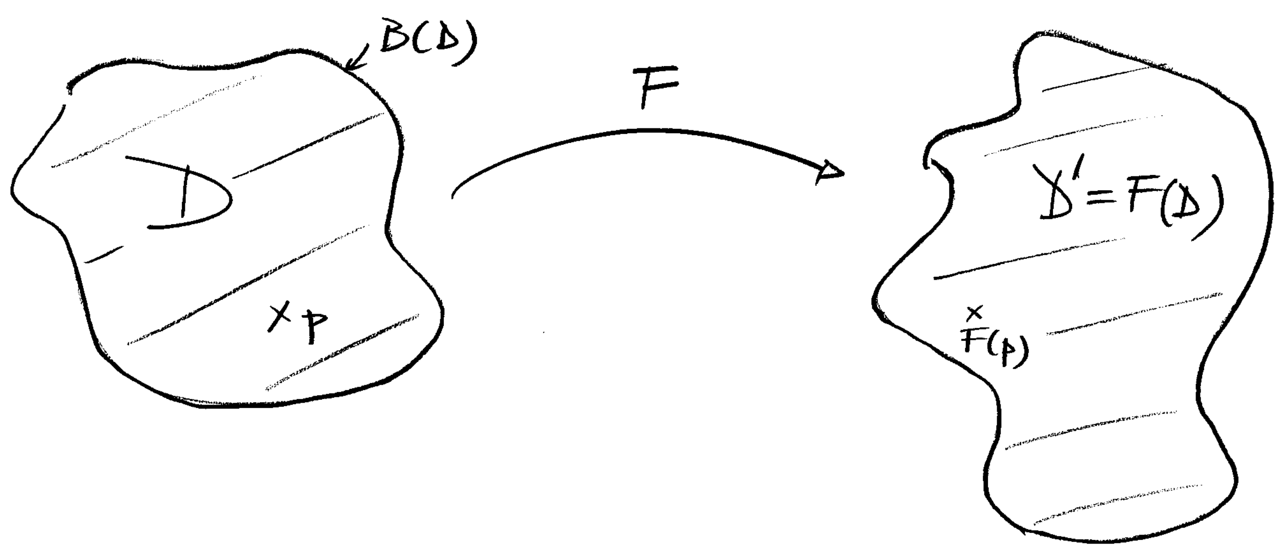
HOW TO FIND THE IMAGE OF A REGION in $\mathbb{R}^2$ under a mapping $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

The general strategy is to:

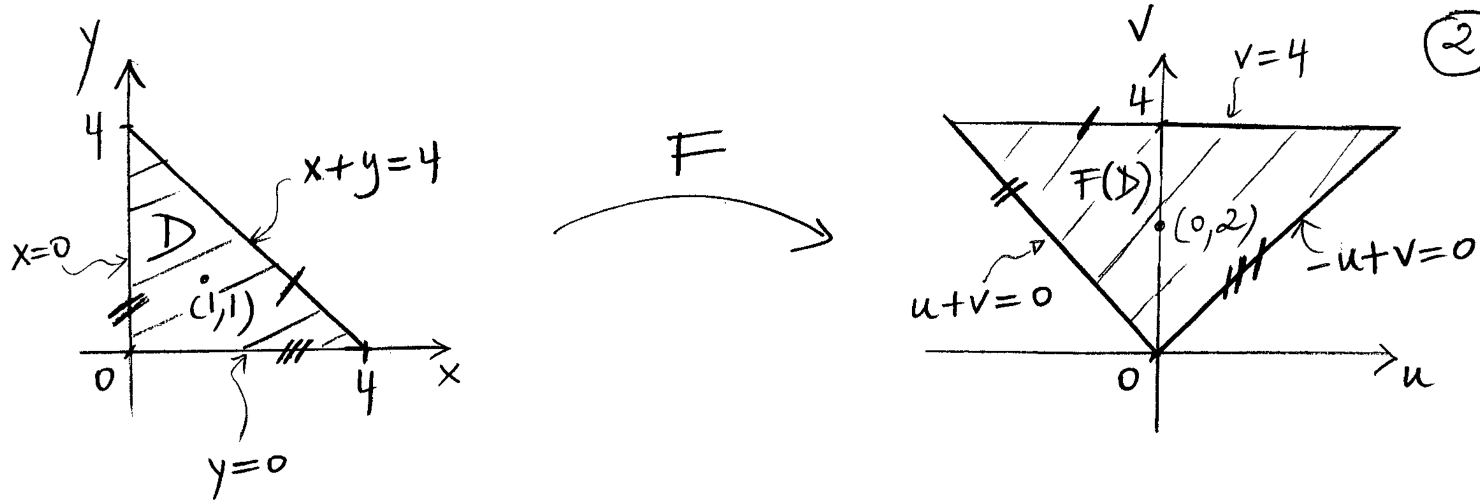
* Find the image under F of the boundary $B(D)$ of the region D.

* Find the image under F of a point OR a curve lying inside D to determine on what side of the image of $B(D)$ the image of D lies.



E.g.: Let D be the triangle in the (x,y)-plane with vertices $(0,0)$, $(0,4)$ and $(0,4)$.
Find the image of D under the mapping

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x-y,\ x+y) = (u,v).$$



* In this case, $B(D)$ is made up of 3 curves:

$$x+y=4, \quad x=0, \quad \text{and} \quad y=0.$$

Now, under F,

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \implies \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{-u+v}{2} \end{cases}$$

So:

$x+y=4$ gets mapped to $\boxed{v=4}$

$x=0$ gets mapped to $\boxed{u+v=0}$

$y=0$ gets mapped to $\boxed{-u+v=0}$

$\Rightarrow$ the boundary of the image $F(D)$ of D under F is made up of the 3 curves:

$$v=4, \quad u+v=0, \quad \text{and} \quad -u+v=0.$$

* Note that $(1,1) \in D$ and $F(1,1) = (0,2) \in F(D)$, so we get the above picture for $F(D)$.

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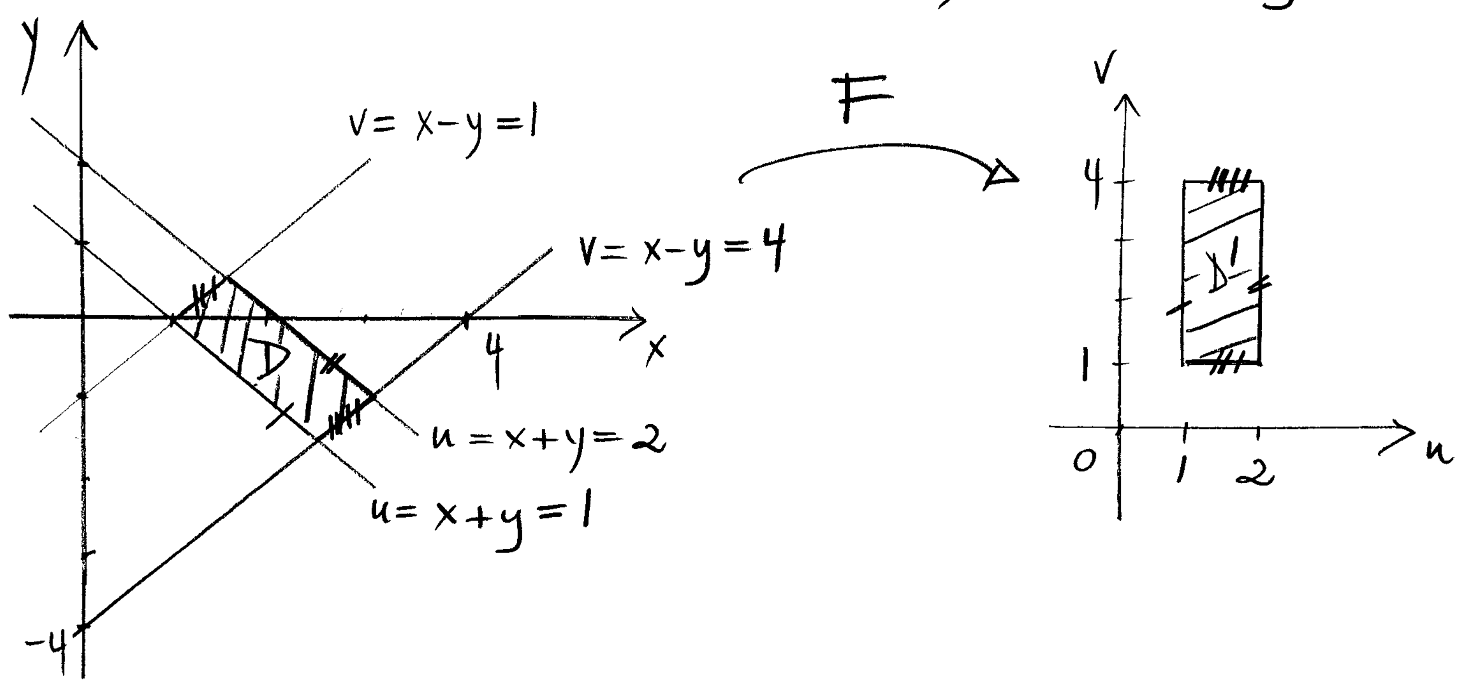
HOWEVER, it is sometimes easier to find the image of D by simply expressing the inequalities defining D (which depend on x & y) in terms of the new variables u & v.

E.g. 1) Find the image of $D = \{1 \le \underbrace{x+y}_{u} \le 2,\ 1 \le \underbrace{x-y}_{v} \le 4\}$ under $F(x,y) = (x+y,\ x-y) = (u,v)$.

$$\Rightarrow D' = F(D) = \{1 \le u \le 2,\ 1 \le v \le 4\}.$$



Note that the boundary of D is made up of the 4 curves:

$$x+y=1, \quad x+y=2, \quad x-y=1, \quad \text{and} \quad x-y=4,$$

which get mapped under F to:

$$u=1, \quad u=2, \quad v=1, \quad \text{and} \quad v=4,$$

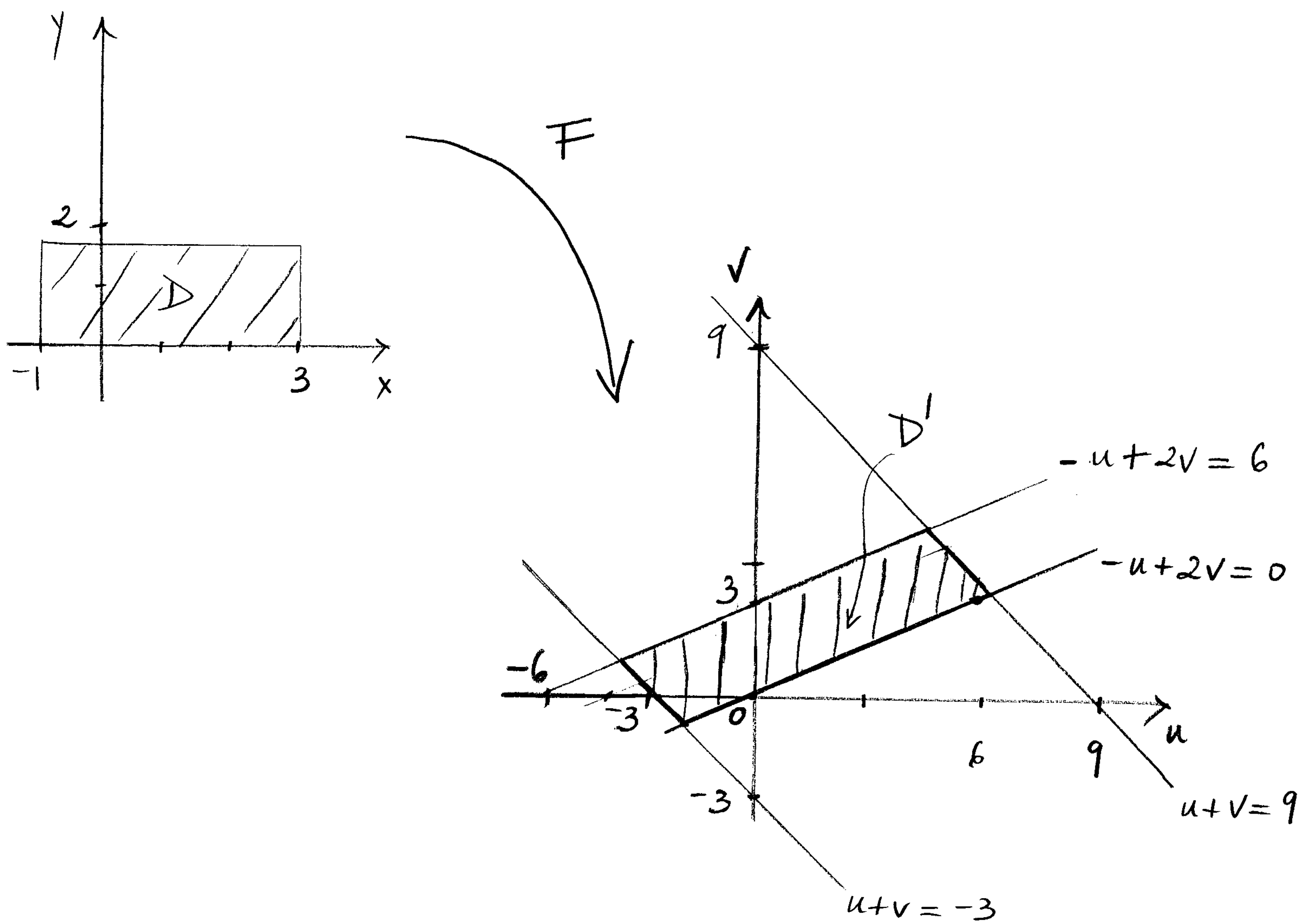
respectively.

2) Find the image of $D = \{-1 \le x \le 3,\ 0 \le y \le 2\}$ under $F(x,y) = (2x-y,\ x+y) = (u,v)$.

HERE: $\begin{cases} u = 2x - y \\ v = x + y \end{cases} \iff \begin{cases} x = \dfrac{u+v}{3} \\ y = \dfrac{-u+2v}{3} \end{cases}$

$\Rightarrow$ the image of D is

$$D' = F(D) = \left\{ -1 \le \frac{u+v}{3} \le 3,\ 0 \le \frac{-u+2v}{3} \le 2 \right\}$$
$$= \{ -3 \le u+v \le 9,\ 0 \le -u+2v \le 6 \}.$$



3) Find the image of $D = \{1 \le x \le 2,\ 2 \le y \le 4\}$ under $F(x,y) = (2x + e^y,\ 2x - e^y) = (u,v)$.

Since the boundary $B(D)$ is made up of the curves $x=1$, $x=2$, $y=2$, and $y=4$, let's try to express x and y as functions of u and v:

$$\left\{ \begin{array}{l} u = 2x + e^y \\ v = 2x - e^y \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x = \frac{u+v}{4} \\ y = \ln\left(\frac{u-v}{2}\right) \end{array} \right\}$$

THUS, D gets mapped to

$$D' = \left\{ 1 \le \frac{u+v}{4} \le 2,\ 2 \le \ln\left(\frac{u-v}{2}\right) \le 4 \right\}$$

$$= \{ 4 \le u+v \le 8,\ 2e^2 \le u-v \le 2e^4 \}.$$

4) Find the image of $D = \{-1 \le x-y \le 3,\ 0 \le x+y \le \frac{\pi}{2}\}$ under $F(x,y) = \left(\sin(x+y),\ \frac{x-y}{2}\right) = (u,v)$.

In D, $-1 \le x-y \le 3$ and $0 \le x+y \le \frac{\pi}{2}$.

So: $$-\frac{1}{2} \le v = \frac{x-y}{2} \le \frac{3}{2}$$

and
$$0 = \sin(0) \le u = \sin(x+y) \le \sin\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow D' = F(D) = \left\{ 0 \le u \le 1,\ -\frac{1}{2} \le v \le \frac{3}{2} \right\}.$$

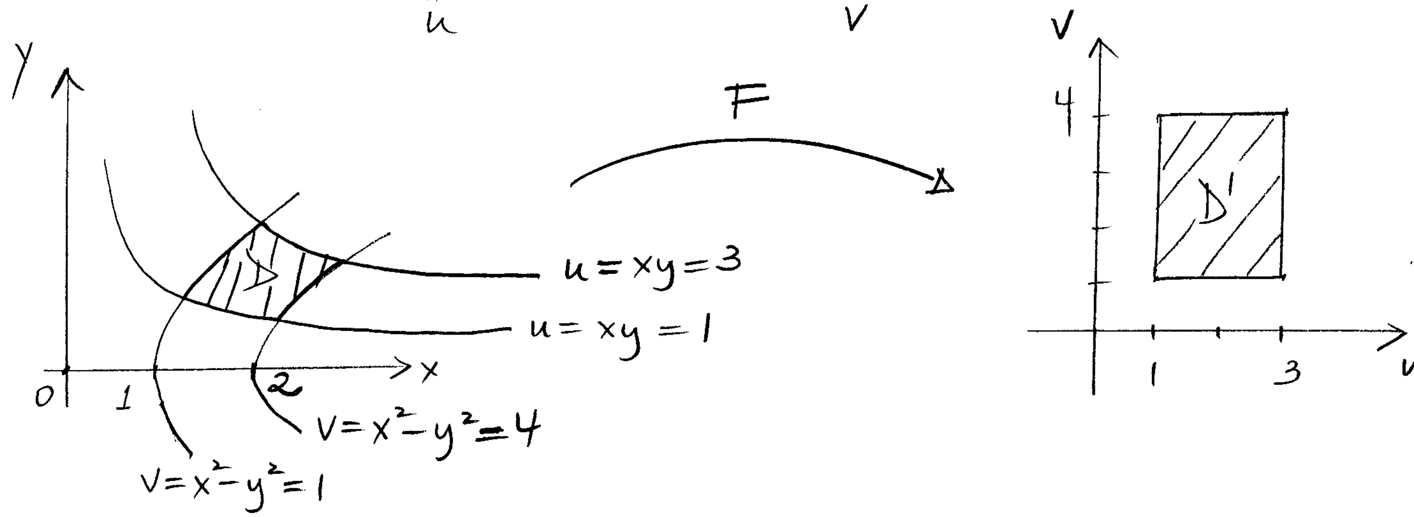
In general, what you will do will depend on the inequalities defining D and the expression of F.

E.g.: Consider $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

$$(x,y) \longmapsto (xy,\ x^2-y^2) = (u,v).$$

(i) Find the image under F of

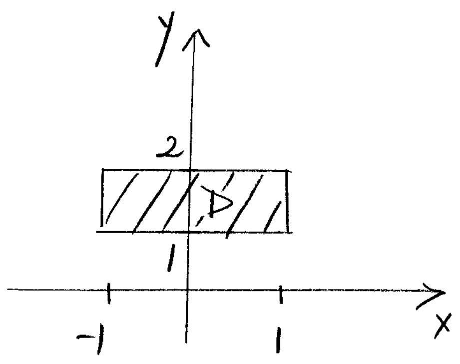
$$D = \{ 1 \le \underbrace{xy}_{=u} \le 3,\ 1 \le \underbrace{x^2-y^2}_{=v} \le 4,\ x \geq 0 \}$$



Since $u = xy$ and $v = x^2-y^2$, D gets mapped to

$$D' = \{ 1 \le u \le 3,\ 1 \le v \le 4 \}.$$

(ii) Find the image of $D = \{ -1 \le x \le 1,\ 1 \le y \le 2 \}$ under F



D bounded by:
$x = -1,\ x = 1,\ y = 0,\ y = 2.$

$\rightsquigarrow$ the simplest solution would be to express x & y as functions of u & v.

Since $u = xy$ and $v = x^2 - y^2$, it is not easy to do this. So, find the image of the curves $x=-1$, $x=1$, $y=0$, $y=2$ under F instead.

$\underline{x=-1}$: $u = -y$, $v = 1-y^2 \Rightarrow v = 1-u^2$ parabola with $u=-y \leq 0$ since $y \geq 0$ in D.

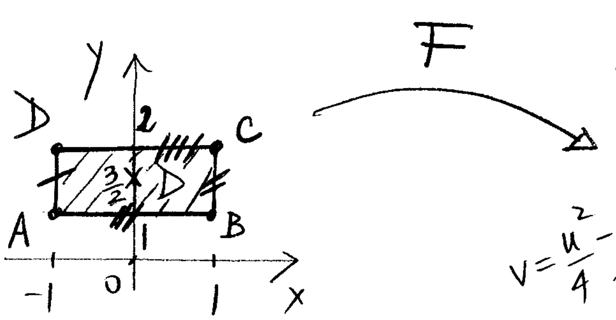
$\underline{x=1}$: $u=y$, $v = 1-y^2 \Rightarrow v = 1-u^2$ parabola with $u = y \geq 0$ since $y \geq 0$ in D.

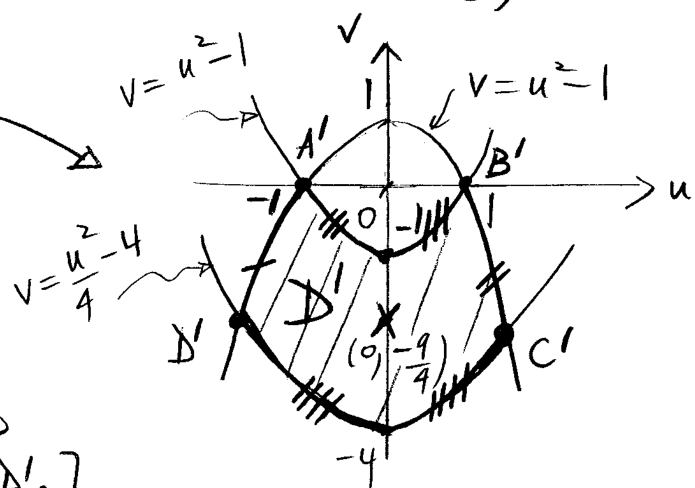
$\Rightarrow$ $x=-1$ and $x=1$ map to the whole parabola $v = 1-u^2$

$\underline{y=1}$: $u = x$, $v = x^2 - 1 \Rightarrow$ parabola $v = u^2 - 1$ with $-1 \leq u = x \leq 1$.

$\underline{y=2}$: $u = 2x$, $v = x^2 - 4 = \left(\frac{u}{2}\right)^2 - 4 \Rightarrow$ parabola $v = \frac{u^2}{4} - 4$ with $-2 \leq u = 2x \leq 2$.

ALSO, $(0, \frac{3}{2}) \in D$ gets mapped to $F(0, \frac{3}{2}) = (0, -\frac{9}{4})$.





[NOTE: The points A, B, C, D get mapped to A', B', C', D'.]

IN THIS CLASS, we're mostly interested in finding mappings that simplify the expression of a function we're trying to integrate (see double and triple integrals) OR that simplify the given region.

Ex. 1) Find a mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the parallelogram

$$D = \{-1 \le \overbrace{2x+y}^{u} \le 4, \quad 0 \le \overbrace{-4x+3y}^{v} \le 2\}$$

into a rectangle.

Set $u = 2x + y$ and $v = -4x + 3y$. Then, F maps D to

$$D' = \{-1 \le u \le 4, \quad 0 \le v \le 2\}.$$

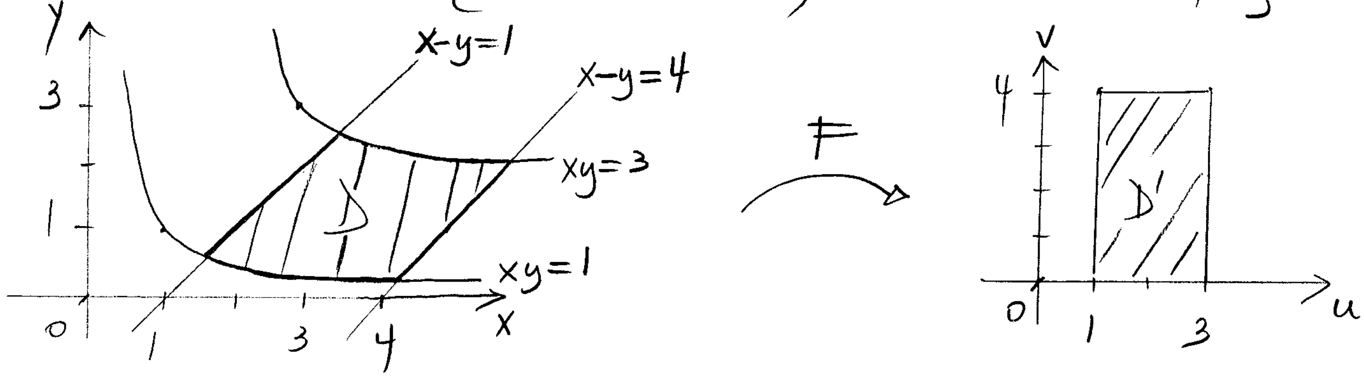
2) Find a mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ that transforms the region

$$D = \{1 \le \overbrace{xy}^{u} \le 3, \quad 1 \le \overbrace{x-y}^{v} \le 4\}$$

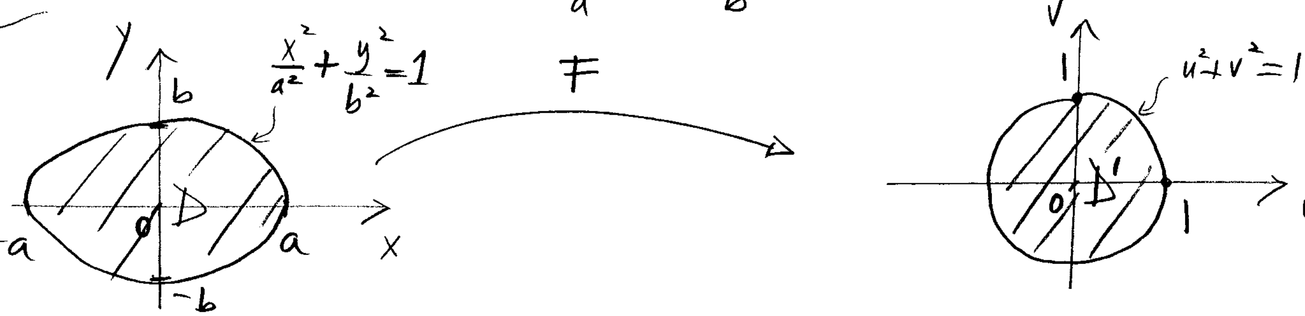
into a rectangle.

Set $u = xy$ and $v = x - y$. Then, F maps D to

$$D' = \{1 \le u \le 3, \quad 0 \le v \le 4\}.$$



3) Find a transformation that takes the region D bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to a unit disc.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \iff u^2 + v^2 = 1$$

with $u = \frac{x}{a}$ & $v = \frac{y}{b}$

So: set $F(x,y) = \left(\frac{x}{a}, \frac{y}{b}\right) = (u, v)$.

4) Find a transformation that maps the ellipse $4x^2 + 12xy + 10y^2 = 5$ to a unit circle.

So: find $F(x,y) = (f(x,y), g(x,y)) = (u,v)$ that maps $4x^2 + 12xy + 10y^2 = 5$ to $u^2 + v^2 = 1$.

IDEA: complete the square!

$$4x^2 + 12xy + 10y^2 = \underbrace{(2x+3y)^2}_{\substack{\shortparallel \\ 4x^2+12xy+9y^2}} + y^2$$

So, $4x^2 + 12xy + 10y^2 = 5 \iff (2x+3y)^2 + y^2 = 5$

$$\iff \left(\frac{2x+3y}{\sqrt{5}}\right)^2 + \left(\frac{y}{\sqrt{5}}\right)^2 = 1 \iff u^2 + v^2 = 1 \text{ with}$$

$u = \frac{2x+3y}{\sqrt{5}}$ and $v = y/\sqrt{5}$.

$$\implies F(x,y) = \left((2x+3y)/\sqrt{5},\ y/\sqrt{5}\right) = (u,v).$$

MAPPINGS $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

The same ideas apply to mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$. And we'll mostly be interested in finding mappings that simplify the expression of functions we integrate OR the 3-dimensional region over which integrate.

Ex. 1) Find a transformation that maps the parallelpiped

$$R = \{ 1 \le \overbrace{x+y+z}^{u} \le 3,\ 0 \le \overbrace{x-2y}^{v} \le 1,\ -1 \le \overbrace{2x+3z}^{w} \le 5 \}$$

to a rectangular box.

Set $F(x,y,z) = (x+y+z,\ x-2y,\ 2x+3z) = (u,v,w)$.
Then F maps R to the rectangular box:

$$R' = \{ 1 \le u \le 3,\ 0 \le v \le 1,\ -1 \le w \le 5 \}.$$

2) Find a transformation that maps the region bounded by the ellipsoid

$$14x^2 + y^2 + 5z^2 - 6xy + 8xz - 4yz = 4 \quad (*)$$

into a unit sphere.

So: find $F(x,y,z) = (f(x,y,z),\ g(x,y,z),\ h(x,y,z)) = (u,v,w)$ that transforms $(*)$ into $u^2 + v^2 + w^2 = 1$.

Recall that

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

$\rightsquigarrow$ USE this to complete the square!

$$14x^2 + y^2 + 5z^2 - 6xy + 8xz - 4yz \qquad (**)$$

$\hookrightarrow$ since there's only one y^2, let's focus on the y's in the expression.

Let's start with

$$(3x - y + 2z)^2$$

because of $-6xy$

because of $-4yz$

because the mixed terms involving y in $(**)$ are negative

Then, $(3x - y + 2z)^2 = 9x^2 + y^2 + 4z^2 - 6xy + 12xz - 4yz$, so that

$$\begin{aligned} &14x^2 + y^2 + 5z^2 - 6xy + 8xz - 4yz \\ &= (3x - y + 2z)^2 + 5x^2 + z^2 - 4xz \\ &= (3x - y + 2z)^2 + (2x - z)^2 + x^2 \end{aligned}$$

$\rightsquigarrow$ complete this square!

because of $-4xz$

since we only have z^2 left

So, the equation of the ellipsoid is:

$$x^2 + (3x - y + 2z)^2 + (2x - z)^2 = 4$$

$$\Leftrightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{3x - y + 2z}{2}\right)^2 + \left(\frac{2x - z}{2}\right)^2 = 1 .$$

$$\Leftrightarrow \quad u^2 + v^2 + w^2 = 1 \quad \text{with}$$

$$u = \frac{x}{2}, \quad v = \frac{3x - y + 2z}{2}, \quad w = \frac{2x - z}{2} .$$

$\rightsquigarrow$ Set

$$F(x,y,z) = \left(\frac{x}{2}, \frac{3x - y + 2z}{2}, \frac{2x - z}{2}\right).$$