Math 237: Suggested problems for the midterm – Fall 2010.

Level sets and cross-sections.

- Problem Set 1: A1 (a), (b).
- Find the domain and range of the following functions f, and sketch level curves and cross-sections of $z = f(x, y)$ for each f.
  (a) $f(x, y) = 3 - x^2 - y^2$
  (b) $f(x, y) = \sqrt{x^2 + y^2}$
  (c) $f(x, y) = x^2 - y^2 + 2$
  (d) $f(x, y) = -\sqrt{4 + x^2 + y^2}$

Limits and continuity.

- State the definition of a limit and of continuity of a function.
- Problem Set 1: A2, A3, B2 (i), (ii), (iii), (iv), (v), (vii).
- Prove that the following limits exists.
  (a) $\displaystyle\lim_{(x,y)\to(1,-2)} \frac{x^2y - \sin y}{\ln x - y}$
  (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^4 + 4y^2 + 2y^3x^2}{x^4 + 2y^2}$
  (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2|x| - 3y^2}{\sqrt{x^2 + y^2}}$
- Problem Set 1: C1.

Partials, linear approximations, and differentiability.

- State the definition of the first and second order partials, of the linear approximation, and of differentiability of a function f of two or three variables.
- Let $f(x, y)$ be a function of two variables. Show that if f is differentiable at (a, b), then f is continuous at (a, b). Is the converse true? Justify your answer.
- Show that if the first order partials f_x and f_y of a function of two variables $f(x, y)$ are continuous at (a, b), then f is continuous at (a, b). Is the converse true? Justify your answer.
- Problem Set 2: A1, A2, A3, A4 (iii).

  Also, use the linear approximation found in A3 (b) to approximate $f(3.02, -0.99)$, where f is the function in A3 (i). Similarly, use the linear approximation found in A3 (b) to approximate $f(0.99, 1.01, -1.02)$, where f is the function in A3 (iii).
- Let
  $$f(x, y) = \begin{cases} \frac{x^2|x|}{\sqrt{x^2+y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$
  Compute the linear approximation of f at $(0, 0)$ and at $(-1, 2)$.
- Problem Set 2: B1 (a), B3 (i), (ii), (iv); Assignment 4: 1 (a).

  Also, in question B3 (iv), determine all the points where the function f is continuous and and all the points where f is differentiable.
- For each of the following functions f, determine all the points where f is continuous and all the points where f is differentiable.

(a) $f(x, y) = x^2 + (3x - 1)e^{y^2}$
(b) $f(x, y) = y\cos(x^2y - 3x)$

- Is the following statement true: "A function of two variables $f(x, y)$ is differentiable at a point (a, b) if and only if its first order partials $f_x(a, b)$ and $f_y(a, b)$ both exist at (a, b)"? Justify your answer.

  Hint: See question B3 (ii) on Problem Set 2.

The Chain Rule.

- Problem Set 3: A2, A3, A4, A6, A7, A17, A19; Assignment 4: 2, 3; all examples in Chapter 6 of the Lecture Notes.
- Suppose that $f(x, y)$ satisfies the Laplace equation $f_{xx} + f_{yy} = 0$. Show that $g(s, t) = f(as + bt, bs - at)$ also satisfies the Laplace equation: $g_{ss} + g_{tt} = 0$.

Directional derivatives and tangent planes.

- State the definition of the directional derivative of a function f of two or three variables.
- State the definition of the tangent plane to the graph of a function of two variables f at the point $(a, b, f(a, b))$.
- What is the equation of the tangent plane to the level surface $f(x, y, z) = k$ of a function of three variables f at the point (a, b, c).
- Problem Set 3: A8, A9, A10, A14, A16, B1.
- Consider the function $f(x, y) = xy^2 - 3e^{x-y}$.
  (a) What are the maximal and minimal rates of change of f at the point (1,1) and in what directions do they occur?
  (b) Is there a direction in which the rate of change of f at the point (1,1) is 7? If your answer is yes, find all possible directions.
  (c) Is there a direction in which the rate of change of f at the point (1,1) is 0? If your answer is yes, find all possible directions.
- In question A7 on Problem Set 3, the velocity vector to the path of the spacecraft at time $t = 0$ is $(2, 1, 2)$. Find the directional derivative of the temperature of the spacecraft at time $t = 0$ in the direction of the velocity. How does this rate of change compare to the rate of change you computed in A7 (b)?

Taylor polynomials and Taylor's Theorem.

- State the definition of second degree Taylor polynomial; also state Taylor's Theorem.
- Problem Set 4: A1 (b), A2, B2, B3, B4.
- Let $f(x, y) = e^{3-x-2y}$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(0,0)}(x, y)$ is at most $3e^3(x^2 + y^2)$ if $x \geq 0$ and $y \geq 0$.
- Let $f(x, y) = \frac{1}{x} + \frac{1}{y}$ for $x > 0$ and $y > 0$. Show that for any (a, b), with $a > 0$ and $b > 0$, we have $f(x, y) > L_{(a,b)}(x, y)$ for all $(x, y) \neq (a, b)$ with $x > 0$ and $y > 0$.
- Let $f(x, y) = \ln(3x - 2y)$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(1,1)}(x, y)$ is at most $\frac{15}{2}3^2[(x - 1)^2 + (y - 1)^2]$ if $1 \leq x \leq 3, 1 \leq y \leq \frac{4}{3}$.