Math 237: Suggested problems for the midterm – Fall 2011.

Level sets, cross-sections, and graphing.

- Problem Set 1: A1 (a), (b).
- Find the domain and range of the following functions f, and sketch level curves and cross-sections of $z = f(x, y)$ for each f. Also sketch the graph $z = f(x, y)$.
  (a) $f(x, y) = 3 - x^2 - y^2$
  (b) $f(x, y) = \sqrt{x^2 + y^2}$
  (c) $f(x, y) = -\sqrt{4 - x^2 - y^2}$
  (d) $f(x, y) = \sqrt{x^2 - y^2}$

Limits and continuity.

- State the definition of a limit and of continuity of a function.
- State and prove the Squeeze Theorem.
- Problem Set 1: A2, A3, B2; Assignment 2: 3.
- Prove that the following limits exists.
  (a) $\lim_{(x,y)\to(1,-2)} \dfrac{x^2 y - \sin y}{\ln x - y}$
  (b) $\lim_{(x,y)\to(0,0)} \dfrac{4x^2 - y^4}{2|x| + y^2}$
  (c) $\lim_{(x,y)\to(0,0)} \dfrac{2x^4 + 4y^2 + 2y^3 x^2}{x^4 + 2y^2}$
  (d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2|x| - 3y^2}{\sqrt{x^2 + y^2}}$
- Problem Set 1: C1.

Partials, linear approximations, and differentiability.

- State the definition of the first and second order partials, of the linear approximation, and of differentiability of a function f of two or three variables.
- Let $f(x, y)$ be a function of two variables. Show that if f is differentiable at (a, b), then f is continuous at (a, b). Is the converse true? Justify your answer.
- Show that if the first order partials f_x and f_y of a function of two variables $f(x, y)$ are continuous at (a, b), then f is continuous at (a, b). Is the converse true? Justify your answer.
- Problem Set 2: A1, A2, A3, A4 (iii).

  Also, use the linear approximation found in A3 (b) to approximate $f(3.02, -0.99)$, where f is the function in A3 (i). Similarly, use the linear approximation found in A3 (b) to approximate $f(0.99, 1.01, -1.02)$, where f is the function in A3 (iii).
- Let

$$f(x, y) = \begin{cases} \frac{x^2|x|}{\sqrt{x^2+y^2}}, & \text{if } (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

  Determine all points where f is continuous. Find the linear approximation of f at both $(0, 0)$ and $(-1, 2)$. Is f differentiable at $(0, 0)$? Compute the directional derivative of f at $(0, 0)$ in the direction $\vec{v} = (1, -3)$.

- Problem Set 2: B3 (i), (ii), (iv); Assignment 3: 2 (a) – (c).

  Also, in question B3 (iv), determine all the points where the function f is continuous and and all the points where f is differentiable.

- For each of the following functions f, determine all the points where f is continuous and all the points where f is differentiable.

  (a) $f(x,y) = x^2 + (3x-1)e^{y^2}$

  (b) $f(x,y) = y\cos(x^2y - 3x)$

  (c) $f(x,y) = \begin{cases} \frac{xy^3}{\sqrt{x^2+y^6}}, & \text{if } (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

- Is the following statement true: "A function of two variables $f(x,y)$ is differentiable at a point (a,b) if and only if its first order partials $f_x(a,b)$ and $f_y(a,b)$ both exist at (a,b)"? Justify your answer.

  Hint: See question B3 (ii) on Problem Set 2.

- Is the following statement true: "If a function of two variables $f(x,y)$ is C^2 about a point (a,b), then it is continuous at (a,b)"? Justify your answer.

- Is the following statement true: "If a function of two variables $f(x,y)$ is C^2 about a point (a,b), then $Hf(a,b)$ is a symmetric 2×2 matrix"? Justify your answer.

The Chain Rule.

- Problem Set 3: A2, A3, A4, A5 (i) – (iv), A6, A7, A17, A18; Assignment 4: 3, 5.

- Suppose that $f(x,y)$ satisfies the Laplace equation $f_{xx} + f_{yy} = 0$. Show that $g(s,t) = f(as+bt, bs-at)$ also satisfies the Laplace equation $g_{ss} + g_{tt} = 0$.

Directional derivatives and tangent planes.

- State the definition of the directional derivative of a function f of two or three variables.

- State the definition of the tangent plane to the graph of a function of two variables f at the point $(a,b,f(a,b))$.

- What is the equation of the tangent plane to the level surface $f(x,y,z) = k$ of a function of three variables f at the point (a,b,c)?

- Problem Set 3: A8, A9, A12, A14, B4 (a), (b); Assignment 5: 1, 2.

  In question A9, also find the direction in which the temperature decreases the fastest. In question 1 on Assignment 5, also determine the slope of the path starting at $(x,y) = (-20,5)$ in the direction $\vec{v} = (2,1)$, and the slope of the steepest path starting at $(-20,5)$.

- Show that the equation of the tangent plane to the hyperboloid

  $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

  at the point (x_0,y_0,z_0) is

  $$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} - \frac{zz_0}{c^2} = -1,$$

  where a,b,c are positive real numbers.

- Consider the function $f(x,y) = xy^2 - 3e^{x-y}$.

(a) What are the maximal and minimal rates of change of f at the point (1,1) and in what directions do they occur?

(b) Is there a direction in which the rate of change of f at the point (1,1) is 7? If your answer is yes, find all possible directions.

(c) Is there a direction in which the rate of change of f at the point (1,1) is 0? If your answer is yes, find all possible directions.

- Is the following statement true: "If a function of two variables $f(x, y)$ is differentiable at a point (a, b), then its directional derivative exists in all directions at (a, b)"? What about the converse? Justify your answers.

  Hint: For the converse, see question 4 on Assignment 5.

Taylor polynomials and Taylor's Theorem.

- State the definition of second degree Taylor polynomial; also state Taylor's Theorem.
- Problem Set 4: A1, A2, B2, B3, B4.
- Let $f(x, y) = e^{3-x-2y}$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(0,0)}(x, y)$ is at most $3e^3(x^2 + y^2)$ if $x \geq 0$ and $y \geq 0$.
- Let $f(x, y) = \frac{1}{x} + \frac{1}{y}$ for $x < 0$ and $y < 0$. Show that for any (a, b), with $a < 0$ and $b < 0$, we have $f(x, y) < L_{(a,b)}(x, y)$ for all $(x, y) \neq (a, b)$ with $x < 0$ and $y < 0$.
- Let $f(x, y) = \ln(3x - 2y)$. Use Taylor's Theorem to show that the error in the linear approximation $L_{(1,1)}(x, y)$ is at most $\frac{15}{2}3^2[(x-1)^2 + (y-1)^2]$ if $1 \leq x \leq 3, 1 \leq y \leq \frac{4}{3}$.