Taylor's Theorem:

Ex. Let $f(x,y) = \frac{1}{x} + \frac{1}{y}$ for $x, y < 0$. Show that

$$f(x,y) < L_{(a,b)}(x,y), \quad \forall\, x, y, a, b < 0. \text{ with } (x,y) \neq (0,0)$$

Note that

$$f(x,y) < L_{(a,b)}(x,y) \iff \underbrace{f(x,y) - L_{(a,b)}(x,y)}_{= R_{1,(a,b)}(x,y)} < 0$$

$$\iff R_{1,(a,b)}(x,y) < 0.$$

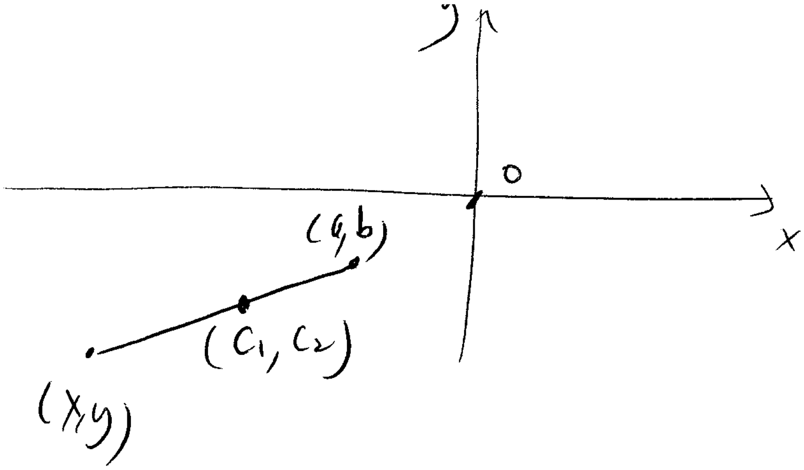
So, we need to check that

$$R_{1,(a,b)}(x,y) < 0, \quad \forall\, x, y, a, b < 0. \text{ with } (x,y) \neq (0,0)$$

By Taylor's Theorem,

$$R_{1,(a,b)}(x,y) = \frac{1}{2!}\Big[f_{xx}(c_1,c_2)(x-a)^2 + 2 f_{xy}(c_1,c_2)(x-a)(y-b) + f_{yy}(c_1,c_2)(y-b)^2\Big]$$

for (c_1, c_2) on line segment joining (a,b) and (x,y).



$\Longrightarrow \quad c_1, c_2 < 0 \quad$ since $\quad a, b, x, y < 0$.

Now, $f_{xx} = \dfrac{1}{x^3}$, $f_{xy} = f_{yx} = 0$, $f_{yy} = \dfrac{1}{y^3}$.

$$\Longrightarrow R_{1,(a,b)}(x,y) = \frac{1}{2!}\left[\frac{1}{c_1^3}(x-a)^2 + \frac{1}{c_2^3}(y-b)^2\right]$$

< 0 if $(x,y) \neq (a,b)$

since $c_1, c_2 < 0$. ∎

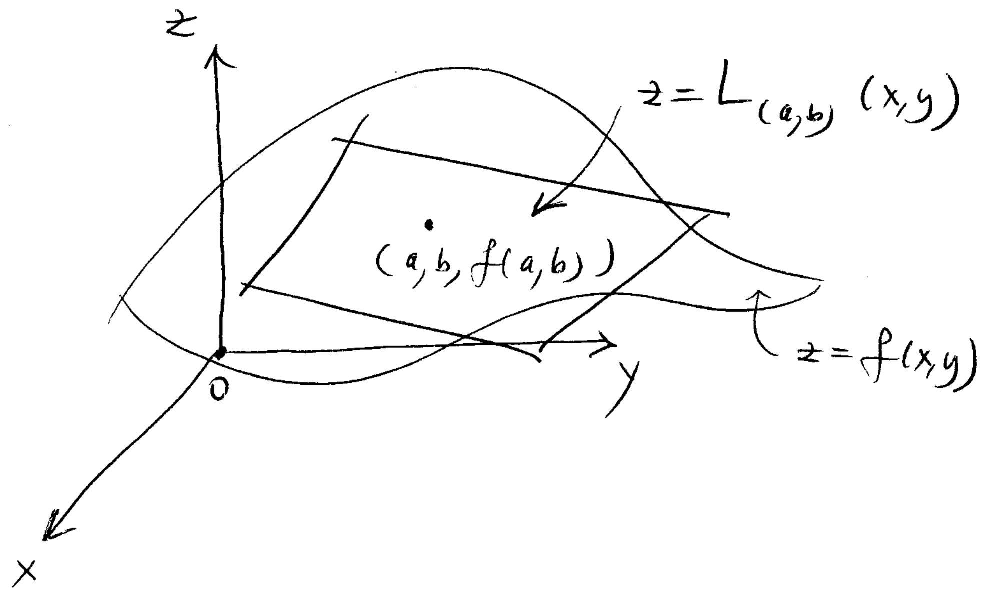
Tangent plane:

* **Graph of a function:**

If $z = f(x,y)$ with f diff. at (a,b), then

$$\boxed{z = L_{(a,b)}(x,y)}$$

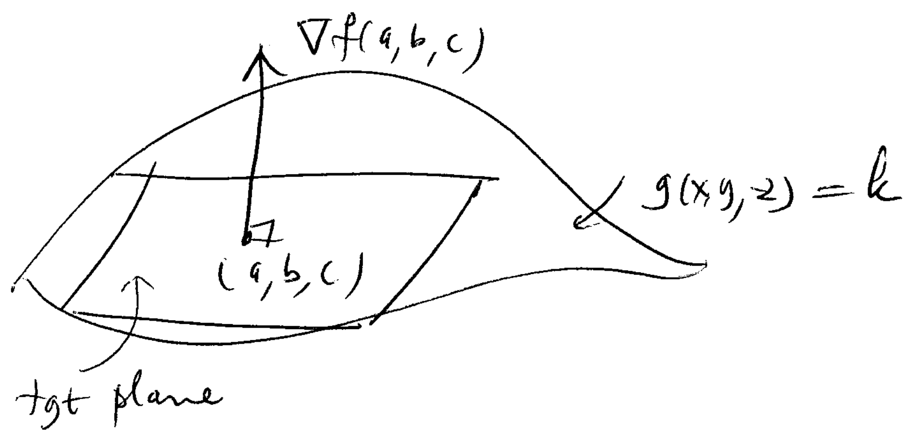
is the tgt plane to $z = f(x,y)$ at $(a,b,f(a,b))$



* **Level surface:**

If $g(x,y,z) = k$, then tgt plane at (a,b,c) is:

$$\boxed{\nabla f(a,b,c) \cdot (x-a,\ y-b,\ z-c) = 0.}$$



ex: 1) Find the tgt plane at $(2, \pi, -4)$ to $z = f(x,y) = x^2 \cos y$.

It's the graph of a fct so use $z = L_{(a,b)}(x,y)$.

$f_x = 2x \cos y$ and $f_y = -x^2 \sin y$

$\Rightarrow$ f_x and f_y are both cont. at $(2,\pi)$

$\Rightarrow$ f is diff. at $(2,\pi)$ and the tgt plane at $(2,\pi,-4)$ is:

$$z = L_{(2,\pi)}(x,y) = f(2,\pi) + f_x(2,\pi)(x-2) + f_y(2,\pi)(y-\pi)$$

$$\Leftrightarrow \quad z = -4 + (-4)(x-2) + (0)(y-\pi)$$

$$\Leftrightarrow \quad \boxed{z = 4 - 4x}.$$

2) Find the tgt plane to $x^2y - ze^{2x+3y} = 3$ at $(-3, 2, 15)$.

It's a level surface: $g(x,y,z) = 3$ with $g(x,y,z) = x^2y - z e^{2x+3y}$

$\Rightarrow$ tgt plane at $(-3,2,15)$ is

$$\nabla(-3,2,15) \cdot (x-(-3), y-2, z-15) = 0.$$

BUT, $\nabla g = (g_x, g_y, g_z) = (2xy - 2ze^{2x+3y}, x^2 - 3ze^{2x+3y}, -e^{2x+3y})$

$\Rightarrow \quad \nabla g(-3,2,15) = (-42,-36,-1)$

and tgt plane is:

$$(-42,-36,-1)\cdot(x+3,\ y-2,\ z-15)=0$$

$$\Leftrightarrow \quad \boxed{42(x+3)+36(y-2)+(z-15)=0}$$

3) The equation of the tgt plane to $x^2/a^2 - y^2/b^2 - z^2/c^2 = -1$ at (x_0, y_0, z_0) is

$$\boxed{xx_0/a^2 - yy_0/b^2 - zz_0/c^2 = -1}.$$

Pf: Here we that the level surface $g(x,y,z)=-1$ where $g(x,y,z) = x^2/a^2 - y^2/b^2 - z^2/c^2$.

Then, $\nabla g = (2x/a^2, -2y/b^2, -2z/c^2)$.

$\Rightarrow$ Tgt plane is

$$\nabla g(x_0,y_0,z_0)\cdot(x-x_0,\ y-y_0,\ z-z_0) = 0$$

$$\Leftrightarrow \left(\cancel{2}x_0/a^2, -\cancel{2}y_0/b^2, -\cancel{2}z_0/c^2\right)\cdot(x-x_0, y-y_0, z-z_0)=0$$

$$\Leftrightarrow xx_0/a^2 - yy_0/b^2 - zz_0/c^2 = x_0^2/a^2 - y_0^2/b^2 - z_0^2/c^2$$

$$= -1$$

(since (x_0,y_0,z_0) is on $x^2/a^2 - y^2/b^2 - z^2/c^2 = -1$).

Chain Rule.

ex. 1) A4 from problem set 3.

Suppose that $f: \mathbb{R}^2 \to \mathbb{R}$ is given, and that $g: \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(s,t) = f(st, s^2 - t^2).$$

If $\nabla f(2,-3) = (4,3)$, find $\nabla g(1,2)$.

We are assuming that f is diff. at $(2,-3)$. Then, $\nabla g(1,2) = (g_s(1,2), g_t(1,2))$.

We have $g(s,t) = f(x(s,t), y(s,t))$ with $x(s,t) = st$ and $y(s,t) = s^2 - t^2$. Therefore,

$$g_s = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= f_x(st, s^2-t^2) \cdot t + f_y(st, s^2-t^2) \cdot 2s$$

$$\Rightarrow g_s(1,2) = f_x(2,-3) \cdot 2 + f_y(2,-3) \cdot (2)$$

$$= 4 \cdot 2 + 3 \cdot 2 = 14$$

since

$(4,3) = \nabla f(2,-3) = (f_x(2,-3), f_y(2,-3))$

Also,

$$g_t = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= f_x(st, s^2 - t^2) \cdot s + f_y(st, s^2 - t^2) \cdot (-2t)$$

$$\Longrightarrow \quad g_t(1, 2) = f_x(2, -3) \cdot (1) + f_y(2, -3)(-4)$$

$$= 4 \cdot 1 + 3 \cdot (-4) = -8.$$

$$\Longrightarrow \quad \boxed{\nabla g(1,2) = (14, -8)}.$$

2) A6 (i) in problem set 3.

If $F(x,y) = y f(x^2 - y^2)$, show that

$$y F_x + x F_y = \frac{x}{y} F.$$

We assume that $f = f(t)$ is diff. at every pt. Then,

$$F_x = \frac{\partial}{\partial x}\left[y f(x^2 - y^2) \right]$$

constant with respect to x!

$$= y \frac{\partial}{\partial x}\left(f(x^2 - y^2) \right)$$

$$= y \left(f'(x^2 - y^2) \cdot 2x \right) = 2xy f'(x^2 - y^2).$$

$$F_y = \frac{\partial}{\partial y}\Big(\underbrace{y}\cdot\underbrace{f(x^2-y^2)}_{\text{product!}}\Big)$$

$$= \frac{\partial}{\partial y}(y)\cdot f(x^2-y^2) + y\cdot\frac{\partial}{\partial y}\big(f(x^2-y^2)\big)$$

product Rule

$$= 1\cdot f(x^2-y^2) + y\cdot\Big(f'(x^2-y^2)\cdot(-2y)\Big)$$

$$= f(x^2-y^2) - 2y^2 f'(x^2-y^2).$$

$$\Longrightarrow \quad yF_x + xF_y = y\big(\cancel{2xy\,f'(x^2-y^2)}\big)$$
$$+\, x\big(f(x^2-y^2) - \cancel{2y^2 f'(x^2-y^2)}\big)$$

$$= x f(x^2-y^2)$$

$$= \frac{x}{y}\big(\underbrace{y\,f(x^2-y^2)}_{=F}\big)$$

$$= \frac{x}{y}F.$$

$$\Longrightarrow \quad \boxed{yF_x + xF_y = \frac{x}{y}F}\ . \ \square$$

3) A18 on problem set 3.

If $u = f(x + g(y))$, where f and g are C^2, show that $u_x u_{xy} = u_y u_{xx}$. Also find u_{yy}.

$$u_x = f'(x + g(y)) \cdot 1$$

$$u_y = f'(x + g(y)) \cdot g'(y).$$

$$u_{xx} = (u_x)_x = \frac{\partial}{\partial x}\left(f'(x + g(y))\right)$$
$$= f''(x + g(y)) \cdot 1.$$

$$u_{xy} = (u_x)_y = \frac{\partial}{\partial y}\left(f'(x + g(y))\right)$$

$$= f''(x + g(y)) \cdot g'(y)$$

So, $$u_x u_{xy} = f'(x + g(y)) \cdot f''(x + g(y)) \cdot g'(y)$$

$$= \left[f'(x + g(y)) \cdot g'(y)\right] \cdot f''(x + g(y))$$

$$= u_y u_{xx}.$$

Also, $$u_{yy} = (u_y)_y = \frac{\partial}{\partial y}\Big(\underbrace{f'(x + g(y)) \cdot g'(y)}_{\text{product}}\Big)$$

$$= \frac{\partial}{\partial y}\left(f'(x + g(y))\right) \cdot g'(y) + f'(x + g(y)) \cdot \frac{\partial}{\partial y}\left(g'(y)\right)$$

$$= f'(x + g(y)) \cdot (g'(y))^2 + f'(x + g(y)) \cdot g''(y).$$

4) Let $g(u,v) = f(\underbrace{u^2-v^2}_{\overset{\shortparallel}{x}}, \underbrace{2uv}_{\overset{\shortparallel}{y}})$. Find g_{uv}.

$$g_u = f_x \cdot x_u + f_y \cdot y_u$$

$$= f_x(u^2-v^2, 2uv)\cdot 2u + f_y(u^2-v^2, 2uv)\cdot(2v)$$

$$g_{uv} = (g_u)_v$$

$$= \frac{\partial}{\partial v}\Big(\underbrace{f_x(u^2-v^2,2uv)\cdot 2u}_{\text{product}}\Big) + \frac{\partial}{\partial v}\Big(\underbrace{f_y(u^2-v^2,2uv)\cdot 2v}_{\text{product}}\Big)$$

product Rule

$$= \frac{\partial}{\partial v}\big(f_x(u^2-v^2,2uv)\big)\cdot 2u + f_x \cdot \cancel{\frac{\partial}{\partial v}(2u)}^{\,0}$$

$$+ \frac{\partial}{\partial v}\big(f_y(u^2-v^2,2uv)\big)\cdot 2v + f_y \cdot \frac{\partial}{\partial v}(2v)$$

$$= \big[(f_x)_x\cdot(-2v) + (f_x)_y\cdot(2u)\big]\cdot 2u + 0$$

$$+ \big[(f_y)_x\cdot(-2v) + (f_y)_y\cdot(2u)\big]\cdot 2v + f_y\cdot 2$$

$$= -4uvf_{xx} + 4u^2 f_{xy} - 4v^2 f_{yx} + 4uvf_{yy} + 2f_y.$$