Chapter 1. Scalar functions.

Functions of 2 variables.

DEF.: A function of 2 variables f is a rule that assigns to each pair of real numbers (x,y) in a subset of $\mathbb{R}^2$ a unique real number denoted by $f(x,y)$, called the value of f at (x,y). We write $f:\mathbb{R}^2\to\mathbb{R}$

If f has a value at (x,y) then it is said to be defined at (x,y). The set of all points in $\mathbb{R}^2$ where f is defined is called the domain of f and is denoted $D(f)$. Moreover, the range of f denoted $R(f)$, is the set of all values that f takes on, i.e.,

$$R(f) := \{ f(x,y) \mid (x,y) \in D(f) \}.$$

IMPORTANT: $D(f) \subseteq \mathbb{R}^2$

$R(f) \subseteq \mathbb{R}$.

ex. 1) $f(x,y)$ = polynomial in x and y.

$D(f) = \mathbb{R}^2$, but $R(f)$ can vary greatly.

e.g. * $f(x,y) = k$ = constant $\Rightarrow R(f) = \{k\}$ = point.

* $f(x,y) = xy^2 \Rightarrow R(f) = \mathbb{R}$

(since, $\forall t \in \mathbb{R}$, $t = f(t,1)$.)

* $f(x,y) = x^2+y^2 \Rightarrow R(f) = [0,+\infty)$.

(Indeed, note that $x^2+y^2 \geq 0$ for all (x,y). Also, $\forall t \geq 0$, $t = f(\sqrt{t}, 0)$, implying that $R(f) = [0,+\infty)$.)

* $f(x,y) = 4 - x^2 - y^2 \Rightarrow R(f) = (-\infty, 4]$.

(Indeed, by the previous example, we know that (x^2+y^2) takes any value in $[0, +\infty)$, so that $-(x^2+y^2)$ takes any value in $(-\infty, 0]$, and $4-(x^2+y^2)$ takes any value in $(-\infty, 4]$.

OR, equivalently, $R(f) = (-\infty, 4]$ since, $\forall t \leq 4$,

$$t = 4 - (\sqrt{t-4})^2 - 0 = f(\sqrt{t-4}, 0).)$$

etc....

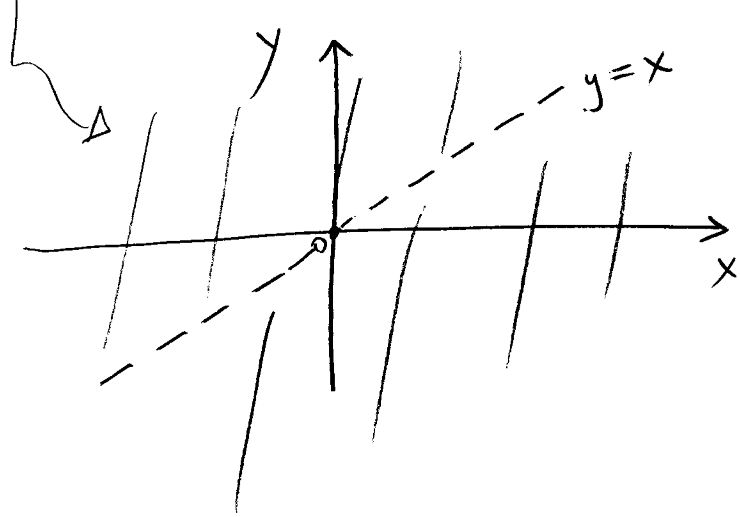
2) Rational functions:

$$f(x,y) = \frac{g(x,y)}{h(x,y)}, \text{ with } g, h \text{ polynomials.}$$

$\rightsquigarrow D(f) = \{(x,y) \in \mathbb{R}^2 \mid h(x,y) \neq 0\}$.

e.g. $f(x,y) = \dfrac{1}{x-y}$. Then,

$$D(f) = \{(x,y) \in \mathbb{R}^2 \mid x-y \neq 0\}.$$



and $R(f) = (-\infty, 0) \cup (0, +\infty) = \mathbb{R} \setminus \{0\}$.

(Indeed, first note that $f(x,y) \neq 0$, $\forall (x,y)$, since otherwise we would have

$$\frac{1}{x-y} = 0 \iff 1 = 0,$$

which is IMPOSSIBLE. Also, $\forall t \neq 0$,

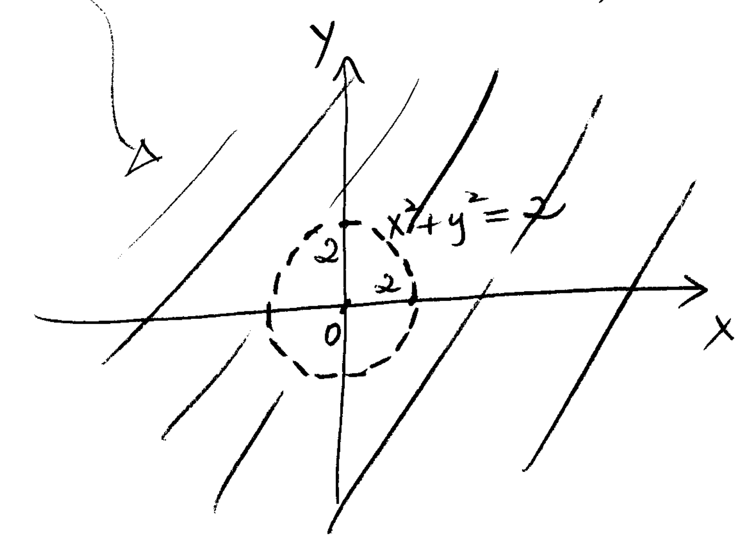
$$t = f(1/t, 0),$$

proving that $R(f) = \mathbb{R} \setminus \{0\}$.)

3) $f(x,y) = \ln(x^2+y^2-4)$.

Note that $f(x,y) = \ln t$ with $t = x^2+y^2-4$. Since $\ln t$ is only defined for $t>0$, this means that

$$D(f) = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2-4>0\}$$
$$= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2>4\}$$
$$= \left(\begin{array}{c}\text{points at a distance}\\ \text{greater than 2 from}\\ (0,0)\end{array}\right)$$



and $R(f) = \mathbb{R}$ (because the range of $H(t) = \ln t$, $t>0$, is $\mathbb{R}$ and

$$t = x^2+y^2-4$$

takes all possible values of $t>0$ for $(x,y)\in D(f)$: $\forall t>0$, $(\sqrt{t},2)\in D(f)$ and

$$t = (\sqrt{t})^2 + (2)^2 - 4 = f(\sqrt{t},2).)$$

Remarks:

1) If $f(x,y) = \dfrac{g(x,y)}{h(x,y)}$, then

$$D(f) = (D(g) \cap D(h)) \setminus \{(x,y) \in \mathbb{R}^2 \mid h(x,y) = 0\}.$$

2) If $f(x,y) = H(g(x,y))$, where $H(t)$ is a function of one variable, then

$$D(f) = \{(x,y) \in D(g) \mid g(x,y) \in D(H)\}.$$

3) To find the range of a 2-variable fct f, it may be useful to fix either x or y, to get a fct of 1 variable, whose range is easier to determine. E.g., consider $f(x,c)$ OR $f(c,y)$.

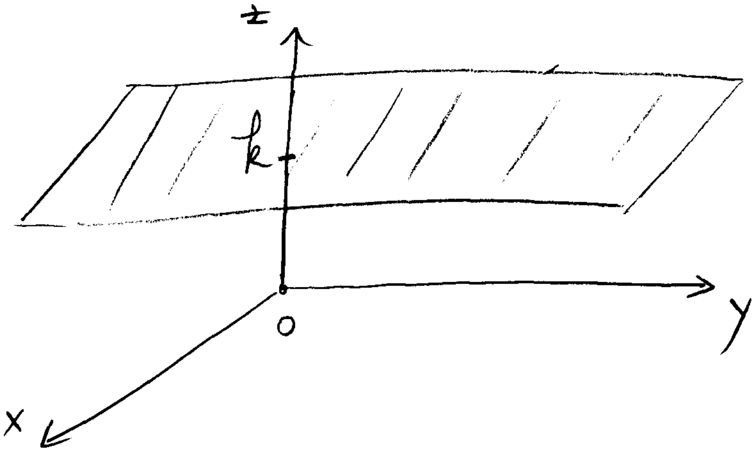
DEF: Given a function of 2 var. f, the set $\{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D(f) \text{ and } z = f(x,y)\} \subseteq \mathbb{R}^3$ is called the graph of f in $\mathbb{R}^3$.

RMK: The graph of a 2 var. fct is a surface in $\mathbb{R}^3$ (i.e., a "2-dimensional set of points") given by the equation

$$z = f(x,y).$$

Ex: 1) $f(x,y) = k$ = constant

$\Rightarrow$ graph: $z = k$ $\leadsto$ horizontal plane

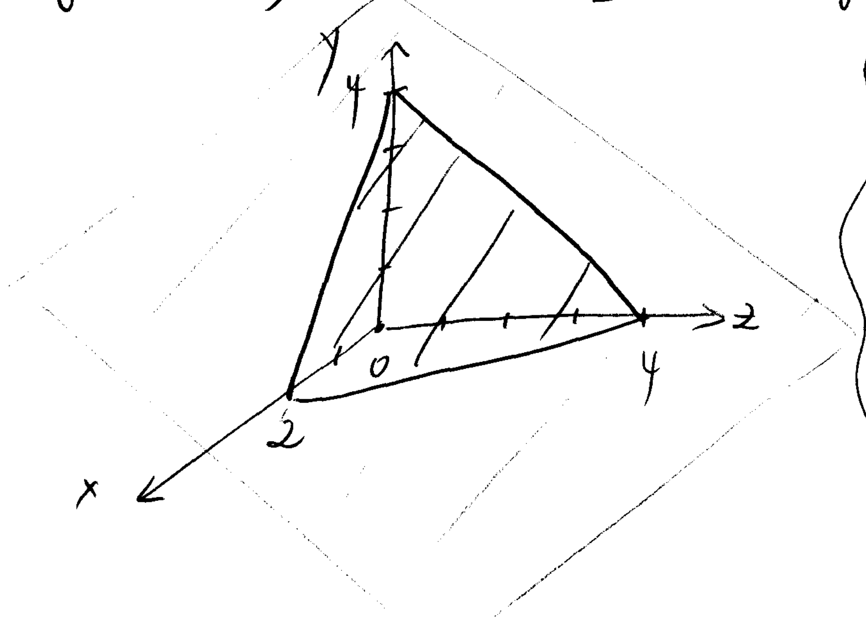


2) More generally,

$f(x,y) = ax + by + c$ = linear function

$\Rightarrow$ graph: $z = ax + by + c$ is a plane.

e.g. $f(x,y) = 4 - 2x - y \Rightarrow$ graph: $z = 4 - 2x - y$



NOTE: To draw plane, find intersection points with coordinate axes, i.e., set $y = z = 0$, $x = z = 0$, and $x = y = 0$

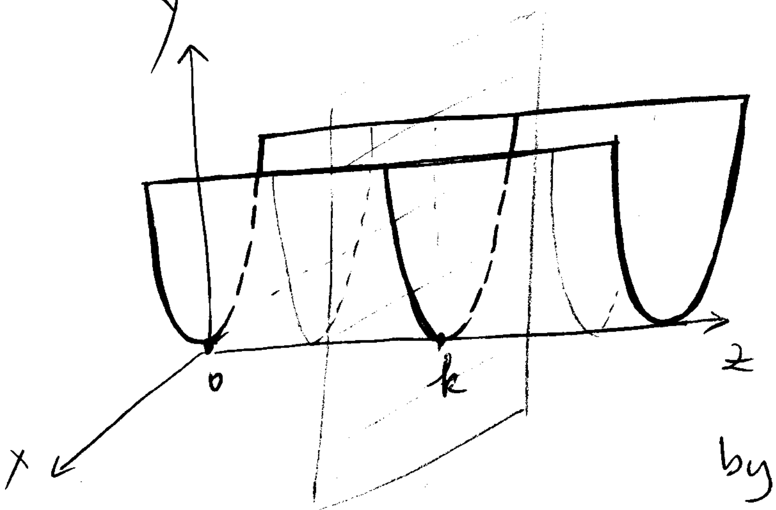
RMK: These planes are NOT vertical.

Vertical planes: $ax + by + c = 0$.
e.g.: $x = k$, $y = k$, $x + 2y = 0$, etc...

3) One variable missing: slide 2D graph along axis of missing variable

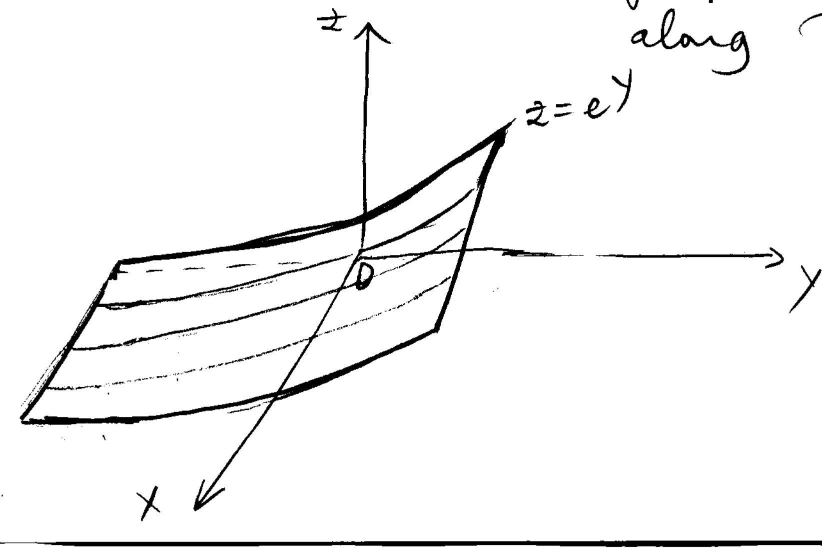
E.g. • $f(x,y)=x^2$ ⇝ y is missing.

graph: $z=x^2$. Since this equation hold $\forall y$, there exist points (x,y,z) on the graph $\forall y$. Moreover, points on the graph with $y=k$ lie on the parabola $z=x^2$ in the vertical plane $y=k$. Thus, since we have the same parabola for all values of y, the graph of $f(x,y)=x^2$ is the surface obtained by "sliding" the graph $z=x^2$ in the xz-plane along the y-axis.



• $f(x,y)=e^y$ ⇝ x missing.

graph: $z=e^y$ ⇝ slide graph $z=e^y$ in yz-plane along the x-axis.



For more general fcts, one can use level sets to draw the graph of a function of 2 variables.

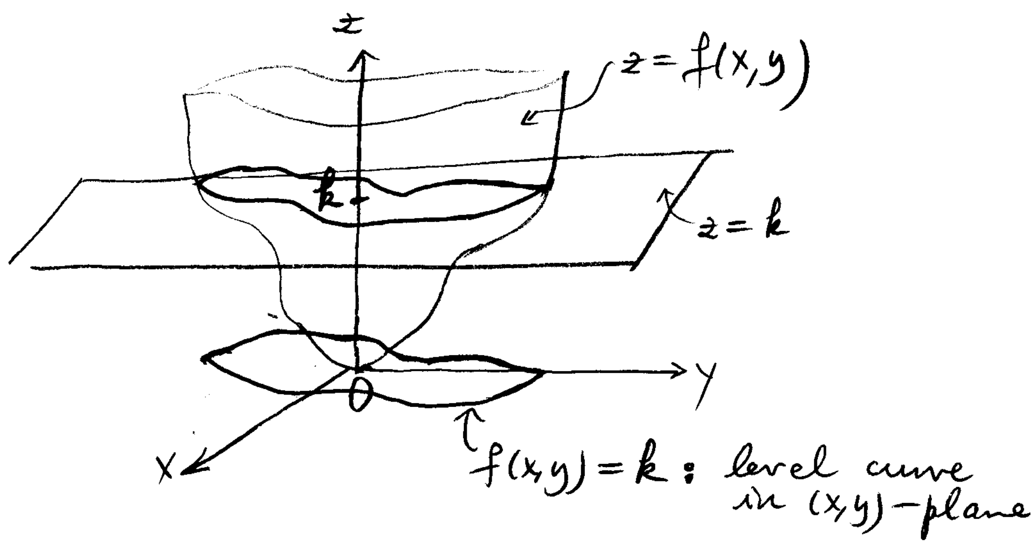
DEF.: The level sets of a function of 2 variables f are the sets of points in $\mathbb{R}^2$ satisfying the equations

$$f(x,y) = k,$$

where k is a constant in the range of f.

IMPORTANT: For a fct of 2 var., the level sets are curves in the (x,y)-plane (!) and are therefore often called level curves.

The level curve $f(x,y) = k$ shows the shape of the graph of f at height k.



NOTE: If $k \notin R(f)$, then the level set

$$f(x,y) = k$$

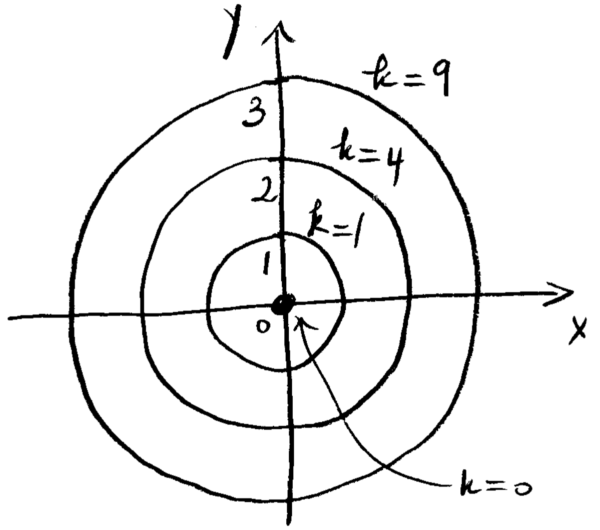
is empty! So, only use $k \in R(f)$.

E.g. 1) $f(x,y) = x^2 + y^2 \rightsquigarrow D(f) = \mathbb{R}^2,\ R(f) = [0,\infty)$.

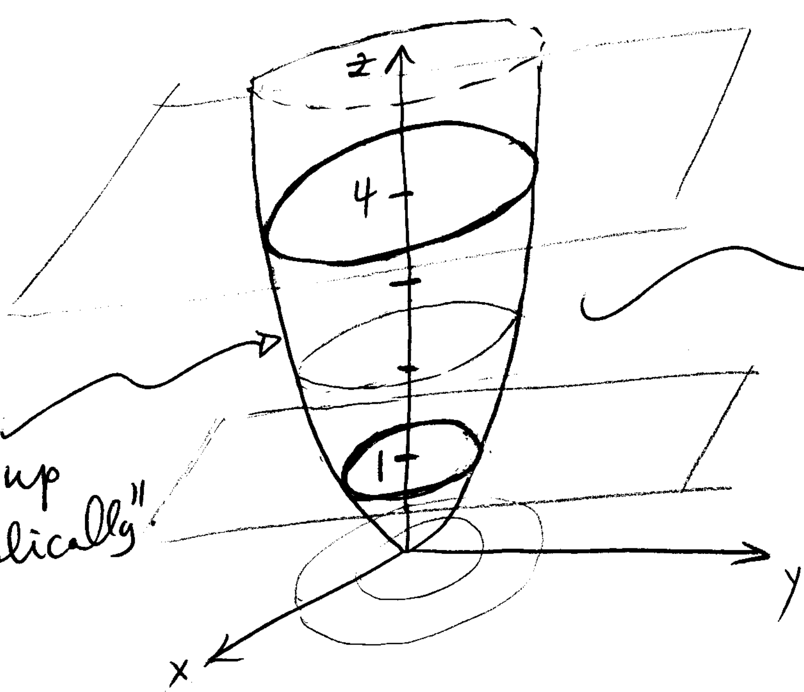
<u>Level sets</u>: $f(x,y) = k,\ k \in R(f)$

$$\iff \quad x^2+y^2 = k,\ k \geq 0.$$

* $\underline{k=0}$: $x^2+y^2=0 \iff (x,y)=(0,0) \rightsquigarrow$ point.
* $\underline{k=1}$: $x^2+y^2=1$: circle centered at $(0,0)$ of radius 1.
* In general, $\underline{k>0}$: $x^2+y^2=k$: circle centered at $(0,0)$ of radius $\sqrt{k}$.



$\Rightarrow$ get concentric circles centered at $(0,0)$.



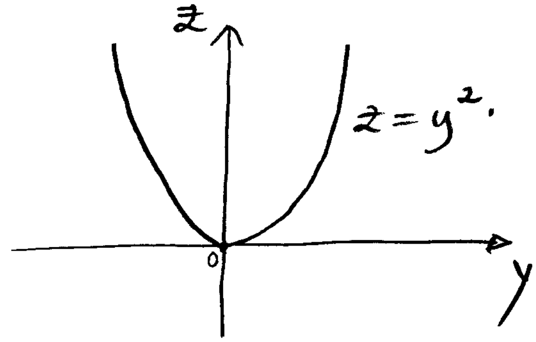
circles stack up "parabolically".

$z = x^2+y^2$: quadric surface called (circular) <u>paraboloid</u>.

Why is the surface $z = x^2 + y^2$ called a paraboloid? Take vertical cross-sections:

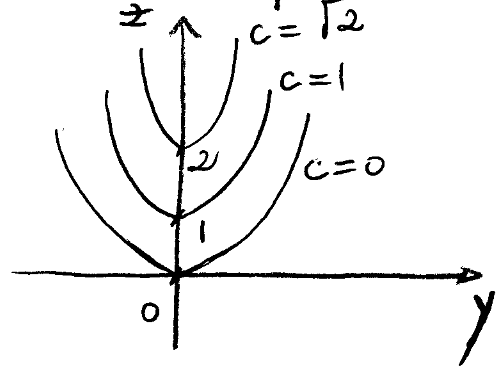
e.g.: * The intersection of $z = x^2+y^2$ with the yz-plane is obtained by setting $x=0$:

$$z = y^2.$$

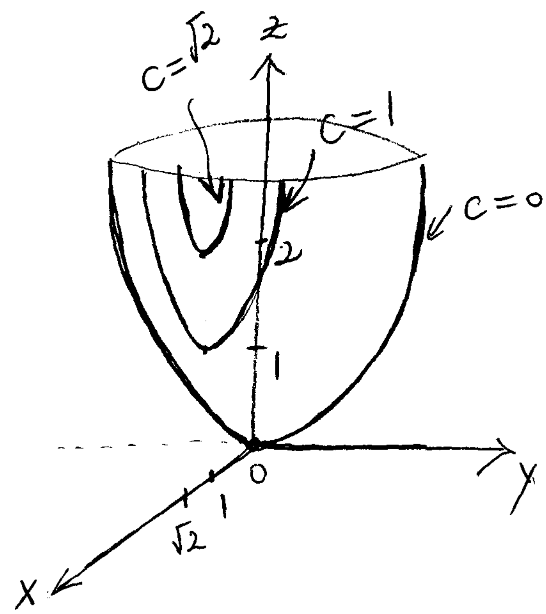


I.e., the trace of the surface in the yz-plane is the parabola $z=y^2$.

* More generally, if we intersect $z=x^2+y^2$ with the vertical plane $x=c$, we get the vertical cross-sections: $z=c^2+y^2$, which are parabolas.

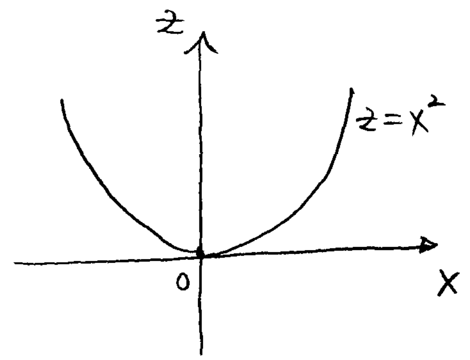


By intersecting the graph $z = x^2+y^2$ with the planes $x=0$, $x=1$, $x = \sqrt{2}$, we see that we indeed get parabolas that are shifting up a c increases:



* One can also take vertical cross-sections of the graph $z = x^2+y^2$ by intersecting it with vertical planes of the form $y = c$.

  ex. $y=0$: $z=x^2$, which is the intersection of the graph with the xy-plane.



NOTE: The traces of $z=x^2+y^2$ show that the radii of the circles making up the graph increase parabolically, hence the name paraboloid.

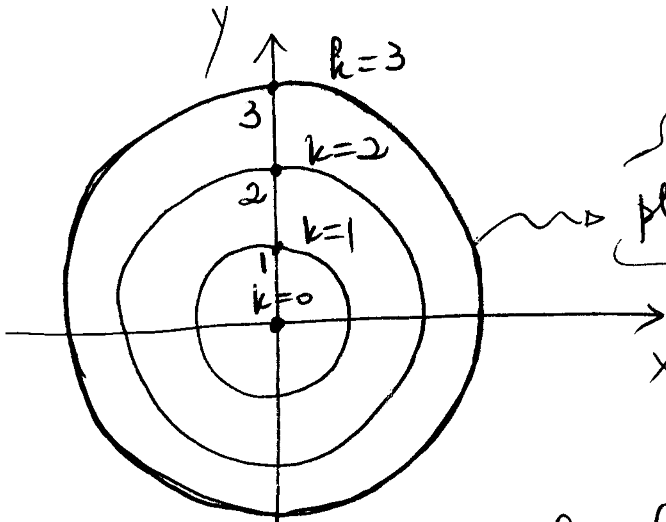
RMK: To get the basic vertical shape of the graph of a simple function, it's often enough to just intersect with the yz-and xz-planes.

2) $f(x,y) = \sqrt{x^2+y^2}$, $D(f) = \mathbb{R}^2$, $R(f) = [0,+\infty)$.

<u>Level sets:</u> $\sqrt{x^2+y^2} = k \iff x^2+y^2 = k^2$.

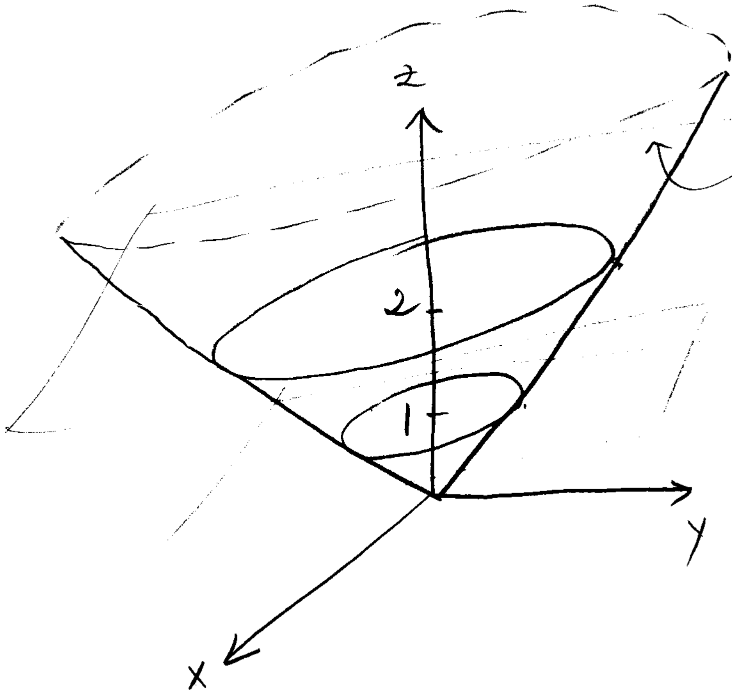
<u>$k=0$:</u> $(0,0)$.

<u>$k>0$:</u> $x^2+y^2=k^2 \rightsquigarrow$ circle centered at $(0,0)$ of radius k.



$\rightsquigarrow$ shape of set of pts on the graph at height $k=3$

Like a topographical map.

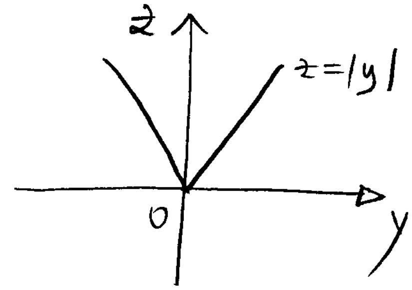


graph $z = \sqrt{x^2+y^2}$: (<u>circular</u>) <u>cone</u>.

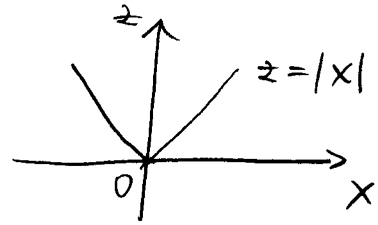
<u>NOTE:</u> In this case, the radius of the circles increases linearly with $z=k$ $\rightsquigarrow$ the circles stack up linearly.

To verify the vertical shape of the graph, let's take two vertical cross-sections:

e.g. $x=0$: $z = \sqrt{y^2} = |y|$



$y=0$: $z = \sqrt{x^2} = |x|$

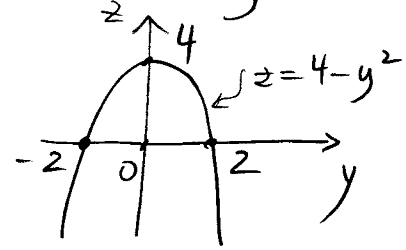


3) $f(x,y) = 4 - x^2 - y^2$, $D(f) = \mathbb{R}^2$, $R(f) = (-\infty, 4]$.

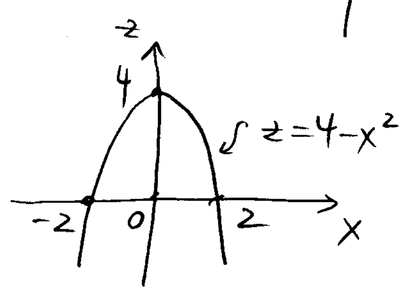
Level sets: $x^2 + y^2 = 4 - k$, $k \le 4$.

* $k=4$: $x^2+y^2=0 \rightsquigarrow (0,0)$ point
* $k=0$: $x^2+y^2=4 \rightsquigarrow \left(\begin{array}{c}\text{intersection} \\ \text{with } (x,y)\text{-plane}\end{array}\right) = \left(\begin{array}{c}\text{circle of} \\ \text{radius } 2 \\ \text{centered at } 0\end{array}\right)$
* $k<0$: $x^2+y^2 = 4-k \rightsquigarrow$ circle centered at $(0,0)$ of radius $\sqrt{4-k}$.

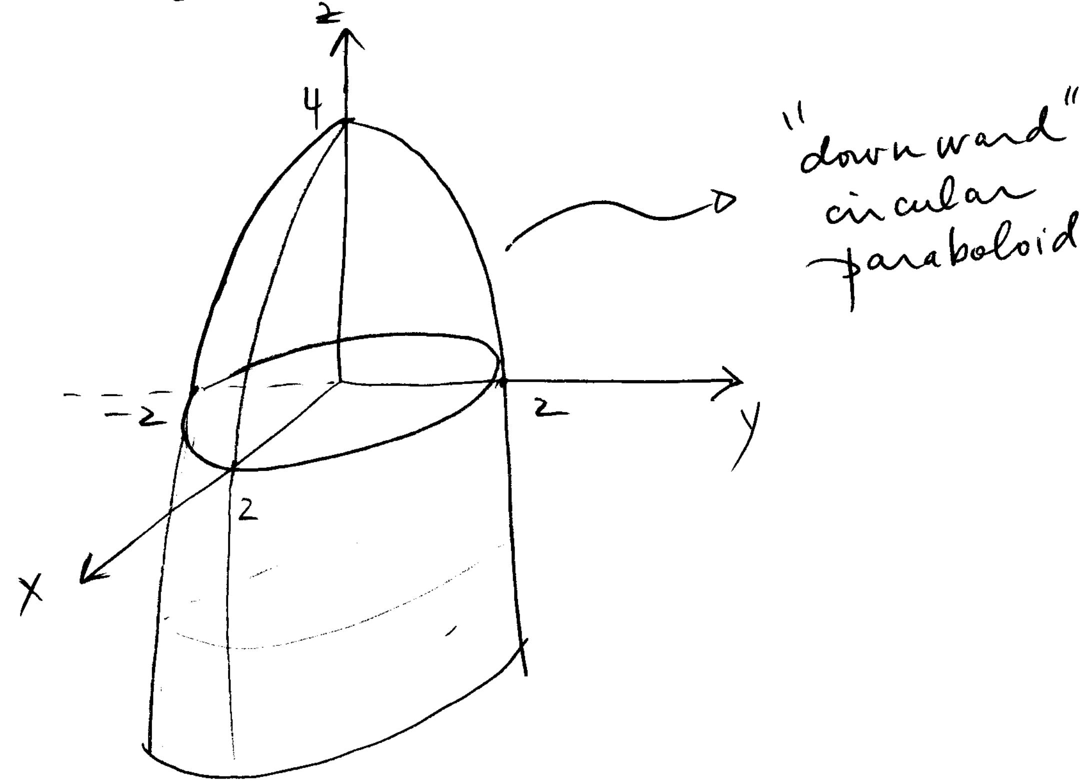
Vertical cross-sections: • $x=0$: $z = 4 - y^2$



• $y=0$: $z = 4 - x^2$

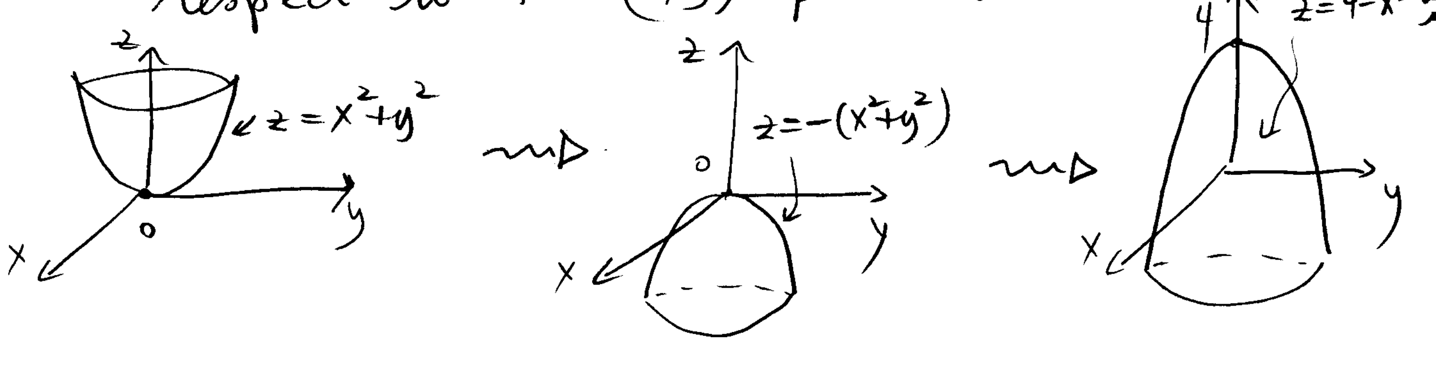


Putting it together we find:



NOTE: • Since $R(f) = (-\infty, 4]$, $z \leq 4$ for points in the graph, which is why there are no points higher than the vertex $(0,0,4)$ on the paraboloid.

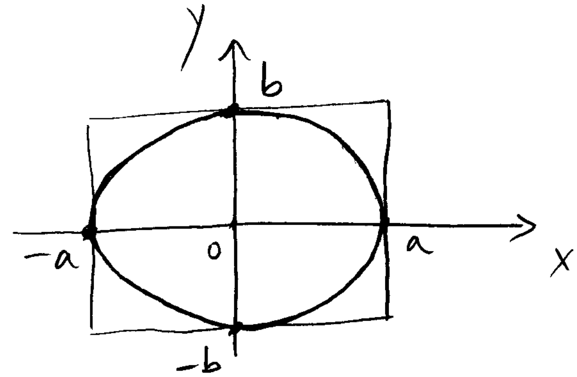
• As with functions of 1 variable, we can also draw the graph $z = 4 - x^2 - y^2$ by thinking of it as $z = 4 - (x^2 + y^2)$, which is a vertical translation by 4 of the reflection $z = -(x^2+y^2)$ of $z = x^2+y^2$ with respect to the (x,y)-plane:



Before looking at more examples, recall that given $a, b > 0$,

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

(<u>ellipses</u>)



and

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

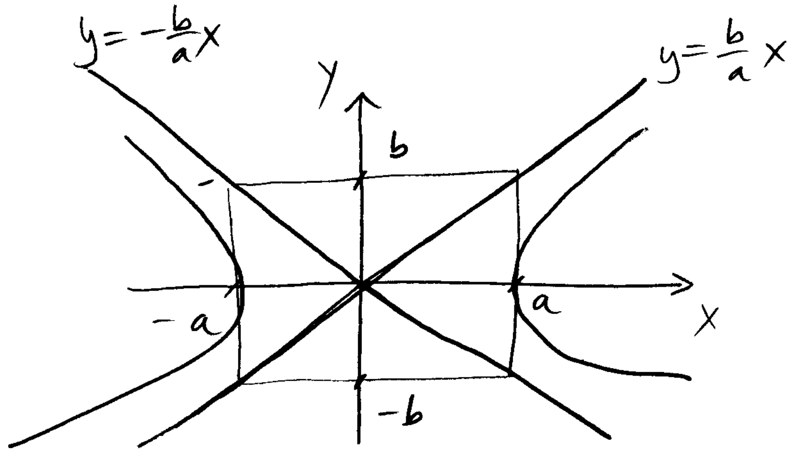
(<u>hyperbolas</u>)

with asymptotes

$$y = \pm \frac{b}{a} x$$

and

<u>x-int.</u>: $x = \pm a$ (NO y-int.)



$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1}$$

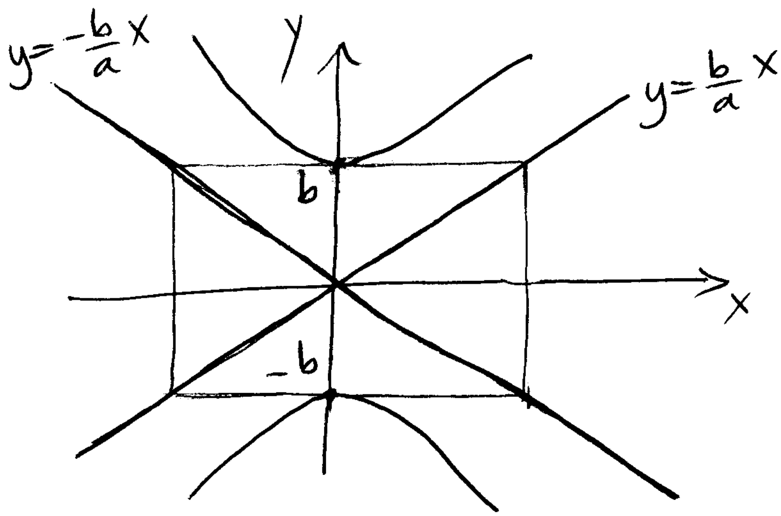
(<u>hyperbolas</u>)

with asymptotes

$$y = \pm \frac{b}{a} x$$

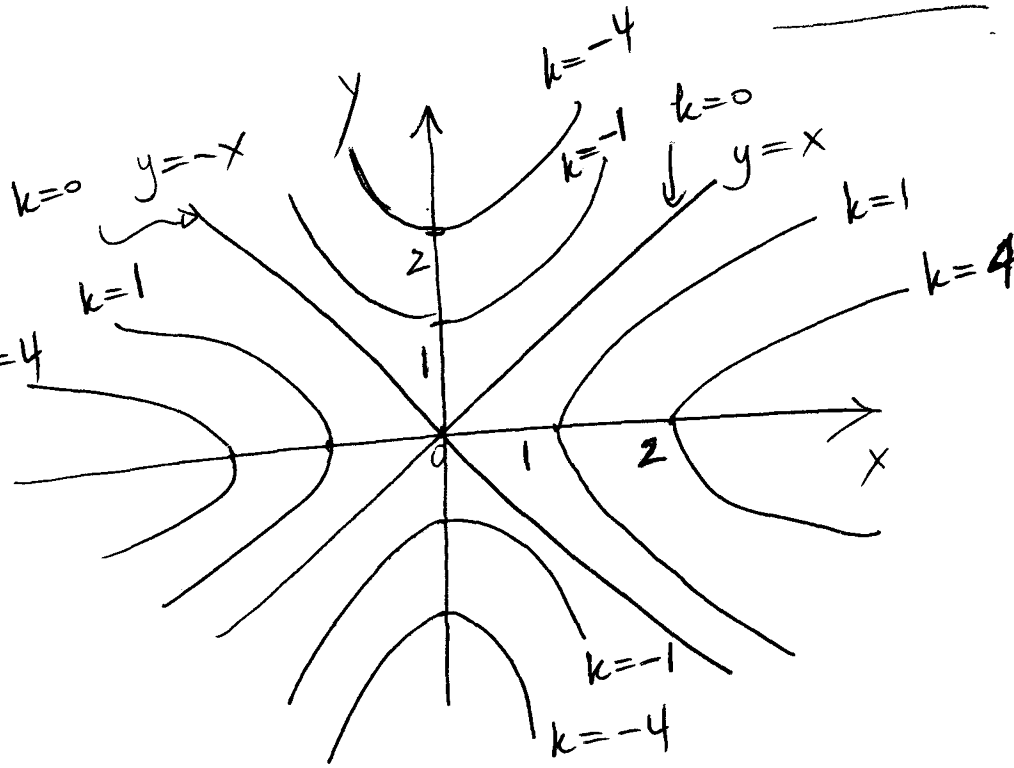
and

<u>y-int.</u>: $y = \pm b$ (NO x-int.)

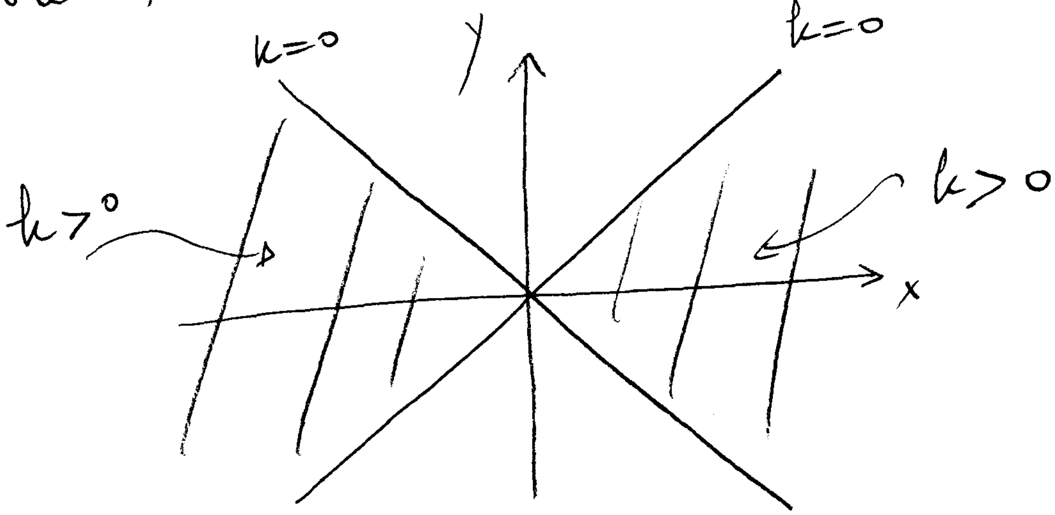


4) $f(x,y) = x^2 - y^2$.

- $D(f) = \mathbb{R}^2$.
- $R(f) = \mathbb{R}$ (since $\forall t \geqslant 0$, $t = f(\sqrt{t}, 0)$, and $\forall t < 0$, $t = f(0, \sqrt{-t})$).
- Level sets: $x^2 - y^2 = k$, $k \in \mathbb{R}$.
  - $k = 0$: $x^2 - y^2 = 0 \iff$ lines $y = \pm x$
  - $k = 1$: $x^2 - y^2 = 1 \rightsquigarrow$ hyperbola with asymp. $y = \pm x$ and x-int. $x = \pm 1$.
  - $k > 0$: $x^2 - y^2 = k \rightsquigarrow$ hyperbola with asymp. $y = \pm x$ and x-int. $x = \pm\sqrt{k}$.
  - $k = -1$: $x^2 - y^2 = -1 \rightsquigarrow$ hyperbola ——— ——— y-int. $y = \pm 1$
  - $k < 0$: $x^2 - y^2 = k \rightsquigarrow$ hyperbola ——— ——— y-int. $y = \pm\sqrt{-k}$

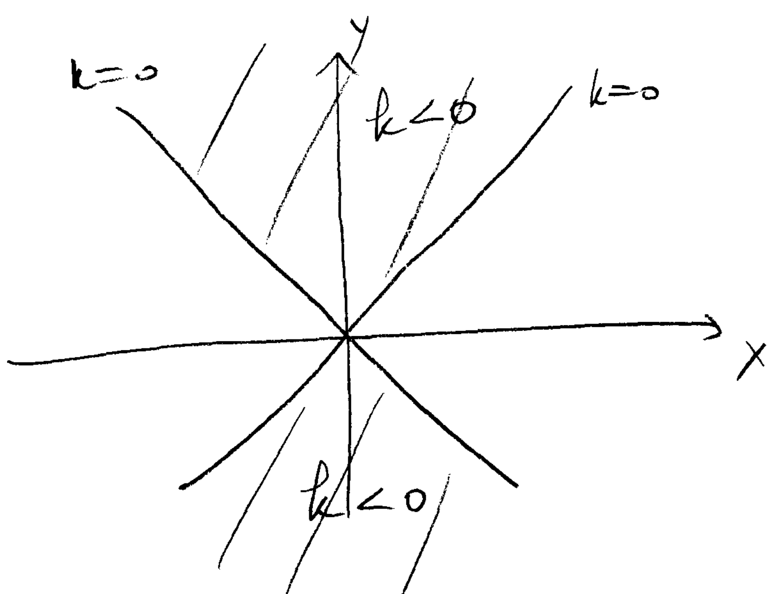


Note that



Level curves $x^2 - y^2 = k$, $k > 0$ live in the above 2 quadrants $\Rightarrow$ points on the graph $z = x^2 - y^2$ with $z \geq 0$ lie above that region in the (x,y)-plane.

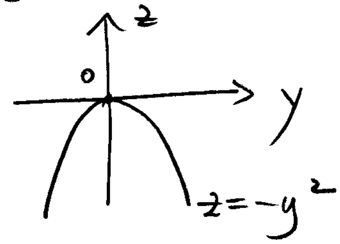
ALSO,



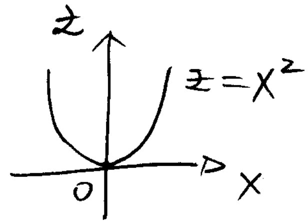
Level curves $x^2 - y^2 = k$, $k < 0$, live in these two quadrants $\Rightarrow$ points on the graph $z = x^2 - y^2$ with $z \leq 0$ lie above this region in the (x,y)-plane.

- Vertical cross-sections:

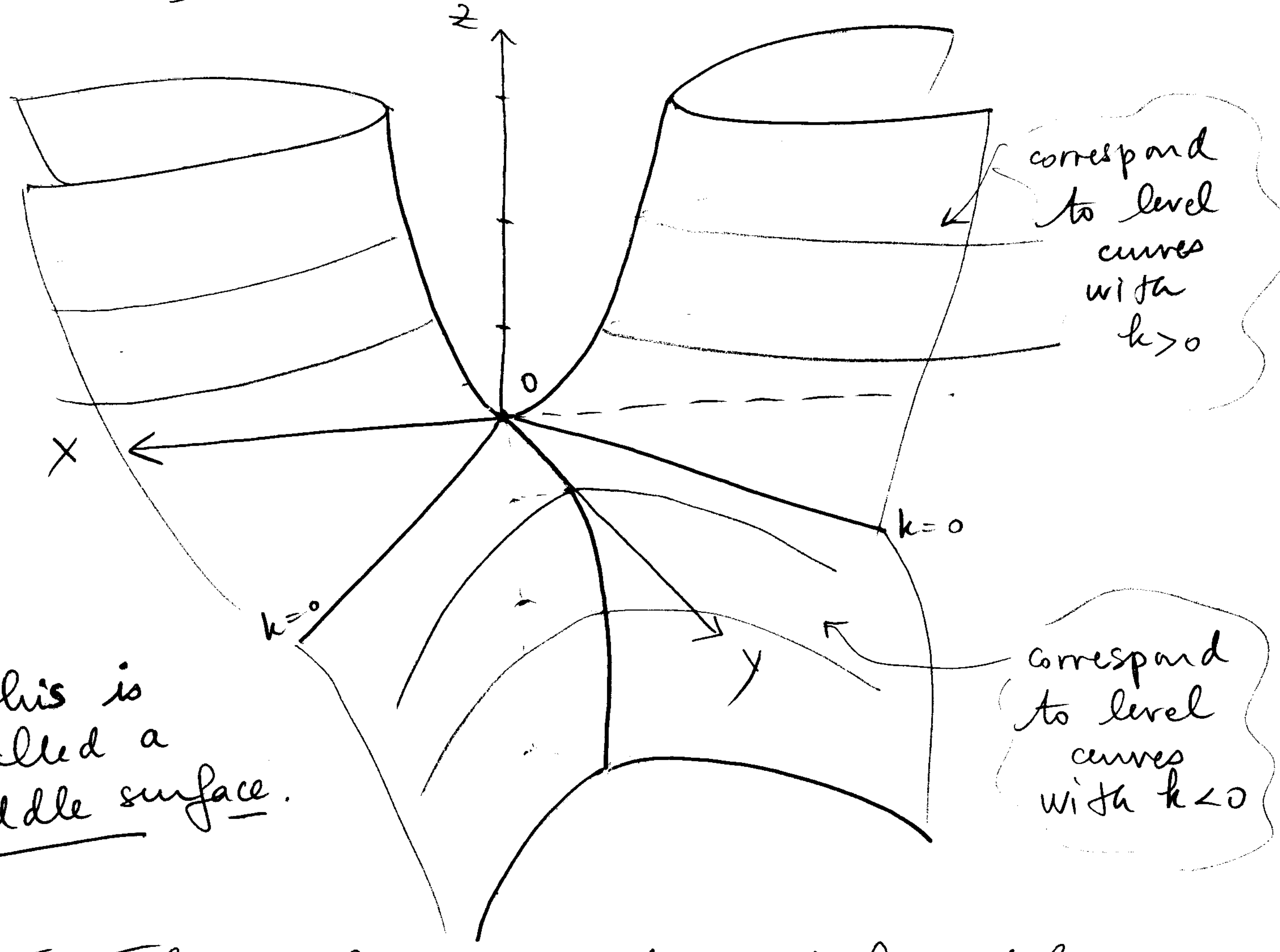
  * $x=0$: $z=-y^2$



  * $y=0$: $z=x^2$



Putting all this together, we get the following surface:

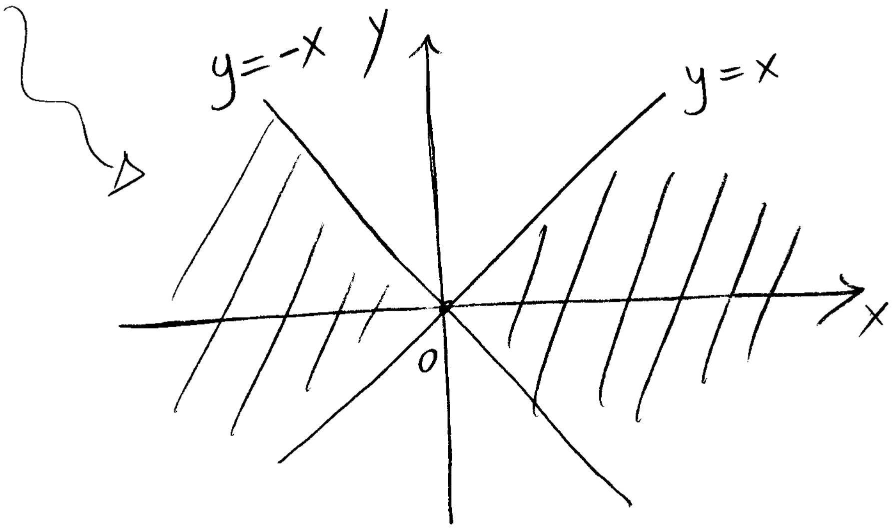


This is called a **saddle surface**.

NOTE: This surface is made up of hyperbolas that stack up "parabolically".

5) $f(x,y) = \sqrt{x^2 - y^2}$.

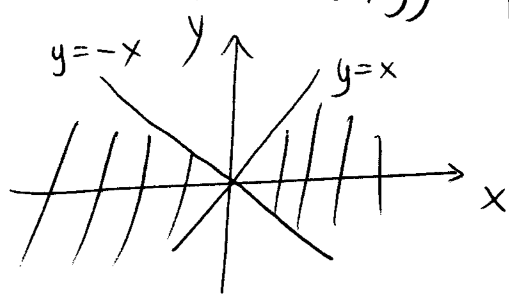
- $D(f) = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 0\}$.



[If $x^2 - y^2 = t \geq 0$, then for $t=0$, $y = \pm x$, and for $t > 0$, (x,y) lies on the parabola $x^2 - y^2 = t > 0$, whose two branches are in the above two quadrants.]

- $R(f) = [0, +\infty)$ (since $x^2 - y^2$ takes all values in $[0, +\infty)$ for $(x,y) \in D(f)$).

This means that points on the graph $z = \sqrt{x^2 - y^2}$ lie above the (x,y)-plane (because $z \geq 0$) and only over points in the (x,y)-plane in the region



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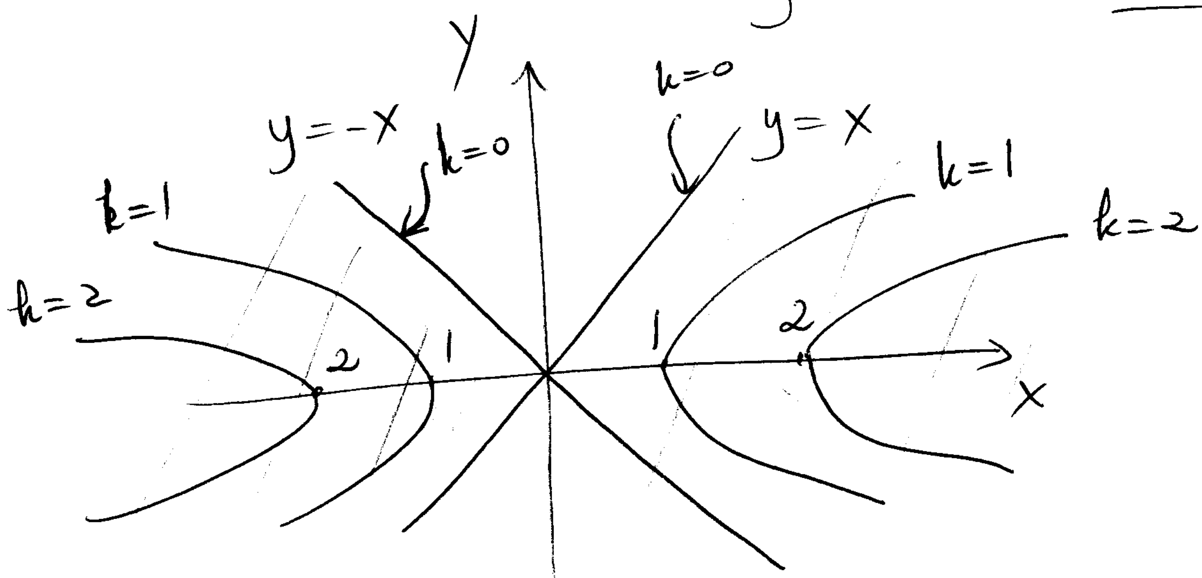
- Level sets: $\sqrt{x^2-y^2} = k,\ k \geq 0$

$\iff x^2 - y^2 = k^2,\ k \geq 0.$

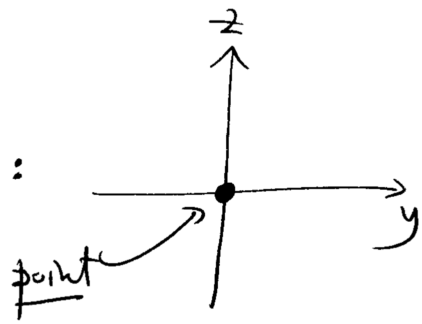
$*$ $k=0$: $x^2-y^2=0 \rightsquigarrow$ lines $y = \pm x$.

$*$ $k>0$: $x^2-y^2=k^2 \rightsquigarrow$ hyperbola with asymp. $y=\pm x$ and x-int. $x = \pm k$.



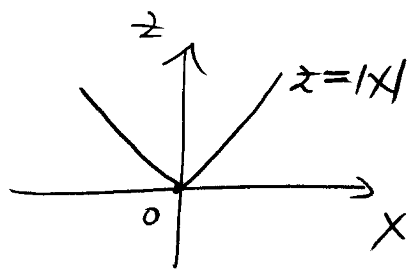
- Vertical cross-sections:

$*$ $x=0$: $z = \sqrt{-y^2} \rightsquigarrow y = z = 0$:

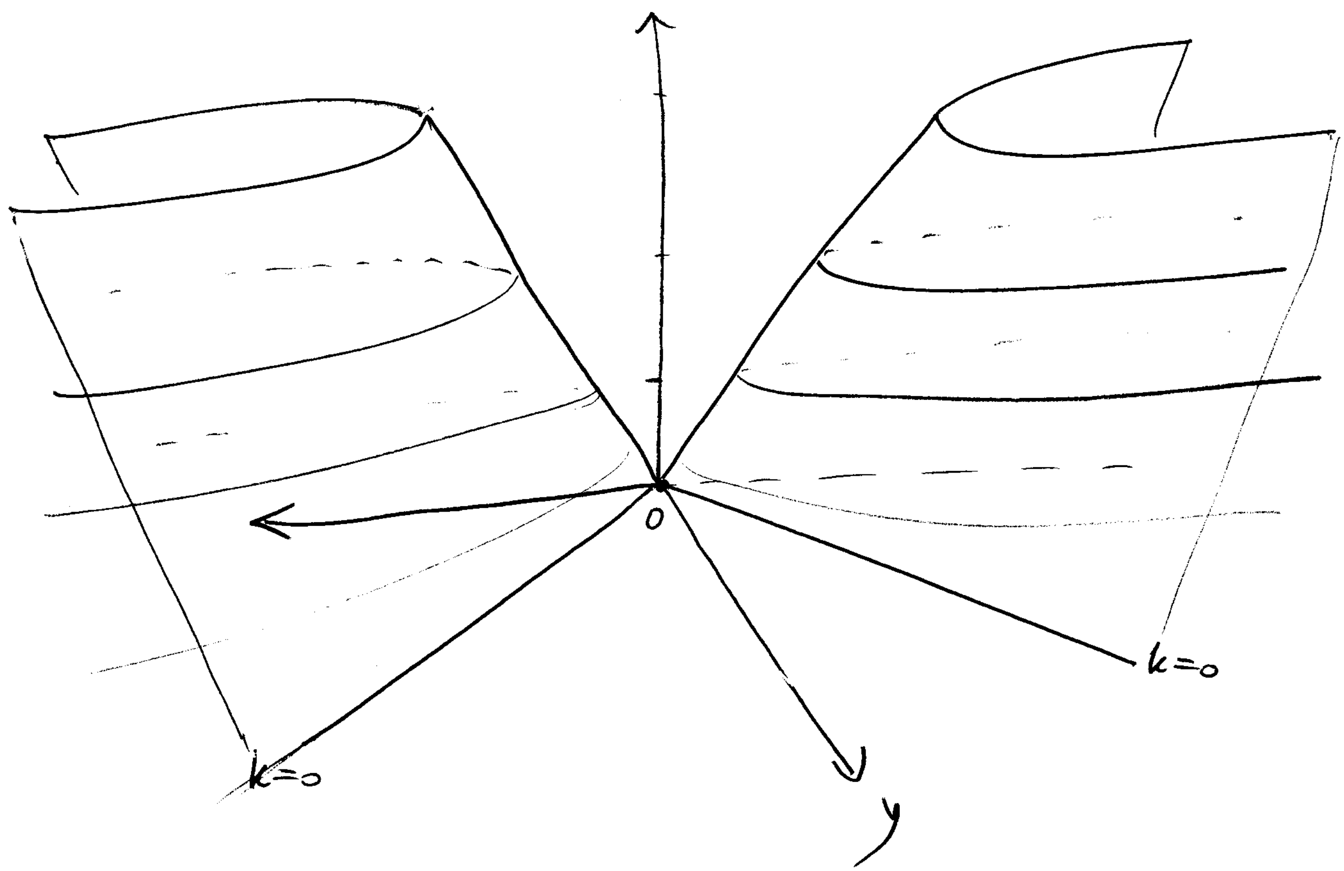


$\Rightarrow$ intersection of $z = \sqrt{x^2-y^2}$ is ONLY a point, which is to be expected since the level sets only intersect the y-axis at the origin.

$*$ $y=0$: $z = \sqrt{x^2} = |x|$



Putting it all together, we get:



NOTE: This surface is made up of hyperbolas that stack up linearly.

RMK: The above examples feature surfaces that are made up of families of:

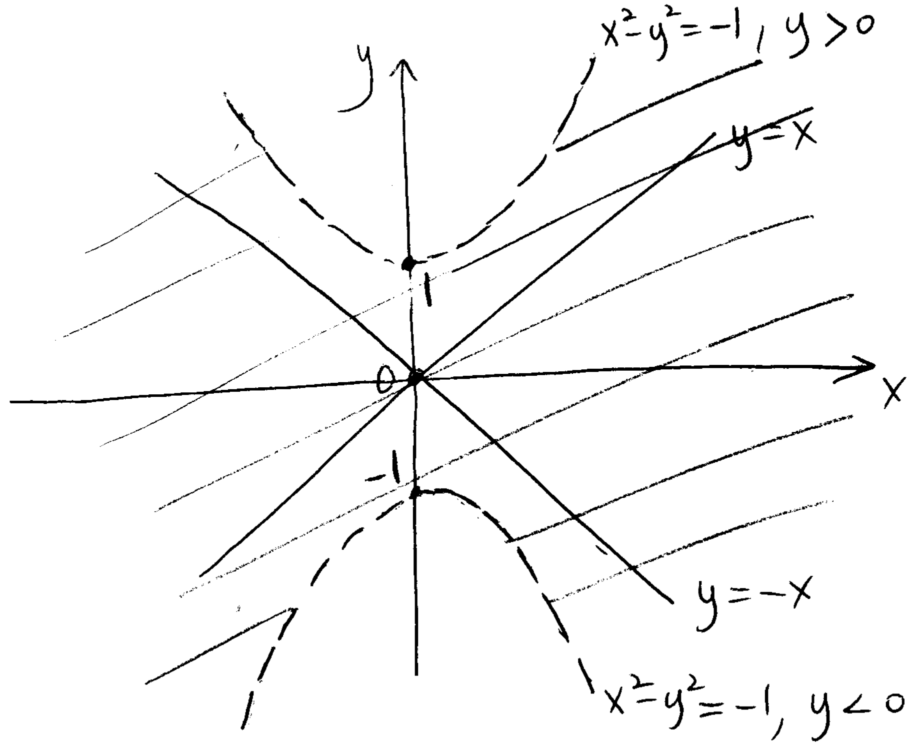
* circles (ellipses): with exceptional curve the center of the circles.

OR

* hyperbolas: with exceptional curves the asymptotes of the hyperbolas.

6) $f(x,y) = \ln(1 + x^2 - y^2)$

* $D(f) = \{(x,y) \in \mathbb{R}^2 \mid 1 + x^2 - y^2 > 0\}$



$\Rightarrow$ $D(f)$ consists of the region in the (x,y)-plane bounded above and below by the two branches of the hyperbola $x^2 - y^2 = -1$.

[Indeed, if $1 + x^2 - y^2 = t > 0$, then

$$x^2 - y^2 = t - 1, \quad t > 0.$$

- For $0 < t < 1$, get $x^2 - y^2 = c$, with $-1 < c = t - 1 < 0$
  $\leadsto$ (x,y) is on a hyperbola with asymp. $y = \pm x$ and y-int. $\boxed{-1 < y = \pm\sqrt{-c} < 1}$
- For $t = 1$, get the lines $y = \pm x$.
- For $t > 1$, (x,y) is on the hyperbola $x^2 - y^2 = c$ with $c = t - 1 > 0$, which has asymp. $y = \pm x$ and x-int.]

* $R(f) = \mathbb{R}$ (since $s = 1 + x^2 - y^2$ takes all values in $(0, +\infty)$ for $(x,y) \in D(f)$ and the range of $\ln(s)$ is $\mathbb{R}$).

* <u>Level sets:</u> $\ln(1 + x^2 - y^2) = k$, $k \in \mathbb{R}$.

$$\Leftrightarrow \boxed{1 + x^2 - y^2 = e^k, \ k = \mathbb{R}}.$$

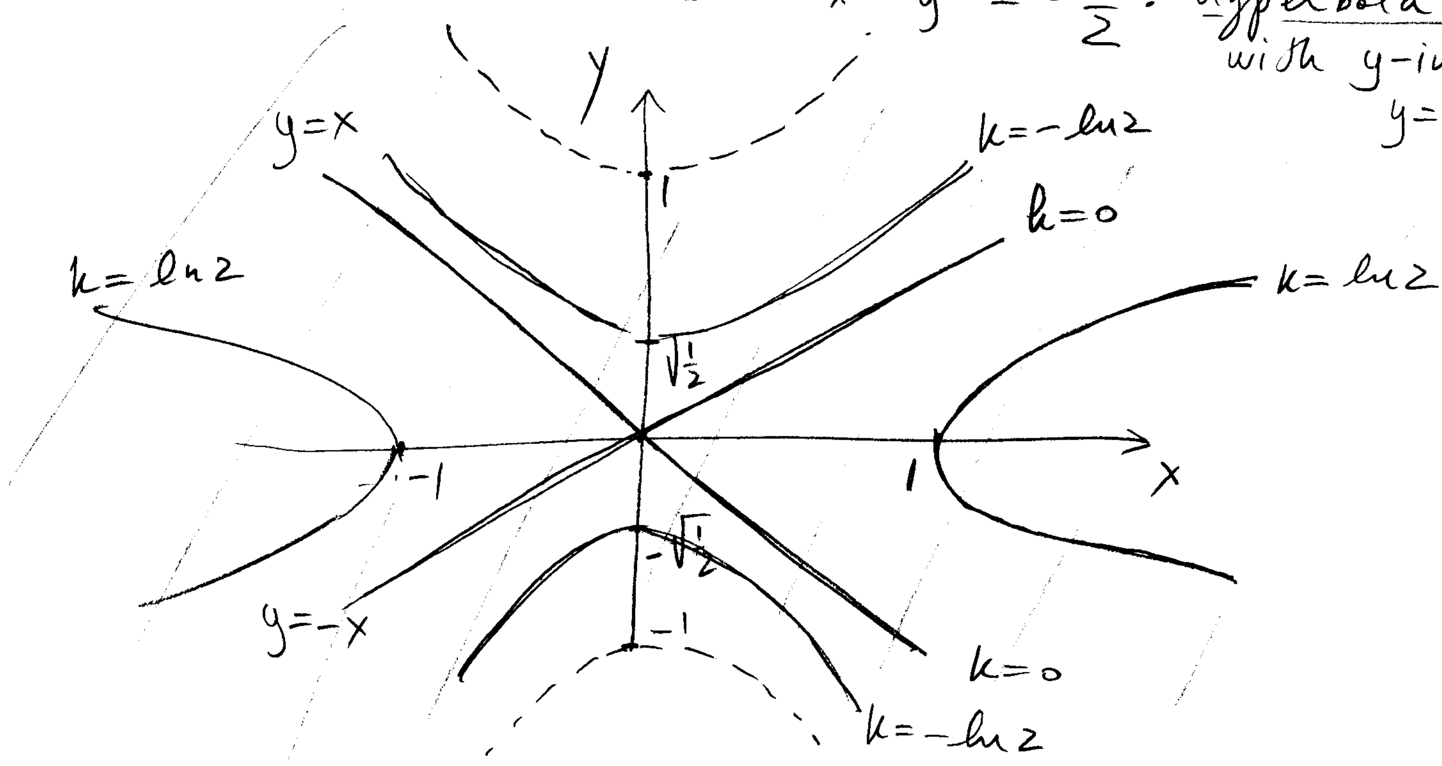
- <u>$k = 0$</u>: $1 + x^2 - y^2 = 1 \Leftrightarrow x^2 - y^2 = 0 \Leftrightarrow$ <u>lines</u> $y = \pm x$.

  $\leadsto$ intersection of the graph $z = \ln(1 + x^2 - y^2)$ is the 2 lines $y = \pm x$.

- <u>$k = \ln 2$</u>: $1 + x^2 - y^2 = e^{\ln 2} = 2 \Leftrightarrow x^2 - y^2 = 1$: <u>hyperbola</u> with x-int. $x = \pm 1$

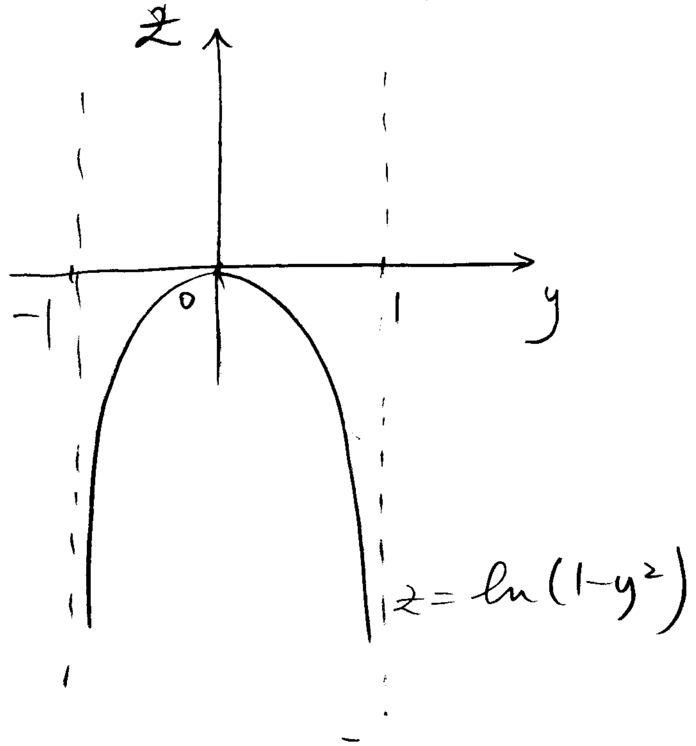
- <u>$k = \ln(\frac{1}{2}) = -\ln 2$</u>: $1 + x^2 - y^2 = e^{-\ln 2} = \frac{1}{2}$

  $\Leftrightarrow x^2 - y^2 = -\frac{1}{2}$: <u>hyperbola</u> with y-int. $y = \pm\sqrt{\frac{1}{2}}$.

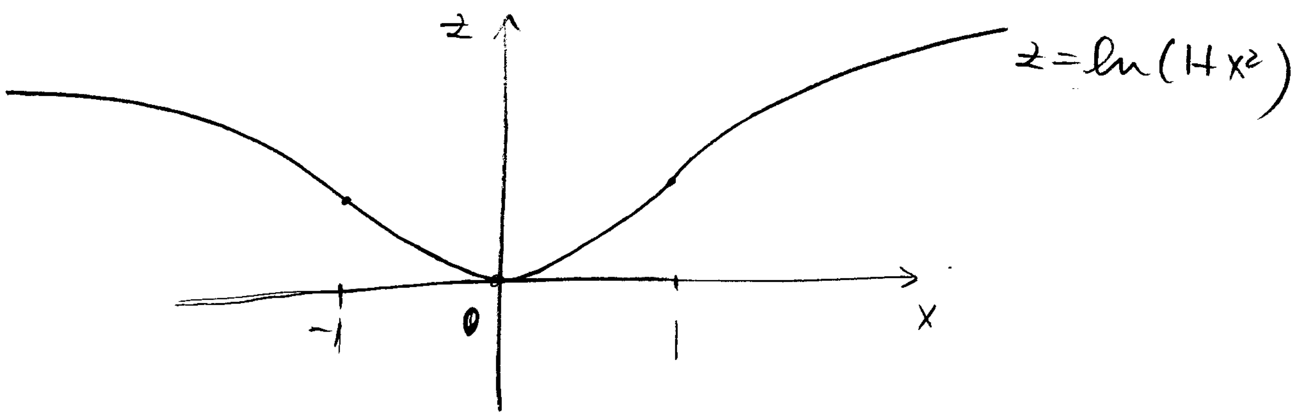


* Vertical cross-sections:

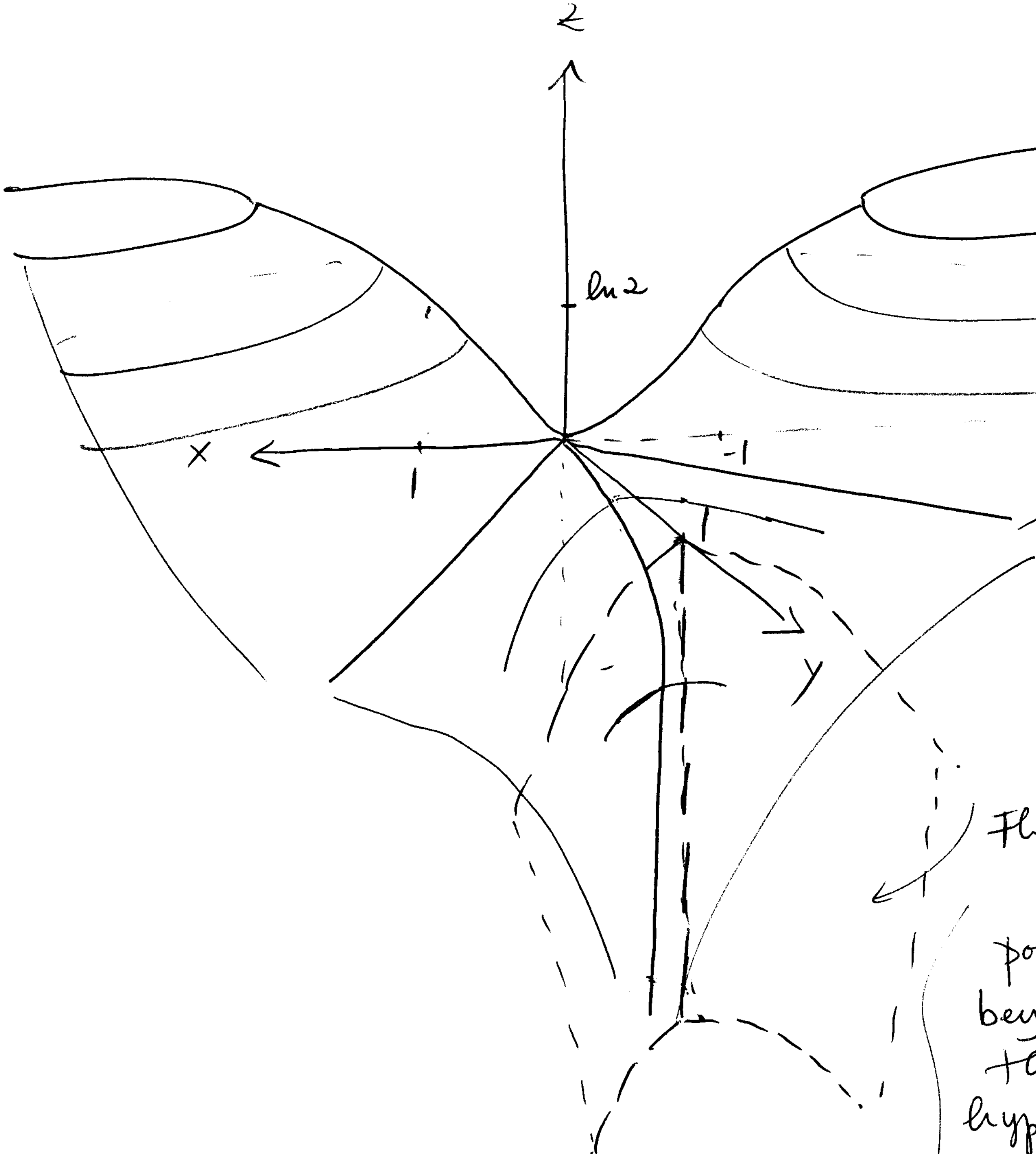
∘ $x=0$: $z = \ln(1-y^2)$, $-1 < y < 1$.



* $y=0$: $z = \ln(1+x^2)$, $x \in \mathbb{R}$

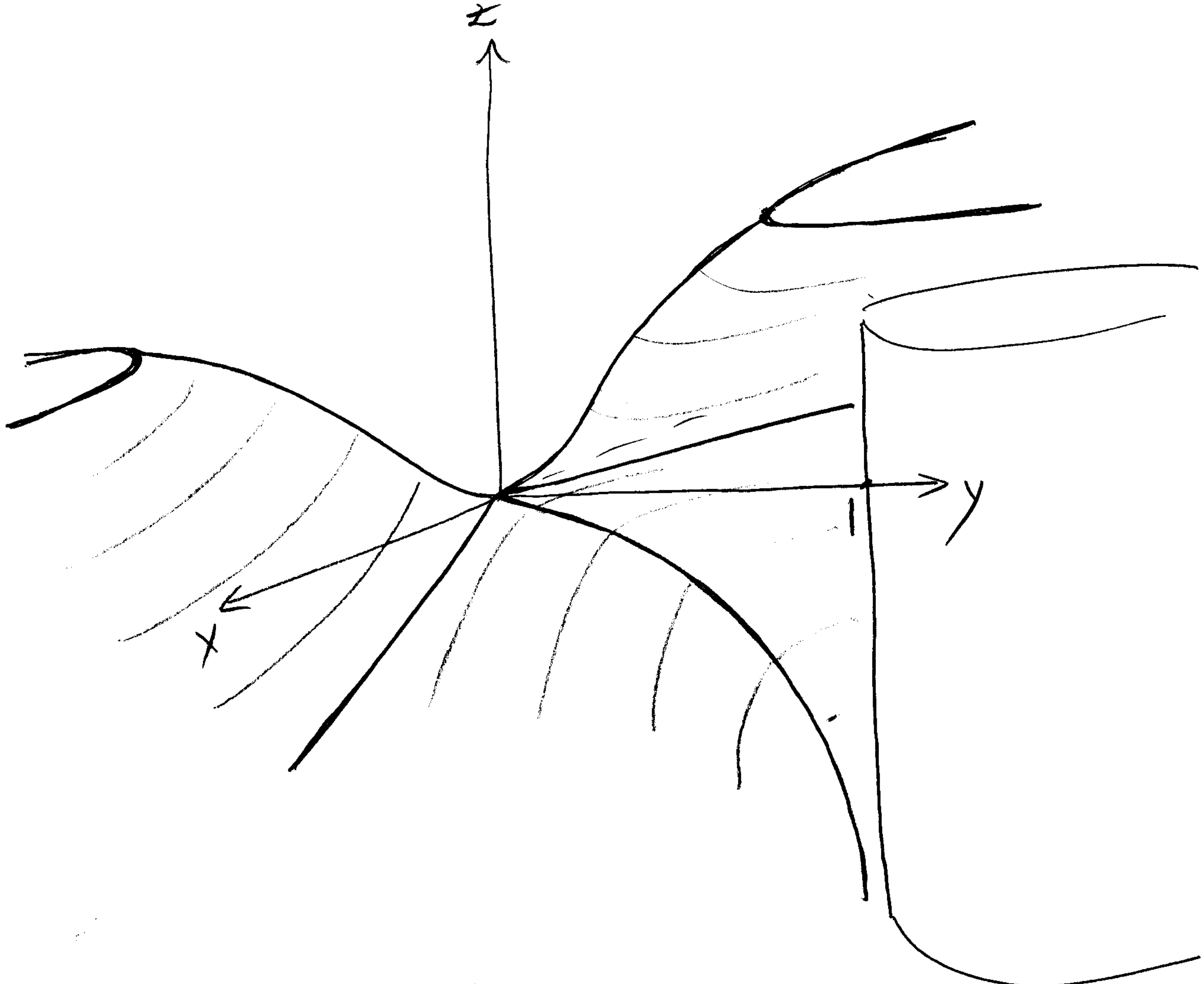


Putting it all together, we get:



There are
NO
points
beyond
the
hyperbola
$x^2 - y^2 = -1, y > 0$

Another view:



RECAP: To draw the graph of a curves:

1) Find $D(f)$, $R(f)$.

2) Draw level curves: $f(x,y) = k$, $k \in R(f)$ to get horizontal distribution of points on the graph.

3) Take some vertical cross-sections, e.g. $x=0$ OR $y=0$, to get the "vertical shape" of the graph.