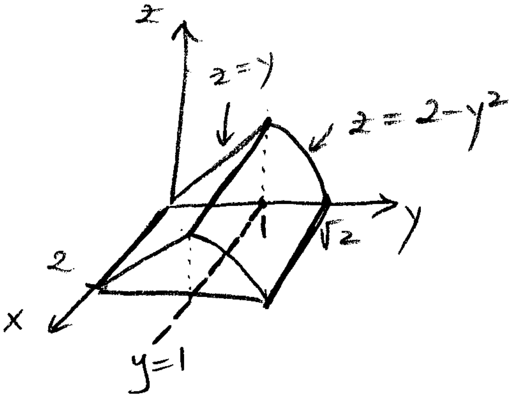
Solutions to the suggested triple integrals

1- $\text{Vol}(R) = \iiint_R 1 \, dV.$

(a) R is the region bounded by $z = y$, $z = 2 - y^2$, $z = 0$, $x = 0$, and $x = 2$.



METHOD 1:

* We see that z is bounded by the surfaces whose equations involve z:

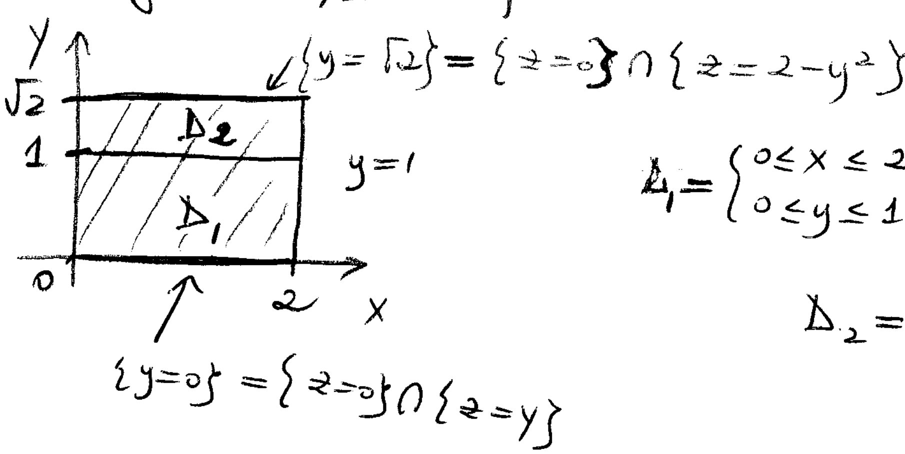
$$\left.\begin{array}{l} z = y \\ z = 2 - y^2 \end{array}\right\} \text{ upper bounds}$$
$$z = 0 \longrightarrow \text{lower bound.}$$

ALSO, the upper bound changes on the line of the intersection of the upper bounds:

$z = y$ and $z = 2 - y^2$. $\iff y = 2 - y^2$
$\iff y^2 + y - 2 = 0$
$\iff y = -2, 1.$

$\rightsquigarrow \boxed{y = 1}$

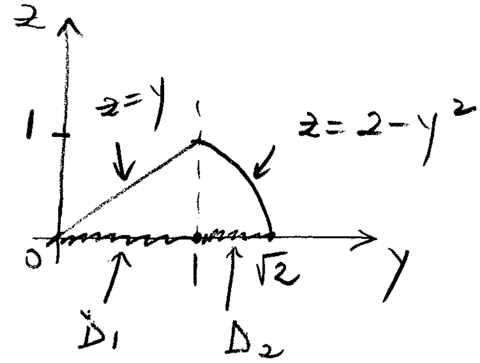
* Bounds of x & y: Project R onto xy-plane.



$$\Delta_1 = \left\{ \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \end{array} \right\} \text{ and}$$

$$\Delta_2 = \left\{ \begin{array}{l} 0 \le x \le 2 \\ 1 \le y \le \sqrt{2} \end{array} \right\}$$

* Bounds for z: take a vertical cross-section.



$\forall (x,y,z) \in R$, if $(x,y) \in R_1$, $\quad 0 \le z \le y$

and if $(x,y) \in R_2$, $\quad 0 \le z \le 2-y^2$

NOTE that we see in the projection of R onto the xy-plane that $z=y$ intersects R_1 and must be the upper bound over R_1, whereas $z = 2-y^2$ intersects R_2 and must be the upper bound over R_2.

So: $\quad R = \left\{ \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \\ 0 \le z \le y \end{array} \right\} \cup \left\{ \begin{array}{l} 0 \le x \le 2 \\ 1 \le y \le \sqrt{2} \\ 0 \le z \le 2-y^2 \end{array} \right\}$

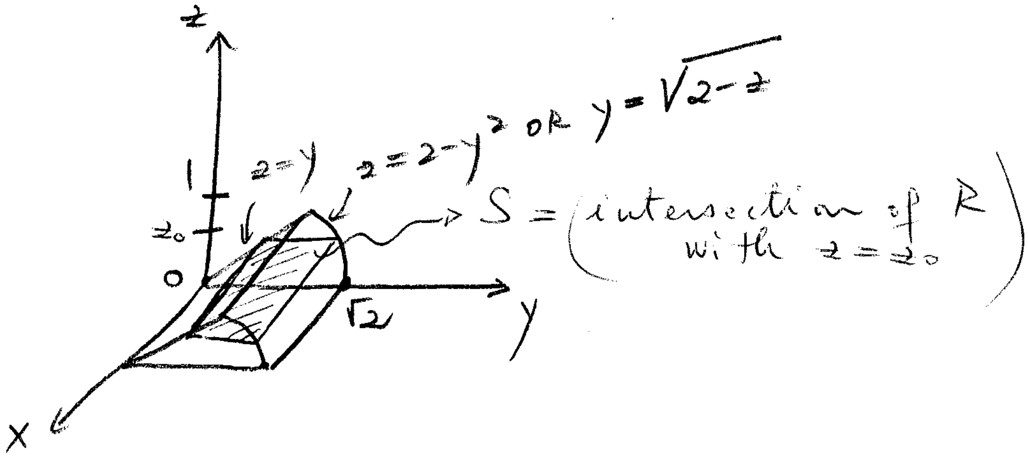
$$\Rightarrow \quad \mathrm{Vol}(R) = \int_0^2 \int_0^1 \int_0^y 1 \, dz\, dy\, dx + \int_0^2 \int_1^{\sqrt{2}} \int_0^{2-y^2} 1 \, dz\, dy\, dx.$$

METHOD 2: We see that in the region R,

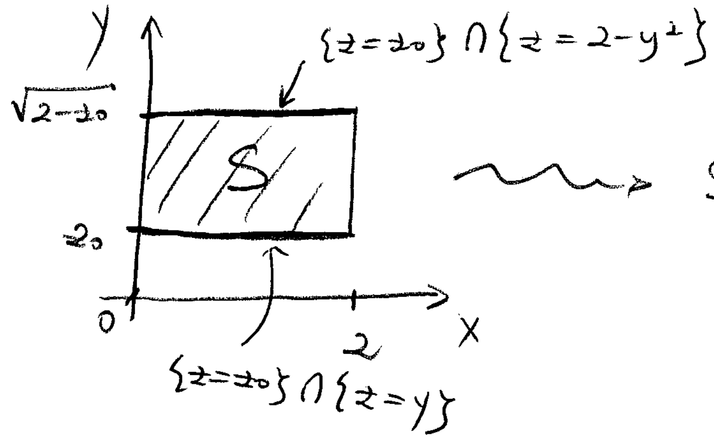
$$\boxed{0 \le z \le 1}.$$

For fixed $z_0 \in [0,1]$, take the horizontal slice

$$S = R \cap \{z = z_0\}.$$



Then, S is the following rectangle:



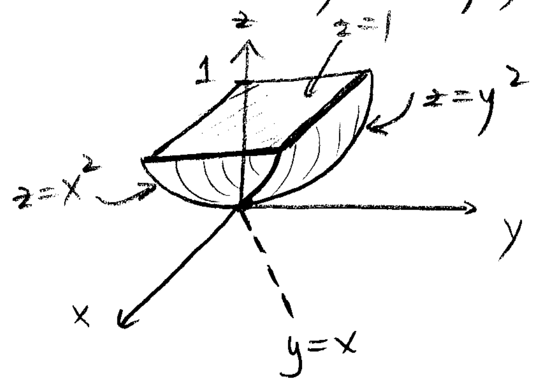
$$\rightsquigarrow\quad S=\left\{\begin{array}{l} 0\le x\le 2\\ z_0\le y\le\sqrt{2-z_0}\end{array}\right\}$$

$$\Downarrow$$

$$R=\left\{\begin{array}{l} 0\le z\le 1\\ 0\le x\le 2\\ z\le y\le\sqrt{2-z}\end{array}\right\}$$

$$\Longrightarrow \quad \operatorname{Vol}(R)=\int_0^1\int_0^2\int_z^{\sqrt{2-z}} 1\,dy\,dx\,dz.$$

(b) R is the region in the first octant bounded by $z = x^2$, $z = y^2$, and $z = 1$.



METHOD 1:

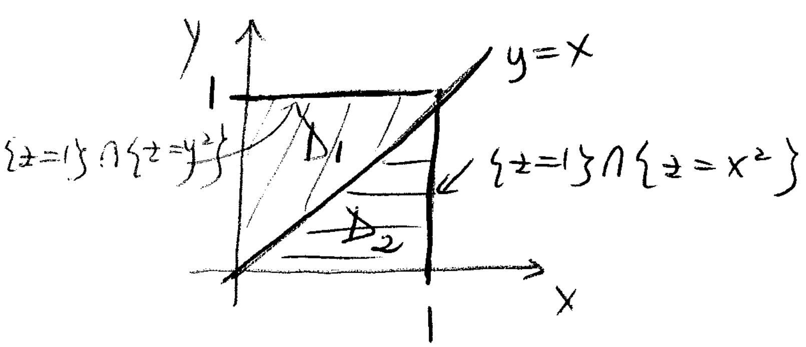
* Again, z is bounded by the surfaces whose equations involve z:

$z = 1 \longrightarrow$ upper bound

$\left.\begin{array}{l} z = x^2 \\ z = y^2 \end{array}\right\}$ lower bound

AND the lower bound changes on the curve of intersection of $z = x^2$ and y^2, so when $x^2 = y^2 \Leftrightarrow \boxed{y = x}$ since we are in the first octant.

* Bounds for x & y: project R onto xy-plane



$D_1 = \left\{ \begin{array}{l} 0 \leq x \leq 1 \\ x \leq y \leq 1 \end{array} \right\}$

and

$D_2 = \left\{ \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq x \end{array} \right\}$

* Bounds for z: $R = \left\{ \begin{array}{l} (x,y) \in D_1 \\ y^2 \leq z \leq 1 \end{array} \right\} \cup \left\{ \begin{array}{l} (x,y) \in D_2 \\ x^2 \leq z \leq 1 \end{array} \right\}$

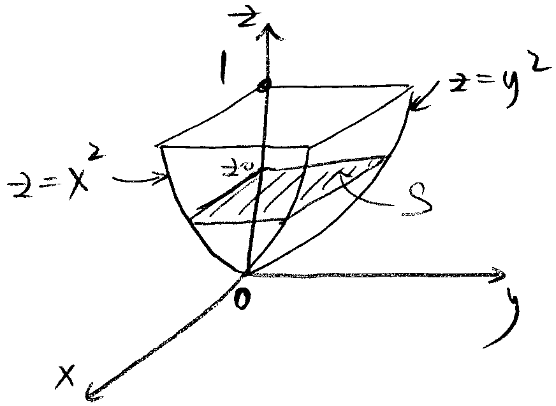
$$\Rightarrow \text{vol}(R) = \int_0^1 \int_x^1 \int_{y^2}^1 1 \, dz \, dy \, dx + \int_0^1 \int_0^x \int_{x^2}^1 1 \, dz \, dy \, dx.$$

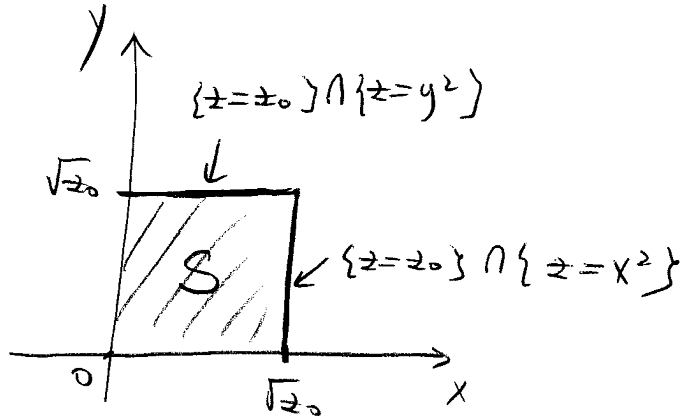
METHOD 2: in R, $\boxed{0 \le z \le 1}$.

For fixed $z_0 \in [0,1]$, the horizontal slice

$$S = \{z = z_0\} \cap R$$

is a square:

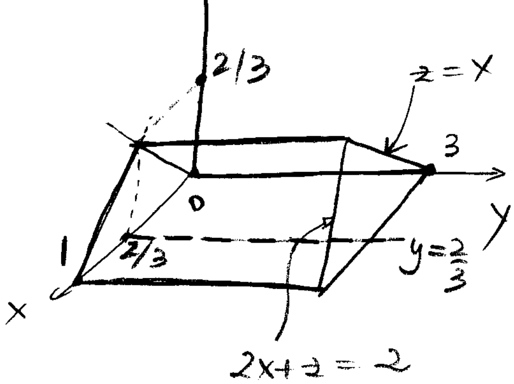




$$\Longrightarrow \quad S = \left\{ \begin{array}{l} 0 \le x \le \sqrt{z_0} \\ 0 \le y \le \sqrt{z_0} \end{array} \right\} \quad \text{and} \quad R = \left\{ \begin{array}{l} 0 \le z \le 1 \\ 0 \le x \le \sqrt{z} \\ 0 \le y \le \sqrt{z} \end{array} \right\}$$

$$\Longrightarrow \quad \mathrm{vol}(R) = \int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{z}} 1 \, dx \, dy \, dz .$$

(c) R is the region bounded by $z = x$, $2x + z = 2$, $y = 0$, $y = 3$, and $z = 0$.



* Bounds for z:

$$\left. \begin{array}{l} z = x \\ 2x + z = 2 \end{array} \right\} \text{ upper bounds}$$

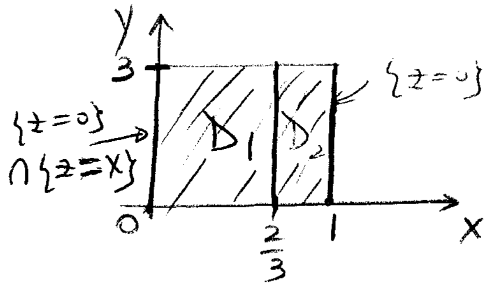
$z = 0 \longrightarrow$ lower bound

AND lower bound changes when $z = x$ and $2x + z = 2$

$\Longleftrightarrow 3x = 2 \Longleftrightarrow \boxed{x = \frac{2}{3}}$

NOTE: $z = 2/3$ at points of intersection.

* Bounds for x & y:



$D_1 = \left\{ \begin{array}{l} 0 \le x \le 2/3 \\ 0 \le y \le 3 \end{array} \right\}$ and $D_2 = \left\{ \begin{array}{l} \frac{2}{3} \le x \le 1 \\ 0 \le y \le 3 \end{array} \right\}$

$$\Rightarrow R = \left\{ \begin{array}{l} (x,y) \in D_1 \\ 0 \le z \le x \end{array} \right\} \cup \left\{ \begin{array}{l} (x,y) \in D_2 \\ 0 \le z \le 2-2x \end{array} \right\}$$

{plane $2x+z=2$ $\Leftrightarrow z = 2-2x$

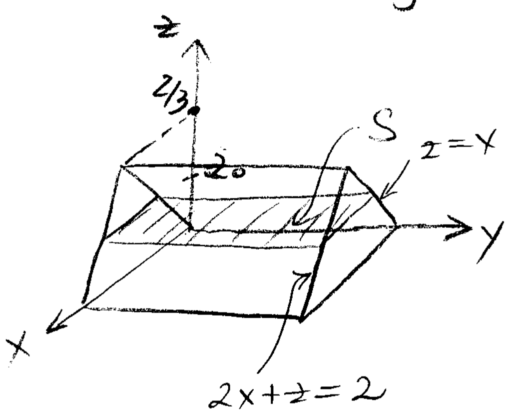
$$\Rightarrow \mathrm{Vol}(R) = \int_0^{\frac{2}{3}} \int_0^3 \int_0^x 1 \, dz \, dy \, dx + \int_{\frac{2}{3}}^1 \int_0^3 \int_0^{2-2x} 1 \, dz \, dy \, dx.$$

METHOD 2: In R, $\boxed{0 \le z \le \frac{2}{3}}$.

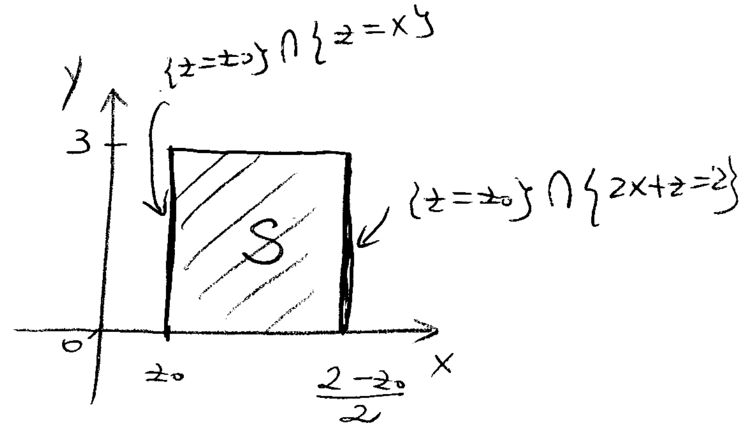
For fixed $z_0 \in [0, \frac{2}{3}]$, the horizontal slice

$$S = R \cap \{ z = z_0 \}$$

is the rectangle:



$\rightsquigarrow$

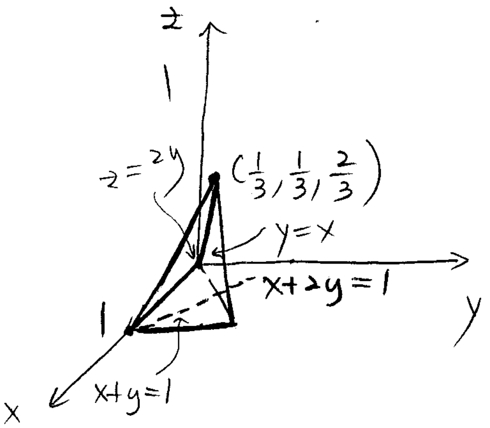


$$\Rightarrow S = \left\{ \begin{array}{l} z_0 \le x \le 1 - z_0/2 \\ 0 \le y \le 3 \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{c} 0 \le z \le 1 \\ z \le x \le 1 - \frac{z}{2} \\ 0 \le y \le 3 \end{array} \right\}$$

$$\Rightarrow \text{vol}(R) = \int_0^1 \int_z^{1-z/2} \int_0^3 1 \, dy \, dx \, dz$$

(d) R is the region bounded by $x + z = 1$, $z = 2y$, $y = x$, and $z = 0$.



R is a prism bounded below by $z=0$, with vertex $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$, which is the point of intersection of the 3 planes $x+z=1$, $z=2y$, $y=x$. The base of R lies on $z=0$, and the other 3 sides lie on the planes $x+z=1$, $z=2y$, and $y=x$.

<u>METHOD 1:</u> * The coordinate z is bounded by the planes whose equations involve z:

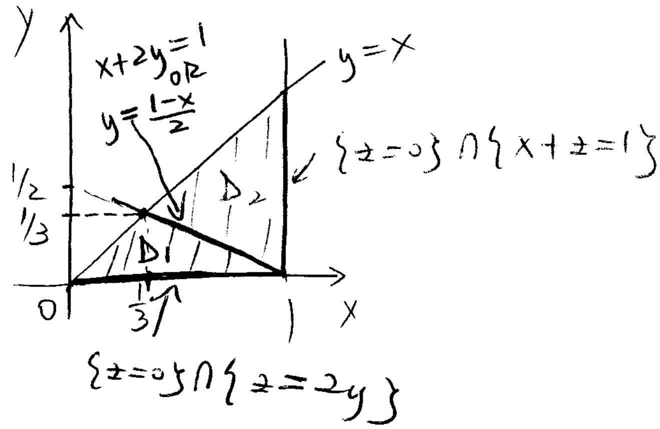
$$\left. \begin{array}{l} x + z = 1 \\ z = 2y \end{array} \right\} \text{ upper bounds}$$

$$z = 0 \longrightarrow \text{lower bound.}$$

AND the upper bound changes on the line of intersection of $x+z=1$ and $z=2y$:

$$z = 1 - x \text{ and } z = 2y \iff 1 - x = 2y \iff \boxed{x + 2y = 1}$$

* Bounds for x & y: project R onto xy-plane



$$D_1 = \left\{ \begin{array}{l} 0 \le y \le \frac{1}{3} \\ y \le x \le 1-2y \end{array} \right\}$$

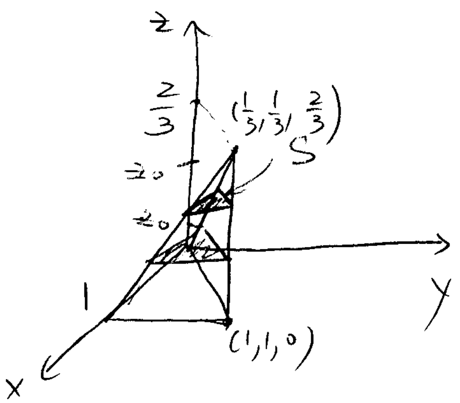
and

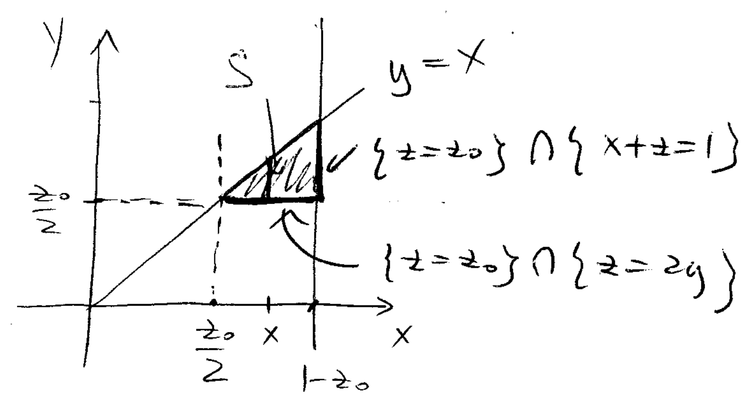
$$D_2 = \left\{ \begin{array}{l} \frac{1}{3} \le x \le 1 \\ \frac{1-x}{2} \le y \le y \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} (x,y) \in D_1 \\ 0 \le z \le 2y \end{array} \right\} \cup \left\{ \begin{array}{l} (x,y) \in D_2 \\ 0 \le z \le 1-x \end{array} \right\}$$

$$\Rightarrow \mathrm{vol}(R) = \int_0^{\frac{1}{3}} \int_y^{1-2y} \int_0^{2y} 1 \, dz \, dx \, dy + \int_{\frac{1}{3}}^{1} \int_{\frac{1-x}{2}}^{x} \int_0^{1-x} 1 \, dz \, dy \, dx$$

METHOD 2: In R, $0 \le z \le \frac{2}{3}$. For fixed $z_0 \in [0, \frac{2}{3}]$, the horizontal slice $S = R \cap \{z = z_0\}$ is the triangle



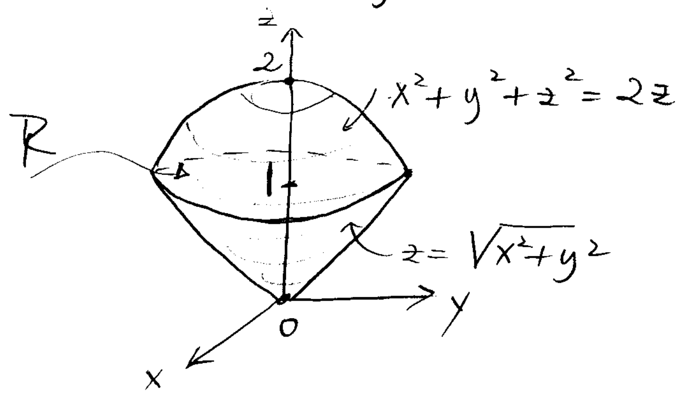


$$\Rightarrow S = \left\{ \begin{array}{l} \frac{z_0}{2} \le x \le 1-z_0 \\ \frac{z_0}{2} \le y \le x \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} 0 \le z \le 2/3 \\ \frac{z_0}{2} \le x \le 1-z_0 \\ \frac{z_0}{2} \le y \le x \end{array} \right\}$$

$$\Rightarrow \quad Vol(R) = \int_0^{2/3} \int_{\frac{z}{2}}^{1-z} \int_{\frac{z}{2}}^{x} 1 \, dy \, dx \, dz$$

2- The "ice cream cone" is bounded by the sphere $x^2+y^2+z^2 = 2z \Leftrightarrow x^2+y^2+(z-1)^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$.



$$Vol(R) = \iiint_R 1 \, dV$$

(a) <u>METHOD 1:</u> * <u>Bounds for z:</u>

We see that z is bounded by the upper-half of the sphere $x^2+y^2+(z-1)^2 = 1 \rightsquigarrow \boxed{z = 1+\sqrt{1-x^2-y^2}}$

AND

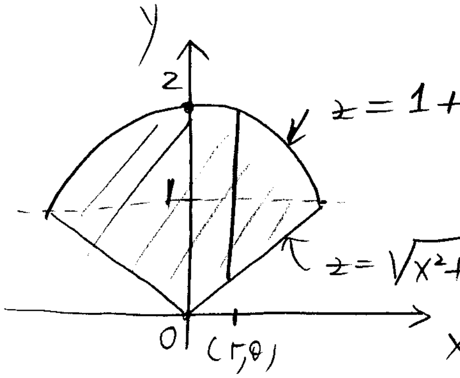
by the cone $\boxed{z = \sqrt{x^2+y^2}}$

Moreover, from the sketch, it's clear that:

$z = 1+\sqrt{1-x^2-y^2} \rightsquigarrow$ upper bound

$z = \sqrt{x^2+y^2} \rightsquigarrow$ lower bound

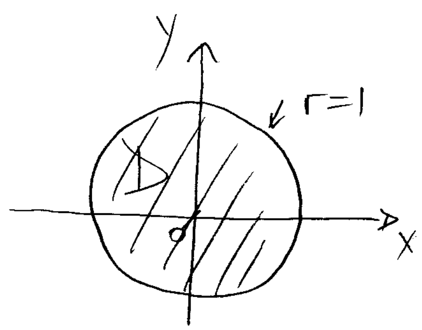
This confirmed by the <u>vertical</u> cross-section



$\Rightarrow$ cone $\leq z \leq$ (upper half sphere)

$\Leftrightarrow$ $\boxed{r \leq z \leq 1 + \sqrt{1-r^2}}$

* Bounds for r & θ: project onto xy-plane.
$\rightsquigarrow$ get a disc bounded by the circle of intersection of the cone $z = r$ and the sphere $z = 1 + \sqrt{1-r^2}$



$\Leftrightarrow$ $r = 1 + \sqrt{1-r^2}$

$\Leftrightarrow$ $(r-1)^2 = 1 - r^2$

$\Rightarrow$ $r = 0, 1$ $\rightsquigarrow$ $\boxed{r = 1}$

since $r=0$ gives $(0,0,0)$.

$$D = \left\{ \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$\Rightarrow R_{r\theta z} = \left\{ \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 1 + \sqrt{1-r^2} \end{array} \right\}$$

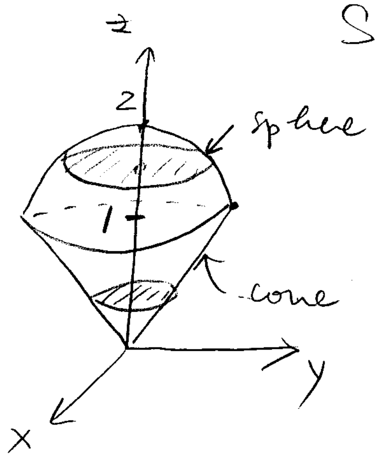
$$\Rightarrow \text{vol} = \iiint_R 1 \, dV = \iiint_{R_{r\theta z}} 1 \cdot \overset{\substack{\text{Jac.} \\ \downarrow}}{r} \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_r^{1+\sqrt{1-r^2}} 1 \cdot r \, dz \, dr \, d\theta = \cdots = \pi .$$

METHOD 2: In R, $\boxed{0 \le z \le 2}$.

For fixed $z_0 \in [0,2]$, consider the horizontal slice

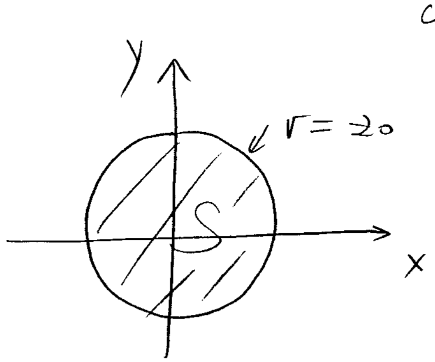
$$S = R \cap \{z = z_0\}.$$



NOTE that the cone $z = r$ and the upper sphere $z = 1 + \sqrt{1-r^2}$ intersect at $r = 1$, so at $z = 1$ (since the intersection line on the cone $z = r$).

We see that the horizontal slice is always a disc, but that:

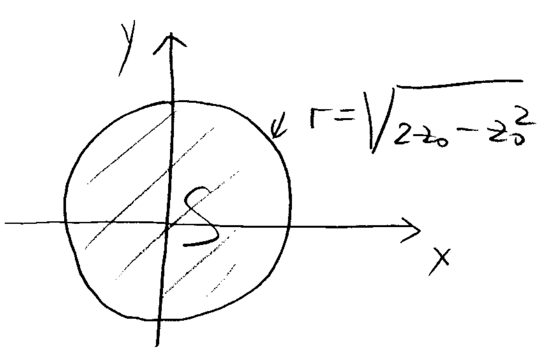
→ if $0 \le z_0 \le 1$: the disc is bounded by the circle of intersection of $z = z_0$ with the cone $z = \sqrt{x^2+y^2} = r \Rightarrow r = z_0$



$$S = \left\{ \begin{array}{l} 0 \le r \le z_0 \\ 0 \le \theta \le 2\pi \end{array} \right\}.$$

→ if $1 \le z_0 \le 2$: the disc is bounded by the circle of intersection of $z = z_0$ with the sphere $x^2+y^2+z^2 = 2z \Leftrightarrow r^2 + z^2 = 2z$

⇝ get $r^2 + z_0^2 = 2z_0 \Leftrightarrow \boxed{r = \sqrt{2z_0 - z_0^2}}$



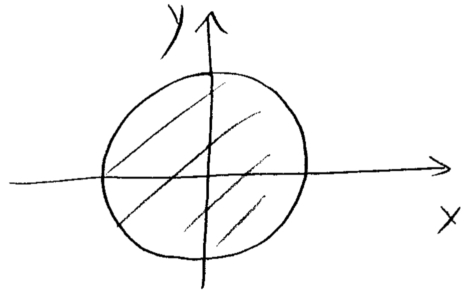
$$S = \left\{ \begin{array}{l} 0 \le r \le \sqrt{2z_0 - z_0^2} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

So: $R = \left\{\begin{array}{l} 0 \le z \le 1 \\ 0 \le r \le z \\ 0 \le \theta \le 2\pi \end{array}\right\} \cup \left\{\begin{array}{l} 1 \le z \le 2 \\ 0 \le r \le \sqrt{2z - z^2} \\ 0 \le \theta \le 2\pi \end{array}\right\}$

$$\Rightarrow \text{vol}(R) = \int_0^1 \int_0^z \int_0^{2\pi} 1 \cdot \underbrace{r}_{\text{Jac.}}\, d\theta\, dr\, dz + \int_1^2 \int_0^{\sqrt{2z-z^2}} \int_0^{2\pi} 1 \cdot \underbrace{r}_{\text{Jac}}\, d\theta\, dr\, dz$$

(b) Using spherical coordinates: $\text{Vol}(R) = \iiint_{\rho\phi\theta} 1 \cdot \underbrace{\rho^2 \sin\phi}_{\text{Jac.}}\, d\rho\, d\phi\, d\theta$

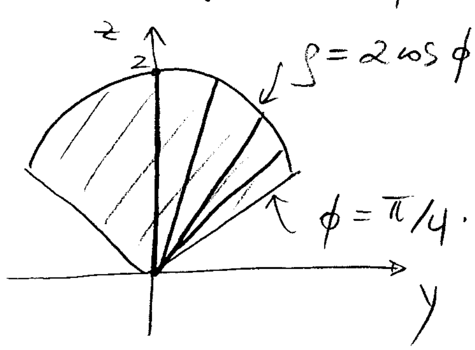
* Bounds for θ: project R onto xy-plane.



$\rightsquigarrow$ Get a disc bounded by the circle of intersection of the cone and the sphere.

$\Rightarrow \boxed{0 \le \theta \le 2\pi}$

* Bounds for ρ & ϕ: take vertical cross-section



$\rightarrow$ Converting the equations to spherical coordinates, we have:

$z = \sqrt{x^2+y^2} \Rightarrow \boxed{\phi = \frac{\pi}{4}}$

$x^2+y^2+z^2 = 2z \Rightarrow \boxed{\rho = 2\cos\phi}$

$\rightarrow$ Taking rays $\phi = \phi_0$ starting at $\phi = 0$, we see that $(0 \le \phi \le \text{cone}) \Leftrightarrow \boxed{0 \le \phi \le \pi/4}$

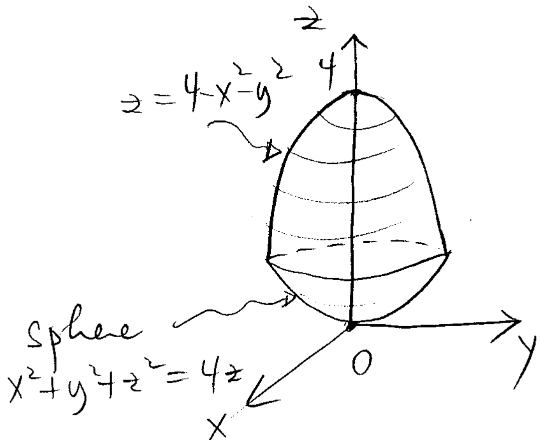
And, the ray $\phi = \phi_0$ intersects R in a strip such that $(0 \le \rho \le \text{sphere}) \Leftrightarrow \boxed{0 \le \rho \le 2\cos\phi}$

So: $R = \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/4 \\ 0 \le \rho \le 2\cos\phi \end{array} \right\}$

$$\Rightarrow \quad vol(R) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta .$$

3- (a) $(\#\text{ of bees}) = \iiint_R 3 \cdot dV = \iiint_{R_{r\theta z}} 3 \cdot \overset{\text{Jac}}{r} \cdot dz\, dr\, d\theta$

where R is below $z = 4 - x^2 - y^2$ and inside $x^2+y^2+z^2 = 4z$.



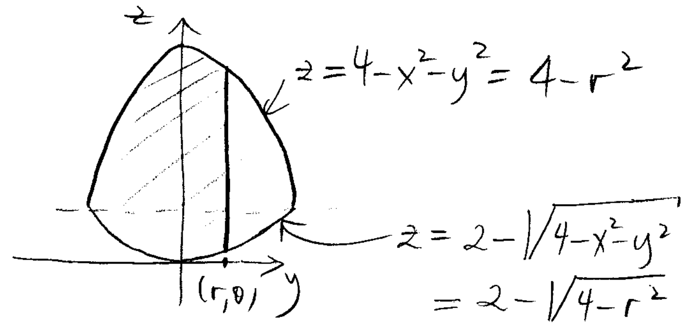
<u>NOTE</u>: The sphere

$x^2+y^2+z^2 = 4z \iff$
$x^2+y^2+(z-2)^2 = 4$

is centered at $(0,0,2)$ of radius 2

<u>METHOD 1</u>:

* <u>Bounds for z</u>: take a vertical cross-section.



$$\underbrace{\left(\text{lower sphere}\right)}_{z=2-\sqrt{4-r^2}} \le z \le \underbrace{\left(\text{paraboloid}\right)}_{z=4-r^2}$$

$$\Rightarrow \boxed{2-\sqrt{4-r^2} \le z \le 4-r^2}$$

* Bounds of r & θ: project R onto xy-plane.

$\rightsquigarrow$ get a disc bounded by the circle of intersection of the paraboloid $z = 4 - x^2 - y^2$ the sphere $x^2+y^2+z^2 = 4z$

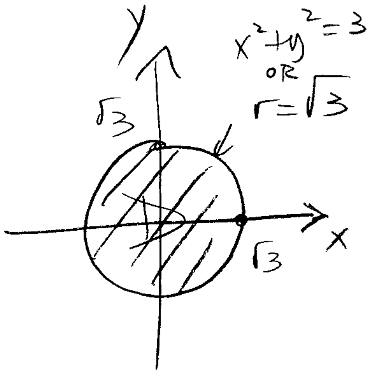
$\Leftrightarrow$ $x^2+y^2 = 4 - z$ and $x^2+y^2 = 4z - z^2$

$\Rightarrow$ $4 - z = 4z - z^2 \Leftrightarrow z^2 - 5z + 4 = 0$

$\Leftrightarrow$ $\boxed{z = 1, 4}$

$\rightsquigarrow$ get circle $x^2+y^2 = \underset{z=1}{\underset{\uparrow}{3}}$ or $\underset{z=4}{\underset{\uparrow}{0}}$

$\Rightarrow$ $\boxed{x^2+y^2 = 3}$ at height $z=1$.



$$D = \left\{ \begin{array}{c} 0 \le r \le \sqrt{3} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

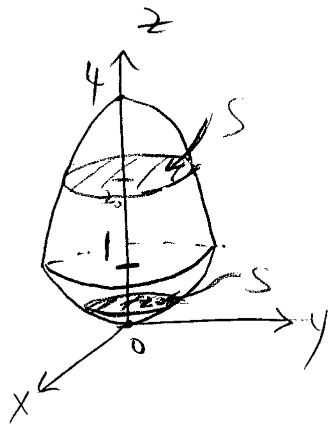
$$\Rightarrow \quad R = \left\{ \begin{array}{c} 2-\sqrt{4-r^2} \le z \le 4-r^2 \\ 0 \le r \le \sqrt{3} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

$$\Rightarrow \left(\#\text{ of bees}\right) = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{2-\sqrt{4-r^2}}^{4-r^2} 3\cdot r \, dz\, dr\, d\theta$$

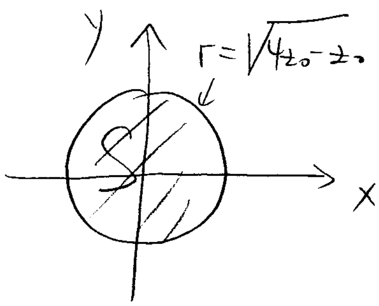
$$= \dots = \frac{37\pi}{2}.$$

METHOD 2: In R, $\boxed{0 \le z \le 4}$.

For $z_0 \in [0,4]$, let $S = R \cap \{z = z_0\}$. Then S is always a disc:



→ if $0 \le z_0 \le 1$: the disc is bounded by the circle of intersection of $z = z_0$ with the sphere $x^2 + y^2 + z^2 = 4z$



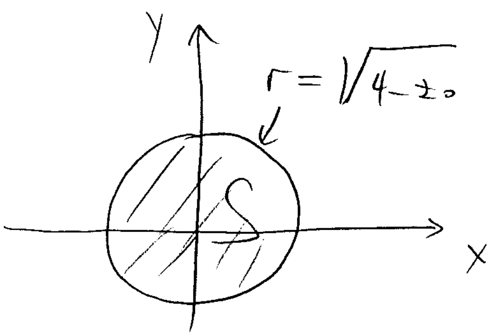
$\rightsquigarrow \quad x^2 + y^2 + z_0^2 = 4z_0$

$\Leftrightarrow \quad x^2 + y^2 = 4z_0 - z_0$

$\rightsquigarrow \quad \boxed{r = \sqrt{4z_0 - z_0}}$

$$S = \left\{ \begin{array}{l} 0 \le r \le \sqrt{4z_0 - z_0^2} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

→ if $1 \le z_0 \le 4$: the disc is bounded by the circle of intersection of $z = z_0$ with the paraboloid $z = 4 - x^2 - y^2$



$\rightsquigarrow \quad z_0 = 4 - x^2 - y^2 \Leftrightarrow x^2 + y^2 = 4 - z_0$

$\rightsquigarrow \quad \boxed{r = \sqrt{4 - z_0}}$

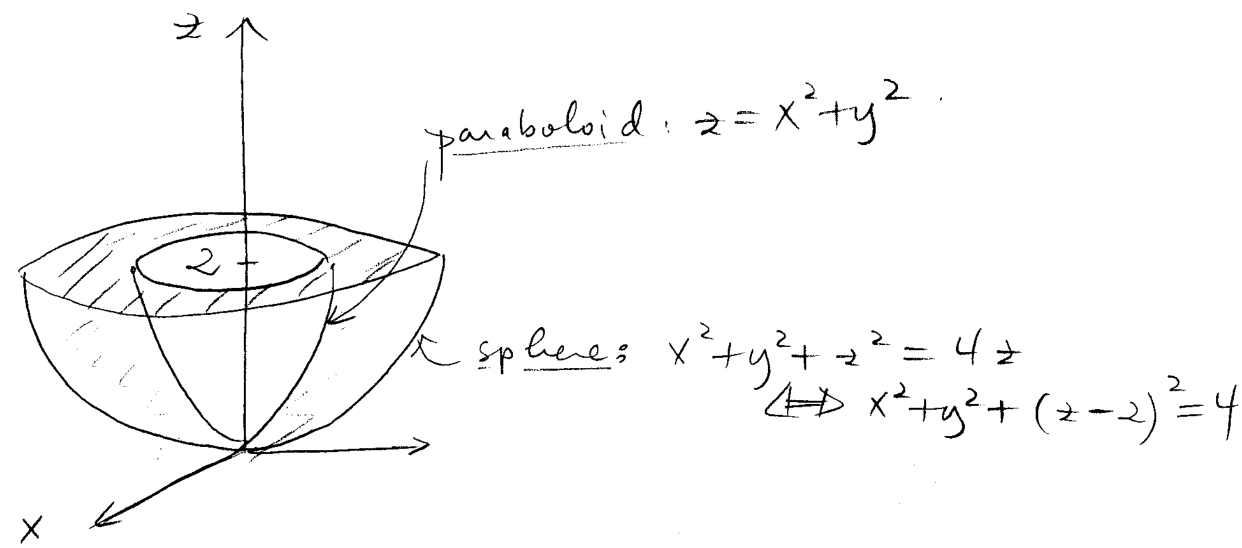
$$S = \left\{ \begin{array}{l} 0 \le r \le \sqrt{4 - z_0} \\ 0 \le \theta \le 2\pi \end{array} \right\}.$$

$$\Rightarrow R = \left\{ \begin{array}{l} 0 \le z \le 1 \\ 0 \le r \le \sqrt{4z - z^2} \\ 0 \le \theta \le 2\pi \end{array} \right\} \cup \left\{ \begin{array}{l} 1 \le z \le 4 \\ 0 \le r \le \sqrt{4 - z} \\ 0 \le \theta \le 2\pi \end{array} \right\}$$

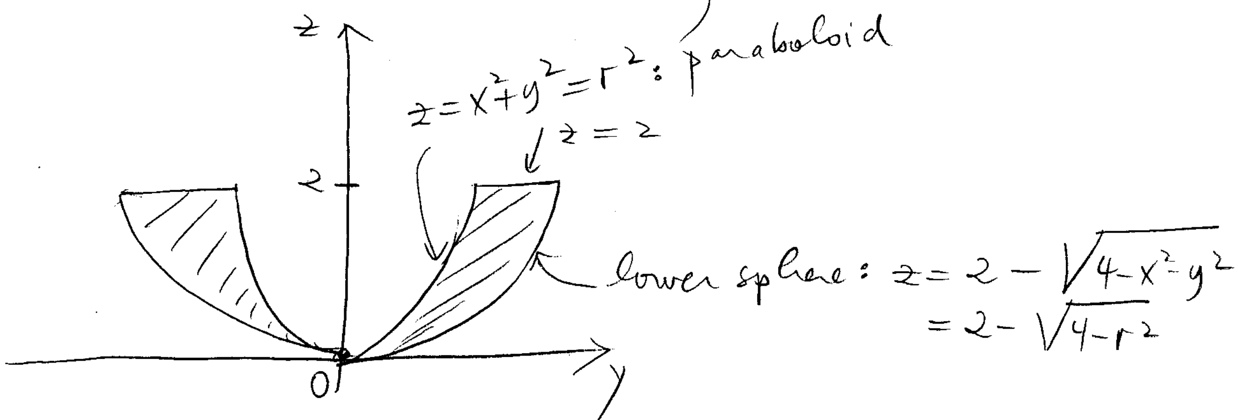
$$\Rightarrow (\#\text{bees}) = \int_0^1 \int_0^{\sqrt{4z - z^2}} \int_0^{2\pi} 3 \cdot r \, dz \, dr \, d\theta + \int_1^4 \int_0^{\sqrt{4 - z}} \int_0^{2\pi} 3 \cdot r \, dz \, dr \, d\theta$$

(b) $\text{Vol}(R) = \iiint_R 1\, dV = \iiint_{R_{r\theta z}} 1 \cdot \overset{\text{Jac.}}{r}\, dz\, dr\, d\theta$.

where R is the region inside the sphere $x^2+y^2+z^2=4z$, outside the paraboloid $z = x^2+y^2$ and below $z=2$.



Taking a vertical cross-section we see:



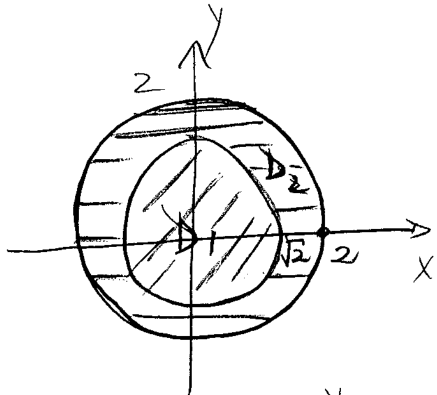
METHOD 1: The bounds for z are determined by the surfaces:

$z = 2-\sqrt{4-r^2}$ $\longrightarrow$ lower bound

$\left.\begin{array}{l} z = r^2 \\ z = 2 \end{array}\right\}$ upper bounds.

We see that the upper bound changes at the intersection of the two upper bounds $z = r^2$ and $z = 2$: at the circle $\boxed{r = \sqrt{2}}$

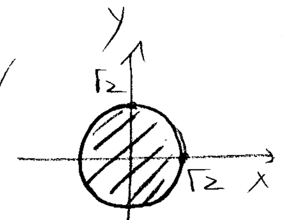
* Bounds for r & θ: project onto xy-plane.

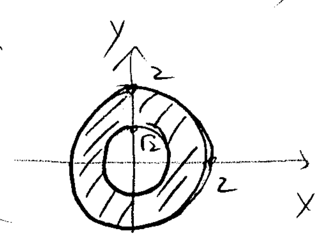


$\rightsquigarrow$ get a disc bounded by the circle of intersection of the sphere $x^2 + y^2 + z^2 = 4z$ with the plane $z = 2$

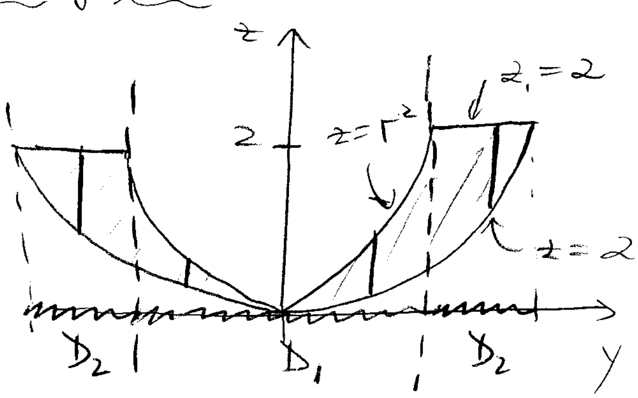
$$\Rightarrow x^2 + y^2 + 4 = 8$$

$$\Leftrightarrow x^2 + y^2 = 4 \Leftrightarrow \boxed{r = 2}$$

$D_1 =$ () $= \left\{ \begin{array}{l} 0 \le r \le \sqrt{2} \\ 0 \le \theta \le 2\pi \end{array} \right\}$

$D_2 =$ () $= \left\{ \begin{array}{l} \sqrt{2} \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right\}$.

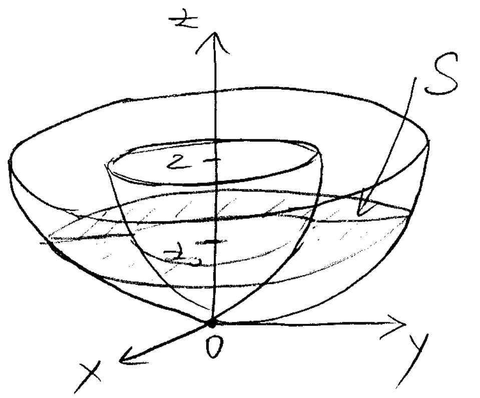
* Bounds for z: Take vertical cross-section.



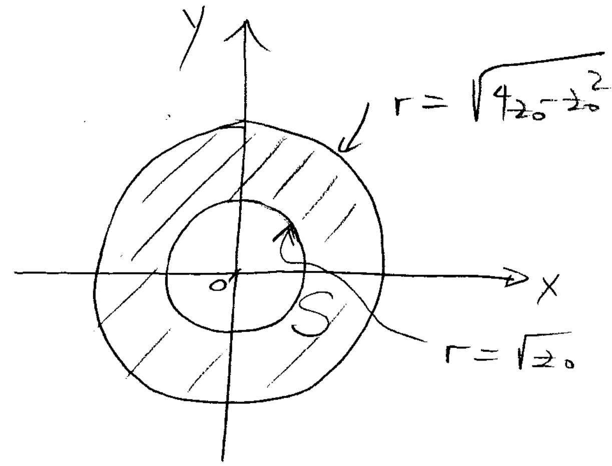
$$\Rightarrow R = \left\{ \begin{array}{l} (x,y) \in D_1 \\ 2 - \sqrt{4 - r^2} \le z \le r^2 \end{array} \right\} \cup \left\{ \begin{array}{l} (x,y) \in D_2 \\ 2 - \sqrt{4 - r^2} \le z \le 2 \end{array} \right\}$$

$$\Rightarrow \text{Vol}(R) = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{2-\sqrt{4-r^2}}^{r^2} 1\cdot r\,dz\,dr\,d\theta + \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_{2-\sqrt{4-r^2}}^{2} 1\cdot r\,dz\,dr\,d\theta$$

METHOD 2: In R, $\boxed{0 \le z \le 2}$. For fixed $z_0 \in [0,2]$, the horizontal slice $S = \{z = z_0\} \cap \{R\}$ is a washer:



$\rightsquigarrow$



S is bounded by the circles of intersection of $z = z_0$ with $z = x^2+y^2$ and $x^2+y^2+z^2 = 4z$:

$$x^2+y^2 = z_0 \iff \boxed{r = \sqrt{z_0}}$$

AND

$$x^2+y^2+z_0^2 = 4z_0 \iff r^2 = 4z_0 - z_0^2 \iff \boxed{r = \sqrt{4z_0 - z_0^2}}$$

$$\Rightarrow S = \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ \sqrt{z_0} \le r \le \sqrt{4z_0 - z_0^2} \end{array} \right\}$$

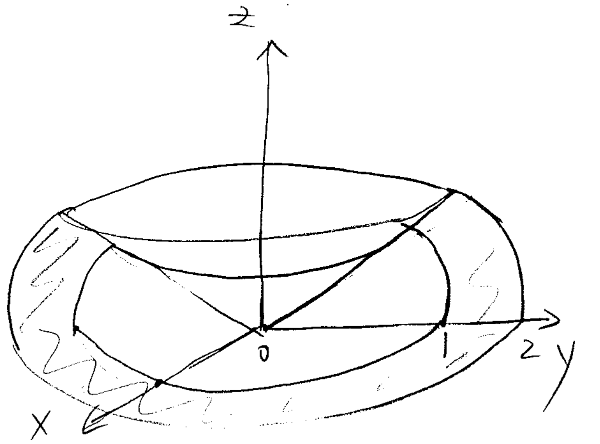
$$\Rightarrow R = \left\{ \begin{array}{l} 0 \le z \le 2 \\ 0 \le \theta \le 2\pi \\ \sqrt{z} \le r \le \sqrt{4z - z^2} \end{array} \right\} \text{ and } \text{Vol}(R) = \int_0^2\int_0^{2\pi}\int_{\sqrt{z}}^{\sqrt{4z-z^2}} 1\cdot r\,dr\,d\theta\,dz$$

$$= \dots = \boxed{\frac{10\pi}{3}}.$$

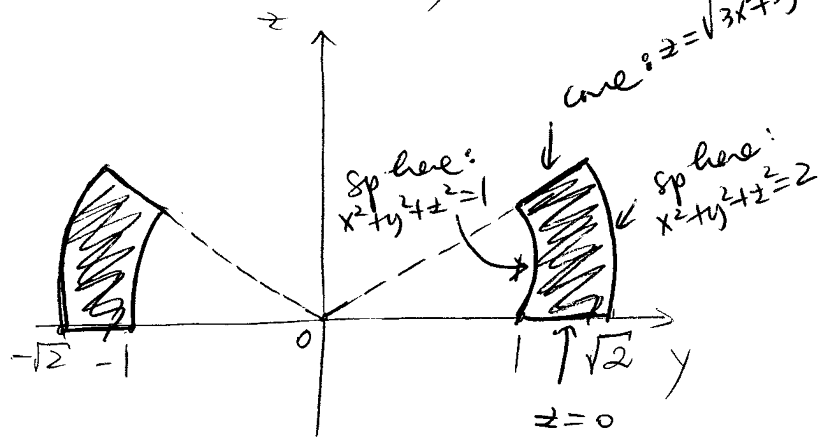
4- (a) $\text{Mass} = \iiint_R \rho(x,y,z)\, dV = \iiint_{R_{\rho\phi\theta}} \frac{1}{\rho} \cdot \overbrace{\rho^2 \sin\phi}^{\text{Jac}}\, d\rho\, d\phi\, d\theta$

$= \iiint_{R_{\rho\phi\theta}} \sin\phi\, d\rho\, d\phi\, d\theta.$

where R lies inside $x^2+y^2+z^2=2$, outside $x^2+y^2+z^2=1$, below $z=\sqrt{3x^2+3y^2}$, above $z=0$, with $x \geq 0$.



Taking a vertical cross-section, we see:



with $x \geq 0$.

Let's convert the equations of the surfaces to spherical coordinates:

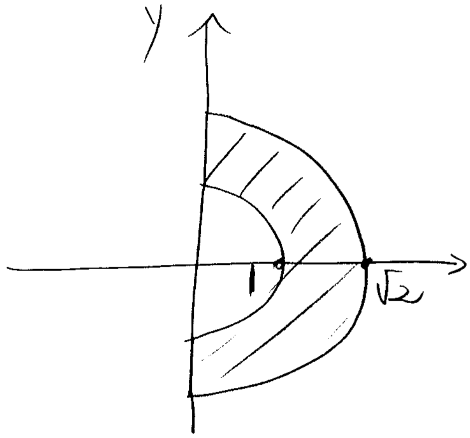
$x^2+y^2+z^2=2 \rightsquigarrow \boxed{\rho=\sqrt{2}}$

$x^2+y^2+z^2=1 \rightsquigarrow \boxed{\rho=1}$

$z=\sqrt{3(x^2+y^2)} \iff \tan\phi = \frac{\sqrt{x^2+y^2}}{z} = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\phi = \frac{\pi}{6}}$

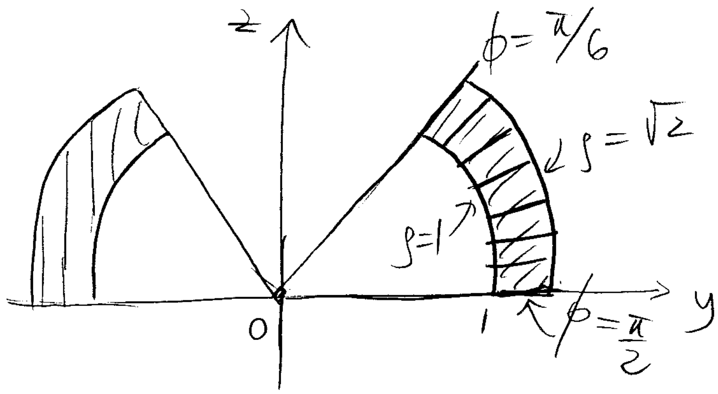
$z=0 \rightsquigarrow \boxed{\phi = \frac{\pi}{2}}$

* Bounds for θ: project onto xy-plane.



$\Longrightarrow \boxed{-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$

* Bounds for ρ & ϕ: Take vertical cross-section.



(cone) $\le \phi \le$ (xy-plane) $\Longleftrightarrow \boxed{\frac{\pi}{6} \le \phi \le \frac{\pi}{2}}$

$\boxed{1 \le \rho \le \sqrt{2}}$.

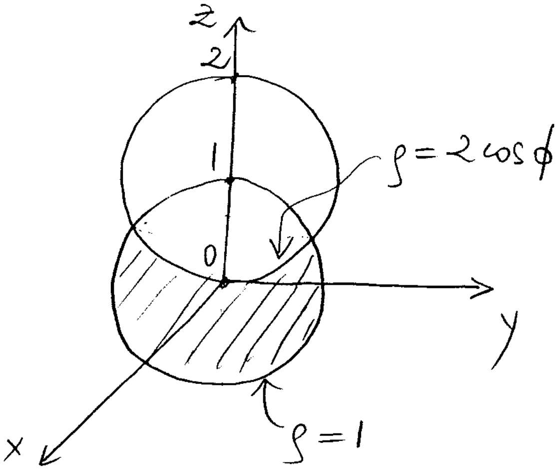
So: $\mathcal{R} = \left\{ \begin{array}{c} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \frac{\pi}{6} \le \phi \le \frac{\pi}{2} \\ 1 \le \rho \le \sqrt{2} \end{array} \right\}$

and

$$\text{Mass} = \int_{-\pi/2}^{\pi/2} \int_{\pi/6}^{\pi/2} \int_{1}^{\sqrt{2}} \sin\phi \, d\rho \, d\phi \, d\theta = \cdots = \boxed{\pi(\sqrt{2}-1)\frac{\sqrt{3}}{2}}$$

(b) Evaluate $\iiint_R z\, dV$, where R is the region inside $x^2+y^2+z^2=1$ and outside $x^2+y^2+z^2=2z$.

* $x^2+y^2+z^2=1 \rightsquigarrow$ sphere of radius 1 centered at $(0,0,0)$.
* $x^2+y^2+z^2=2z \iff x^2+y^2+(z-1)^2=1$
  $\rightsquigarrow$ sphere of radius 1 centered at $(0,0,1)$.

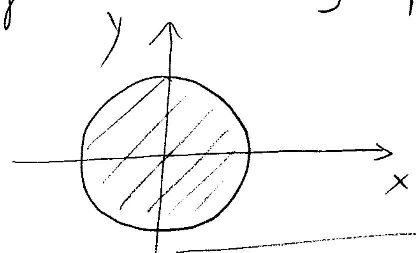


In spherical coordinates:

* $x^2+y^2+z^2=1 \implies \boxed{\rho=1}$
* $x^2+y^2+z^2=2z \iff \rho^2=2\rho\cos\phi$
  $\iff \cancel{\rho=0}$ OR $\boxed{\rho=2\cos\phi}$

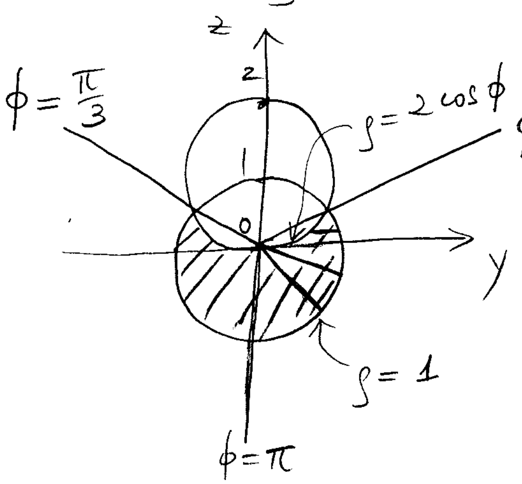
only gives $(0,0,0)$ which is obtained from $\rho=2\cos\phi$ at $\phi=\frac{\pi}{2}$.

Bounds for θ: project region onto xy-plane.



$\implies \boxed{0\le\theta\le 2\pi}$.

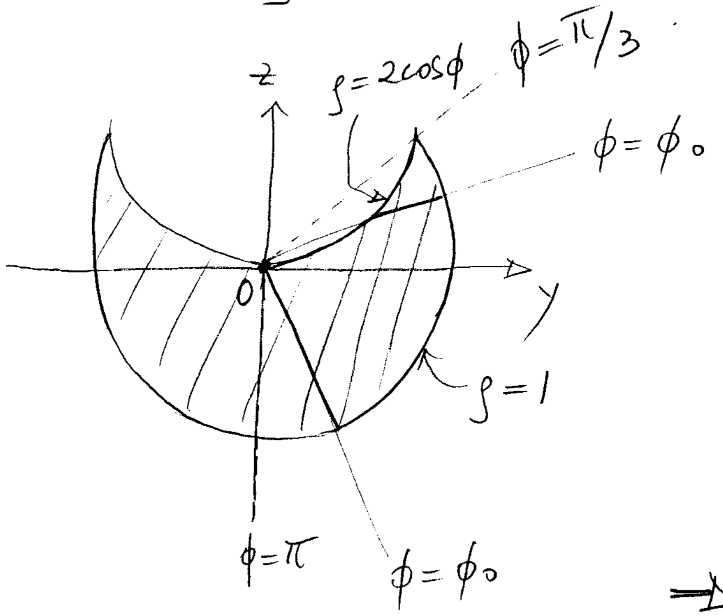
Bounds for ρ & ϕ: take a vertical cross-section.



* We see that points in R have ϕ at most π and at least equal to the angle of intersection of the 2 sphere: $\rho=1$ and $\rho=2\cos\phi$
  $\iff 1=2\cos\phi \iff \cos\phi=1/2$
  $\iff \phi=\frac{\pi}{3}$ since $\phi\in[0,\pi]$.

$\Rightarrow$ $\boxed{\frac{\pi}{3} \leq \phi \leq \pi}$ for points in R.

* Moreover, any radial ray corresponding to an angle $\frac{\pi}{3} \leq \phi_0 \leq \pi$ intersects R as follows:



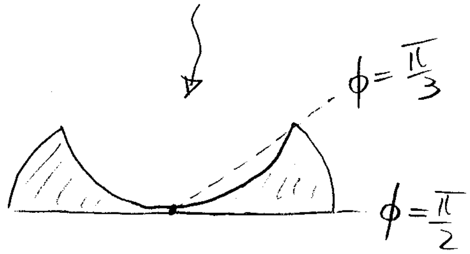
We see that if $\frac{\pi}{3} \leq \phi_0 \leq \frac{\pi}{2}$, then

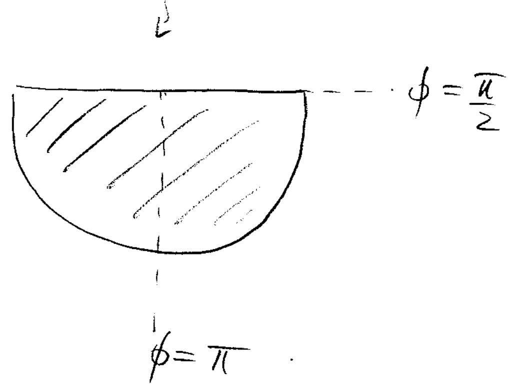
$$\boxed{2\cos\phi \leq \rho \leq 1}$$

BUT, if $\frac{\pi}{2} \leq \phi_0 \leq \pi$, then $\boxed{0 \leq \rho \leq 1}$

$\Rightarrow$ LOWER BOUND CHANGES!

So: $R_{\rho\phi\theta} = \left\{ \begin{array}{c} 0 \leq \theta \leq 2\pi \\ \frac{\pi}{3} \leq \phi \leq \frac{\pi}{2} \\ 2\cos\phi \leq \rho \leq 1 \end{array} \right\} \cup \left\{ \begin{array}{c} 0 \leq \theta \leq 2\pi \\ \frac{\pi}{2} \leq \phi \leq \pi \\ 0 \leq \rho \leq 1 \end{array} \right\}$





$\rightsquigarrow$ will need 2 integrals.

So, $$\iiint_R z\,dV = \iiint_{R_{\rho\phi\theta}} \overset{z}{(\rho\cos\phi)}\cdot \overset{\text{Jac}}{\rho^2\sin\phi}\cdot d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi/2}\int_{2\cos\phi}^{1} (\rho\cos\phi)\cdot\rho^2\sin\phi\cdot d\rho\, d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi}\int_{0}^{1} (\rho\cos\phi)\cdot\rho^2\sin\phi\cdot d\rho\, d\phi\, d\theta.$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\cdot\cos\phi\sin\phi \Big|_{\rho=2\cos\phi}^{\rho=1} \right] d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \left[\frac{\rho^4}{4}\cos\phi\sin\phi \Big|_{\rho=0}^{\rho=1} \right] d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{4}\left(\cos\phi\sin\phi - 16\cos^5\phi\sin\phi\right) d\phi\, d\theta$$

$$+ \int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{4}\left(\cos\phi\sin\phi\right) d\phi\, d\theta$$

$$= \int_0^{2\pi}\frac{1}{4}\left[-\frac{\cos^2\phi}{2} + \frac{16}{6}\cos^6\phi \Big|_{\phi=\frac{\pi}{3}}^{\phi=\frac{\pi}{2}}\right] d\theta + \int_0^{2\pi}\frac{1}{4}\left[-\frac{\cos^2\phi}{2}\Big|_{\phi=\frac{\pi}{2}}^{\phi=\pi}\right] d\theta$$

$$= \int_0^{2\pi}\frac{1}{4}\left(\frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{16}{6}\left(\frac{1}{2}\right)^6\right) d\theta + \int_0^{2\pi}\frac{1}{4}\left(-\frac{(-1)^2}{2}\right) d\theta = \frac{2\pi}{4}\left(\frac{1}{8} - \frac{1}{24} - \frac{1}{2}\right)$$

$$= -5\pi/24.$$