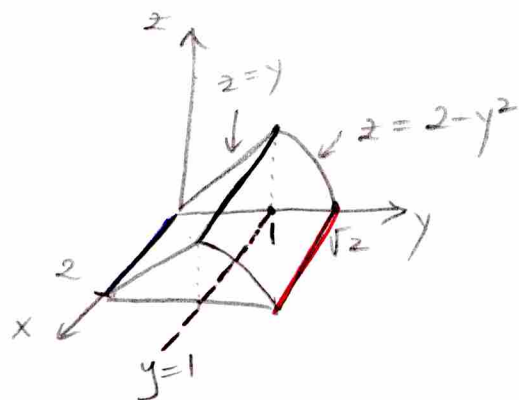


# Solutions to the suggested triple integrals

(1)

1.  $\text{Vol}(R) = \iiint_R 1 \, dV.$

(a)  $R$  is the region bounded by  $z = y$ ,  $z = 2 - y^2$ ,  $z = 0$ ,  $x = 0$ , and  $x = 2$ .



## METHOD 1:

\* We see that  $z$  is bounded by the surfaces whose equations involve  $z$ :

$$\left. \begin{array}{l} z = y \\ z = 2 - y^2 \end{array} \right\} \text{upper bounds}$$

$$z = 0 \rightarrow \text{lower bound.}$$

ALSO, the upper bound changes on the line of the intersection of the upper bounds:

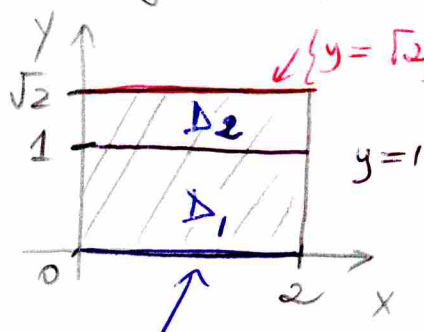
$$z = y \text{ and } z = 2 - y^2 \quad \Leftrightarrow \quad y = 2 - y^2$$

$$\Leftrightarrow \quad y^2 + y - 2 = 0$$

$$\Leftrightarrow \quad y = -2, \quad \textcircled{1}.$$

$\leadsto$   $y = 1$

\* Bounds of  $x$  &  $y$ : Project  $R$  onto  $xy$ -plane.



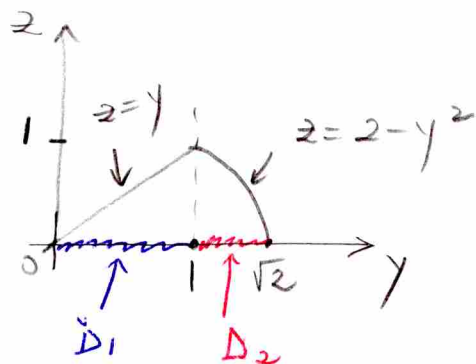
$$\Delta_1 = \left\{ \begin{array}{l} 0 \leq x \leq 2 \\ 0 \leq y \leq 1 \end{array} \right\} \text{ and}$$

$$\Delta_2 = \left\{ \begin{array}{l} 0 \leq x \leq 2 \\ 1 \leq y \leq \sqrt{2} \end{array} \right\}$$

$$\{y = \sqrt{2}\} = \{z = 0\} \cap \{z = 2 - y^2\}$$

\* Bounds for  $z$ : take a vertical cross-section.

(2)



$\forall (x,y,z) \in R$ , if  $(x,y) \in R_1$ ,  $0 \leq z \leq y$   
 and  
 if  $(x,y) \in R_2$ ,  $0 \leq z \leq 2-y^2$

NOTE that we see in the projection of  $R$  onto the  $xy$ -plane that  $z=y$  intersects  $R_1$  and must be the upper bound over  $R_1$ , whereas  $z=2-y^2$  intersects  $R_2$  and must be the upper bound over  $R_2$ .

So:  $R = \left\{ \begin{array}{l} 0 \leq x \leq 2 \\ 0 \leq y \leq 1 \\ 0 \leq z \leq y \end{array} \right\} \cup \left\{ \begin{array}{l} 0 \leq x \leq 2 \\ 1 \leq y \leq \sqrt{2} \\ 0 \leq z \leq 2-y^2 \end{array} \right\}$

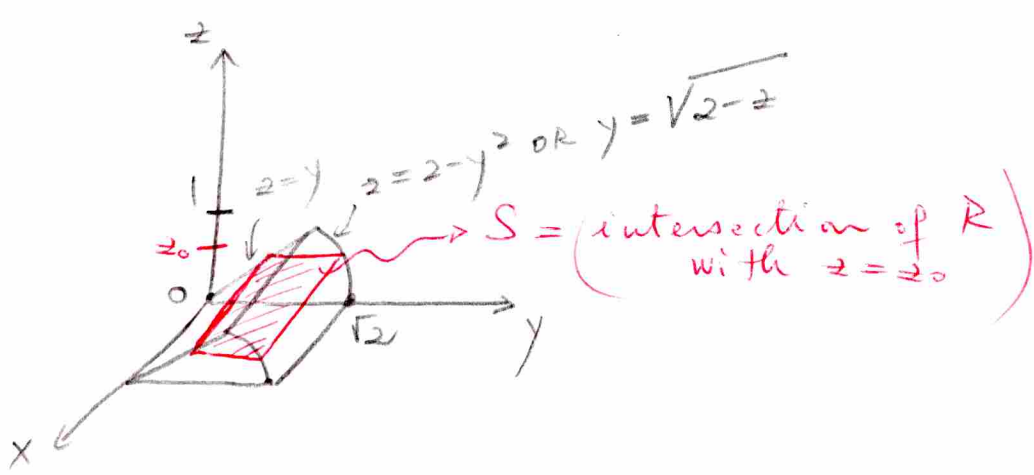
$\Rightarrow \text{Vol}(R) = \int_0^2 \int_0^1 \int_0^y 1 \, dz \, dy \, dx + \int_0^2 \int_1^{\sqrt{2}} \int_0^{2-y^2} 1 \, dz \, dy \, dx$

METHOD 2: We see that in the region  $R$ ,

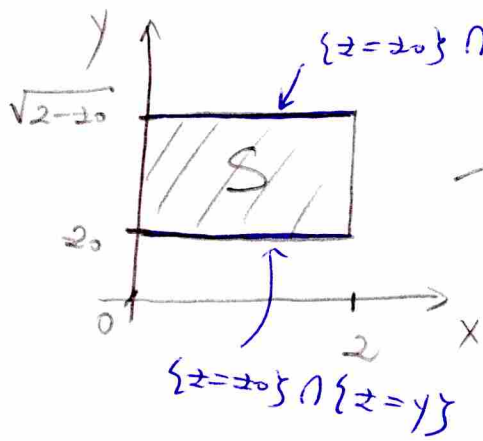
$0 \leq z \leq 1$

For fixed  $z_0 \in [0,1]$ , take the horizontal slice

$S = R \cap \{z = z_0\}$ .



Then,  $S$  is the following rectangle:

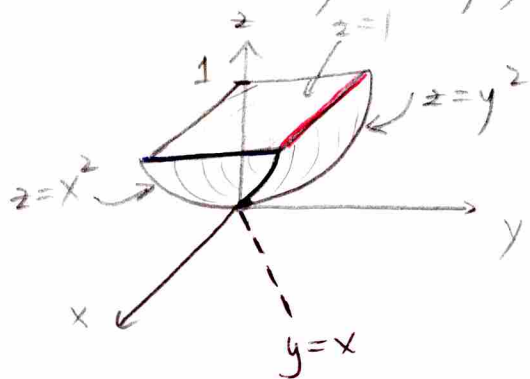


$$S = \left\{ \begin{array}{l} 0 \leq x \leq 2 \\ z_0 \leq y \leq \sqrt{2 - z_0} \end{array} \right\}$$

$$R = \left\{ \begin{array}{l} 0 \leq z \leq 1 \\ 0 \leq x \leq 2 \\ z \leq y \leq \sqrt{2 - z} \end{array} \right\}$$

$$\Rightarrow \text{Vol}(R) = \int_0^1 \int_0^2 \int_z^{\sqrt{2-z}} 1 \, dy \, dx \, dz$$

(b)  $R$  is the region in the first octant bounded by  $z = x^2$ ,  $z = y^2$ , and  $z = 1$ .



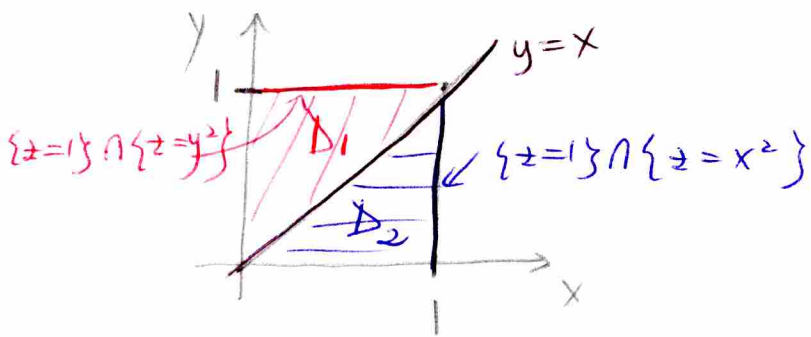
METHOD 1:

\* Again,  $z$  is bounded by the surfaces whose equations involve  $z$ :

$$\begin{aligned} z=1 &\rightarrow \text{upper bound} \\ \left. \begin{aligned} z=x^2 \\ z=y^2 \end{aligned} \right\} &\text{lower bound} \end{aligned}$$

AND the lower bound changes on the curve of intersection of  $z = x^2$  and  $z = y^2$ , so when  $x^2 = y^2 \iff \boxed{y=x}$  since we are in the first octant.

\* Bounds for  $x \neq y$ : project  $R$  onto  $xy$ -plane



$$D_1 = \begin{cases} 0 \leq x \leq 1 \\ x \leq y \leq 1 \end{cases}$$

and

$$D_2 = \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq x \end{cases}$$

\* Bounds for  $z$ :  $R = \left\{ (x,y) \in D_1 \right\} \cup \left\{ (x,y) \in D_2 \right\}$   
 $\left\{ y^2 \leq z \leq 1 \right\} \cup \left\{ x^2 \leq z \leq 1 \right\}$

$$\Rightarrow \text{vol}(R) = \int_0^1 \int_x^1 \int_{y^2}^1 1 \, dz \, dy \, dx + \int_0^1 \int_0^x \int_{x^2}^1 1 \, dz \, dy \, dx.$$

METHOD 2: in  $\mathbb{R}$ ,

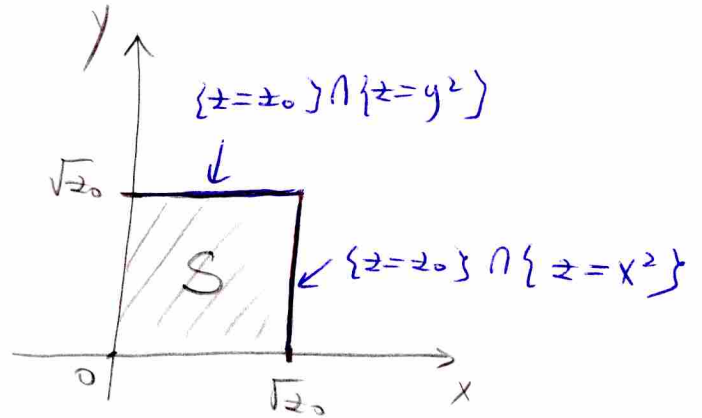
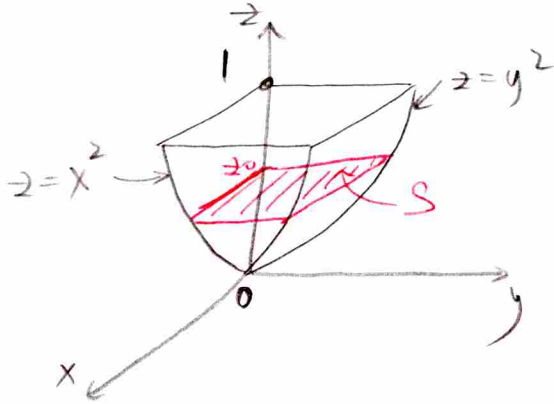
$$0 \leq z \leq 1.$$

(5)

For fixed  $z_0 \in [0, 1]$ , the horizontal slice

$$S = \{z = z_0\} \cap R$$

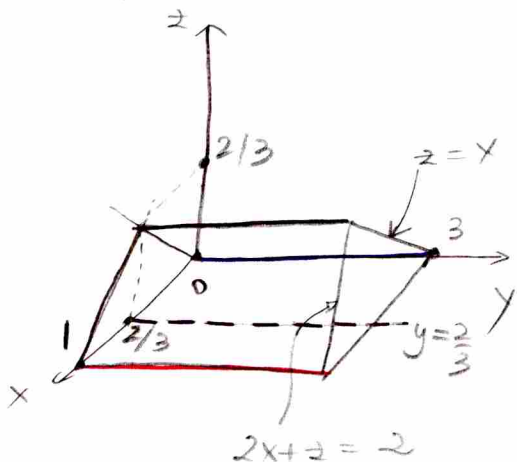
is a square:



$$\Rightarrow S = \left\{ \begin{array}{l} 0 \leq x \leq \sqrt{z_0} \\ 0 \leq y \leq \sqrt{z_0} \end{array} \right\} \text{ and } R = \left\{ \begin{array}{l} 0 \leq z \leq 1 \\ 0 \leq x \leq \sqrt{z} \\ 0 \leq y \leq \sqrt{z} \end{array} \right\}$$

$$\Rightarrow \text{vol}(R) = \int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{z}} 1 \cdot dx \, dy \, dz.$$

(c)  $R$  is the region bounded by  $z = x$ ,  $2x + z = 2$ ,  $y = 0$ ,  $y = 3$ , and  $z = 0$ .



+ Bounds for  $z$ :

$$\left. \begin{array}{l} z = x \\ 2x + z = 2 \end{array} \right\} \text{ upper bounds}$$

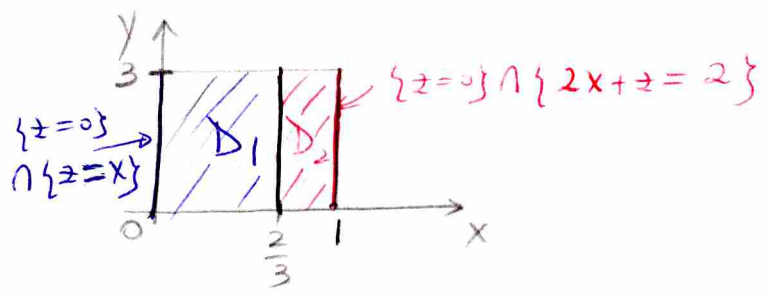
$$z = 0 \rightarrow \text{lower bound}$$

AND lower bound changes when  $z = x$  and  $2x + z = 2$

$$\Leftrightarrow 3x = 2 \Leftrightarrow \boxed{x = \frac{2}{3}}$$

NOTE:  $z = 2/3$  at points of intersection.

\* Bounds for x & y:



$$D_1 = \left\{ \begin{array}{l} 0 \leq x \leq 2/3 \\ 0 \leq y \leq 3 \end{array} \right\}$$

and  $D_2 = \left\{ \begin{array}{l} 2/3 \leq x \leq 1 \\ 0 \leq y \leq 3 \end{array} \right\}$

$$\Rightarrow R = \left\{ \begin{array}{l} (x,y) \in D_1 \\ 0 \leq z \leq x \end{array} \right\} \cup \left\{ \begin{array}{l} (x,y) \in D_2 \\ 0 \leq z \leq 2-2x \end{array} \right\}$$

plane  $2x+z=2$   
 $\Rightarrow z=2-2x$

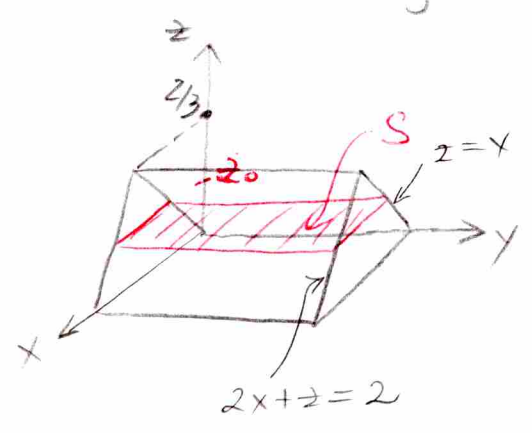
$$\Rightarrow \text{Vol}(R) = \int_0^{2/3} \int_0^3 \int_0^x 1 \, dz \, dy \, dx + \int_{2/3}^1 \int_0^3 \int_0^{2-2x} 1 \, dz \, dy \, dx.$$

METHOD 2: In  $R$ ,  $0 \leq z \leq \frac{2}{3}$ .

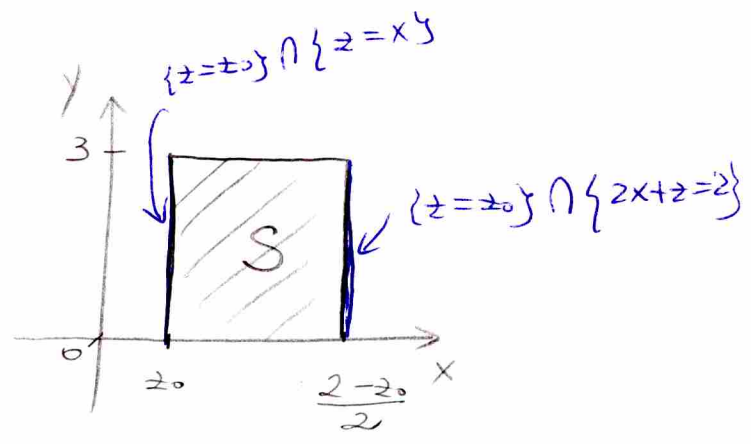
For fixed  $z_0 \in [0, \frac{2}{3}]$ , the horizontal slice

$$S = R \cap \{z = z_0\}$$

is the rectangle:



~>



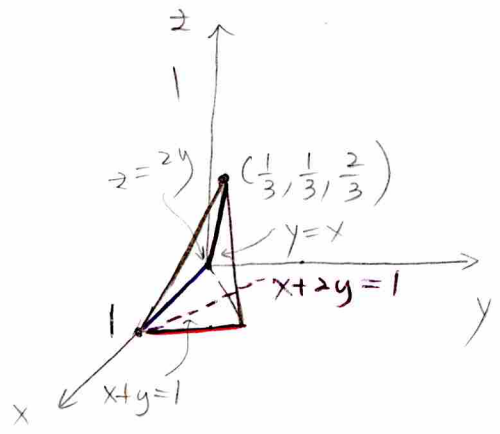
$$\Rightarrow S = \left\{ \begin{array}{l} z_0 \leq x \leq 1 - z_0/2 \\ 0 \leq y \leq 3 \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} 0 \leq z \leq 1 \\ z \leq x \leq 1 - \frac{z}{2} \\ 0 \leq y \leq 3 \end{array} \right\}$$

$$\Rightarrow \text{vol}(R) = \int_0^1 \int_z^{1-\frac{z}{2}} \int_0^3 1 \, dy \, dx \, dz$$



(d) R is the region bounded by  $x+z=1$ ,  $z=2y$ ,  $y=x$ , and  $z=0$ .



R is a prism bounded below by  $z=0$ , with vertex  $(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$  which is the point of intersection of the 3 planes  $x+z=1$ ,  $z=2y$ ,  $y=x$ . The base of R lies on  $z=0$ , and the other 3 sides lie on the planes  $x+z=1$ ,  $z=2y$ , and  $y=x$ .

METHOD 1: \* The coordinate  $z$  is bounded by the planes whose equations involve  $z$ :

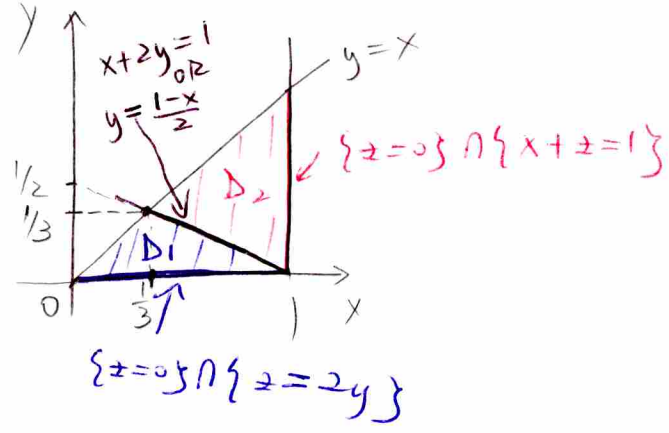
$$\left. \begin{array}{l} x+z=1 \\ z=2y \end{array} \right\} \text{ upper bounds}$$

$$z=0 \rightarrow \text{lower bound.}$$

AND the upper bound changes on the line of intersection of  $x+z=1$  and  $z=2y$ :

$$z=1-x \text{ and } z=2y \Leftrightarrow 1-x=2y \Leftrightarrow \boxed{x+2y=1}$$

\* Bounds for  $x \neq y$ : project  $R$  onto  $xy$ -plane



$$D_1 = \left\{ \begin{array}{l} 0 \leq y \leq \frac{1}{3} \\ y \leq x \leq 1-2y \end{array} \right\}$$

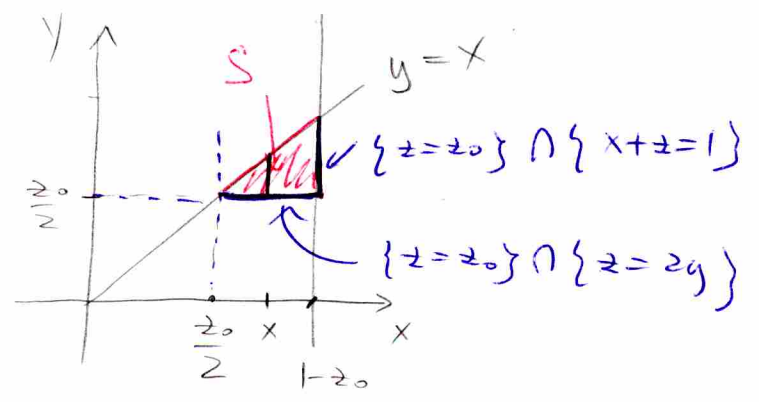
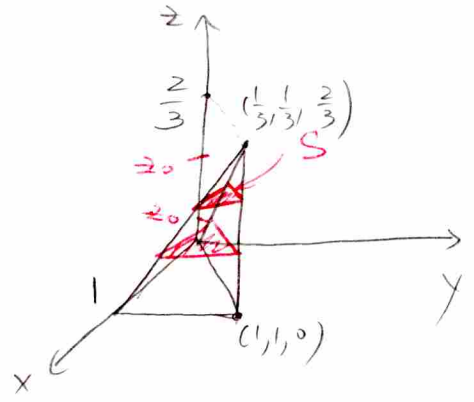
and

$$D_2 = \left\{ \begin{array}{l} \frac{1}{3} \leq x \leq 1 \\ \frac{1-x}{2} \leq y \leq y \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} (x,y) \in D_1 \\ 0 \leq z \leq 2y \end{array} \right\} \cup \left\{ \begin{array}{l} (x,y) \in D_2 \\ 0 \leq z \leq 1-x \end{array} \right\}$$

$$\Rightarrow \text{vol}(R) = \int_0^{\frac{1}{3}} \int_y^{1-2y} \int_0^{2y} 1 \, dz \, dx \, dy + \int_{\frac{1}{3}}^1 \int_{\frac{1-x}{2}}^x \int_0^{1-x} 1 \, dz \, dy \, dx$$

METHOD 2: In  $R$ ,  $0 \leq z \leq \frac{2}{3}$ . For fixed  $z_0 \in [0, \frac{2}{3}]$ , the horizontal slice  $S = R \cap \{z = z_0\}$  is the triangle



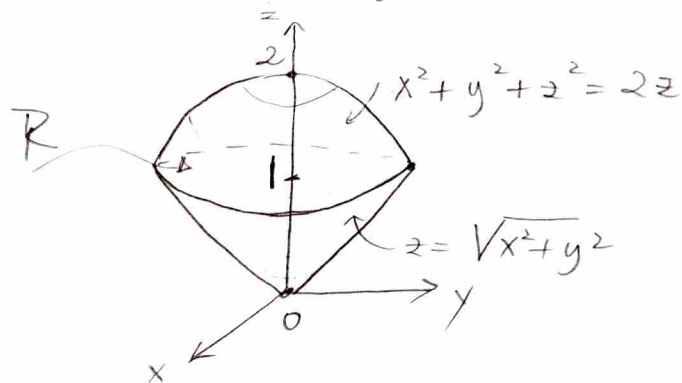
$$\Rightarrow S = \left\{ \begin{array}{l} \frac{z_0}{2} \leq x \leq 1-z_0 \\ \frac{z_0}{2} \leq y \leq x \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} 0 \leq z \leq \frac{2}{3} \\ \frac{z}{2} \leq x \leq 1-z_0 \\ \frac{z}{2} \leq y \leq x \end{array} \right\}$$



$$\Rightarrow \text{Vol}(R) = \int_0^{2/3} \int_{\frac{z}{2}}^{1-z} \int_{\frac{z}{2}}^x 1 \, dy \, dx \, dz$$

2- The "ice cream cone" is bounded by the sphere  $x^2 + y^2 + z^2 = 2z \Leftrightarrow x^2 + y^2 + (z-1)^2 = 1$  and the cone  $z = \sqrt{x^2 + y^2}$ .



$$\text{Vol}(R) = \iiint_R 1 \, dV$$

(a) METHOD 1:  $\times$  Bounds for  $z$ :

We see that  $z$  is bounded by the upper-half of the sphere  $x^2 + y^2 + (z-1)^2 = 1 \Rightarrow z = 1 + \sqrt{1 - x^2 - y^2}$

AND

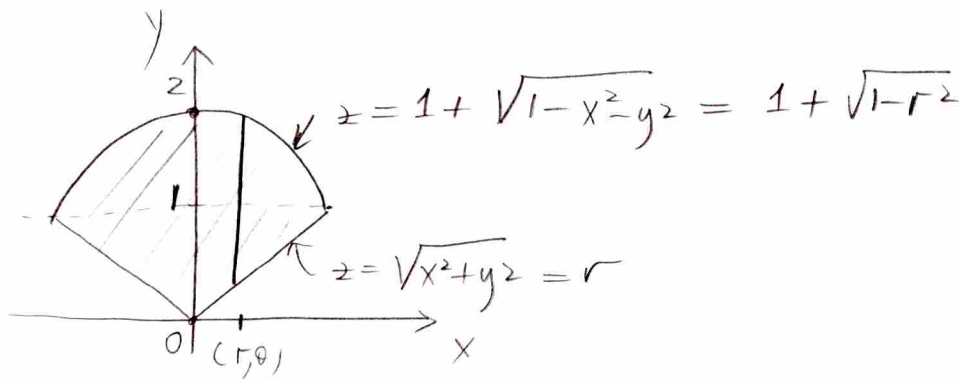
by the cone  $z = \sqrt{x^2 + y^2}$ .

Moreover, from the sketch, it's clear that:

$$z = 1 + \sqrt{1 - x^2 - y^2} \rightarrow \text{upper bound}$$

$$z = \sqrt{x^2 + y^2} \rightarrow \text{lower bound}$$

This confirmed by the vertical cross-section

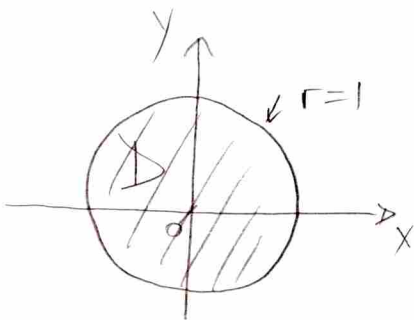


⇒ cone  $z \leq z \leq$  (upper half sphere)

⇔  $\boxed{r \leq z \leq 1 + \sqrt{1 - r^2}}$

\* Bounds for  $r \neq \theta$ : project onto  $xy$ -plane.

⇒ get a disc bounded by the circle of intersection of the cone  $z = r$  and the sphere  $z = 1 + \sqrt{1 - r^2}$



⇔  $r = 1 + \sqrt{1 - r^2}$

⇔  $(r - 1)^2 = 1 - r^2$

⇒  $r = 0, 1 \Rightarrow \boxed{r = 1}$

since  $r = 0$  gives  $(0, 0, 0)$ .

$D = \left\{ \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$

⇒  $R_{r\theta z} = \left\{ \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 1 + \sqrt{1 - r^2} \end{array} \right\}$

Jac.

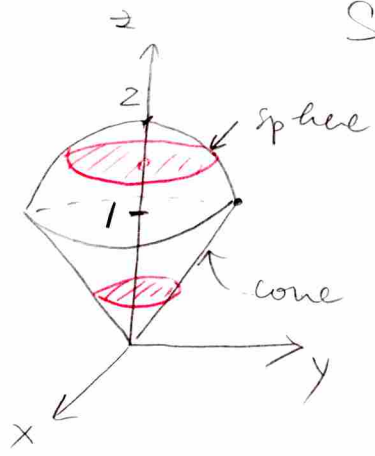
⇒  $vol = \iiint_{R_{r\theta z}} 1 \, dV = \iiint_{R_{r\theta z}} 1 \cdot r \, dz \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^1 \int_r^{1 + \sqrt{1 - r^2}} 1 \cdot r \, dz \, dr \, d\theta = \dots = \boxed{\pi}$ .

METHOD 2: In  $\mathbb{R}^3$ ,  $0 \leq z \leq 2$ .

For fixed  $z_0 \in [0, 2]$ , consider the horizontal slice

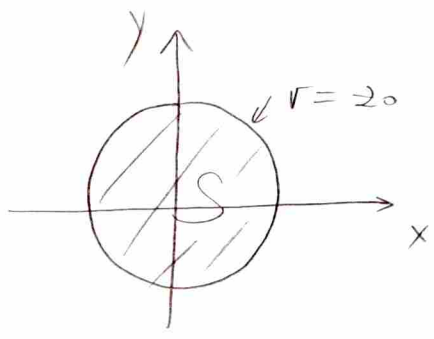
$$S = \mathbb{R}^3 \cap \{z = z_0\}.$$



NOTE that the cone  $z = r$  and the upper sphere  $z = 1 + \sqrt{1-r^2}$  intersect at  $r = 1$ , so at  $z = 1$  (since the intersection line on the cone  $z = r$ ).

We see that the horizontal slice is always a disc, but that:

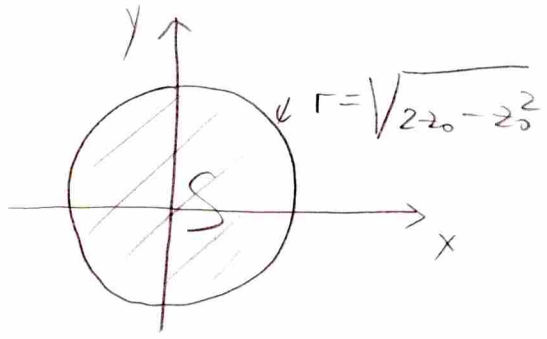
→ if  $0 \leq z_0 \leq 1$ : the disc is bounded by the circle of intersection of  $z = z_0$  with the cone  $z = \sqrt{x^2 + y^2} = r \Rightarrow r = z_0$



$$S = \left\{ \begin{array}{l} 0 \leq r \leq z_0 \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

→ if  $1 \leq z_0 \leq 2$ : the disc is bounded by the circle of intersection of  $z = z_0$  with the sphere  $x^2 + y^2 + z^2 = 2z \Rightarrow r^2 + z^2 = 2z$

$\Rightarrow$  get  $r^2 + z_0^2 = 2z_0 \Rightarrow r = \sqrt{2z_0 - z_0^2}$



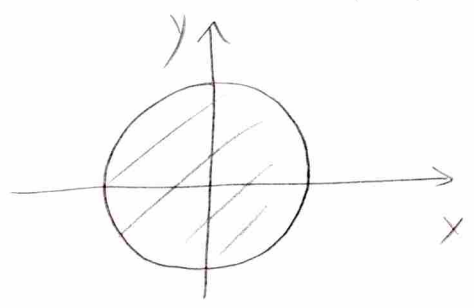
$$S = \left\{ \begin{array}{l} 0 \leq r \leq \sqrt{2z_0 - z_0^2} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

So:  $R = \left\{ \begin{array}{l} 0 \leq z \leq 1 \\ 0 \leq r \leq z \\ 0 \leq \theta \leq 2\pi \end{array} \right\} \cup \left\{ \begin{array}{l} 1 \leq z \leq 2 \\ 0 \leq r \leq \sqrt{2z - z^2} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$

$\Rightarrow \text{vol}(R) = \int_0^1 \int_0^z \int_0^{2\pi} 1 \cdot r \, d\theta \, dr \, dz + \int_1^2 \int_0^{\sqrt{2z - z^2}} \int_0^{2\pi} 1 \cdot r \, d\theta \, dr \, dz$

(b) Using spherical coordinates:  $\text{Vol}(R) = \iiint 1 \cdot \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

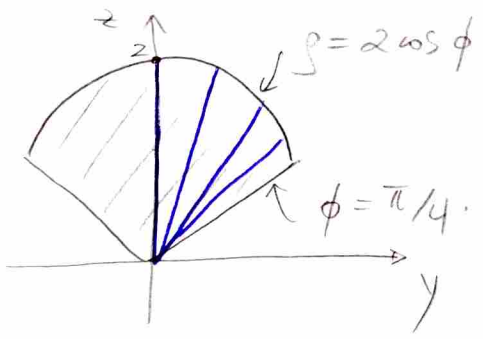
\* Bounds for  $\theta$ : project  $R$  onto  $xy$ -plane.  $\uparrow$  Jac.



$\leadsto$  Get a disc bounded by the circle of intersection of the cone and the sphere.

$\Rightarrow \boxed{0 \leq \theta \leq 2\pi}$

\* Bounds for  $\rho$  &  $\phi$ : take vertical cross-section



$\rightarrow$  Converting the equations to spherical coordinates, we have:

$z = \sqrt{x^2 + y^2} \Rightarrow \boxed{\phi = \frac{\pi}{4}}$

$x^2 + y^2 + z^2 = 2z \Rightarrow \boxed{\rho = 2 \cos \phi}$

$\rightarrow$  Taking rays  $\phi = \phi_0$  starting at  $\phi = 0$ , we see that  $(0 \leq \phi \leq \text{cone}) \Leftrightarrow \boxed{0 \leq \phi \leq \pi/4}$

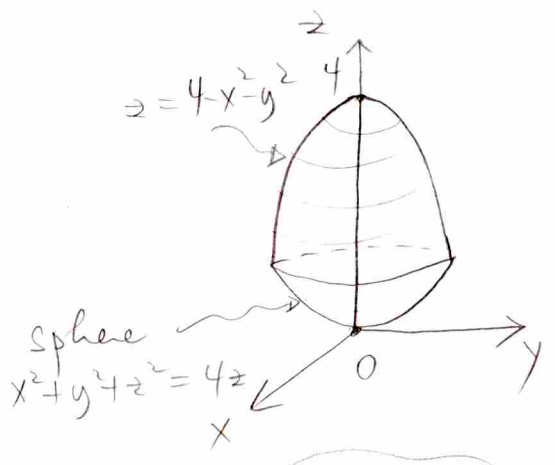
And, the ray  $\phi = \phi_0$  intersects  $R$  in a strip such that  $(0 \leq \rho \leq \text{sphere}) \Leftrightarrow \boxed{0 \leq \rho \leq 2 \cos \phi}$

So:  $R = \left\{ \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/4 \\ 0 \leq \rho \leq 2\cos\phi \end{array} \right\}$

$\Rightarrow \text{vol}(R) = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2\cos\phi} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

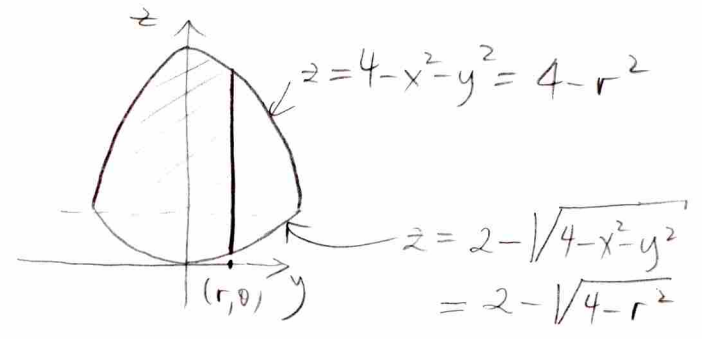
3- (a) (# of bees) =  $\iiint_R 3 \cdot dV = \iiint_{R_{r\theta z}} 3 \cdot \overset{\text{Jac}}{r} \, dz \, dr \, d\theta$

where  $R$  is below  $z = 4 - x^2 - y^2$  and inside  $x^2 + y^2 + z^2 = 4z$ .



METHOD 1:

\* Bounds for z: take a vertical cross-section.

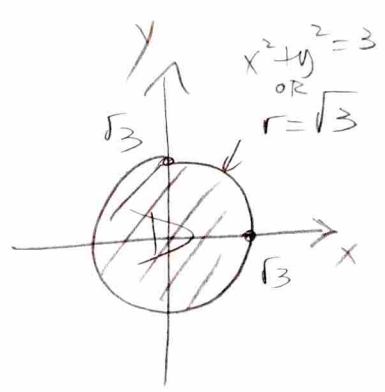


NOTE: The sphere  $x^2 + y^2 + z^2 = 4z$   $\Leftrightarrow$   $x^2 + y^2 + (z-2)^2 = 4$  is centered at  $(0,0,2)$  of radius 2

(lower sphere)  $\leq z \leq$  (paraboloid)  
 $z = 2 - \sqrt{4 - r^2} \leq z \leq 4 - r^2$

$\Rightarrow \boxed{2 - \sqrt{4 - r^2} \leq z \leq 4 - r^2}$

\* Bounds of  $r \neq \theta$ : project  $R$  onto  $xy$ -plane.



$\leadsto$  get a disc bounded by the circle of intersection of the paraboloid  $z = 4 - x^2 - y^2$  the sphere  $x^2 + y^2 + z^2 = 4z$

$\Leftrightarrow x^2 + y^2 = 4 - z$  and  $x^2 + y^2 = 4z - z^2$

$\Rightarrow 4 - z = 4z - z^2 \Leftrightarrow z^2 - 5z + 4 = 0$

$\Leftrightarrow \boxed{z = 1, 4}$

$$D = \left\{ \begin{array}{l} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$\leadsto$  get circle  $x^2 + y^2 = 3$  or  $0$

$\uparrow$                      $\uparrow$   
 $z=1$                  $z=4$

$\Rightarrow \boxed{x^2 + y^2 = 3}$  at height  $z=1$ .

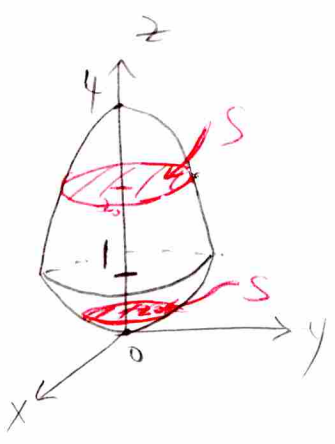
$$\Rightarrow R = \left\{ \begin{array}{l} 2 - \sqrt{4 - r^2} \leq z \leq 4 - r^2 \\ 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$\Rightarrow (\# \text{ of bees}) = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{2 - \sqrt{4 - r^2}}^{4 - r^2} 3 \cdot r \, dz \, dr \, d\theta$$

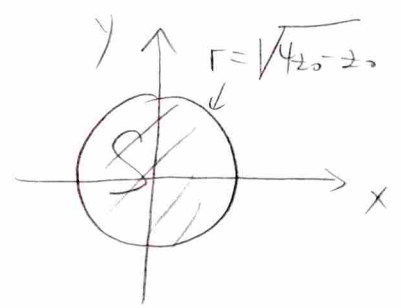
$$= \dots = \left( \frac{37\pi}{2} \right)$$

METHOD 2: In  $R$ ,  $\boxed{0 \leq z \leq 4}$ .

For  $z_0 \in [0, 4]$ , let  $S = R \cap \{z = z_0\}$ . Then  $S$  is always a disc:



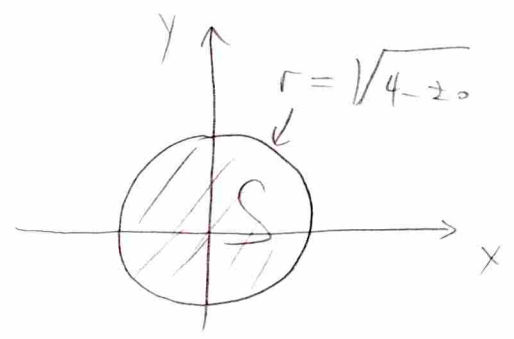
→ if  $0 \leq z_0 \leq 1$ : the disc is bounded by the circle of intersection of  $z = z_0$  with the sphere  $x^2 + y^2 + z^2 = 4z$



$$\begin{aligned} \leadsto x^2 + y^2 + z_0^2 &= 4z_0 \\ \Leftrightarrow x^2 + y^2 &= 4z_0 - z_0 \\ \leadsto \boxed{r = \sqrt{4z_0 - z_0}} \end{aligned}$$

$$S = \left\{ \begin{array}{l} 0 \leq r \leq \sqrt{4z_0 - z_0^2} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

→ if  $1 \leq z_0 \leq 4$ : the disc is bounded by the circle of intersection of  $z = z_0$  with the paraboloid  $z = 4 - x^2 - y^2$



$$\begin{aligned} \leadsto z_0 &= 4 - x^2 - y^2 \Leftrightarrow x^2 + y^2 = 4 - z_0 \\ \leadsto \boxed{r = \sqrt{4 - z_0}} \end{aligned}$$

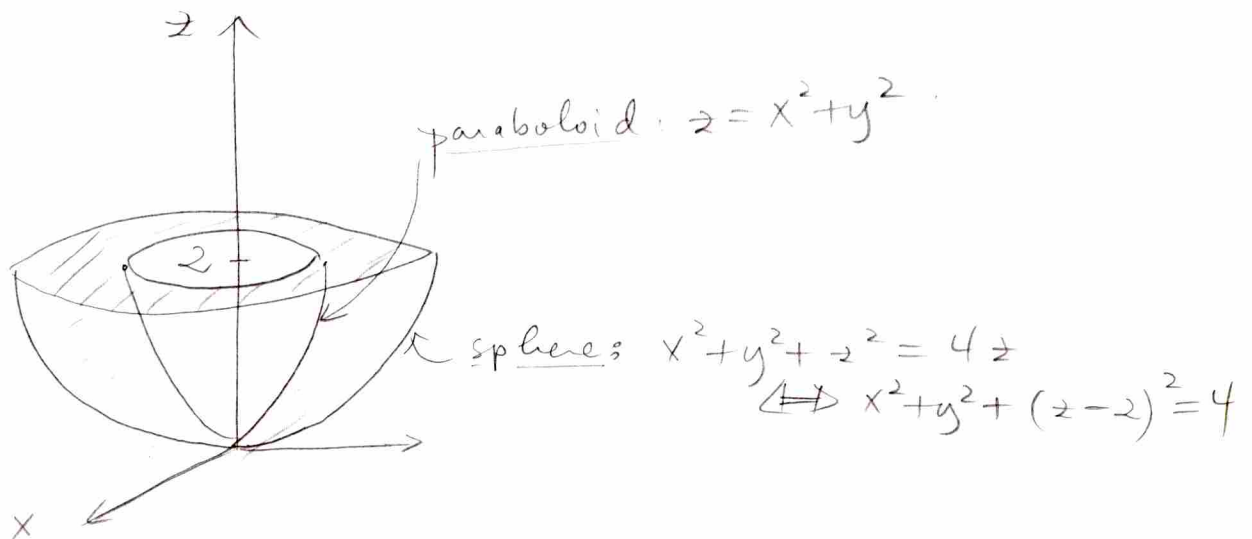
$$S = \left\{ \begin{array}{l} 0 \leq r \leq \sqrt{4 - z_0} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} 0 \leq z \leq 1 \\ 0 \leq r \leq \sqrt{4z - z^2} \\ 0 \leq \theta \leq 2\pi \end{array} \right\} \cup \left\{ \begin{array}{l} 1 \leq z \leq 4 \\ 0 \leq r \leq \sqrt{4 - z} \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

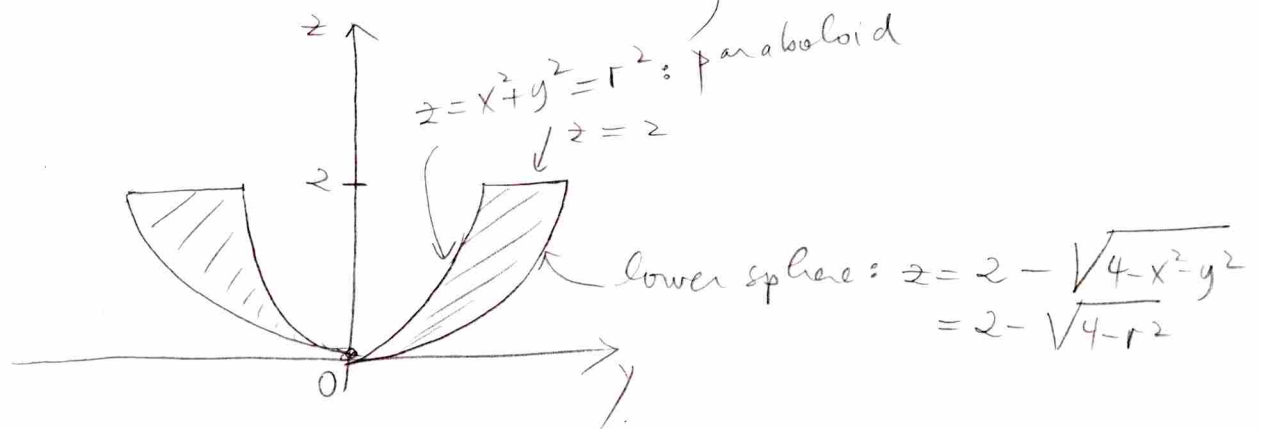
$$\Rightarrow (\# \text{ bees}) = \int_0^1 \int_0^{\sqrt{4z - z^2}} \int_0^{2\pi} 3 \cdot r \, dz \, dr \, d\theta + \int_1^4 \int_0^{\sqrt{4 - z}} \int_0^{2\pi} 3 \cdot r \, dz \, dr \, d\theta$$

$$(b) \text{Vol}(R) = \iiint_R 1 \, dV = \iiint_{R_{r\theta z}} 1 \cdot r \, dz \, dr \, d\theta \quad \text{Jac.}$$

where  $R$  is the region inside the sphere  $x^2 + y^2 + z^2 = 4z$ , outside the paraboloid  $z = x^2 + y^2$  and below  $z = 2$ .



Taking a vertical cross-section we see:



METHOD 1: The bounds for  $z$  are determined by the surfaces:

$$z = 2 - \sqrt{4 - r^2} \rightarrow \text{lower bound}$$

$$\left. \begin{array}{l} z = r^2 \\ z = 2 \end{array} \right\} \text{upper bounds.}$$



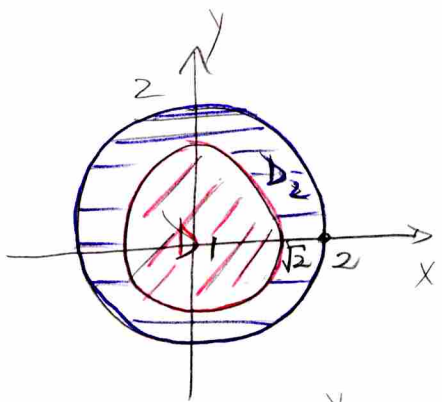
We see that the upper bound changes at the intersection of the two upper bounds  $z = r^2$  and  $z = 2$  : at the circle  $r = \sqrt{2}$

\* Bounds for  $r \neq 0$ : project onto  $xy$ -plane.

$\leadsto$  get a disc bounded by the circle of intersection of the sphere  $x^2 + y^2 + z^2 = 4z$  with the plane  $z = 2$

$$\Rightarrow x^2 + y^2 + 4 = 8$$

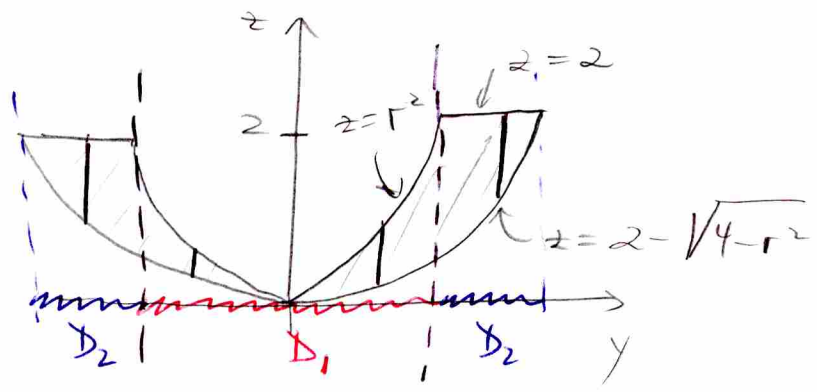
$$\Leftrightarrow x^2 + y^2 = 4 \Leftrightarrow r = 2$$



$$D_1 = \left( \text{disc of radius } \sqrt{2} \right) = \left\{ \begin{array}{l} 0 \leq r \leq \sqrt{2} \\ 0 \leq \theta \leq 2\pi \end{array} \right.$$

$$D_2 = \left( \text{annulus between } r = \sqrt{2} \text{ and } r = 2 \right) = \left\{ \begin{array}{l} \sqrt{2} \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{array} \right.$$

\* Bounds for  $z$ : Take vertical cross-section.

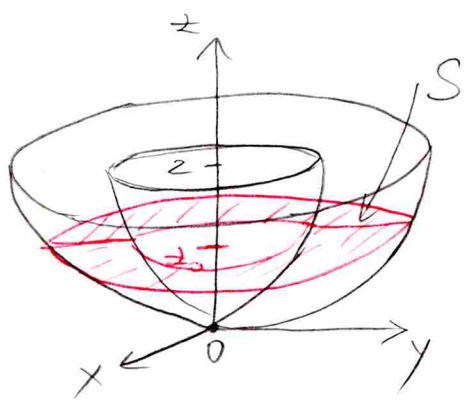


$$\Rightarrow R = \left\{ (x,y) \in D_1 \mid 2 - \sqrt{4 - r^2} \leq z \leq r^2 \right\} \cup \left\{ (x,y) \in D_2 \mid 2 - \sqrt{4 - r^2} \leq z \leq 2 \right\}$$

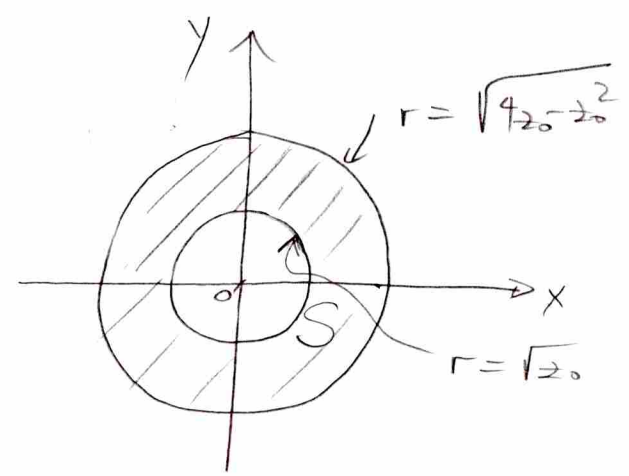
$$\Rightarrow \text{Vol}(R) = \int_0^{2\pi} \int_0^{\sqrt{z}} \int_{2-\sqrt{4-r^2}}^{r^2} 1 \cdot r \, dz \, dr \, d\theta + \int_0^{2\pi} \int_{\sqrt{z}}^2 \int_{2-\sqrt{4-r^2}}^z 1 \cdot r \, dz \, dr \, d\theta$$

METHOD 2: In  $R$ ,  $0 \leq z \leq 2$ . For fixed  $z_0 \in [0, 2]$ ,

the horizontal slice  $S = \{z = z_0\} \cap \{R\}$  is a washer:



$\rightsquigarrow$



$S$  is bounded by the circles of intersection of  $z = z_0$  with  $z = x^2 + y^2$  and  $x^2 + y^2 + z^2 = 4z$ :

$$x^2 + y^2 = z_0 \iff r = \sqrt{z_0}$$

AND

$$x^2 + y^2 + z_0^2 = 4z_0 \iff r^2 = 4z_0 - z_0^2 \iff r = \sqrt{4z_0 - z_0^2}$$

$$\Rightarrow S = \left\{ \begin{array}{l} 0 \leq \theta \leq 2\pi \\ \sqrt{z_0} \leq r \leq \sqrt{4z_0 - z_0^2} \end{array} \right\}$$

$$\Rightarrow R = \left\{ \begin{array}{l} 0 \leq z \leq 2 \\ 0 \leq \theta \leq 2\pi \\ \sqrt{z} \leq r \leq \sqrt{4z - z^2} \end{array} \right\}$$

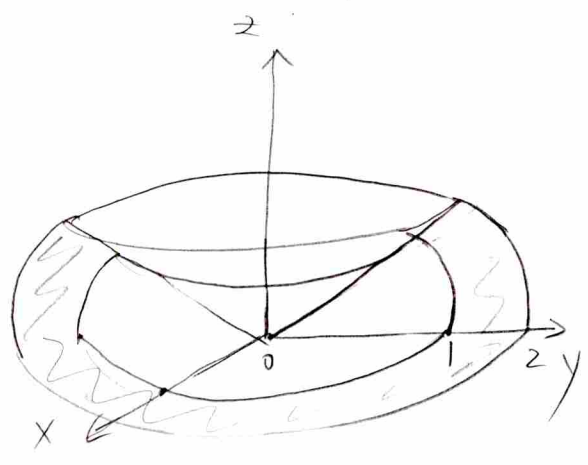
$$\text{and Vol}(R) = \int_0^2 \int_0^{2\pi} \int_{\sqrt{z}}^{\sqrt{4z-z^2}} 1 \cdot r \, dr \, d\theta \, dz$$

$$= \dots = \frac{10\pi}{2}$$

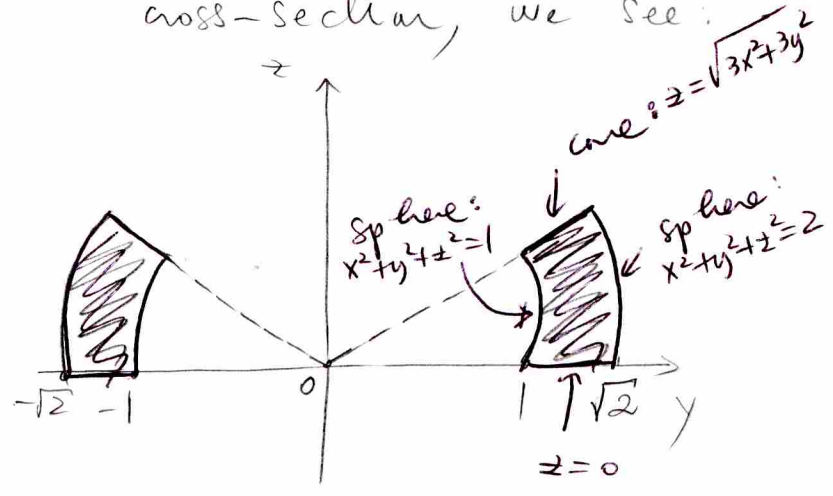
$$4- (a) \text{ Mass} = \iiint_R f(x,y,z) dV = \iiint_{R_{\rho\phi\theta}} \frac{1}{\rho} \cdot \overset{\text{Jac}}{\rho^2 \sin\phi} d\rho d\phi d\theta$$

$$= \iiint_{R_{\rho\phi\theta}} \sin\phi d\rho d\phi d\theta.$$

where  $R$  lies inside  $x^2+y^2+z^2=2$ , outside  $x^2+y^2+z^2=1$ , below  $z=\sqrt{3x^2+3y^2}$ , above  $z=0$ , with  $x \geq 0$ .



Taking a vertical cross-section, we see:



Let's convert the equations of the surfaces to spherical coordinates:

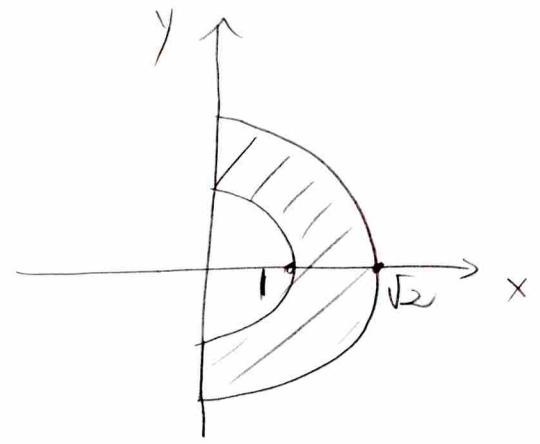
$$x^2+y^2+z^2=2 \rightsquigarrow \boxed{\rho = \sqrt{2}}$$

$$x^2+y^2+z^2=1 \rightsquigarrow \boxed{\rho = 1}$$

$$z = \sqrt{3(x^2+y^2)} \iff \tan\phi = \frac{\sqrt{x^2+y^2}}{z} = \frac{1}{\sqrt{3}} \implies \boxed{\phi = \frac{\pi}{6}}$$

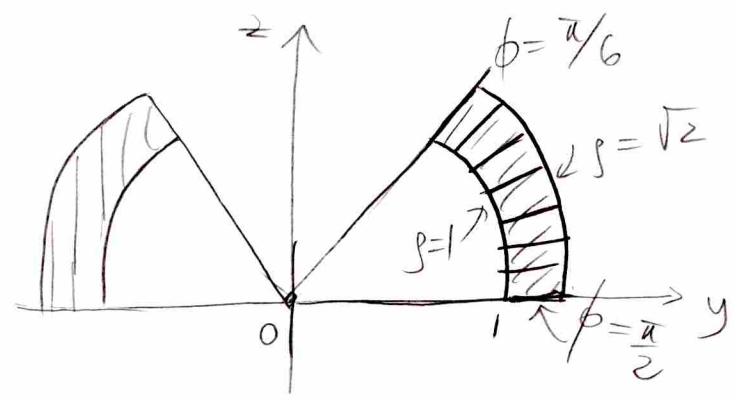
$$z=0 \rightsquigarrow \boxed{\phi = \frac{\pi}{2}}$$

\* Bounds for  $\theta$ : project onto  $xy$ -plane.



$$\Rightarrow \boxed{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$$

\* Bounds for  $\rho$  &  $\phi$ : Take vertical cross-section.



$$(\text{cone}) \leq \phi \leq (\text{xy-plane}) \Leftrightarrow \boxed{\frac{\pi}{6} \leq \phi \leq \frac{\pi}{2}}$$

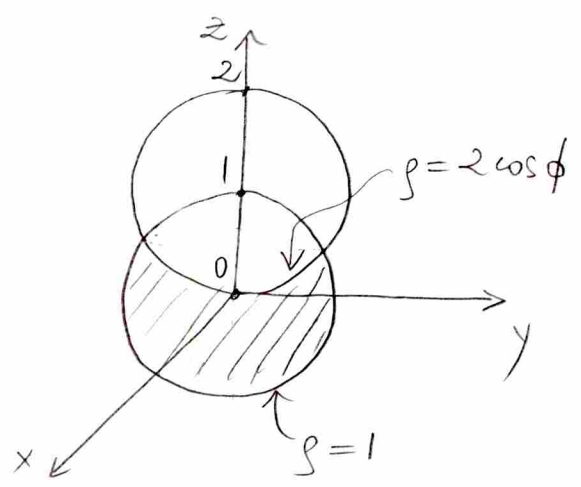
$$\boxed{1 \leq \rho \leq \sqrt{2}}$$

So: 
$$\mathcal{R} = \left\{ \begin{array}{l} -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \frac{\pi}{6} \leq \phi \leq \frac{\pi}{2} \\ 1 \leq \rho \leq \sqrt{2} \end{array} \right\}$$

and 
$$\text{Mass} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \sin \phi \, d\rho \, d\phi \, d\theta = \dots = \boxed{\frac{\pi(\sqrt{2}-1)\sqrt{3}}{2}}$$

(b) Evaluate  $\iiint_R z \, dV$ , where  $R$  is the region inside  $x^2 + y^2 + z^2 = 1$  and outside  $x^2 + y^2 + z^2 = 2z$ .

- \*  $x^2 + y^2 + z^2 = 1 \rightsquigarrow$  sphere of radius 1 centered at  $(0, 0, 0)$ .
- \*  $x^2 + y^2 + z^2 = 2z \iff x^2 + y^2 + (z-1)^2 = 1 \rightsquigarrow$  sphere of radius 1 centered at  $(0, 0, 1)$ .

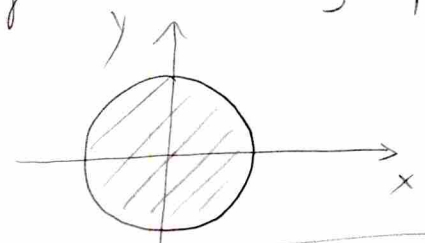


In spherical coordinates:

- \*  $x^2 + y^2 + z^2 = 1 \implies \boxed{\rho = 1}$
- \*  $x^2 + y^2 + z^2 = 2z \iff \rho^2 = 2\rho \cos \phi \iff \rho = 0 \text{ OR } \boxed{\rho = 2 \cos \phi}$

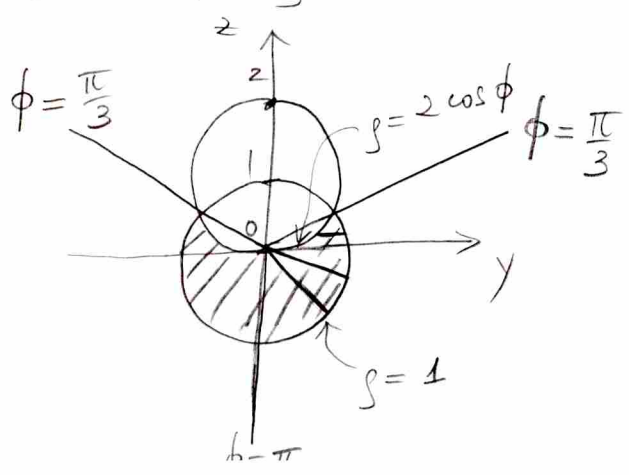
only gives  $(0, 0, 0)$  which is obtained from  $\rho = 2 \cos \phi$  at  $\phi = \frac{\pi}{2}$ .

Bounds for  $\theta$ : project region onto  $xy$ -plane.



$\implies \boxed{0 \leq \theta \leq 2\pi}$

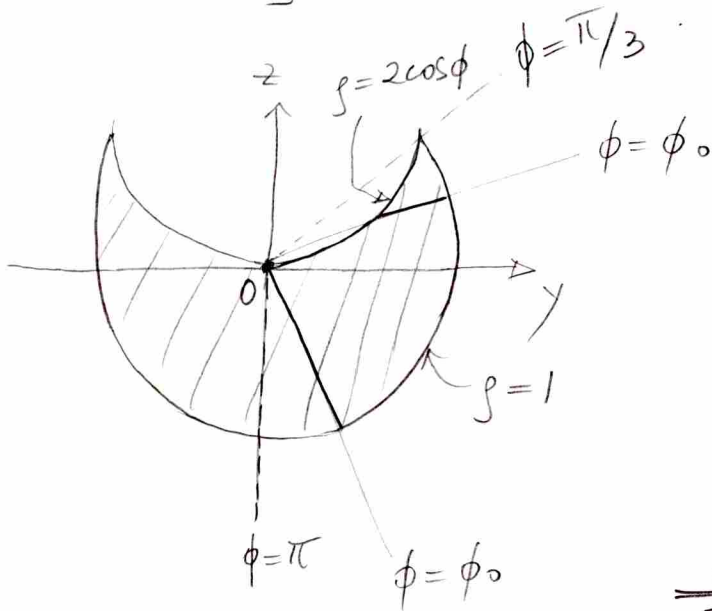
Bounds for  $\rho$  &  $\phi$ : take a vertical cross-section.



- \* We see that points in  $R$  have  $\phi$  at most  $\pi$  and at least equal to the angle of intersection of the 2 spheres:  $\rho = 1$  and  $\rho = 2 \cos \phi$
- $\iff 1 = 2 \cos \phi \iff \cos \phi = 1/2$
- $\iff \phi = \frac{\pi}{3}$  since  $\phi \in [0, \pi]$ .

$\Rightarrow \boxed{\frac{\pi}{3} \leq \phi \leq \pi}$  for points in  $R$ .

Moreover, any radial ray corresponding to an angle  $\frac{\pi}{3} \leq \phi_0 \leq \pi$  intersects  $R$  as follows:



We see that if  $\frac{\pi}{3} \leq \phi_0 \leq \frac{\pi}{2}$ , then

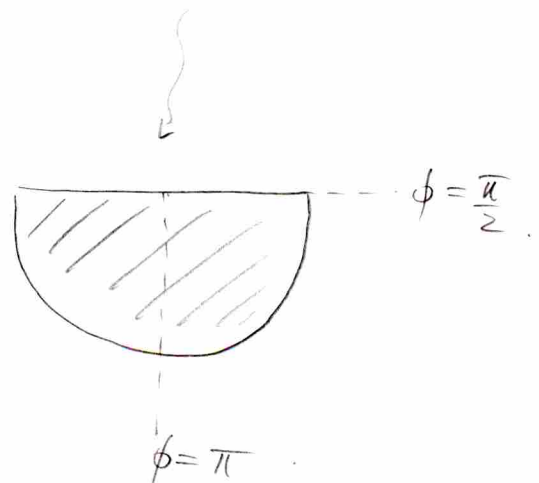
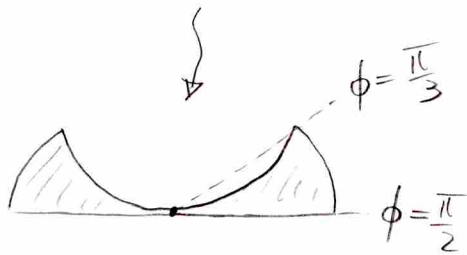
$$\boxed{2 \cos \phi \leq \rho \leq 1}$$

BUT, if  $\frac{\pi}{2} \leq \phi_0 \leq \pi$ ,

then  $\boxed{0 \leq \rho \leq 1}$

$\Rightarrow$  LOWER BOUND CHANGES!

So:  $R_{\phi, \theta} = \left\{ \begin{array}{l} 0 \leq \theta \leq 2\pi \\ \frac{\pi}{3} \leq \phi \leq \frac{\pi}{2} \\ 2 \cos \phi \leq \rho \leq 1 \end{array} \right\} \cup \left\{ \begin{array}{l} 0 \leq \theta \leq 2\pi \\ \frac{\pi}{2} \leq \phi \leq \pi \\ 0 \leq \rho \leq 1 \end{array} \right\}$



$\Rightarrow$  will need 2 integrals.

$$\begin{aligned}
 S_0, \quad \iiint_R z \, dV &= \iiint_R (\rho \cos \phi) \cdot \rho^2 \sin \phi \cdot d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\pi/2} \int_{2\cos\phi}^1 (\rho \cos \phi) \cdot \rho^2 \sin \phi \cdot d\rho \, d\phi \, d\theta \\
 &\quad + \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^1 (\rho \cos \phi) \cdot \rho^2 \sin \phi \cdot d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left[ \frac{\rho^4}{4} \cos \phi \sin \phi \Big|_{\rho=2\cos\phi}^{\rho=1} \right] d\phi \, d\theta \\
 &\quad + \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \left[ \frac{\rho^4}{4} \cos \phi \sin \phi \Big|_{\rho=0}^{\rho=1} \right] d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{4} (\cos \phi \sin \phi - 16 \cos^5 \phi \sin \phi) d\phi \, d\theta \\
 &\quad + \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{1}{4} (\cos \phi \sin \phi) d\phi \, d\theta \\
 &= \int_0^{2\pi} \frac{1}{4} \left[ -\frac{\cos^2 \phi}{2} + \frac{16}{6} \cos^6 \phi \Big|_{\phi=\frac{\pi}{3}}^{\phi=\frac{\pi}{2}} \right] d\theta + \int_0^{2\pi} \frac{1}{4} \left[ -\frac{\cos^2 \phi}{2} \Big|_{\phi=\frac{\pi}{2}}^{\phi=\pi} \right] d\theta \\
 &= \int_0^{2\pi} \frac{1}{4} \left( \frac{1}{2} \left( \frac{1}{2} \right)^2 - \frac{16}{6} \left( \frac{1}{2} \right)^6 \right) d\theta + \int_0^{2\pi} \frac{1}{4} \left( -\frac{(-1)^2}{2} \right) d\theta = \frac{2\pi}{4} \left( \frac{1}{8} - \frac{1}{24} - \frac{1}{2} \right) \\
 &= \frac{-5\pi}{4} \dots
 \end{aligned}$$