$$\lim_{(x,y)\to(a,b)} \frac{p(x,y)}{q(x,y)}.$$

$\rightarrow$ If $p(x,y)$ is a sum of monomials $x^m y^n$: consider each $\frac{x^m y^n}{q(x,y)}$ seperately to get a sense of the convergence of $\frac{p(x,y)}{q(x,y)}$ OR factor.

e.g: 1) $\lim_{(x,y)\to(0,0)} \frac{x^3y - xy^4}{x^2+y^2} = 0$?

$$\frac{x^3y - xy^4}{x^2+y^2} = \frac{x^3y}{x^2+y^2} - \frac{xy^4}{x^2+y^2}$$

$\frac{x^3y}{x^2+y^2} \to 0$ since $\frac{3}{2}+\frac{1}{2}>1$

$\frac{xy^4}{x^2+y^2} \to 0$ since $\frac{1}{2}+\frac{4}{2}>1$

$\rightsquigarrow$ converges to 0.

Let's now prove this using Squeeze Thm:

$$\left|\frac{x^3y - xy^4}{x^2+y^2} - 0\right| = \frac{|x^3y - xy^4|}{x^2+y^2} \leq \frac{|x|^3|y| + |x|\cdot y^4}{x^2+y^2}$$

$$= \frac{x^2|x|\cdot|y| + |x|\cdot y^2 y^2}{x^2+y^2} \leq \frac{\cancel{(x^2+y^2)}|x|\cdot|y| + |x|\cancel{(x^2+y^2)}y^2}{\cancel{x^2+y^2}}$$

$$= |x|\cdot|y| + |x|y^2 \to 0 \quad \text{as } (x,y)\to(0,0)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3y - xy^4}{x^2+y^2} = 0.$$

2) $\lim_{(x,y)\to(0,0)} \frac{x^3y - xy}{x^4+y^2} = 0$?

$$\frac{x^3y - xy}{x^4+y^2} = \frac{x^3y}{x^4+y^2} - \frac{xy}{x^4+y^2}$$

$\frac{x^3y}{x^4+y^2} \to 0$ since $3/4+1/2>1$

$\frac{xy}{x^4+y^2} \to$ DNE since $\frac{1}{4}+\frac{1}{2}<1$

$\rightsquigarrow$ probably DNE.

Along $y=x$,

$$\lim_{x\to 0} \frac{x^4-x^2}{x^4+x^2} = \lim_{x\to 0} \frac{x^2-1}{x^2+1} = -1 \neq 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3y-xy}{x^4+y^2} \neq 0.$$

3) $$\lim_{(x,y)\to(0,2)} \frac{x^4y-2x^4}{(x^2+(y-2)^2)^{3/2}} = \lim_{(x,y)\to(0,2)} \frac{x^4(y-2)}{(x^2+(y-2)^2)^{3/2}}$$

like $\frac{|s|^a|t|^b}{|s|^c+|t|^d}$ with $a=4, b=1, c=d=2\left(\frac{3}{2}\right)$

$\Rightarrow \frac{4}{3}+\frac{1}{3}>1$, so converges to 0

Using Squeeze Thm:

$$\left|\frac{x^4(y-2)}{(x^2+(y-2)^2)^{3/2}} - 0\right|$$

$$= \frac{x^4|y-2|}{(x^2+(y-2)^2)^{3/2}} = \frac{x^2\cdot x^2\cdot\sqrt{(y-2)^2}}{(x^2+(y-2)^2)^{3/2}}$$

$$\leq \frac{x^2\cdot\cancel{(x^2+(y-2)^2)}\cdot\cancel{\sqrt{x^2+(y-2)^2}}}{\cancel{(x^2+(y-2)^2)^{3/2}}}$$

$$= x^2 \to 0 \quad \text{as } (x,y)\to(0,2)$$

$$\Rightarrow \lim_{(x,y)\to(0,2)} \frac{x^4y-2x^4}{(x^2+(y-2)^2)^{3/2}} = 0.$$

$\rightarrow$ If $p(x,y)$ is the difference of square roots, try conjugation: if $p(x,y) = \sqrt{a} - \sqrt{b}$, then

$$\frac{p(x,y)}{q(x,y)} = \frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{q(x,y)(\sqrt{a} + \sqrt{b})}$$

e.g. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\overbrace{\sqrt{1+y^2} - \sqrt{1+x^4+y^2}}^{\to 0}}{\underbrace{2x^2+y^4}_{\to 0}} \rightsquigarrow \frac{0}{0}$

$$= \lim_{(x,y)\to(0,0)} \frac{\left(\sqrt{1+y^2} - \sqrt{1+x^4+y^2}\right)}{2x^2+y^4} \cdot \frac{\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)}{\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\left(\sqrt{1+y^2}\right)^2 - \left(\sqrt{1+x^4+y^2}\right)^2}{(2x^2+y^4)\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-x^4}{(2x^2+y^4)\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)} = 0$$

since

$$\left| \frac{-x^4}{(2x^2+y^4)\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)} - 0 \right| = \frac{x^2 \cdot x^2}{(2x^2+y^4)\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)}$$

$$\leqslant \frac{\cancel{(2x^2+y^4)}\, x^2}{\cancel{(2x^2+y^4)}\left(\sqrt{1+y^2} + \sqrt{1+x^4+y^2}\right)} = \frac{x^2}{\sqrt{1+y^2} + \sqrt{1+x^4+y^2}}$$

$$\xrightarrow[(x,y)\to(0,0)]{} \frac{0}{1+1} = 0.$$

→ If $p(x,y)$ involves $\sin(\cdot)$, $\cos(\cdot)$, $e^{(\cdot)}$, $\ln(\cdot)$, try using the MVT to find an upper bound for $|p(x,y)|$.

e.g. $$\lim_{(x,y)\to(0,0)} \frac{e^{3x^2y}-1}{\sqrt{x^2+y^2}} = 0. \quad \rightsquigarrow \frac{0}{0}$$

By the MVT, $(e^t - e^0) = e^c(t-0)$ for some c between 0 and t. For t close to 0, c is also close to c and $e^c \simeq 1$, so that $e^c < 2$. Thus, for t close to 0,

$$|e^t - 1| = |e^c| \cdot |t-0| < 2|t|.$$

$$\Rightarrow \left| e^{3x^2y} - 1 \right| < 2|3x^2y| = 6x^2|y|$$

and

$$\left| \frac{e^{3x^2y}-1}{\sqrt{x^2+y^2}} - 0 \right| < \frac{6x^2|y|}{\sqrt{x^2+y^2}} = \frac{6x^2\sqrt{y^2}}{\sqrt{x^2+y^2}}$$

$$\leq \frac{6x^2\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{x^2+y^2}}} = 6x^2 \to 0 \text{ as } (x,y)\to(0,0)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{e^{3x^2y}-1}{\sqrt{x^2+y^2}} = 0.$$

Suppose that $f(x,y)$ satisfies

$$|f(x,y)-f(1,0)| \leq |x-1|\cdot|y| \quad (*)$$

for all (x,y) near $(1,0)$. Show that f is diff. at $(1,0)$.

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* $f_x(1,0) = \lim_{h\to 0} \frac{f(1+h,0)-f(1,0)}{h}$.

By $(*)$, $|f(1+h,0)-f(1,0)| \leq |(h+1)-1|\cdot|0| = 0$ for h near 0, so that

$$\left|\frac{f(1+h,0)-f(1,0)}{h}\right| \leq \frac{0}{|h|} = 0,$$

proving that $\lim_{h\to 0}\frac{f(1+h,0)-f(1,0)}{h} = 0$.

$$\Rightarrow \boxed{f_x(1,0)=0}.$$

* $f_y(1,0) = \lim_{h\to 0}\frac{f(1,h)-f(1,0)}{h} = 0$ since, by $(*)$,

$$\left|\frac{f(1,h)-f(1,0)}{h}\right| \leq \frac{|1-1|\cdot|h|}{|h|} = 0 \text{ near } 0,$$

$$\Rightarrow \boxed{f_y(1,0)=0}.$$

* $R_{1,(1,0)}(x,y) = f(x,y) - L_{(1,0)}(x,y) = f(x,y)-f(1,0)$ (since $f_x(1,0)=f_y(1,0)=0$).

$\Rightarrow \lim_{(x,y)\to(1,0)} \frac{R_{1,(1,0)}(x,y)}{\sqrt{(x-1)^2+y^2}} = 0$ since

$$\left|\frac{f(x,y)-f(1,0)}{\sqrt{(x-1)^2+y^2}} - 0\right| \leq \frac{|x-1|\cdot|y|}{\sqrt{(x-1)^2+y^2}} \leq \frac{|x-1|\cdot\sqrt{(x-1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}$$

$|y| = \sqrt{y^2}$

$= |x-1| \to 0$ as $(x,y)\to(1,0)$.

$\Rightarrow f$ is diff. at $(1,0)$.