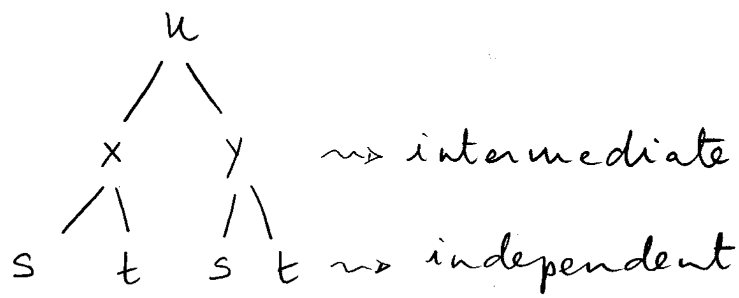
CHAIN RULE: a few more examples.

Ex. 1) $u(s,t) = f(x(s,t), y(s,t))$ where $x(s,t) = st^2$ and $y(s,t) = s^2+t$. Find $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s^2}, \frac{\partial^2 u}{\partial t^2}$.

HERE: $u(s,t) = f(st^2, s^2+t)$ and $f(x,y)$:



* $$\frac{\partial u}{\partial s} = f_x(x(s,t), y(s,t)) \cdot \frac{\partial x}{\partial s} + f_y(x(s,t), y(s,t)) \cdot \frac{\partial y}{\partial s}$$

$$= f_x(st^2, s^2+t) \cdot t^2 + f_y(st^2, s^2+t) \cdot (2s)$$

* $$\frac{\partial u}{\partial t} = f_x(x(s,t), y(s,t)) \cdot \frac{\partial x}{\partial t} + f_y(x(s,t), y(s,t)) \cdot \frac{\partial y}{\partial t}$$

$$= f_x(st^2, s^2+t) \cdot (2st) + f_y(st^2, s^2+t) \cdot 1$$

* $$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\Big[f_x(st^2, s^2+t) \cdot t^2 + \overbrace{f_y(st^2, s^2+t) \cdot (2s)}^{\text{product of 2 fcts of } s:}\Big]$$ USE PRODUCT RULE!

$$= t^2 \frac{\partial}{\partial s}\left[f_x(st^2, s^2+t)\right] + \frac{\partial}{\partial s}\left[f_y(st^2, s^2+t)\right] \cdot (2s) + f_y(st^2, s^2+t) \cdot 2.$$

NOW, to compute $\frac{\partial}{\partial s}\left[f_x(st^2, s^2+t)\right]$, REMEMBER that f_x is a function of x and y that we are composing with $x(s,t) = st^2$ and $y(s,t) = s^2+t$:

$$f_x(\underbrace{st^2}_{\substack{\parallel \\ x}}, \underbrace{s^2+t}_{\substack{\parallel \\ y}}).$$

Thus, by the Chain Rule,

$$\frac{\partial}{\partial s}\left[f_x(st^2, s^2+t)\right] = (f_x)_x(st^2, s^2+t) \cdot \frac{\partial}{\partial s}(st^2) + (f_x)_y(st^2, s^2+t) \cdot \frac{\partial}{\partial s}(s^2+t)$$

$$= f_{xx}(st^2, s^2+t) \cdot t^2 + f_{xy}(st^2, s^2+t) \cdot (2s)$$

Similarly,

$$\frac{\partial}{\partial s}\left[f_y(st^2, s^2+t)\right] = (f_y)_x(st^2, s^2+t) \cdot \frac{\partial}{\partial s}(st^2) + (f_y)_y(st^2, s^2+t) \cdot \frac{\partial}{\partial s}(s^2+t)$$

$$= f_{yx}(st^2, s^2+t) \cdot t^2 + f_{yy}(st^2, s^2+t) \cdot (2s).$$

Putting it all together, we get:

$$\frac{\partial^2 u}{\partial s^2} = t^2\left[f_{xx}(st^2, s^2+t)\cdot t^2 + f_{xy}(st^2, s^2+t)\cdot(2s)\right]$$

$$+\left[f_{yx}(st^2, s^2+t)\cdot t^2 + f_{yy}(st^2, s^2+t)\cdot(2s)\right]\cdot(2s)$$

$$+2f_y(st^2, s^2+t).$$

$$* \ \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left[f_x(st^2, s^2+t)\cdot(2st) + f_y(st^2, s^2+t)\cdot 1\right]$$

Product of 2 fcts of t : USE PRODUCT RULE!

$$= \frac{\partial}{\partial t}\left[f_x(st^2, s^2+t)\right]\cdot(2st) + f_x(st^2, s^2+t)\cdot(2s)$$

$$+\frac{\partial}{\partial t}\left[f_y(st^2, s^2+t)\right]$$

$$= \left[(f_x)_x(st^2, s^2+t)\cdot\frac{\partial}{\partial t}(st^2) + (f_x)_y(st^2, s^2+t)\cdot\frac{\partial}{\partial t}(s^2+t)\right]\cdot(2st)$$

$$+f_x(st^2, s^2+t)\cdot(2s)$$

$$+\left[(f_y)_x(st^2, s^2+t)\cdot\frac{\partial}{\partial t}(st^2) + (f_y)_y(st^2, s^2+t)\cdot\frac{\partial}{\partial t}(s^2+t)\right]$$

$$= \left[f_{xx}(st^2, s^2+t)\cdot(2st) + f_{xy}(st^2, s^2+t)\cdot 1\right]\cdot(2st)$$

$$+f_x(st^2, s^2+t)\cdot(2s) + \left[f_{yx}(st^2, s^2+t)\cdot(2st) + f_{yy}(st^2, s^2+t)\cdot 1\right].$$

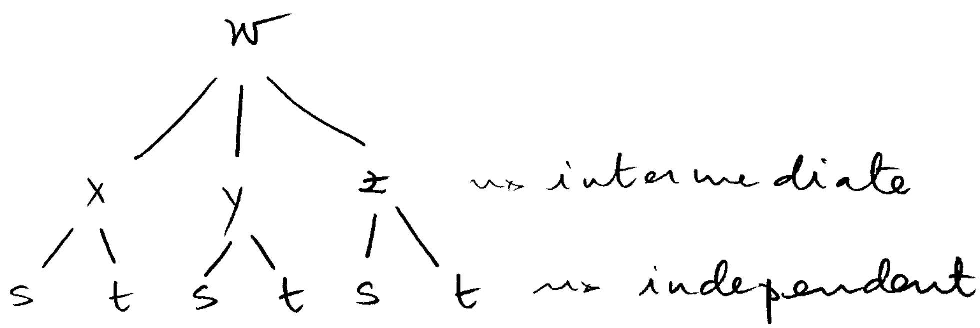
NOTE: In this problem, we assume that f, f_x and f_y are differentiable in order to be able to use the Chain Rule.

2) $w(s,t) = F(t^4+1,\ s^2-2t,\ e^{3ts})$.

Find $\dfrac{\partial w}{\partial s}$, $\dfrac{\partial w}{\partial t}$, $\dfrac{\partial^2 w}{\partial s\,\partial t}$.

HERE: $w(s,t) = F(\underbrace{t^4+1}_{=x},\ \underbrace{s^2-2t}_{=y},\ \underbrace{e^{3ts}}_{=z})$

$\rightsquigarrow$ can assume that $w(s,t)$ is the composition of $F(x,y,z)$ with $x(s,t) = t^4+1$, $y(s,t) = s^2-2t$, and $z(s,t) = e^{3ts}$.



$$* \ \frac{\partial w}{\partial s} = F_x \cdot \cancelto{0}{\frac{\partial x}{\partial s}} + F_y \cdot \frac{\partial y}{\partial s} + F_z \cdot \frac{\partial z}{\partial s}$$

$$= F_y(t^4+1, s^2-2t, e^{3ts}) \cdot (2s) + F_z(t^4+1, s^2-2t, e^{3ts}) \cdot (3te^{3ts})$$

$$* \ \frac{\partial w}{\partial t} = F_x \cdot \frac{\partial x}{\partial t} + F_y \cdot \frac{\partial y}{\partial t} + F_z \cdot \frac{\partial z}{\partial t}$$

$$= F_x \cdot (4t^3) + F_y \cdot (-2) + F_z \cdot (3se^{3ts})$$

$$* \ \frac{\partial^2 w}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial t}\right) = \frac{\partial}{\partial s}\left[F_x \cdot 4t^3 - 2F_y + F_z \cdot (3s\,e^{3ts})\right]$$

Product of 2 fcts of s: PRODUCT RULE

$$= \frac{\partial}{\partial s}\left[F_x(t^4+1,\ s^2-2t,\ e^{3ts})\right]\cdot 4t^3$$

$$-2\frac{\partial}{\partial s}\left[F_y(t^4+1,\ s^2-2t,\ e^{3ts}\right] \ +$$

$$\frac{\partial}{\partial s}\left[F_z(t^4+1,\ s^2-2t,\ e^{3ts})\right]\cdot(3se^{3ts})$$

$$+F_z(t^4+1,\ s^2-2t,\ e^{3ts})\cdot\frac{\partial}{\partial s}(3se^{3ts})$$

$$= \left[\cancel{(F_x)_x\cdot 0} + (F_x)_y\cdot(2s) + (F_x)_z\cdot(3te^{3ts})\right]\cdot 4t^3$$

$$-2\left[\cancel{(F_y)_x\cdot 0} + (F_y)_y\cdot(2s) + (F_y)_z\cdot(3te^{3ts})\right]$$

$$+\left[\cancel{(F_z)_x\cdot 0} + (F_z)_y\cdot(2s) + (F_z)_z(3te^{3ts})\right]\cdot(3se^{3ts})$$

$$+F_z\cdot\left(3e^{3ts} + 3s\cdot(3te^{3ts})\right).$$

$$= \left[F_{xy}\cdot(2s) + F_{xz}\cdot(3te^{3ts})\right]\cdot 4t^3$$

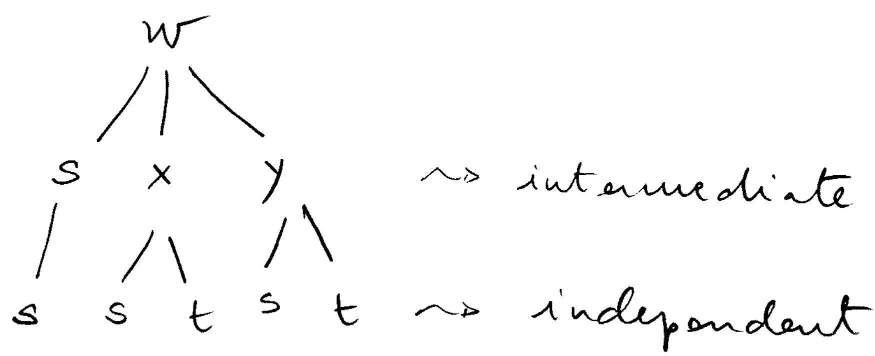
$$-2\left[F_{yy}\cdot(2s) + F_{yz}\cdot(3te^{3ts})\right]$$

$$+\left[F_{zy}\cdot(2s) + F_{zz}(3te^{3ts})\right]\cdot(3se^{3ts})$$

$$+F_z\cdot(3e^{3ts})(1+st).$$

3) If $w(s,t) = F(s, x(s,t), y(s,t))$, find w_s and w_t, where $F(s,x,y)$, $x(s,t)$ and $y(s,t)$ are differentiable.

HERE: $w(s,t)$ is the composition of $F(s,x,y)$ with $x(s,t)$ and $y(s,t)$.



$\Rightarrow$ s is BOTH an intermediate and independent variable. THUS,

$$w_s = F_s \cdot \overset{=1}{\frac{\partial s}{\partial s}} + F_x \cdot \frac{\partial x}{\partial s} + F_y \cdot \frac{\partial y}{\partial s}$$

$$= F_s(s, x(s,t), y(s,t)) + F_x(s, x(s,t), y(s,t)) \cdot \frac{\partial x}{\partial s} + F_y(s, x(s,t), y(s,t)) \cdot \frac{\partial y}{\partial s}.$$

and

$$w_t = F_s \cdot \overset{=0}{\frac{\partial s}{\partial t}} + F_x \cdot \frac{\partial x}{\partial t} + F_y \cdot \frac{\partial y}{\partial t}$$

$$= F_x(s, x(s,t), y(s,t)) \cdot \frac{\partial x}{\partial t} + F_y(s, x(s,t), y(s,t)) \cdot \frac{\partial y}{\partial t}.$$

4) Let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be a function of one variable that is differentiable up to order (at least) 2 (i.e., g' and g'' exist). Define $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ by

$$f(x,y)=g(x/y),\ \text{for } y\neq 0.$$

Verify that f is a solution of the differential equation:

$$f_{xy}=f_{yx}.$$

$\sim$

$$* \ f_x = g'(x/y)\cdot\frac{\partial}{\partial x}\left(x/y\right) = \frac{1}{y}\,g'(x/y)$$

$$* \ f_{xy} = \frac{\partial}{\partial y}\left(\frac{1}{y}\,g'(x/y)\right)$$

$$= \frac{\partial}{\partial y}\left(g'(x/y)\right)$$

$$=\left(\frac{-1}{y^2}\right)\cdot g'(x,y) + \frac{1}{y}\left[g''(x/y)\cdot\frac{\partial}{\partial y}\left(x/y\right)\right]$$

$$= -\frac{1}{y^2}\cdot g'(x,y) - \frac{x}{y^3}\cdot g''(x/y).$$

$$* \ f_y = g'(x/y)\cdot\frac{\partial}{\partial y}\left(x/y\right) = -\frac{x}{y^2}\cdot g'(x/y)$$

$$* \ f_{yx} = \frac{\partial}{\partial x}\left[\frac{-x}{y^2}\cdot g'(x/y)\right]$$

$$= \frac{\partial}{\partial x}\left(g'(x/y)\right)$$

$$= \frac{-1}{y^2}\cdot g'(x/y) + \left(\frac{-x}{y^2}\right)\cdot\left[g''(x/y)\cdot\frac{\partial}{\partial x}\left(x/y\right)\right]$$

$$= -\frac{1}{y^2}\cdot g'(x/y) - \frac{x}{y^3}\cdot g''(x/y) = f_{xy}.$$

$\boxtimes$