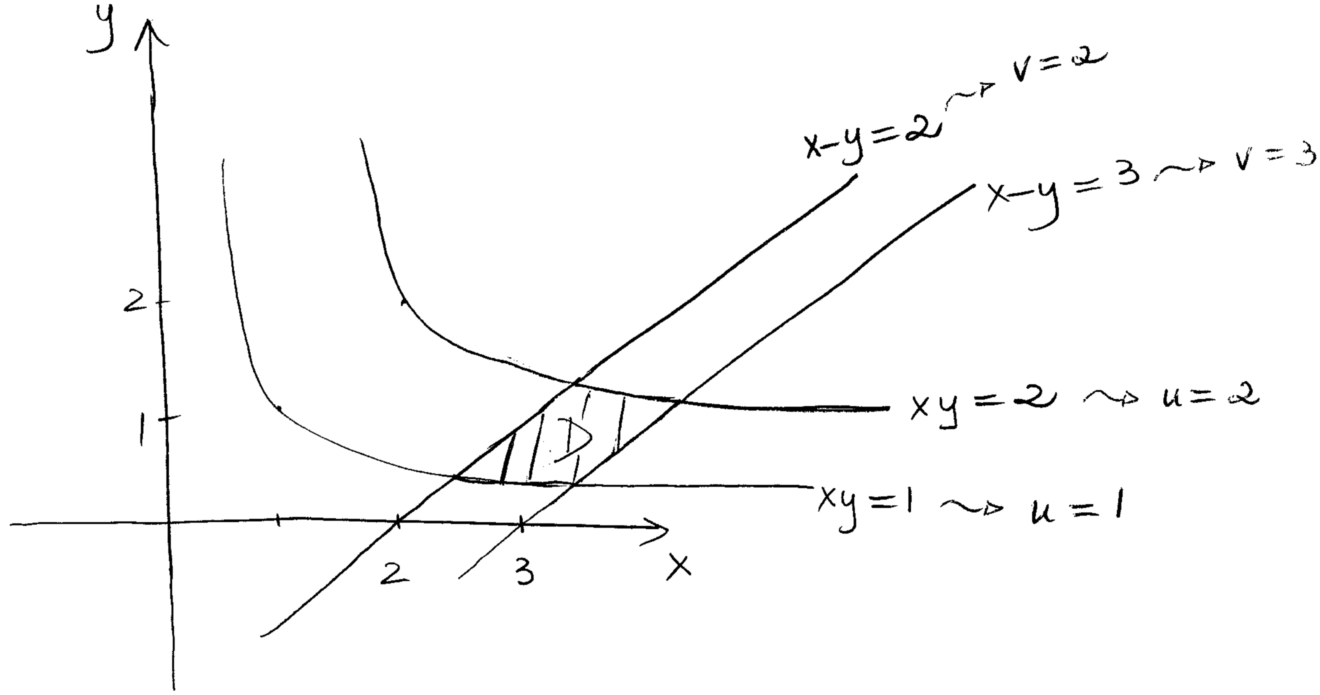
Ex. 1) Find $\iint_D x+y \, dA$, where D is the region bounded by $\underbrace{xy}_{u} = 1$, $\underbrace{xy}_{u} = 2$, $\underbrace{x-y}_{v} = 2$, $\underbrace{x-y}_{v} = 3$, $x,y \geq 0$.



Let $u = xy$ and $v = x-y$. Then, since any point in D can be obtained by intersecting a unique hyperbola in the family $u = xy = k$, $1 \leq k \leq 2$, with a unique line in the family $v = x-y = \ell$, $2 \leq \ell \leq 3$, the mapping

$$F(x,y) = (xy,\ x-y) = (u,v)$$

maps D in a one-to-one fashion onto

$$D' = \{ (u,v) \mid 1 \leq u \leq 2,\ \ 2 \leq v \leq 3 \}.$$

Note that $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_x \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & -1 \end{vmatrix} = -(x+y) < 0$ on D

and it is to be expected that $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ on D since F is invertible on D.

ALSO,

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{-(x+y)}.$$

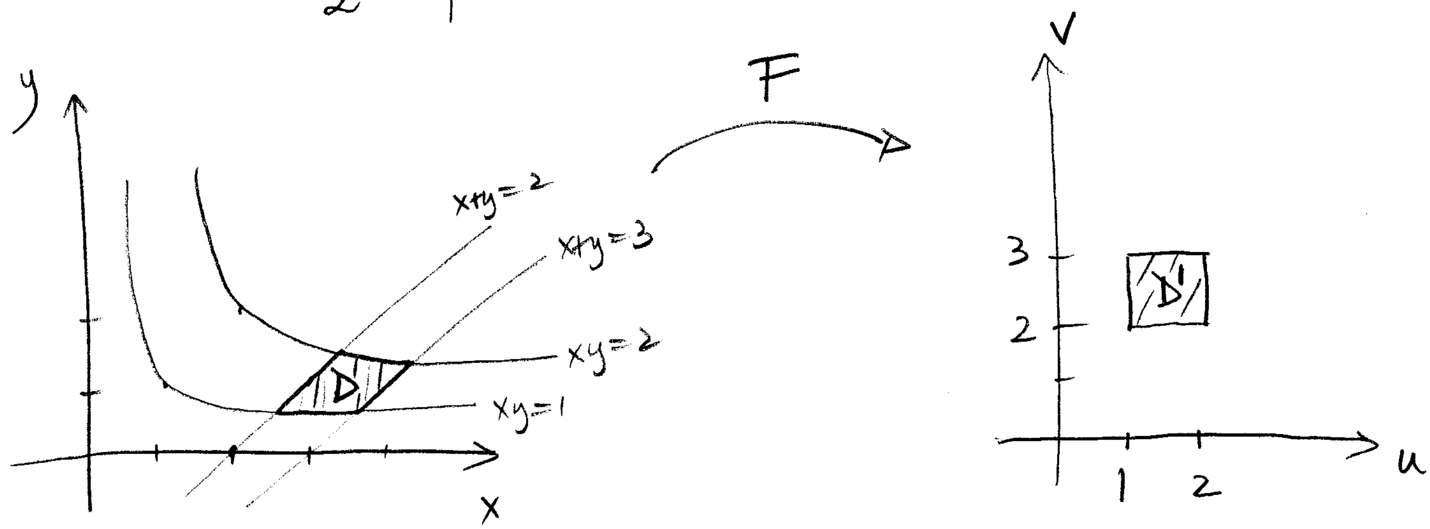
$$\Longrightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{(x+y)}, \quad \text{since } x+y>0 \text{ on } D.$$

THUS, by the Change of variable formula,

$$\iint_D x+y\,dA = \iint_{D'} (x+y)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_{D'} \cancel{(x+y)}\cdot\frac{1}{\cancel{(x+y)}}\,du\,dv = \iint_{D'} 1\,du\,dv$$

$$= \int_2^3\int_1^2 1\,du\,dv = \boxed{1}.$$



NOTE: Since $F(x,y) = (xy, x-y) = (u,v)$ maps D onto D' injectively, it must have an inverse, although in this case, finding the inverse it not completely straightforward.

$$\left.\begin{array}{l} u = xy \\ v = x-y \end{array}\right\} \longrightarrow$$ To find F^{-1}, we must express x & y as fcts of u & v.

In D, $x, y \neq 0$, so $y = \dfrac{u}{x} \Leftrightarrow v = x - \dfrac{u}{x}$

$$\Leftrightarrow xv = x^2 - u$$

$$\Leftrightarrow x^2 - xv - u = 0$$

Solving for x the quadratic equation $x^2 - vx - u = 0$ $\rightarrow$ $\Leftrightarrow x = \dfrac{v \pm \sqrt{v^2+4u}}{2}$

with $u, v > 0$.

NOW, since $x > 0$ in D and $u,v>0$ in D',

$$\sqrt{v^2+4u} > v \Rightarrow \frac{v - \sqrt{v^2+4u}}{2} < 0 \text{ in } D'$$

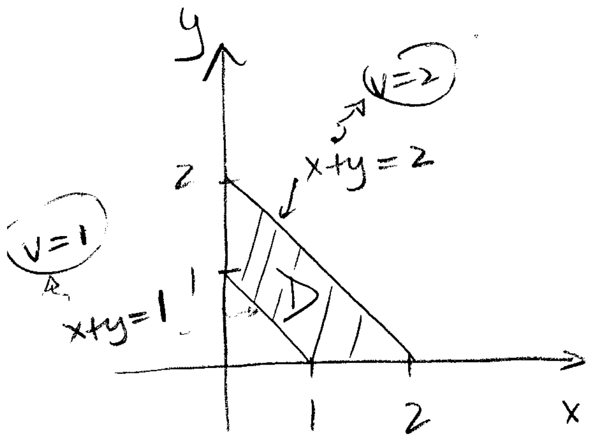
$$\Rightarrow x = \frac{v + \sqrt{v^2+4u}}{2} \text{ and } y = \frac{2u}{v+\sqrt{v^2+4u}}.$$

THUS, $F^{-1}(u,v) = \left(\dfrac{v+\sqrt{v^2+4u}}{2}, \dfrac{2u}{v+\sqrt{v^2+4u}} \right) = (x,y)$ on D'.

CLEARLY, for this example it is EASIER to compute $\partial(x,y)/\partial(u,v)$ as $1/\left(\partial(u,v)/\partial(x,y)\right)$!!

2) Evaluate $I = \iint_D \frac{e^{\frac{x-y}{x+y}}}{x+y}\, dA$, where D is the region bounded by $x+y=1$, $x+y=2$, $x=0$, $y=0$.

Difficult to integrate like this: try changing variables.



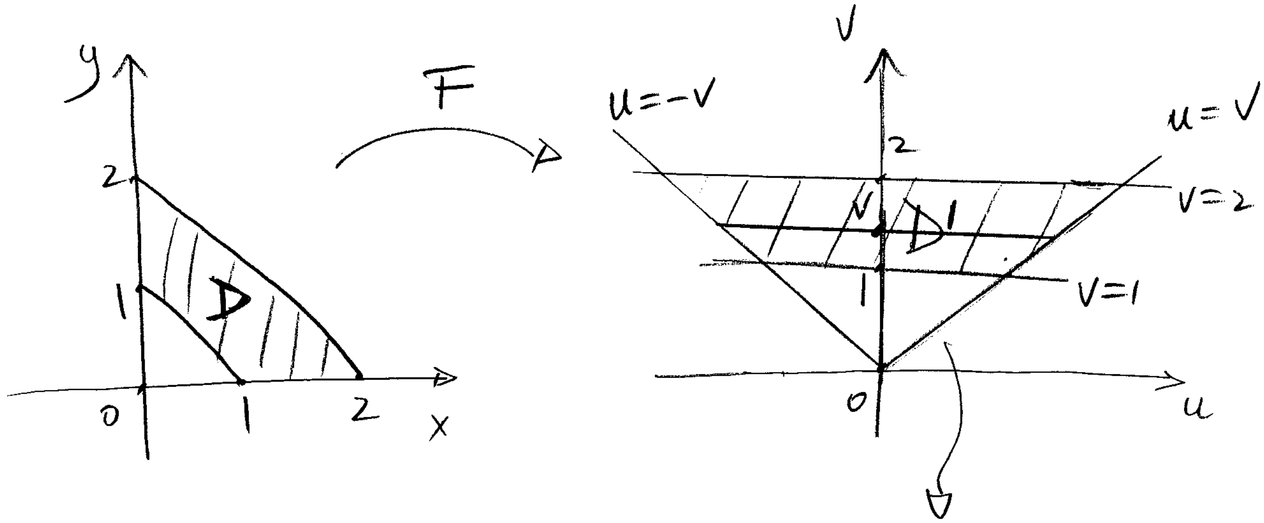
There are 2 natural changes of variables for this problem.

VERSION 1: Set $u = x-y$ and $v = x+y$.

In terms of these new variables, the integrand is:
$$\frac{e^{u/v}}{v}.$$

Moreover, the image of D under the change of variable mapping
$$F(x,y) = (x-y,\ x+y) = (u,v)$$
is bounded by:

$$x+y=1 \longrightarrow v=1$$
$$x+y=2 \longrightarrow v=2$$
$$x=0 \longrightarrow u=-y,\ v=y \iff u=-v$$
$$y=0 \longrightarrow u=x,\ v=x \iff u=v.$$



Note that F maps D in a one-to-one fashion onto D' since F is invertible on D, with inverse:

$$D' = \left\{ \begin{array}{l} 1 \le v \le 2 \\ -v \le u \le v \end{array} \right\}$$

$1 \le v \le 2 \to$ indep. $\Downarrow$ last variable to integrate

$$F^{-1}(u,v) = \left(\frac{u+v}{2}, \frac{-u+v}{2} \right) = (x,y).$$

To express the integral I in terms of the new variables u & v, we also need

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

THUS,

$$I = \iint_{D'} \frac{e^{u/v}}{v} \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_1^2 \int_{-v}^{v} \frac{e^{u/v}}{v} \cdot \left| \frac{1}{2} \right| du\,dv$$

$$= \frac{1}{2}\int_1^2 \left(\int_{-v}^{v} \frac{e^{u/v}}{v}\,du\right) dv = \frac{1}{2}\int_1^2 \left[e^{u/v}\Big|_{u=-v}^{u=v} \right] dv$$

$$= \frac{1}{2}\int_1^2 (e^1 - e^{-1})\,dv = \frac{(e-e^{-1})}{2}\int_1^2 dv$$

$$= \frac{(e-e^{-1})}{2}\left[v\Big|_1^2 \right] = \boxed{\frac{e-e^{-1}}{2}}.$$

NOTE: Finding the inverse F^{-1} of F was not necessary to compute $\frac{\partial(x,y)}{\partial(u,v)}$ since $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$.

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = 1/2.$$

VERSION 2: Another possible change of variable for this problem is:

$$u = \frac{x-y}{x+y} \quad \text{and} \quad v = x+y$$

so that the integrand becomes $\frac{e^u}{v}$.

In this case, the image of D under the change of variable mapping

$$F(x,y) = \left(\frac{x-y}{x+y}, x+y \right) = (u,v)$$

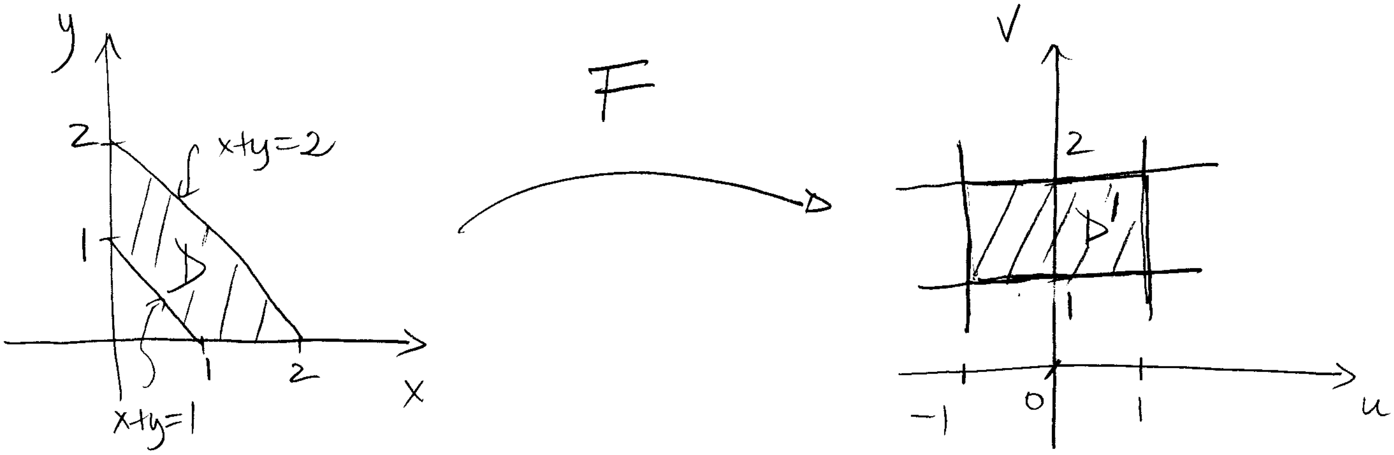
is bounded by:

$x+y=1 \rightsquigarrow v=1$

$x+y=2 \rightsquigarrow v=2$

$x=0 \rightsquigarrow u=-1,\ v=x \Rightarrow u=-1$

$y=0 \rightsquigarrow u=1,\ v=y \Rightarrow u=1$



$D' = \{(u,v) \mid -1 \le u \le 1,\ 1 \le v \le 2\}$

Now, note that:

$u = \frac{x-y}{x+y}$ and $v = x+y$

$\Rightarrow uv = x-y$, so that $x = \frac{v+uv}{2}$ and $y = \frac{v-uv}{2}$.

F is therefore invertible on D with inverse:

$$F^{-1}(u,v) = \left(\frac{v+uv}{2}, \frac{v-uv}{2} \right) = (x,y).$$

THUS,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{v}{2} & \frac{1-u}{2} \\ \frac{-v}{2} & \frac{1+u}{2} \end{vmatrix} = \frac{1}{4}(2v) = \frac{v}{2} > 0$$

on D'.

$\Longrightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{v}{2}$ on D', and by the change of variable formula,

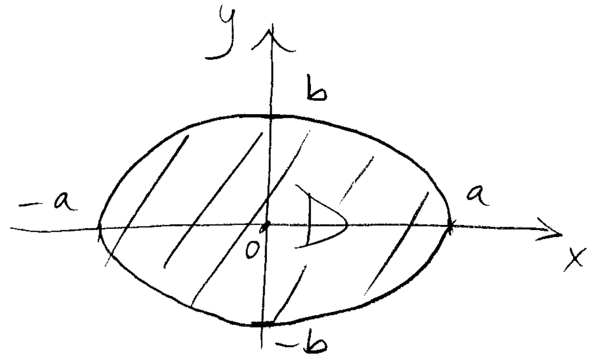
$$I = \iint_{D'} \frac{e^u}{v} \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_1^2 \int_{-1}^1 \frac{e^u}{\cancel{v}} \cdot \frac{\cancel{v}}{2}\, du\, dv = \frac{1}{2} \int_1^2 \left(e^u \Big|_{-1}^{1} \right) dv$$

$$= \frac{(e-e^{-1})}{2} \int_1^2 dv = \boxed{\frac{e-e^{-1}}{2}}.$$

NOTE: Even though the second change of variable gives a simpler integrand (e^u/v instead of $e^{u/v}/v$), computing $\frac{\partial(x,y)}{\partial(u,v)}$ is complicated in the second case.

3) Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

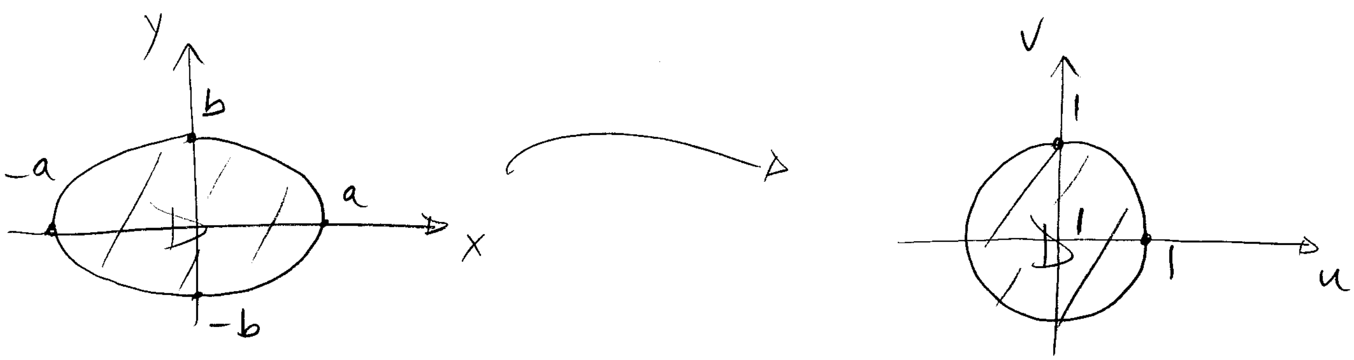


IDEA: Transform the region D bounded by the ellipse into a disc centered at the origin of radius 1:

$$D' = \{ (u,v) \mid u^2 + v^2 \leq 1 \}.$$

So, need to transform $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ into

$u^2 + v^2 = 1 \implies$ set $u = \frac{x}{a}$ and $v = \frac{y}{a}$.



ALSO, since the change of variable mapping

$$F(x,y) = (x/a,\ y/b) = (u,v)$$

has inverse

$$F^{-1}(u,v) = (au,\ bv) = (x,y),$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab,$$

$$\begin{pmatrix}\text{Area of}\\ \text{ellipse}\end{pmatrix} = \iint 1 \; dx\,dy$$

$$= \iint 1 \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \iint 1 \cdot ab \; du\,dv$$

$$= ab \cdot \underbrace{\left(\iint 1 \; du\,dv \right)}_{= \begin{pmatrix}\text{area of disc}\\ \text{of radius } 1\end{pmatrix}}$$

$$= ab\pi .$$

$\Longrightarrow$ Area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $ab\pi$.

NOTE: If $a=b$, the ellipse is a circle and the area is $a^2\pi$.

4) Find a linear transformation that maps the ellipse $x^2+4xy+5y^2=4$ onto a unit circle. Hence, show that the area enclosed by the ellipse equals 4π.

<u>IDEA</u>: Transform $x^2+4xy+5y^2=4$

into $u^2+v^2=1$

$\rightsquigarrow$ Since u^2+v^2 is a sum of two squares, try completing the square in the expression

$$x^2+4xy+5y^2.$$

There are 2 ways of doing this:

→ with respect to the x:

$$x^2+4xy+5y^2 = (x+2y)^2 - 4y^2 + 5y^2$$
$$= (x+2y)^2 + y^2$$

So that $x^2+4xy+5y^2=4$

$$\Leftrightarrow (x+2y)^2 + y^2 = 4 \rightsquigarrow \text{need 1 for unit circle}$$

(divide equation by 4)

$$\Leftrightarrow \left(\frac{x+2y}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$$

$\iff u^2+v^2=1$ with $u=\dfrac{x+2y}{2}$, $v=\dfrac{y}{2}$.

OR

→ with respect to the y:

$$x^2+4xy+5y^2 = x^2+5\left[\frac{4}{5}xy+y^2\right]$$
$$= x^2+5\left(\frac{2x}{5}+y\right)^2-\frac{4x^2}{5}$$
$$= \frac{x^2}{5}+5\left(\frac{2x}{5}+y\right)^2$$

So that $x^2+4xy+5y^2=4$

$$\iff \frac{x^2}{5}+5\left(\frac{2x}{5}+y\right)^2=4$$

$$\iff \left(\frac{x}{2\sqrt5}\right)^2+\left(\frac{x}{\sqrt5}+\frac{\sqrt5 y}{2}\right)^2=1$$

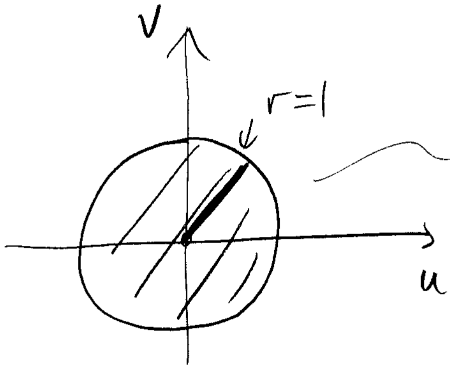
$\iff u^2+v^2=1$ with $u=\dfrac{x}{2\sqrt5}$, $v=\dfrac{x}{\sqrt5}+\dfrac{\sqrt5 y}{2}$.

⟹ NOTE: The first change of variable is much nicer. This is because we completed the square using the variable, x, that had the constant 1 in front of its square, x^2.

So, if we set $u = \dfrac{x+2y}{2}$, $v = \dfrac{y}{2}$,

the ellipse $x^2 + 4xy + 5y^2 = 4$ becomes $u^2 + v^2 = 1$,

and the region enclosed by the ellipse in the (x,y)-plane becomes the unit disc:



[In polar coordinates:

$$\left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{array} \right\}.$$
]

Also, the change of variable mapping

$$F(x,y) = \left(\frac{x+2y}{2}, \frac{y}{2} \right) = (u,v)$$

has inverse

$$F^{-1}(u,v) = (2u - 4v, 2v) = (x,y)$$

so that $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = 4.$$

$$\Longrightarrow \begin{pmatrix} \text{area of} \\ \text{ellipse} \end{pmatrix} = \iint 1 \, dx\,dy$$

$$\underset{\text{change of variable}}{=} \iint 1 \cdot \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \iint 1 \cdot 4 \, du\,dv$$

$$= 4\left(\underbrace{\iint_{\circ} 1 \, du\, dv}_{\substack{\| \\ \text{area of} \\ \text{unit} \\ \text{disc}}}\right) = 4(\pi) = 4\pi.$$

[OR, you can compute $\iint_{\circ} 1 \, du\, dv$ using polar coordinates:

$$\iint_{\circ} 1 \, du\, dv = \int_0^{2\pi}\int_0^1 1 \cdot \overset{\text{Jac.}}{r} \, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\Big|_0^1\right] d\theta$$

$$= \frac{1}{2}\left(\int_0^{2\pi} d\theta\right) = \pi.]$$

$u = r\cos\theta$
$v = r\sin\theta$
$\frac{\partial(u,v)}{\partial(r,\theta)} = r$

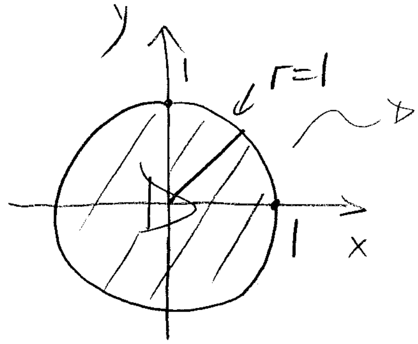
NOTE: Sometimes one needs to use more than one change of variable:

$$(x,y) \longmapsto \left(\frac{x+2y}{2}, \frac{y}{2}\right) = (u,v)$$

and $(r,\theta) \longmapsto (r\cos\theta, r\sin\theta) = (u,v)$.

5) Evaluate $I = \iint_D e^{x^2+y^2}\, dA$, where D is the unit disc:

$$D = \{ (x,y) \mid x^2+y^2 \le 1 \}.$$



$$D = \begin{Bmatrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{Bmatrix}$$

in polar coordinates

Note that $e^{x^2+y^2}$ cannot be integrated with respect to x or y

$\Rightarrow$ IDEA: Since the region of integration is a disc AND $x^2+y^2=r^2$ in polar coordinates: try using polar coordinates.

$$I = \iint_D e^{x^2+y^2}\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^1 e^{r^2} \cdot r \, dr\, d\theta = \int_0^{2\pi} \left[\int_0^1 e^{r^2} \cdot r\, dr \right] d\theta$$

Jac!

$u = r\cos\theta$
$v = r\sin\theta$
$\frac{\partial(x,y)}{\partial(u,v)} = r$

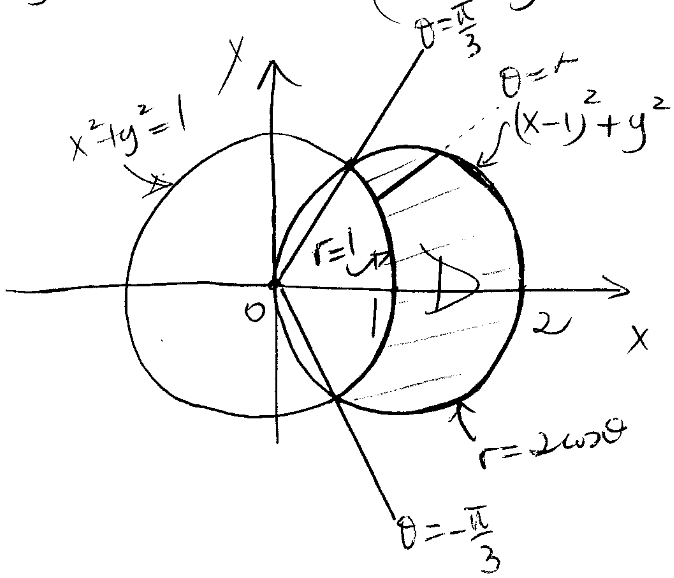
$$= \int_0^{2\pi} \left(\frac{e^{r^2}}{2} \Big|_{r=0}^{r=1} \right) d\theta = \int_0^{2\pi} \left(\frac{e-1}{2} \right) d\theta$$

$$= \frac{(e-1)}{2} \int_0^{2\pi} d\theta = \boxed{\pi(e-1)}.$$

6) Evaluate $I = \iint_D \frac{x}{\sqrt{x^2+y^2}}\, dA$, where D

is the region inside $x^2+y^2=2x$ and outside $x^2+y^2=1$.

The region D is bounded by two circles:
$x^2+y^2=1$ and $\left(x^2+y^2=2x \iff (x-1)^2+y^2=1\right)$.



This region is much easier to describe in polar coordinates:

$x^2+y^2=1 \iff r=1$

$x^2+y^2=2x \iff r=2\cos\theta$

(since $x^2+y^2=r^2$ and $x=r\cos\theta$)

Intersection points:

The curves $r=1$ and $r=2\cos\theta$ intersect in the 1st and 4th quadrants at points (r,θ) where

$\left(r=1 \;\&\; r=2\cos\theta,\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]\right)$

$\iff 2\cos\theta=1,\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

$\iff \cos\theta=\frac{1}{2},\ \theta\in[-\frac{\pi}{2},\frac{\pi}{2}] \iff \theta=-\frac{\pi}{3},\frac{\pi}{3}$.

ALSO: every ray $\theta=\alpha$, $-\frac{\pi}{3}\le\alpha\le\frac{\pi}{3}$, intersects D in a line segment whose points have radius r:

$$1\le r\le 2\cos\alpha.$$

THUS, in polar coordinates,

$$D = \left\{ \begin{array}{c} -\frac{\pi}{3} \le \theta \le \frac{\pi}{3} \\ 1 \le r \le 2\cos\theta \end{array} \right\}.$$

$$\Rightarrow I = \iint_D \frac{x}{\sqrt{x^2+y^2}}\, dx\, dy$$

$x = r\cos\theta$
$y = r\sin\theta$
$\frac{\partial(x,y)}{\partial(u,v)} = r$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} \frac{r\cos\theta}{\cancel{r}} \cdot \cancel{r}\, dr\, d\theta$$

Jac!

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\theta \left(\int_1^{2\cos\theta} r\, dr \right) d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \cos\theta \left[\frac{r^2}{2} \Big|_{r=1}^{r=2\cos\theta} \right] d\theta = \int_{-\frac{\pi}{3}}^{\pi/3} \cos\theta \Big[2\underbrace{\cos^2\theta}_{(1-\sin^2\theta)} - \frac{1}{2} \Big] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2(1-\sin^2\theta)\cos\theta\, d\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\pi/3} \cos\theta\, d\theta$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

θ	u
$\pi/3$	$\sqrt{3}/2$
$-\frac{\pi}{3}$	$-\sqrt{3}/2$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} 2(1-u^2)\, du - \frac{1}{2}\left[\sin\theta \Big|_{-\frac{\pi}{3}}^{\pi/3} \right]$$

$$= \left(2u - \frac{2}{3}u^3 \Big|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) \right) = \sqrt{3}$$