Chapter 9. Critical points.

Let f be of class C^1.

DEF. (a, b) is a critical point (c.p.) of $f(x,y)$ if

$$f_x(a,b) = f_y(a,b) = 0$$

or, equivalently, if

$$\nabla f(a,b) = (0,0).$$

NOTE: At a c.p. (a,b),

$$L_{(a,b)}(x,y) = f(a,b) + \cancel{f_x(a,b)}^{0}(x-a) + \cancel{f_y(a,b)}^{0}(y-b)$$

$$= f(a,b)$$

So the tgt plane to the graph $z = f(x,y)$ at $(a, b, f(a,b))$ is horizontal: $\boxed{z = f(a,b)}$.

DEF.: Given a fct $f: \mathbb{R}^2 \to \mathbb{R}$, a point (a,b) is:

(i) a local maximum of f if

$$f(x,y) \leq f(a,b),$$

$\forall (x,y)$ in a neighbourhood of (a,b).

(ii) a local minimum of f if

$$f(x,y) \geq f(a,b),$$

$\forall (x,y)$ in a neighbourhood of (a,b).

(B)

THM: Let f be of class C^1. If (a,b) is a local max. or min. of f, then (a,b) is a C.P. of f.

Pf: Let $g(x) := f(x,y)$. Then, if (a,b) is a local max. or min. of f, $x=a$ is a local max. or min. of $g(x)$.
$\Rightarrow f_x(a,b) = g'(a) = 0$. Similarly, if $h(y) := f(a,y)$, then $f_y(a,b) = h'(b) = 0$. ☒

NOTE: Even though all local max./min. are C.P., not all C.P. are local max. or min.

e.g.: $f(x,y) = x^2 - y^2 \rightsquigarrow \nabla f = (2x, -2y) = (0,0)$
$\Leftrightarrow (x,y) = (0,0)$.

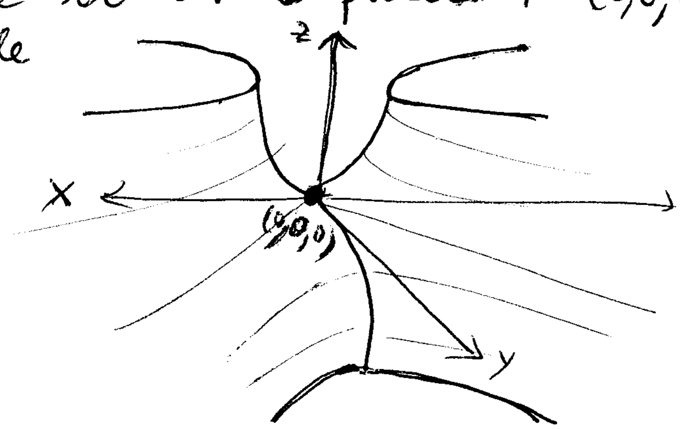
So, $(0,0)$ is a C.P. of f. BUT it's not a local max. or min. since

$$f(x,0) = x^2 > 0 = f(0,0), \quad \forall x.$$

and

$$f(0,y) = -y^2 < 0 = f(0,0), \quad \forall y.$$

$\rightsquigarrow$ The origin $(0,0)$ is called a saddle point since it corresponds to $(0,0,0)$ on the saddle surface $z = x^2 - y^2$



DEF: A C.P. that is not a local max. or min. is called a saddle point of f.

How can one determine whether a c.p. is a local max., min., or a saddle point? (c)

Can one use $L_{(a,b)}(x,y)$? We have seen that at a c.p. the linear approximation is a constant fct:

$$L_{(a,b)}(x,y) = f(a,b).$$

The linear approximation will therefore not give us a good approximation for $f(x,y)$ at $(x,y) \neq (a,b)$.

<u>IDEA</u>: Use $P_{2,(a,b)}(x,y)$!

At a c.p. (a,b),

$$P_{2,(a,b)}(x,y) = \underbrace{f(a,b)}_{= L_{(a,b)}(x,y)} + \frac{1}{2!}\underbrace{\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2 \right]}_{= Q(x-a,\, y-b)}.$$

So,

$$f(x,y) \simeq P_{2,(a,b)}(x,y) = f(a,b) + \frac{1}{2!} Q(x-a,\, y-b).$$

* if $Q(x-a, y-b) > 0$, $\forall (x,y) \neq (a,b)$, then (a,b) = local <u>MIN</u>.
* if $Q(x-a, y-b) < 0$, $\forall (x,y) \neq (a,b)$, then (a,b) = local <u>MAX</u>.
* if $Q(x-a, y-b)$ take both positive and negative values at points near (a,b), then (a,b) = <u>saddle point</u>.

$Q(x-a, y-b)$ is a special case of what is called a <u>quadratic form</u>, which we'll now study.

Quadratic forms:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \quad \text{symmetric matrix}$$

Suppose that $\det A \neq 0$.

DEF.:

all cases where $\det A > 0$ (since $a_{11}=0 \Rightarrow \det A < 0$)

(i) A is called <u>positive definite</u> if $\det A > 0$ and $a_{11} > 0$.

(ii) A is called <u>negative definite</u> if $\det A > 0$ and $a_{11} < 0$.

(iii) A is called <u>indefinite</u> if $\det A < 0$.

NOTE: If $\det A > 0$, then $a_{11} \neq 0$ because when $a_{11} = 0$, $\det A = -a_{12}^2 < 0$. So, if $\det A > 0$, can only have $a_{11} > 0$ <u>OR</u> $a_{11} < 0$.

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e.g.: 1) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ positive definite

2) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ indefinite

3) $\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ negative definite

4) $\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ positive definite

etc...

One can associate to any symmetric matrix A a degree 2 polynomial as follows:

Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$, set

$$Q(u,v) := a_{11}u^2 + 2a_{12}uv + a_{22}v^2$$

$= \underline{\text{quadratic form}}$.

e.g.: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightsquigarrow Q(u,v) = u^2 + 4uv + 4v^2$.

NOTE: (a) $Q(0,0) = 0$ for any A.

(b) If $a_{11} \neq 0$, then

$$Q(u,v) = a_{11}\left(u + \frac{a_{12}}{a_{11}}\right)^2 + \frac{\det A}{a_{11}} v^2 \qquad (*)$$

completing the square

PROPOSITION: Suppose that A is symmetric with $\det A \neq 0$. Then,

(i) A is positive definite $\Longleftrightarrow Q(u,v) > 0, \ \forall (u,v) \neq (0,0)$

(ii) A is negative definite $\Longleftrightarrow Q(u,v) < 0, \ \forall (u,v) \neq (0,0)$

(iii) A is indefinite $\Longleftrightarrow Q(u,v) > 0$ for some (u,v)

AND

$Q(u,v) > 0$ for some (u,v)

Pf: (i) ($\Rightarrow$) Suppose that A is positive definite. Then, $a_{11} > 0$ and $\det A > 0$, so that

$$Q(u,v) = \underbrace{a_{11}}_{>0}\underbrace{\left(u + \frac{a_{12}}{a_{11}}\right)^2}_{\geq 0} + \underbrace{\frac{\det A}{a_{11}}}_{>0}\underbrace{v^2}_{>0} > 0 \quad \text{if } (u,v) \neq (0,0)$$

by (*)

($\Leftarrow$) If $Q(u,v) > 0$ for all $(u,v) \neq (0,0)$, then

$$a_{11} = Q(1,0) > 0$$

and

$$\det A = \frac{1}{\underbrace{a_{11}}_{>0}} \overbrace{Q(-a_{12}, a_{11})}^{>0} > 0$$

$\Rightarrow$ A is positive definite.

(ii) The proof of (ii) is similar to (i).

(iii) ($\Rightarrow$) Suppose that $\det A < 0$.

* if $a_{11} = a_{22} = 0$, then $\det A = -a_{12}^2$, so that $a_{12} \neq 0$ since $\det A \neq 0$.

  $\Rightarrow$ $Q(1,1) = 2a_{12}$ and $Q(1,-1) = -2a_{12}$ must have opposite signs (since $a_{12} \neq 0$)

  $\Rightarrow$ $Q(u,v)$ takes both positive and negative values.

* if $a_{11} \neq 0$: $Q(1,0) = \underbrace{a_{11}}_{\neq 0}$ and $Q(-a_{12}, a_{11}) = \overbrace{\det A}^{<0} \cdot \underbrace{a_{11}}_{\neq 0}$ have opposite signs.

* if $a_{22} \neq 0$: $Q(0,1) = \underbrace{a_{22}}_{\neq 0}$ and $Q(a_{22}, -a_{12}) = \overbrace{\det A}^{<0} \cdot \underbrace{a_{22}}_{\neq 0}$ have opposite signs.

($\Leftarrow$) If $Q(u,v)$ takes both positive values, then cannot have $\det A > 0$ (by (i) and (ii))
$\Rightarrow \det A < 0$, since $\det A \neq 0$. $\boxtimes$

The Hessian matrix.

Let $f(x,y)$ be a fct of 2-var. whose partial derivatives exist up to order 2. We define:

$$Hf(x,y) := \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

= Hessian matrix of f.

If f_{xy} and f_{yx} are continuous so that $f_{xy} = f_{yx}$, then

$$Hf(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

is a symmetric matrix.

e.g. $f(x,y) = x^3y + 2xy^2 - e^x$.

$$Hf(x,y) = \begin{bmatrix} 6xy - e^x & 3x^2 + 4y \\ 3x^2 + 4y & 4x \end{bmatrix}$$

Second Derivative Test.

Let $f(x,y)$ be a fct of class C^2 in a neighbourhood of (a,b), and assume that (a,b) is a C.P. of f. Then,

(i) If $Hf(a,b)$ is positive definite, (a,b) is a <u>local minimum</u> of f.

(ii) If $Hf(a,b)$ is negative definite, (a,b) is a <u>local maximum</u> of f.

(iii) If $Hf(a,b)$ is indefinite, (a,b) is a <u>saddle point</u> of f.

(iv) Otherwise, the test is inconclusive.

<u>Sketch of proof:</u> At a C.P. of f,

$$f_x(a,b) = f_y(a,b) = 0$$

So that

$$P_{2,(a,b)}(x,y) = f(a,b) + \frac{1}{2!} Q(x-a, y-b),$$

where Q is the quadratic form associated to $Hf(a,b)$. [Indeed, since $Hf(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{bmatrix}$, its associated quadratic form is

$$Q(u,v) = f_{xx}(a,b)u^2 + 2f_{xy}(a,b)uv + f_{yy}(a,b)v^2.]$$

Now, near (a,b),

$$f(x,y) \simeq P_{2,(a,b)}(x,y) = f(a,b) + \frac{1}{2!} Q(x-a, y-b).$$

(i) If $Hf(a,b)$ is positive definite, then

$$Q(x-a,\ y-b) > 0 \quad \text{for all } (x,y) \neq (a,b)$$

and

$$f(x,y) > f(a,b)$$

$\Rightarrow$ (a,b) is a local MIN.

(ii) If $Hf(a,b)$ is negative definite, then

$$Q(x-a,\ y-b) < 0 \quad \text{for all } (x,y) \neq (a,b)$$

and

$$f(x,y) < f(a,b)$$

$\Rightarrow$ (a,b) is a local MAX.

(iii) If $Hf(a,b)$ is indefinite, then $Q(x-a,\ y-b)$ takes both positive and negative values, so that (a,b) is not a local max. or min.

$\Rightarrow$ (a,b) is a saddle point.

$\boxed{8}$

Ex. 1) $f(x,y) = x^3 + y^3 - 3x^2 + 3y^2$.

Classify all C.P. of f.

* C.P.: $\nabla f(x,y) = (0,0)$.

$$\begin{cases} f_x = 3x^2 - 6x = 0 \Rightarrow x = 0, 2 \\ f_y = 3y^2 + 6y \Rightarrow y = 0, -2 \end{cases}$$

$\Rightarrow$ get 4 C.P.: $(0,0)$, $(0,-2)$, $(2,0)$, $(2,-2)$.

* Testing the C.P.: use the Second Derivative Test.

$$f_{xx} = 6x - 6, \quad f_{xy} = 0, \quad f_{yy} = 6y + 6.$$

$$\Rightarrow Hf(x,y) = \begin{bmatrix} 6x-6 & 0 \\ 0 & 6y+6 \end{bmatrix}$$

At (0,0): $Hf(0,0) = \begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix}$ indefinite since $\det(Hf(0,0)) = -36 < 0$.

$\Rightarrow (0,0)$ is a saddle point.

At (0,−2): $Hf(0,-2) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$ negative definite since $a_{11} = -6 < 0$ and $\det(Hf(0,-2)) = 36 > 0$

$\Rightarrow (0,-2)$ is a local MAX.

At (2,0): $Hf(2,0) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ positive definite since $a_{11} > 0$ and $\det(Hf(2,0)) = 36 > 0$

$\Rightarrow (2,0)$ is a local MIN.

At (2,−2): $Hf(2,-2) = \begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$ indefinite since $\det(Hf(2,-2)) < 0$

$\Rightarrow (2,-2)$ is a saddle point.

2) $f(x,y) = x^3 + 3xy - y^3$. Classify the c.p.

* c.p.: $\nabla f = (3x^2+3y,\ 3x-3y^2) = (0,0)$

$$\Leftrightarrow \begin{cases} 3x^2+3y=0 \\ 3x-3y^2=0 \end{cases} \Leftrightarrow \begin{cases} y=-x^2 \\ x=y^2 \end{cases}$$

$$\Rightarrow y = -x^2 = -(y^2)^2 \Leftrightarrow y = -y^4$$

$$\Leftrightarrow \boxed{\begin{array}{c} y=0 \text{ or } y=-1 \\ \text{and } x=y^2 \end{array}}$$

$\Rightarrow$ get 2 c.p.: $(0,0)$ and $(1,-1)$.

* Test the c.p.: $f_{xx} = 6x$, $f_{xy} = 3$, $f_{yy} = -6y$.

$$\Rightarrow Hf = \begin{pmatrix} 6x & 3 \\ 3 & -6y \end{pmatrix}$$

At $(0,0)$: $Hf(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \rightsquigarrow \det = -9 < 0$ indefinite.

$\Rightarrow (0,0)$ is a saddle point

At $(1,-1)$: $Hf(1,-1) = \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \rightsquigarrow \det = 27 > 0$ and $a_{11} = 6 > 0$

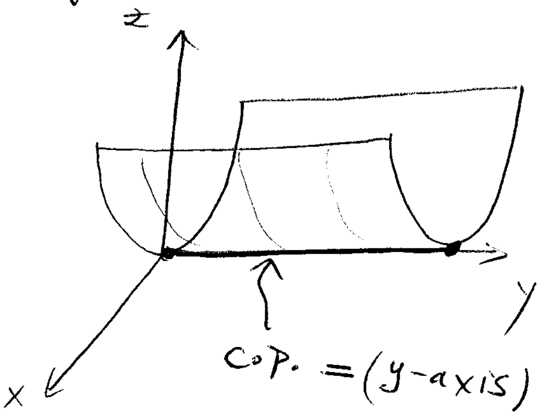
positive def.

$\Rightarrow (1,-1)$ is a local MIN.

(9)

REMARK: Suppose that (a,b) is a C.P. of $f(x,y)$ such that $\det(Hf(a,b)) = 0$. Then the 2^{nd} derivative test is INCONCLUSIVE. In that case, one needs to test further since ANYTHING CAN HAPPEN.

E.g.: 1) $f(x,y) = x^2 \rightsquigarrow$ graph is $z = x^2$.



* C.P.: $\nabla f = (2x, 0) = (0,0)$

$\Leftrightarrow \quad x = 0$

$\rightsquigarrow \{(0,y) \mid y \in \mathbb{R}\} = \boxed{y\text{-axis}}$

$\Rightarrow$ whole line of C.P.

* Classify C.P.: $Hf(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \det = 0, \ \forall (x,y)$

$\Rightarrow$ 2^{nd} derivative test is INCONCLUSIVE

Nonetheless, all C.P. are minima (from the graph).

2) $f(x,y) = -x^2 + y^3$. Classify the C.P.

* C.P.: $\nabla f = (-2x, 3y^2) = (0,0) \Leftrightarrow \boxed{(x,y) = (0,0)}$

* Classify C.P.: $Hf(x,y) = \begin{pmatrix} -2 & 0 \\ 0 & 6y \end{pmatrix}$

At $(0,0)$, $Hf(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \det = 0$ at $(0,0)$

$\Rightarrow$ 2^{nd} derivative test is INCONCLUSIVE.

BUT, $(0,0)$ is not a local max. or min. since

$$f(0,y) = y^3 > 0 = f(0,0) \quad \text{if } y>0$$

and

$$f(0,y) = y^3 < 0 = f(0,0) \quad \text{if } y<0,$$

so there are points near $(0,0)$ where f takes values both $> f(0,0)$ and $< f(0,0)$.

$\Longrightarrow$ $(0,0)$ is a <u>saddle point</u>.

In the above examples, we were able to classify the c.p. even though the 2^{nd} derivative test failed. However, it is in general difficult c.p. when the 2^{nd} derivative test fails because there isn't a general method that works.