Examples of regions in $\mathbb{R}^3$ in cylindrical and spherical coordinates.

* Cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}, \qquad \theta \in [0, 2\pi] \text{ or } [-\pi, \pi]$$

$$r = \sqrt{x^2+y^2} \geq 0.$$

$$\tan\theta = y/x.$$

→ To find the bounds for r & θ: project the region on the xy-plane.
→ To find the bounds for z: take a vertical cross-section + take vertical strips.

* Spherical coordinates:

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}, \qquad \theta \in [0, 2\pi], \quad \phi \in [0, \pi]$$

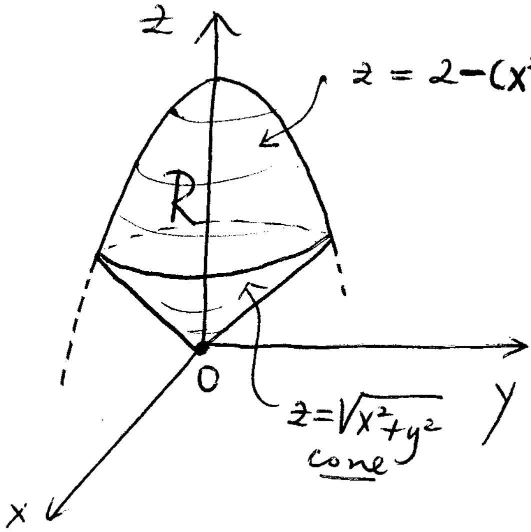
$$\rho = \sqrt{x^2+y^2+z^2} \geq 0.$$

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}.$$

→ To find the bounds for θ: project the region on the xy-plane.
→ To find the bounds for ρ & ϕ: take a vertical cross-section + take radial strips.

ex. 1) Let R be the region in $\mathbb{R}^3$ below $z = 2 - x^2 - y^2$ and above $z = \sqrt{x^2+y^2}$.

Describe the region in cylindrical coordinates.



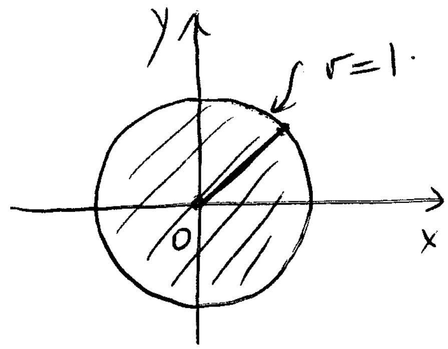
* Convert equations of surfaces into cylindrical coordinates:

paraboloid: $z = 2-(x^2+y^2) \iff \boxed{z = 2 - r^2}$.

cone: $z = \sqrt{x^2+y^2} \iff \boxed{z = r}$.

* Bounds for r and θ: project R on the xy-plane.

$\leadsto$ get a disc centered at O whose boundary is the circle of intersection of the sphere and the paraboloid.

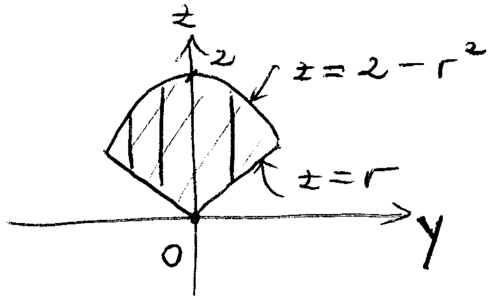


Intersection of the 2 surfaces:

$$z = 2 - r^2 \text{ and } z = r \iff 2 - r^2 = r$$
$$\iff r^2 + r - 2 = 0$$
$$\iff r = 1, \cancel{-2}$$
$$\Rightarrow \boxed{r = 1} \text{ since } r \geq 0.$$

Therefore, $\left\{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{array}\right\}$

* <u>Bounds for z</u>: Take a vertical cross-section: with yz-plane.

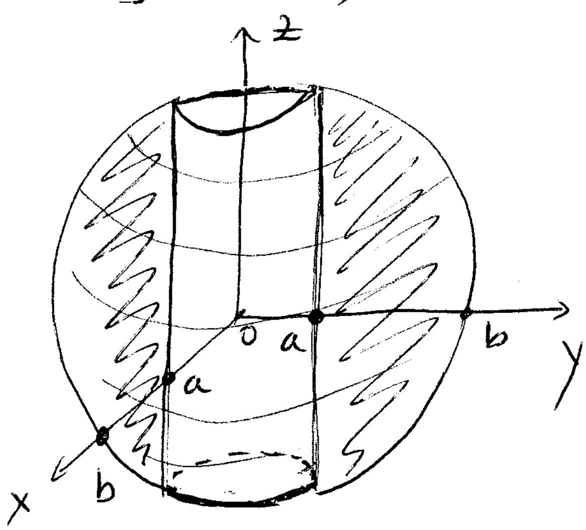


cone $\le z \le$ paraboloid

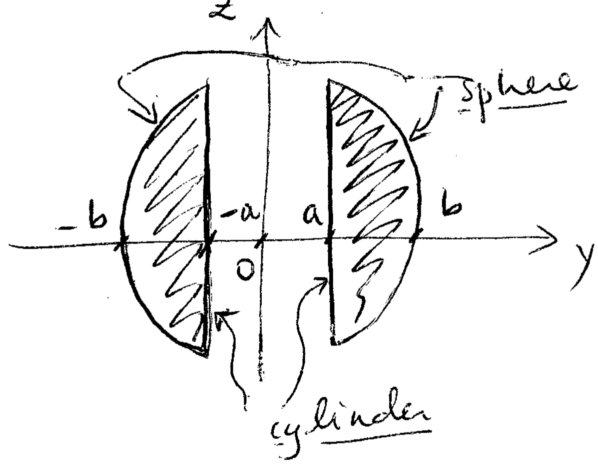
$\Longleftrightarrow \boxed{r \le z \le 2 - r^2}$

Therefore, $R = \left\{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ r \le z \le 2 - r^2 \end{array}\right\}$

2) Let R be the region inside the sphere $x^2 + y^2 + z^2 = b^2$ and outside the cylinder $x^2 + y^2 = a^2$, with $a < b$. Describe the region in <u>both</u> spherical and cylindrical coordinates.



To better visualise R, let's take a vertical cross-section: with yz-plane.



→ Cylindrical coordinates:

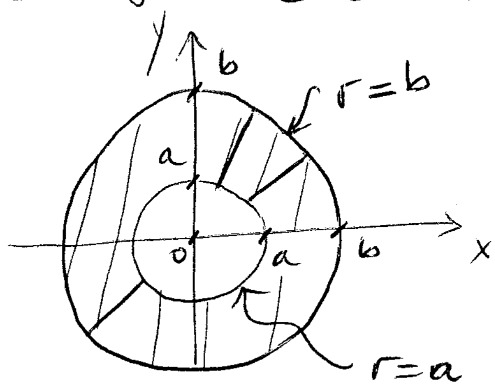
* Convert equations of surfaces into cylindrical coordinates:

sphere: $x^2+y^2+z^2=b^2 \iff r^2+z^2=b^2$
$\iff z^2=b^2-r^2$
$\iff z=\pm\sqrt{b^2-r^2}$

$\rightsquigarrow$ $\boxed{z=\sqrt{b^2-r^2}}$: upper $\frac{1}{2}$-sphere
$\boxed{z=-\sqrt{b^2-r^2}}$: lower $\frac{1}{2}$-sphere

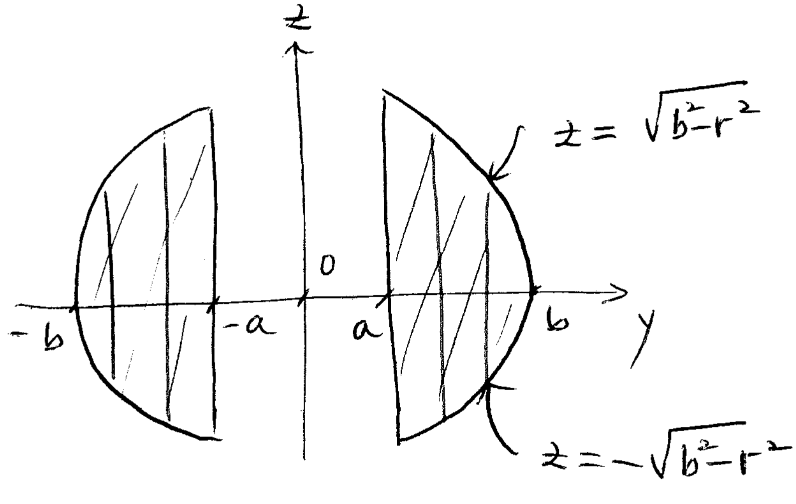
cylinder: $x^2+y^2=a^2 \iff \boxed{r=a}$

* Bounds for r & θ: project R on xy-plane.



$$\begin{Bmatrix} 0\le\theta\le 2\pi \\ a\le r\le b \end{Bmatrix}$$

* Bounds for z: Take vertical cross-section.



lower $\frac{1}{2}$-sphere $\leq z \leq$ upper $\frac{1}{2}$-sphere

$$\Leftrightarrow \boxed{-\sqrt{b^2-r^2} \leq z \leq \sqrt{b^2-r^2}}$$

Thus, $$R = \left\{ \begin{array}{c} 0 \leq \theta \leq 2\pi \\ a \leq r \leq b \\ -\sqrt{b^2-r^2} \leq z \leq \sqrt{b^2-r^2} \end{array} \right\}$$

→ Spherical coordinates:

* Convert equations of surfaces into spherical coordinates:

Sphere: $x^2+y^2+z^2 = b^2 \Leftrightarrow \boxed{\rho = b}$

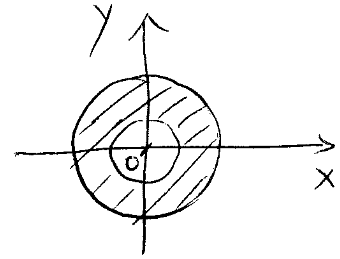
cylinder: $x^2+y^2 = a^2 \Leftrightarrow \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = a^2$

$\Leftrightarrow \rho^2\sin^2\phi = a^2$

$\Leftrightarrow \rho^2 = a^2/\sin^2\phi$

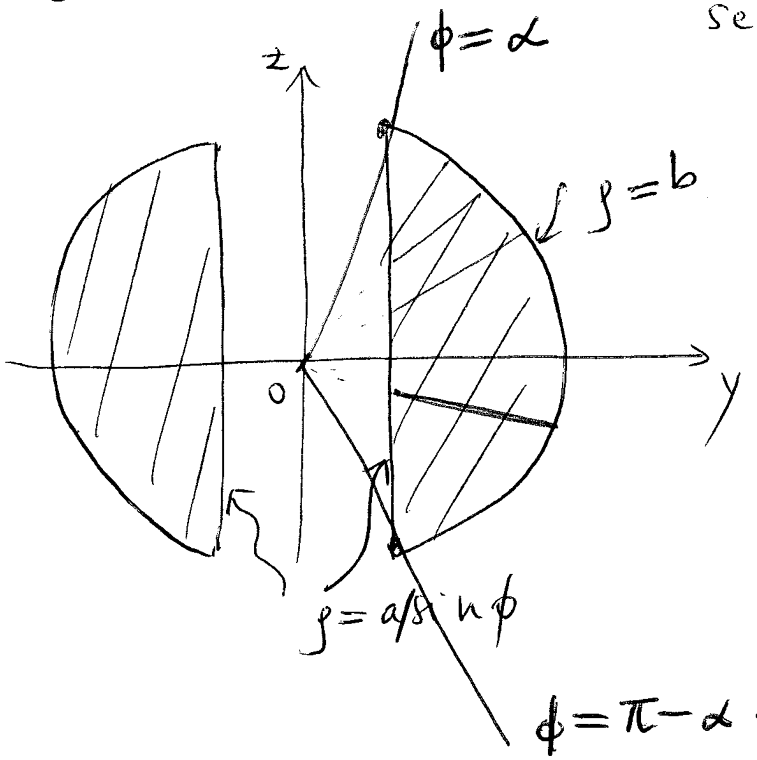
$\Rightarrow \boxed{\rho = a/\sin\phi}$ (since $\sin\phi \geq 0$ for $\phi \in [0,\pi]$).

* Bounds for θ: project R on xy-plane.



$$0 \le \theta \le 2\pi .$$

* Bounds for ρ and ϕ: take vertical cross-section, e.g., with yz-plane.



⇝ Note that ϕ is bounded by the angle of intersection of the sphere and the cylinder: $\rho = b$ and $\rho = a \sin\phi$

$$\Leftrightarrow \quad b = a/\sin\phi \quad \Leftrightarrow \quad \sin\phi = a/b$$

$$\Leftrightarrow \quad \phi = \arcsin(a/b).$$

$$= \alpha \text{ and } (\pi - \alpha)$$

⇝ $\forall \quad \alpha \le \phi \le \pi - \alpha,$

$$\text{cylinder} \le \rho \le \text{sphere}$$

$$\Leftrightarrow \quad a/\sin\phi \le \rho \le b . \quad \Leftrightarrow \quad a\csc\phi \le \rho \le b$$

$$\Rightarrow \quad \alpha \le \phi \le \pi - \alpha$$
$$a\csc\phi \le \rho \le b$$

and

$$R = \left\{ \begin{array}{c} 0 \le \theta \le 2\pi \\ \alpha \le \phi \le \pi - \alpha \\ a\csc\phi \le \rho \le b \end{array} \right\}.$$

NOTE: In cylindrical coordinates, the cylinder $x^2+y^2=a^2$ only appears in the bounds of r. Whereas in spherical coordinates, it contributes to both the bounds of ϕ and ρ.

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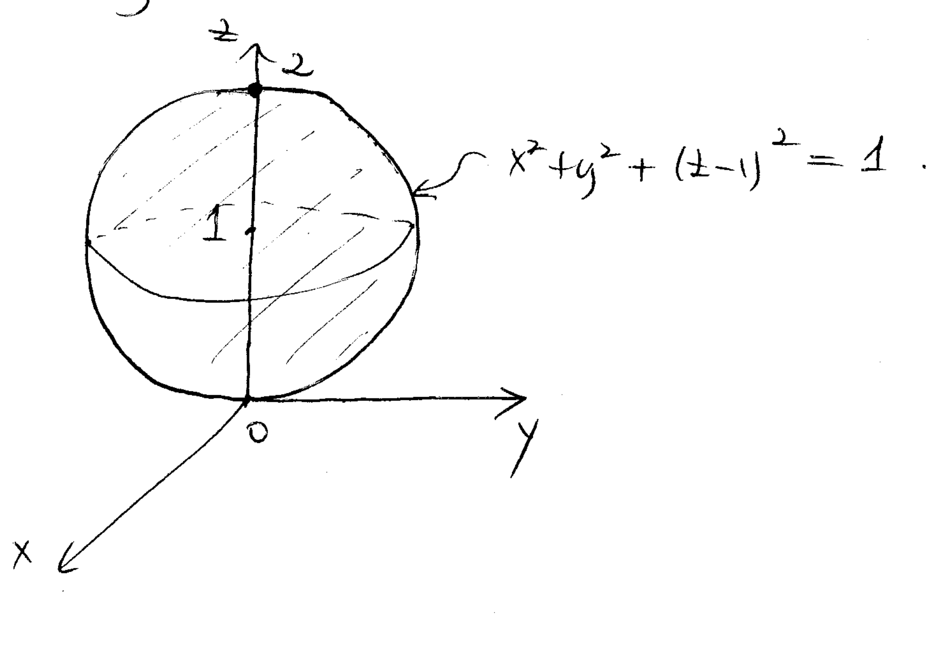
3) Let R be the region bounded by the sphere $x^2+y^2+(z-1)^2=1$. Describe R in both spherical and cylindrical coordinates.



→ Cylindrical coordinates:

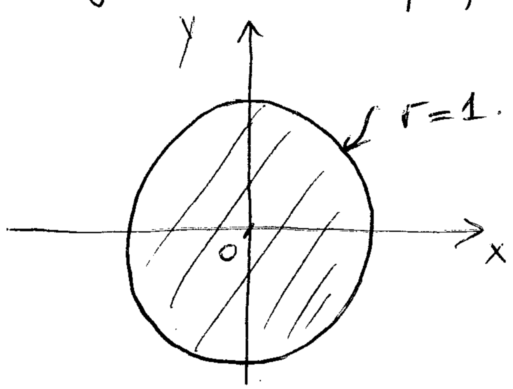
* Convert equations to cylindrical coordinates:

$$\underbrace{x^2+y^2}_{= r^2}+(z-1)^2=1 \iff r^2+(z-1)^2=1$$
$$\iff (z-1)^2=1-r^2$$
$$\iff z-1=\pm\sqrt{1-r^2}$$
$$\iff z=1\pm\sqrt{1-r^2}$$

⇝ $z=1+\sqrt{1-r^2}$: upper $\frac{1}{2}$-sphere

$z=1-\sqrt{1-r^2}$: lower $\frac{1}{2}$-sphere

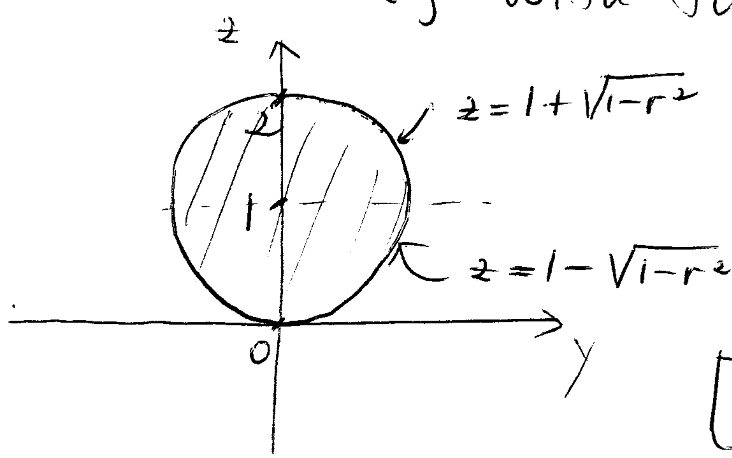
* Bounds for r & θ: project R on xy-plane:



get a disc centered at O whose radius is the radius of the sphere.

$$\left\{\begin{array}{l} 0\le\theta\le 2\pi \\ 0\le r\le 1 \end{array}\right\}$$

* Bounds for z: take a vertical cross-section, e.g. with the yz-plane.



$$\begin{pmatrix}\text{lower}\\ \frac{1}{2}\text{-sphere}\end{pmatrix}\le z\le\begin{pmatrix}\text{upper}\\ \frac{1}{2}\text{-sphere}\end{pmatrix}$$

$$\Updownarrow$$

$$\boxed{1-\sqrt{1-r^2}\le z\le 1+\sqrt{1-r^2}}$$

Thus,

$$R = \left\{ \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 1-\sqrt{1-r^2} \le z \le 1+\sqrt{1-r^2} \end{array} \right\}$$

→ Spherical coordinates:

* Convert equations to spherical coordinates:

$$x^2+y^2+(z-1)^2=1 \iff \underbrace{x^2+y^2+z^2}_{=\rho^2} = 2\underbrace{z}_{=\rho\cos\phi}$$

$$\iff \rho^2 = 2\rho\cos\phi$$

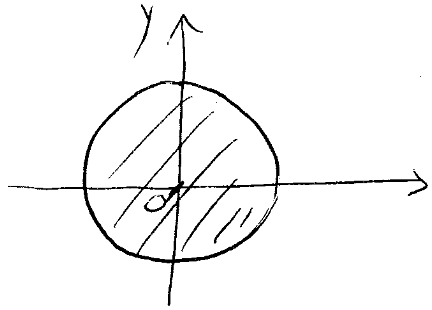
$$\iff \rho = 2\cos\phi \quad \text{or} \quad \rho = 0.$$

also gives $(0,0,0)$ for $\phi = \pi/2$

only gives $(0,0,0)$

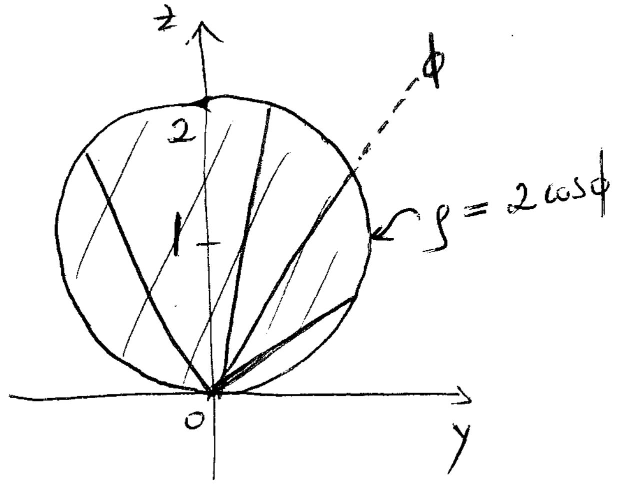
$$\iff \boxed{\rho = 2\cos\phi}$$

* Bounds for θ: project R on xy-plane



$\longrightarrow \quad 0 \le \theta \le 2\pi.$

* Bounds for ρ and ϕ: take vertical cross-section.



$$\begin{cases} 0 \le \phi \le \pi/2 \\ 0 \le \rho \le 2\cos\phi \end{cases}$$

$\rightsquigarrow$ since only have points above xy-plane

sphere

Thus, $R = \left\{ \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \\ 0 \le \rho \le 2\cos\phi . \end{array} \right\}$

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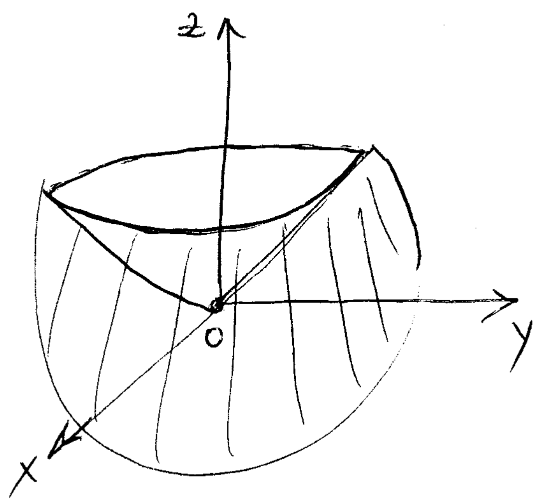
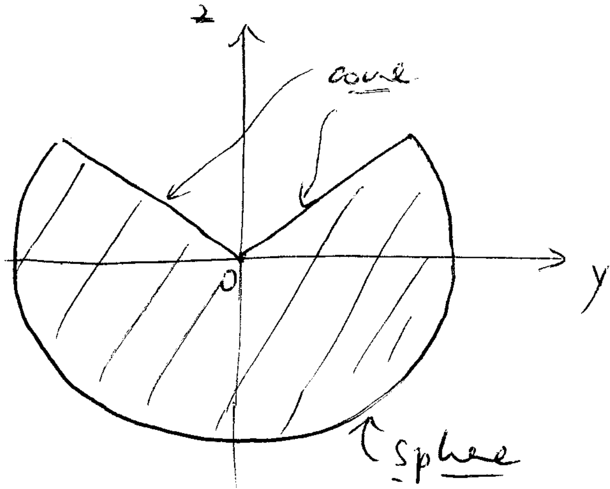
4) Let R be the region inside the sphere $x^2+y^2+z^2=1$ below the cone $z=\sqrt{3}\sqrt{x^2+y^2}$. Describe R in spherical coordinates.



To better visualize the region, let's take a vertical cross-section with the yz-plane:



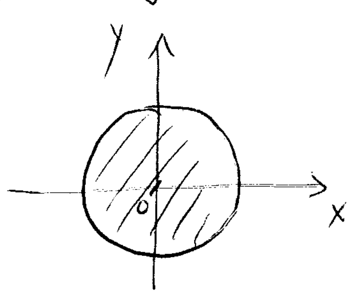
* Convert equations to spherical coordinates:

sphere: $x^2+y^2+z^2 = 1 \iff \boxed{\rho = 1}$

cone: $z = \sqrt{3}\sqrt{x^2+y^2} \iff \tan\phi = \dfrac{\sqrt{x^2+y^2}}{z} = 1/\sqrt{3}$

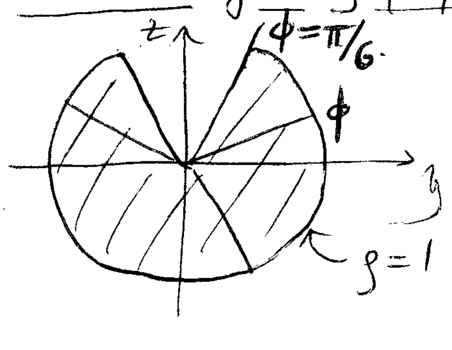
$\Rightarrow \boxed{\phi = \pi/6}$

* Bounds for θ: project R on xy-plane.



$0 \le \theta \le 2\pi$

* Bounds for ρ & ϕ: take vertical cross-section.



cone $\leftarrow \dfrac{\pi}{6} \le \phi \le \pi$

$0 \le \rho \le 1 \rightsquigarrow$ sphere

Thus,

$$R = \left\{ \begin{array}{c} 0 \le \theta \le 2\pi \\ \pi/6 \le \phi \le \pi \\ 0 \le \rho \le 1 \end{array} \right\}$$

<u>NOTE</u>: We see that in spherical coordinates, the sphere <u>only</u> bounds ρ, whereas the cone <u>only</u> bounds ϕ. This is to be expected since the equation of the sphere $\rho = 1$ only involves ρ and the equation for the cone $\phi = \pi/6$ only involves ϕ.