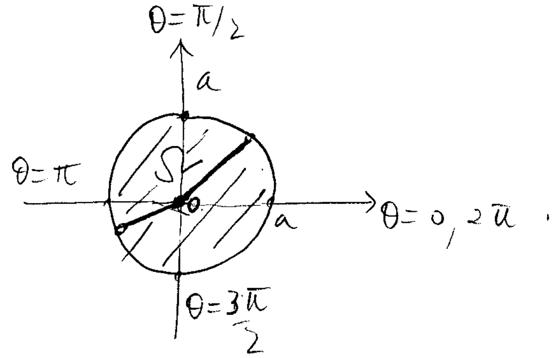
Ex. 1) Verify that the area of region Ω lying inside the circle $r=a$ (of radius a centered at the origin) is πa^2.



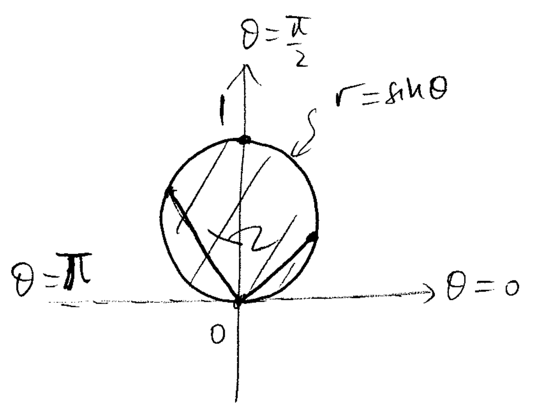
The boundary of Ω is the circle $r=a$, with $0 \le \theta \le 2\pi$.
Also, for any point (r,θ) in Ω,

$$0 \le r \le a.$$

THUS,

$$\text{area}(\Omega) = \int_0^{2\pi} \frac{(a)^2}{2}\, d\theta = \frac{a^2}{2}\left[\theta\right]_0^{2\pi} = a^2\pi.$$

2) Verify that the area of the region Ω lying inside the circle $r=\sin\theta$ is $\pi/4$.



In this case, Ω only lies in the 1st and 2nd quadrant, so that we only need to consider

$$0 \le \theta \le \pi.$$

Moreover, the boundary of Ω is the circle $r=\sin\theta$, with $0\le\theta\le\pi$, and $\forall (r,\theta)\in\Omega$,

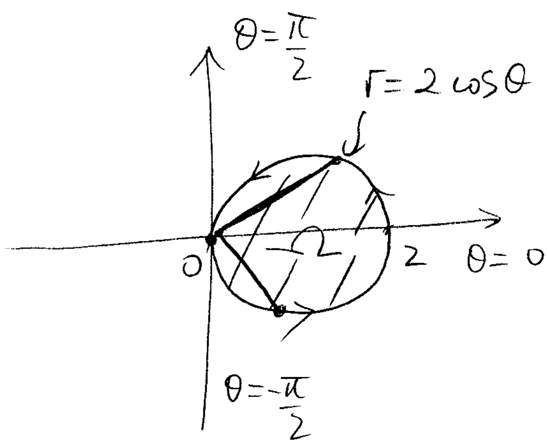
$$0 \le r \le \sin\theta.$$

THUS,

$$\text{area}(\Omega) = \int_0^{\pi} \frac{(\sin\theta)^2}{2}\, d\theta = \int_0^{\pi} \frac{1}{2}\cdot\left[\frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{\pi}{4}.$$

3) Verify that the area of the region Ω lying inside the circle $r = 2\cos\theta$ is π.



HERE, Ω only lies in the 1^{st} and 4^{th} quadrants, so that we only need to consider

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Since the boundary of Ω is the circle $r = 2\cos\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and $\forall (r,\theta) \in \Omega$,

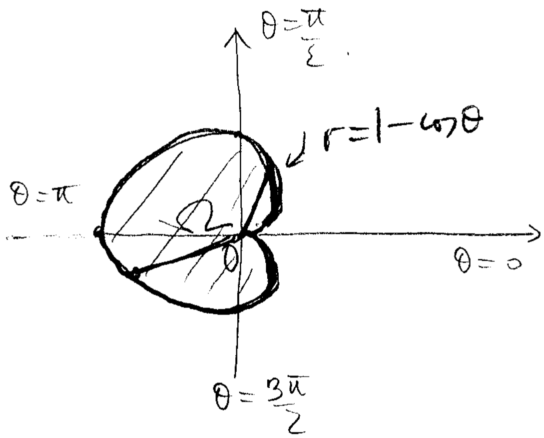
$$0 \le r \le 2\cos\theta,$$

we have:

$$\text{area}(\Omega) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cdot(2\cos\theta)^2\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cancel{2}}\left[\cancel{4}\left(\frac{1+\cos 2\theta}{\cancel{2}}\right)\right] d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi.$$

4) Find the area of the region Ω bounded by $r = 1 - \cos\theta$.



Since Ω has points in every quadrant, we need to consider

$$0 \le \theta \le 2\pi.$$

Moreover, for every point $(r, \theta) \in \Omega$,

$$0 \le r \le 1 - \cos\theta$$

with $0 \le \theta \le 2\pi$.

THUS,

$$\text{area}(\Omega) = \int_0^{2\pi} \frac{(1 - \cos\theta)^2}{2}\, d\theta$$

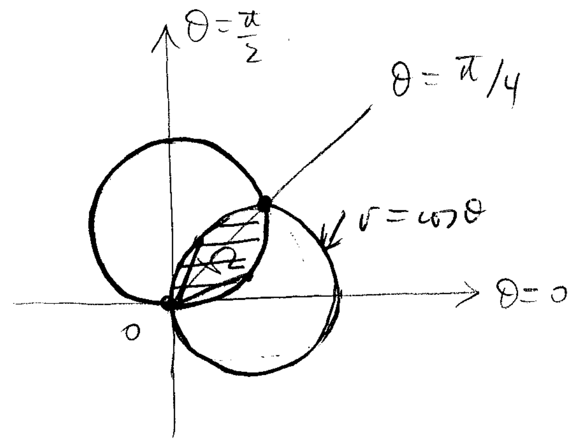
$$= \int_0^{2\pi} \frac{1}{2} \cdot \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \cdot \left(1 - 2\cos\theta + \frac{(1 + \cos 2\theta)}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= \frac{3\pi}{2}.$$

5) Find the area of the region Ω inside both $r=\sin\theta$ and $r=\cos\theta$.



* Intersection points: if (r,θ) lies on both curves, then: $(r=\sin\theta$ and $r=\cos\theta) \Leftrightarrow \sin\theta=\cos\theta$.

Since the region Ω is in the first quadrant, it is enough to find angles $\theta\in[0,\frac{\pi}{2}]$ for which $\sin\theta=\cos\theta \Rightarrow$ the only possibility is $\theta=\frac{\pi}{4}$, so that $(\frac{\sqrt{2}}{2},\frac{\pi}{4})$ is an intersection point. The other intersection point is the origin $O=(0,0),(0,\frac{\pi}{2})$.

Now since $r=\sin\theta$ and $r=\cos\theta$ have intersection points $(\frac{\sqrt{2}}{2},\frac{\pi}{4})$ and $O=(0,0)=(0,\frac{\pi}{2})$, this means that the boundary of Ω changes at $\theta=0,\frac{\pi}{4}$, and $\frac{\pi}{2}$.

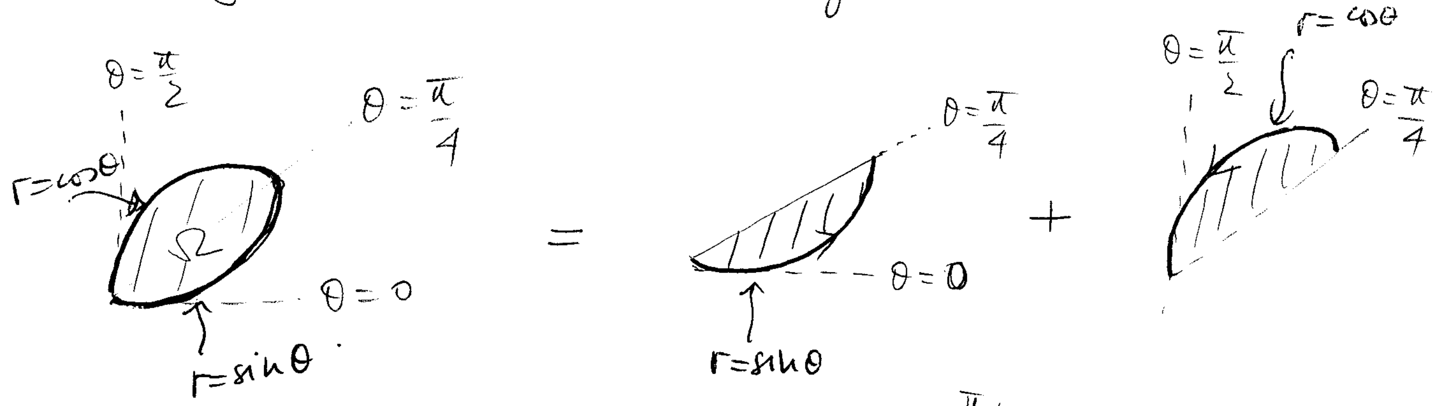
* Bounds for r: any ray $\theta=\alpha$, with $0\le\alpha\le\frac{\pi}{4}$, intersects Ω in a line segments whose points (r,θ) are such that

$$0\le r\le\sin\theta.$$

BUT, if $\frac{\pi}{4}\le\alpha\le\frac{\pi}{2}$, then $\theta=\alpha$ intersects Ω in a line segment whose points are such that

$$0\le r\le\cos\theta.$$

Note that the lower bound of r is always 0, but that the upper bound changes at $\theta = \frac{\pi}{4}$. We therefore need two integrals.



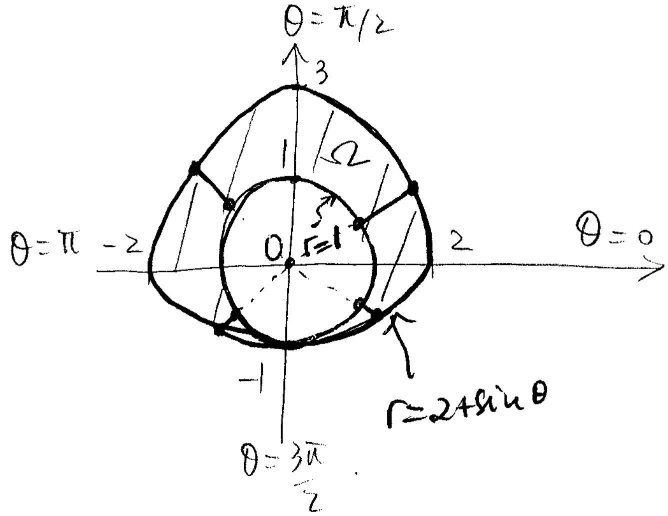
$$= \int_0^{\pi/4} \frac{(\sin\theta)^2}{2}\, d\theta + \int_{\frac{\pi}{4}}^{\pi/2} \frac{(\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \cdot \left(\frac{1 - \cos 2\theta}{2} \right) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \cdot \left(\frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4} + \frac{1}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/2}$$

$$= \cdots = \frac{\pi}{8} - \frac{1}{4}.$$

6) Find the area of the region Ω inside $r = 2+\sin\theta$ and outside $r=1$.

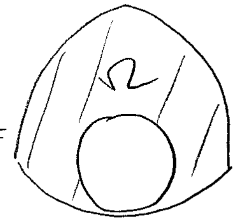


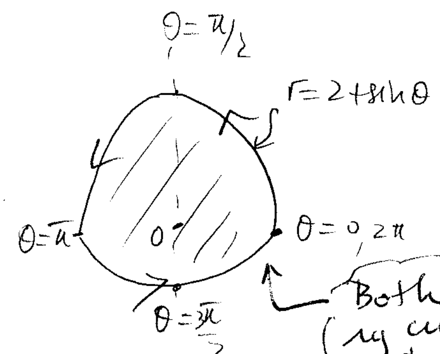
NOTE that there are points in every quadrant, So that we need to consider points (r,θ) with $0 \le \theta \le 2\pi$.

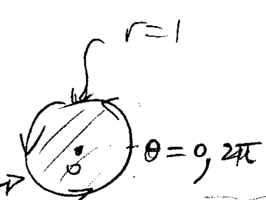
Moreover, for every (r,θ) in Ω, we have:

$$\underline{1} \le r \le 2+\sin\theta,$$

since every ray $\theta = \alpha$, $0 \le \alpha \le 2\pi$, intersects Ω in a line segment whose closest point from the origin O lies on $r=1$ and whose furthest point from the origin lies on $r = 2+\sin\theta$. Since the lower bound of r is not 0, we should consider the area of Ω as the difference of two areas:

area(Ω) = 

= 

— 

Both boundary curves have points with θ from 0 to 2π

The bounds of integration are 0 to 2π since both curves $r = 2+\sin\theta$ and $r=1$ have points with $0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \frac{(2+\sin\theta)^2}{2}\, d\theta - \int_0^{2\pi} \frac{(1)^2}{2}\, d\theta$$

$$= \int_0^{2\pi} \frac{(4+4\sin\theta+\sin^2\theta)}{2}\, d\theta - \int_0^{2\pi} \frac{1}{2}\, d\theta$$

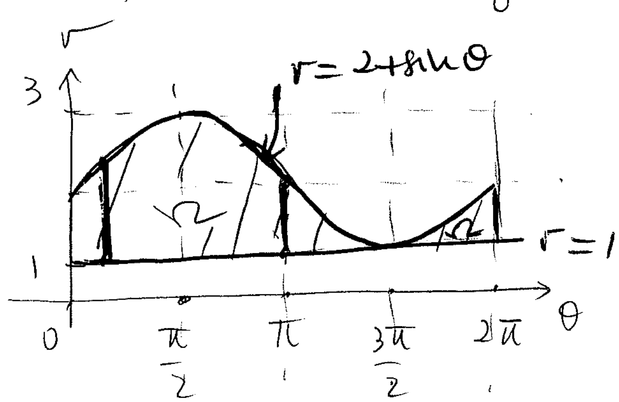
$$= \int_0^{2\pi} \left(2 + 2\sin\theta + \frac{\sin^2\theta}{2}\right) d\theta \;-\; \left[\frac{\theta}{2}\Big|_0^{2\pi}\right]$$

$$= \int_0^{2\pi} \left[2 + 2\sin\theta + \frac{1}{2}\cdot\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta \;-\; \pi$$

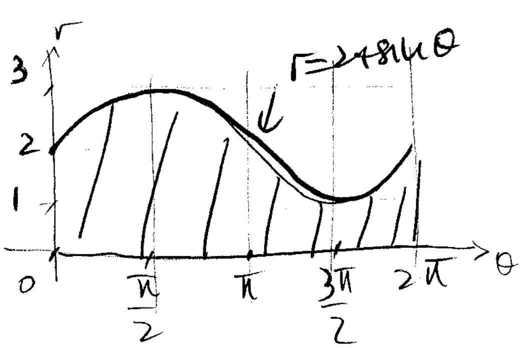
$$= \left[2\theta - 2\cos\theta + \frac{1}{2}\cdot\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)\right]_0^{2\pi} \;-\; \pi$$

$$= \cdots = \frac{7\pi}{2}$$

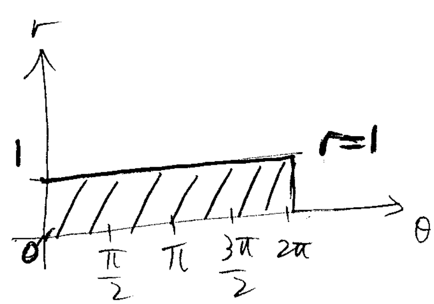
<u>NOTE</u>: If one considers the graphs of $r = 2+\sin\theta$ and $r=1$ in the (θ, r)-plane, it is clear why the area is given by the above integrals.



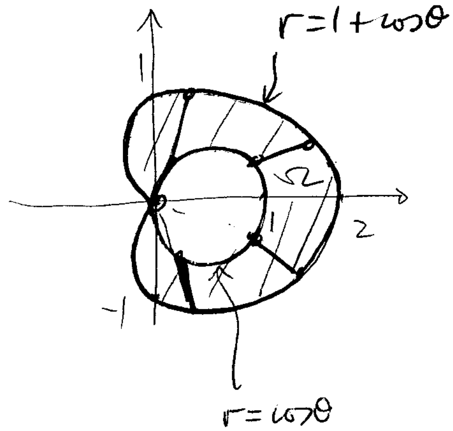
$=$



$-$



7) Find the area of the region Ω inside $r = 1+\cos\theta$ and outside $r = \cos\theta$.



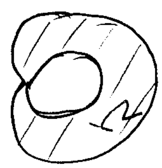
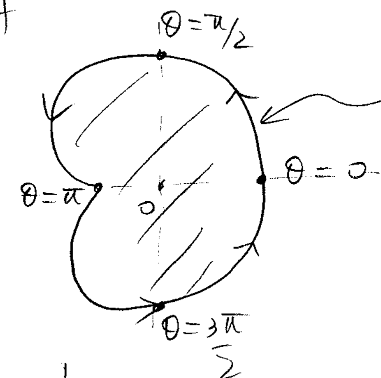
Since there are points in Ω in every quadrant, we need to consider

$$0 \le \theta \le 2\pi.$$

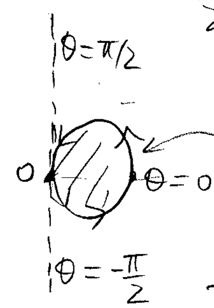
Moreover for every (r,θ) in the part of Ω lying in the 1st and 4th quadrants,

$$\cos\theta \le r \le 1+\cos\theta.$$

$\Rightarrow$ it is therefore better to think of area(Ω) as a difference of 2 areas.

area(Ω) =  = 

$r = 1+\cos\theta$ with $0\le\theta\le 2\pi$

$-$ 

$r=\cos\theta$ with $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$= \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\cos\theta)^2}{2}\,d\theta$$

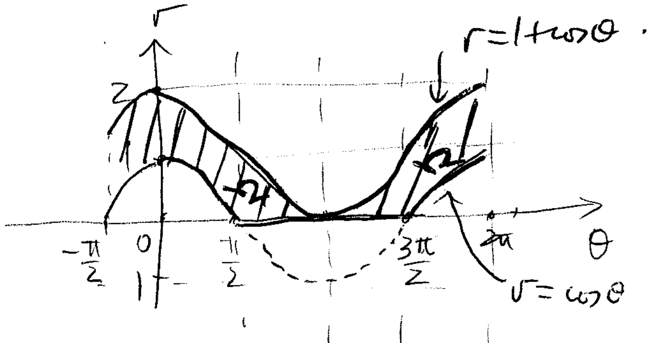
$$= \int_0^{2\pi} \frac{1}{2}\cdot\left[1+2\cos\theta+\cos^2\theta\right]d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\cdot\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\cdot\left[1 + 2\cos\theta + \frac{(1+\cos 2\theta)}{2}\right] d\theta \; - \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1}{4}\cdot(1+\cos 2\theta)\, d\theta$$

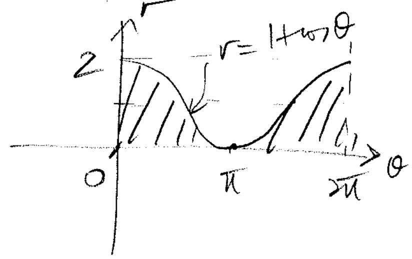
$$= \frac{1}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} - \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\pi/2}$$

$$= \cdots = \frac{5\pi}{4}.$$

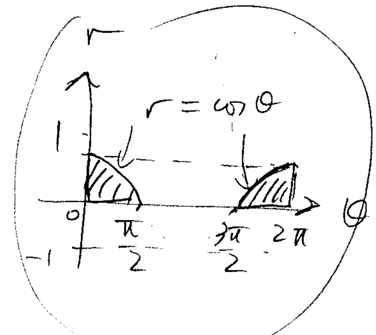
NOTE: If one considers the graphs of $r = 1+\cos\theta$ and $r = \cos\theta$ in the (θ, r)-plane, it is clear why the area is given by the above integrals:



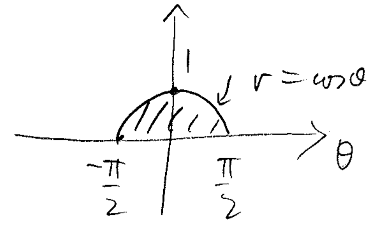
$\Rightarrow$ area(Ω) =



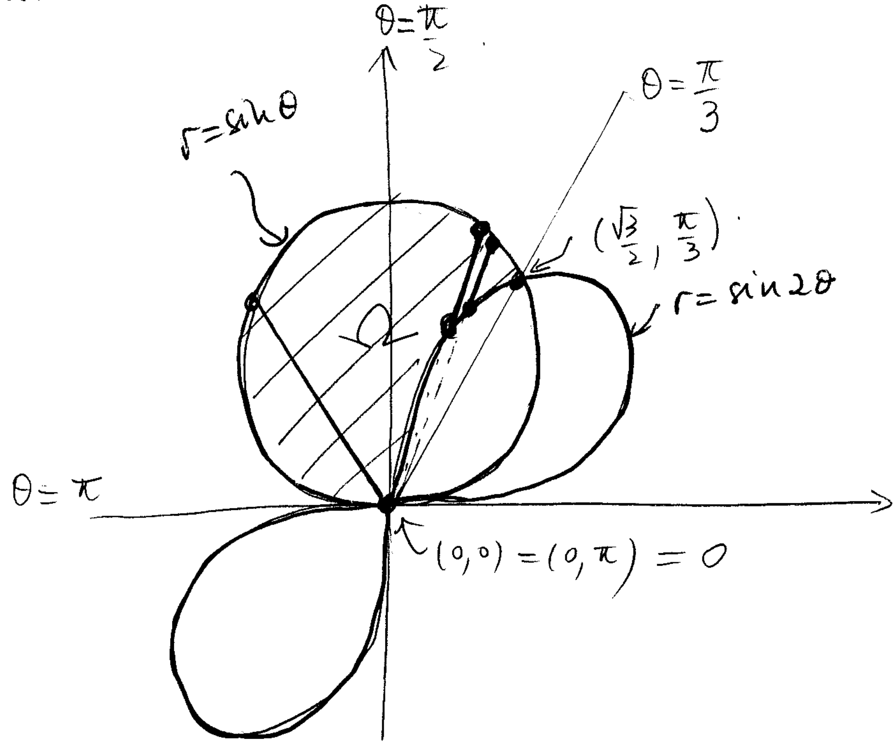
—



$\parallel$



8) Find the area of the region Ω outside $r = \sin 2\theta$ and inside $r = \sin\theta$.



* Intersection points: if (r, θ) lies on the two curves, then: $(r = \sin 2\theta \text{ and } r = \sin\theta) \Longleftrightarrow \sin 2\theta = \sin\theta$

$\Longleftrightarrow 2\sin\theta\cos\theta = \sin\theta$ (since $\sin 2\theta = 2\sin\theta\cos\theta$)

$\Longleftrightarrow \sin\theta(1 - 2\cos\theta) = 0$

$\Longleftrightarrow \sin\theta = 0$ OR $\cos\theta = 1/2$.

From the picture, it is clear that the intersection points lie the 1st and 2nd quadrants: it is therefore enough to look for angles θ between 0 and π:

$\rightsquigarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi. \Rightarrow r = \sin 0 = \sin\pi = 0$: $(0,0) \,\$\, (0,\pi)$

$\rightsquigarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}. \Rightarrow r = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$: $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$.

The region we are considering therefore has points whose angles θ are between $\frac{\pi}{3}$ and π. (since the region starts, counterclockwise, at the point $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and ends at the origin $O = (0, \pi)$)!

* Bounds for r: if we consider a ray $\theta = \alpha$, with $\frac{\pi}{3} \le \alpha \le \frac{\pi}{2}$, it intersects Ω in a line segment

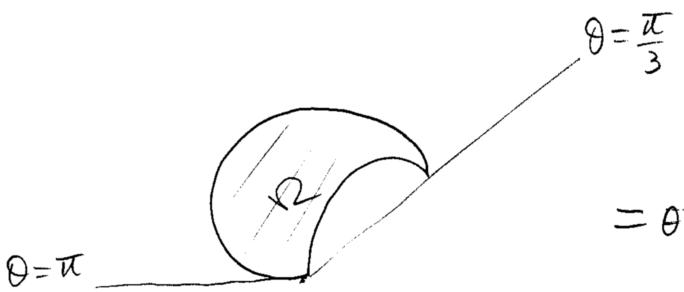
on which points (r, θ) have radius

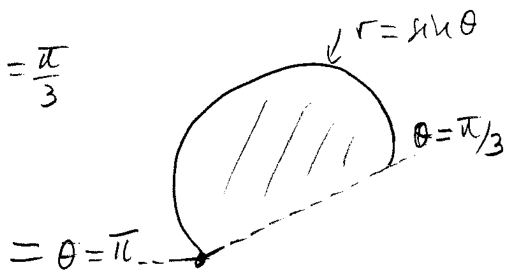
$$\sin 2\theta \le r \le \sin\theta.$$

But, if $\frac{\pi}{2} \le \alpha \le \pi$, the ray $\theta = \alpha$ intersects Ω in a line segment whose points have radius

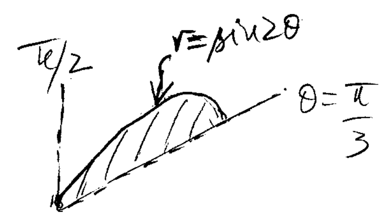
$$0 \le r \le \sin\theta.$$

$\Rightarrow$ The lower bound of r is not always 0, meaning that is better to think of the area of Ω as being the difference of 2 areas:





$-$



$$= \int_{\frac{\pi}{3}}^{\pi} \frac{(\sin\theta)^2}{2}\, d\theta$$

$$- \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(\sin 2\theta)^2}{2}\, d\theta.$$

So,

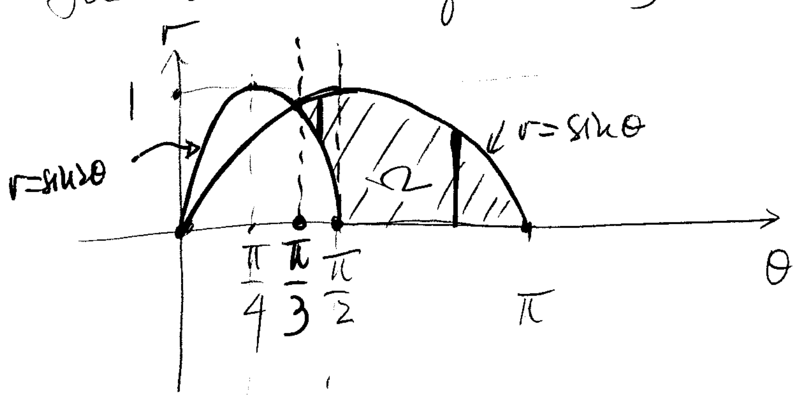
$$\text{area}(\Omega) = \int_{\frac{\pi}{3}}^{\pi} \frac{(\sin\theta)^2}{2}\, d\theta \;-\; \int_{\pi/3}^{\pi/2} \frac{(\sin 2\theta)^2}{2}\, d\theta$$

$$= \int_{\frac{\pi}{3}}^{\pi} \frac{1}{2}\cdot\left(\frac{1-\cos 2\theta}{2}\right) d\theta \;-\; \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}\cdot \frac{(1-\cos 4\theta)}{2}\, d\theta$$

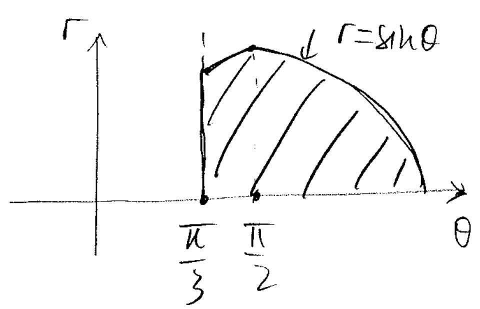
$$= \left[\frac{1}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\frac{\pi}{3}}^{\pi} \;-\; \left[\frac{1}{4}\left(\theta - \frac{\sin 4\theta}{4}\right)\right]_{\frac{\pi}{3}}^{\pi/2}$$

$$= \cdots = \boxed{\frac{\pi}{8}\left(\pi + \frac{3\sqrt{3}}{2}\right)}$$

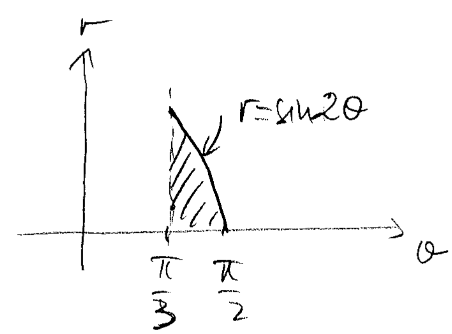
NOTE: If one considers the graphs of $r = \sin 2\theta$ and $r = \sin\theta$ in the (θ, r)-plane, it is clear why the area is given by the above integrals:



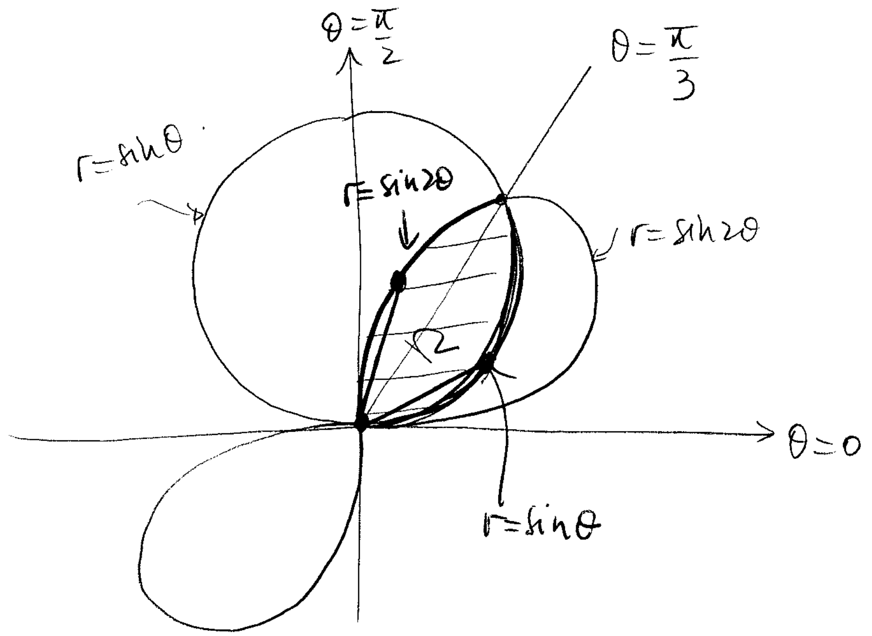
=



−



9) Find the area of the region Ω bounded by both $r = \sin 2\theta$ and $r = \sin\theta$.



* Intersection points: by example 8, if (r,θ) lies on both curves, **then** $\sin\theta = 0$ OR $\cos\theta = 1/2$. HOWEVER, this time, all points in the region lies in the 1^{st} quadrant, so that θ will take values from 0 to $\frac{\pi}{2}$. Moreover, it is clear from the picture that θ takes every values in $[0, \frac{\pi}{2}]$ and that the intersection points are $O = (0,0)$ and $(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$, indicating that the boundary of the region will change at $\theta = \frac{\pi}{3}$.

* Bounds for r: if one considers a ray $\theta = \alpha$, with $0 \le \alpha \le \frac{\pi}{3}$, it intersects Ω in a line segment on which points (r,θ) have a radius

$$0 \le r \le \sin\theta.$$

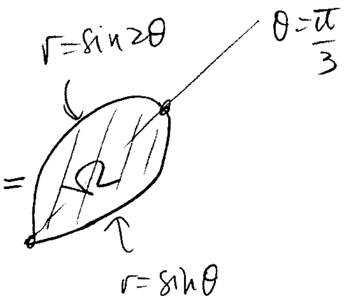
But, if $\frac{\pi}{3} \le \alpha \le \frac{\pi}{2}$, the ray $\theta = \alpha$ intersects Ω in points for which

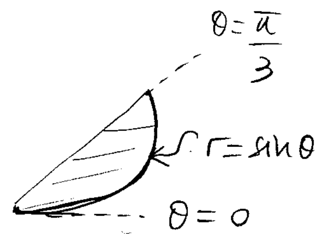
$$0 \le r \le \sin 2\theta.$$

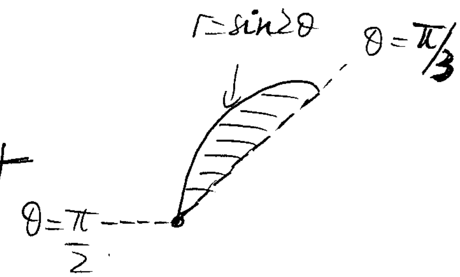
So, although the lower bound of r is always 0, the upper bound changes at $\theta = \frac{\pi}{3}$:

In Ω,
$$0 \le r \le \sin\theta \quad , \text{ if } \; 0 \le \theta \le \frac{\pi}{3},$$
$$0 \le r \le \sin 2\theta, \quad \text{if } \; \frac{\pi}{3} \le \theta \le \frac{\pi}{2}.$$

$\Longrightarrow$ we need two integrals:

area(Ω) = 

= 

+ 

$$= \int_0^{\pi/3} \frac{(\sin\theta)^2}{2}\, d\theta + \int_{\pi/3}^{\pi/2} \frac{(\sin 2\theta)^2}{2}\, d\theta$$

$$= \int_0^{\pi/3} \frac{1}{2}\cdot\left(\frac{1-\cos 2\theta}{2}\right) d\theta + \int_{\frac{\pi}{3}}^{\pi/2} \frac{1}{2}\cdot\frac{(1-\cos 4\theta)}{2}\, d\theta$$

$$= \left[\frac{1}{4}\cdot\left(\theta - \frac{\sin 2\theta}{4}\right)\right]_0^{\pi/3} + \left[\frac{1}{4}\cdot\left(\theta - \frac{\sin 4\theta}{4}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \dots = \frac{\pi}{6} - \frac{\sqrt{3}}{16}.$$

NOTE: As in example 8, the need to use 2 integrals is clear if we consider the graphs of $r=\sin 2\theta$ and $r=\sin\theta$ in the (θ, r)-plane: