Chapter 10 - Optimisation problems.

10.1 The Extreme Value Theorem.

In optimisation problems, one is interested in finding the largest or smallest possible value of a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ on some specific subset $S \subseteq \mathbb{R}^2$, rather than finding the local max./min. points of f.

DEF.: Given a function $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and a set $S \subseteq \mathbb{R}^2$:

(i) A point $(a,b) \in S$ is a maximum point of f on S if

$$f(x,y) \leq f(a,b), \quad \forall (x,y) \in S.$$

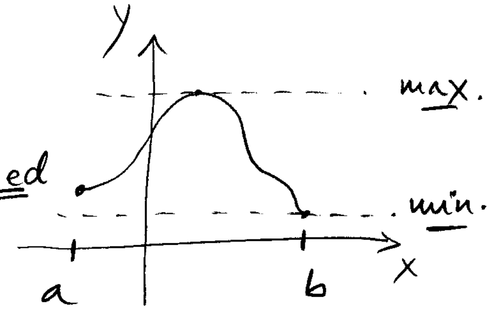
The number $f(a,b)$ is called the maximum value of f on S.

2) A point $(a,b) \in S$ is a minimum point of f on S if

$$f(x,y) \geq f(a,b), \quad \forall (x,y) \in S.$$

The number $f(a,b)$ is called the minimum value of f on S.

1-var. (EXTREME VALUE THEOREM)



If $f: \mathbb{R} \to \mathbb{R}$ is continuous on the closed & bounded interval $I = [a,b]$, then $\exists c_1, c_2 \in I$ such that

$$f(c_1) \leq f(x) \leq f(c_2), \quad \forall x \in I.$$

In particular, the max. and min. of f on I is either a local max. or local min. OR the value of f at one of the endpoints (i.e. boundary) points of I.

So, if is continuous AND I is closed and bounded, then f has BOTH a max. and a min. on I. To find the max./min.:

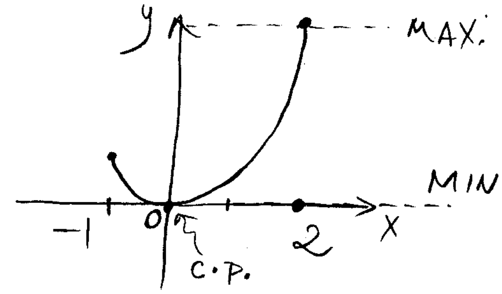
* find the c.p. of f in I.
* compare the values of f at the c.p. and the boundary points a & b.

Ex. Find the max./min. of f on $I = [-1, 2]$.

Since f is cont. and I is closed and bounded, f has a max./min. on I.

* c.p. of f on I: $f'(x) = 2x = 0 \Leftrightarrow \boxed{x = 0}$
* Compare values of f:

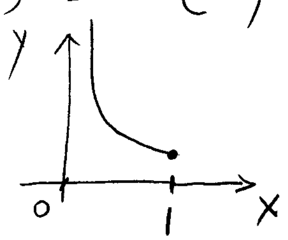
x	C.P.: $x = 0$	Boundary: -1	2
$f(x)$	0	1	4
	MIN.		MAX.



$\Rightarrow$ max. is $f(2) = 4$ and min. is $f(0) = 0$.

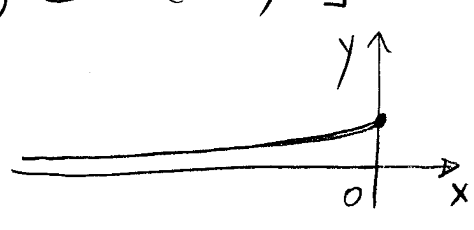
NOTE: If I is not closed or bounded OR if f is discontinuous at some point in I or on the boundary of I, then f may not have a max. or min. on I.

E.g. 1) $I = (0, 1]$ and $f(x) = \frac{1}{x}$ ~> I is NOT closed (and f is discont. at $x = 0$)



min $= 1$, NO max.

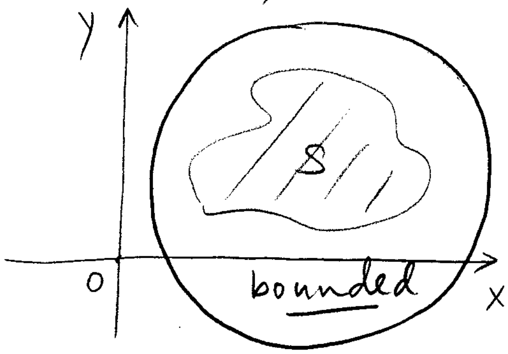
2) $I = (-\infty, 0]$ and $f(x) = e^x$ ~> here I is not bounded.

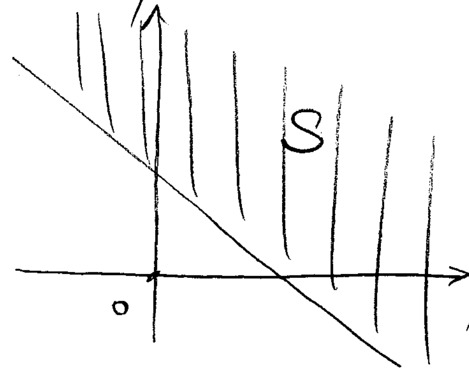


NO min., max. $= 1$.

2-var. Let S be a subset of $\mathbb{R}^2$.

(i) S is bounded if S is contained in some closed disc in $\mathbb{R}^2$, i.e., if "one can draw a circle around S".



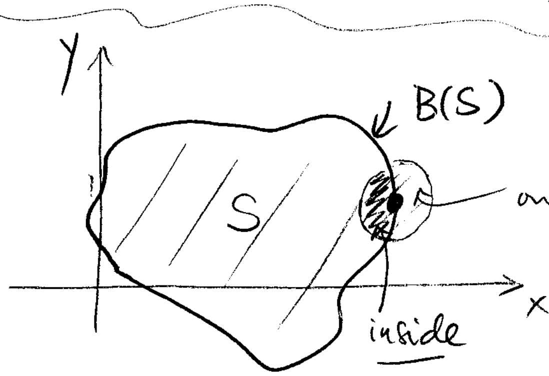


not bounded

(ii) $B(S)$:= boundary of S = (edge of S)

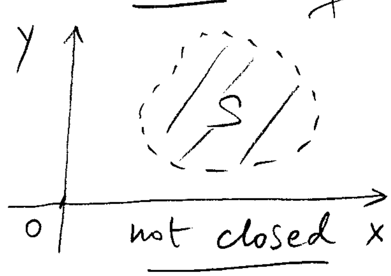
→ a point (a,b) is on the boundary of S if any disc centered at (a,b) contains both points inside S and points outside S.

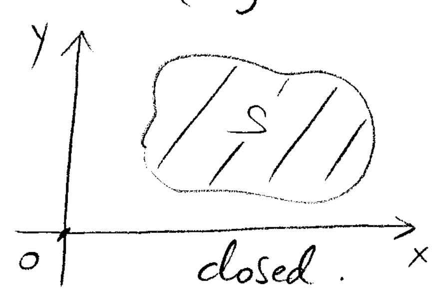
e.g.:



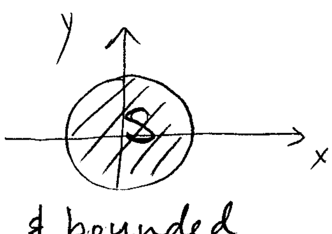
NOTE: $B(S)$ is a curve.

(iii) S is closed if it contains $B(S)$:

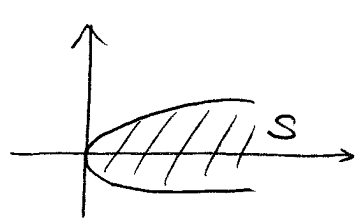




e.g.:



closed & bounded with $B(S) =$ 



$B(S) =$  included in S

$\Rightarrow$ S closed

BUT, S not bounded.

THM: (EXTREME VALUE THEOREM). (EVT)

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $S \subseteq \mathbb{R}^2$ a subset. If f is continuous on S and S is closed and bounded, then there exist (c_1, d_1) and (c_2, d_2) in S such that

$$f(c_1, d_1) \leq f(x,y) \leq f(c_2, d_2), \quad \forall (x,y) \in S.$$

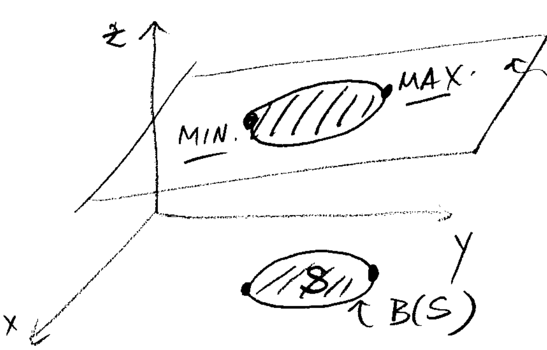
In particular, the max. and min. of f on S occur at c.p. of f inside S or on $B(S)$.

ALGORITHM for find max./min.:

1) Find c.p. of f in S. (No need to test them!)
2) Find max. and min. of f on $B(S)$.
3) Compare values found in 1) and 2).

NOTE: The max. or min. may not occur at c.p.

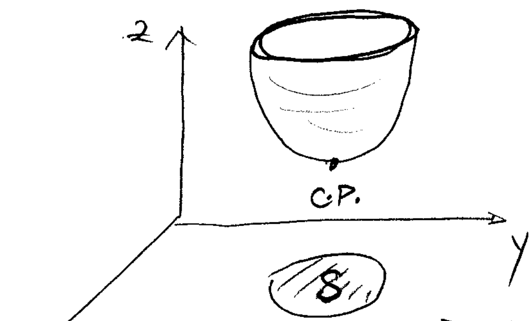
E.g.: 1)



$z = f(x,y)$: plane

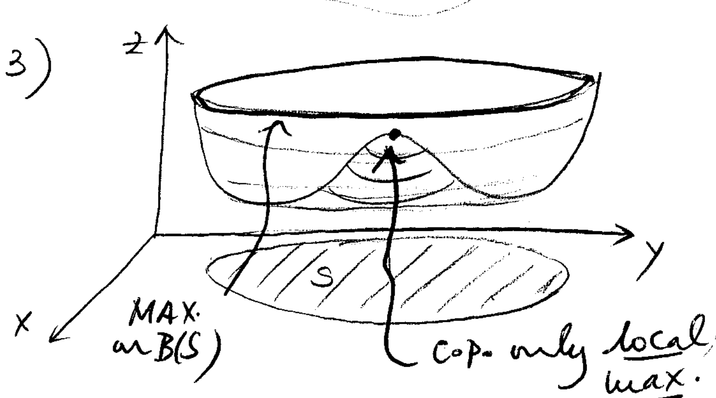
f has NO c.p., but max./min. of f occur on $B(S)$.

2)

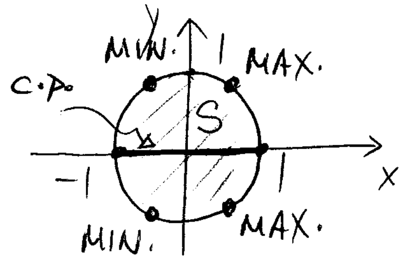


max. occurs on $B(S)$

3)



Ex. 1) Find the max. and min. of $f(x,y) = xy^2$ on
$$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}.$$



S is closed & bounded with
$$B(S) = \{x^2 + y^2 = 1\}.$$
$+\ f$ is cont. on S
$\Rightarrow f$ has max./min. on S by EVT.

* C.P. inside S: $\nabla f = (y^2, 2xy) = (0,0) \Leftrightarrow \boxed{y = 0}$

$\Rightarrow$ get line segment of C.P. on x-axis:
$$\{(x,0) \mid -1 \le x \le 1\}.$$

* on B(S): $x^2 + y^2 = 1 \Leftrightarrow y^2 = 1 - x^2,\ -1 \le x \le 1$

$\Rightarrow f$ restricted to $B(S)$ is:
$$g(x) := f(x,y) = x(1 - x^2),\ -1 \le x \le 1.$$

$y = \pm \frac{2}{\sqrt{3}}$ on $B(S)$

To find max./min. of g on $[-1, 1]$:

$\rightarrow$ C.P. of g: $g'(x) = 1 - 3x^2 = 0 \Rightarrow \boxed{x = \pm 1/\sqrt{3}}$

$\rightarrow$ Compare values of g:

x	C.P.: $-1/\sqrt{3}$	$1/\sqrt{3}$	Boundary: -1	1
$g(x)$	$-2/3\sqrt{3}$	$2/3\sqrt{3}$	0	0
	MIN.	MAX.		

$\Rightarrow$ max. of f on $B(S)$ is $2/3\sqrt{3}$ at $(x,y) = (1/\sqrt{3}, \pm 2/\sqrt{3})$

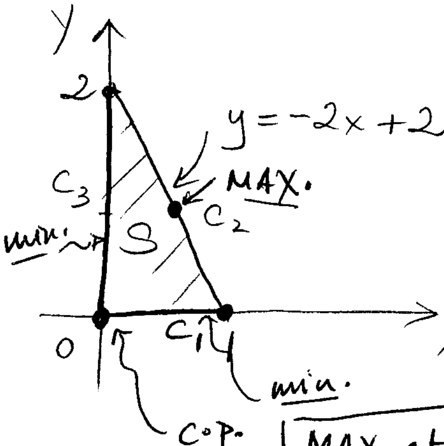
min. of f on $B(S)$ is $-2/3\sqrt{3}$ at $(x,y) = (-1/\sqrt{3}, \pm 2/\sqrt{3})$

* Compare all values:

(x,y)	C.P.: $(x,0)$	B(S): $(1/\sqrt{3}, \pm 2/\sqrt{3})$	$(-1/\sqrt{3}, \pm 2/\sqrt{3})$
$f(x,y)$	0	$2/3\sqrt{3}$	$-2/3\sqrt{3}$
		MAX.	MIN.

2) Find max./min. of $f(x,y) = xy$ on the region S bounded by the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$.

By EVT, since S is closed and bounded and f is cont. on S, f has a max./min. on S.



$$\boxed{\begin{array}{l}\text{MAX. at } (1/2, 1) \\ \text{MIN. on } C_1 \,\&\, C_3\end{array}}$$

* c.p. inside S: $\nabla f = (y, x) = (0,0)$

$\Longrightarrow (x,y) = (0,0)$ is the ONLY c.p. of f and $(0,0) \in S$.

* on $B(S)$: here $B(S) = C_1 \cup C_2 \cup C_3$.

→ On C_1: $(x,y) = (x,0)$, with $0 \le x \le 1$.

$\Longrightarrow f(x,y) = f(x,0) = 0, \quad \forall (x,y) \in C_1$.

→ On C_2: $(x,y) = (x, -2x+2)$, $0 \le x \le 1$.

$\rightsquigarrow f(x,y) = f(x, -2x+2) = x(-2x+2) =: g(x)$.

(i) c.p. of g for $0 \le x \le 1$: $g'(x) = -4x + 2 = 0 \Longrightarrow x = 1/2$ ↳ $y = 1$ on C_2

(ii) Compare values of g:

x	c.p.: $1/2$	Boundary: 0	1
$g(x)$	max. $1/2$	0	0

min.

$\Longrightarrow$ max. of f on C_2 is $1/2$ at $(x,y) = (1/2, 1)$.
min. of f on C_2 is 0 at $(x,y) = (0,2)$ and $(1,0)$.

→ On C_3: $(x,y) = (0,y)$, $0 \le y \le 2$.

$\Longrightarrow f(x,y) = f(0,y) = 0, \quad \forall (x,y) \in C_3$.

* Compare all values:

(x,y)	c.p.: $(0,0)$	C_1	C_2: $(1/2, 1)$	$(0,2)$	$(1,0)$	C_3
$f(x,y)$	0 MIN.	0 MIN.	$1/2$ MAX.	0 MIN.	0 MIN.	0 MIN.