CHOOSING THE ORDER OF INTEGRATION FOR $\iint_D f(x,y)\,dA$

1) First consider $f(x,y)$:

→ if you can ONLY integrate f with respect to x, then set:

$$\iint_D f(x,y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,\underline{\underline{dx}} \right] dy$$

↳ i.e., first integrate with respect to x!

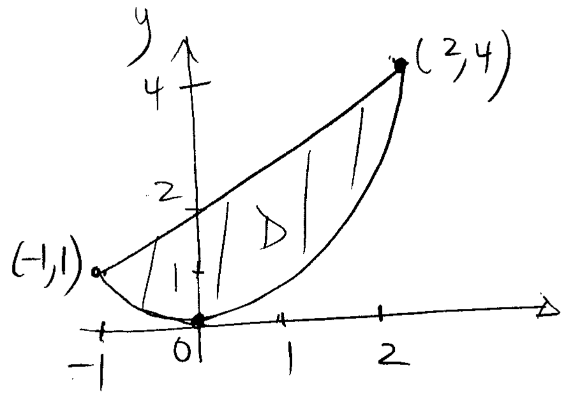
→ if you can ONLY integrate f with respect to y, then set:

$$\iint_D f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,\underline{\underline{dy}} \right] dx$$

↳ i.e., first integrate with respect to y

2) If you can easily integrate f with respect to BOTH x and y, then choose the order of integration that gives you the simplest description of D [i.e., the description that gives you the fewest integrals.]

Ex. 1) Compute $I = \iint_D x \, dA$, where D is the region bounded by $y = x^2$, $x - y = -2$.

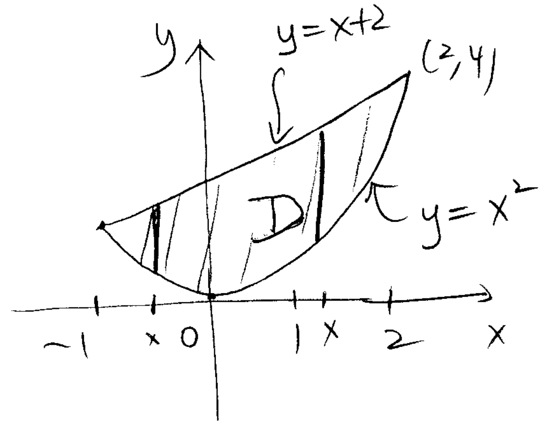


HERE: $f(x,y) = x$ can be easily integrated with respect to BOTH x & y.

Let describe D in terms of type I and type II regions to see which type has the nicest description:

→ TYPE I: x is independent.

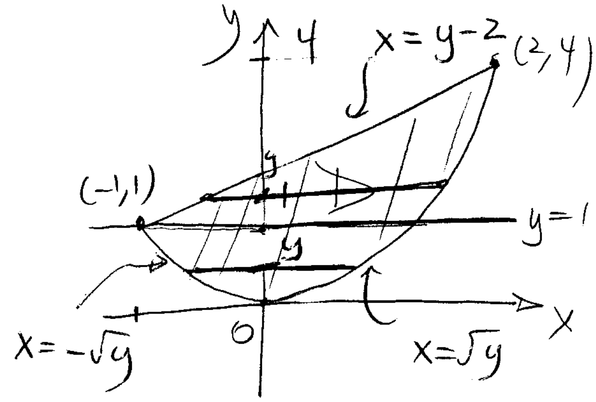
For any $-1 \le x \le 2$, the lower bound for y is on the parabola $y = x^2$ and the upper bound for y is on the line $y = x+2$.



$$\Rightarrow D = \left\{ \begin{array}{l} -1 \le x \le 2 \\ x^2 \le y \le x+2 \end{array} \right\}$$

↳ only need one integral in the order $dy\,dx$.

→ TYPE II: y is independent.



Now, $0 \le y \le 4$, but the lower bound for x changes at $y = 1$ from the negative branch of the parabola $x = -\sqrt{y}$ to the line $x = y - 2$.

BUT, the upper bound is always the positive branch of the parabola $x = \sqrt{y}$.

$$\Rightarrow D = \left\{ \begin{array}{c} 0 \le y \le 1 \\ -\sqrt{y} \le x \le \sqrt{y} \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \le y \le 4 \\ y-2 \le x \le \sqrt{y} \end{array} \right\}$$

⇝ need 2 integrals (in the order $dx\,dy$)

$\Rightarrow$ order $dy\,dx$ is BETTER!

$$I = \int_{-1}^{2}\int_{x^2}^{x+2} x \, dy\, dx = \int_{-1}^{2} \left[xy \Big|_{y=x^2}^{y=x+2} \right] dx$$

$$= \int_{-1}^{2} \left(x(x+2) - x^3 \right) dx = \int_{-1}^{2} \left(x^2 + 2x - x^3 \right) dx$$

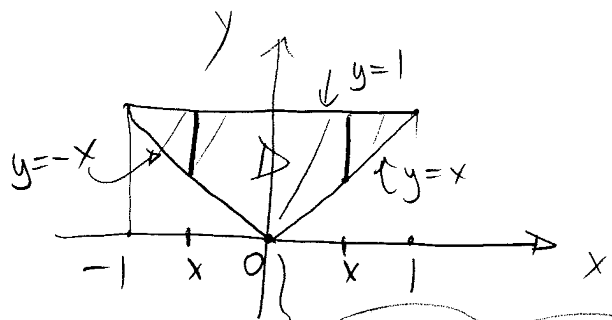
$$= x^3/3 + x^2 - x^4/4 \Big|_{-1}^{2} = 9/4 .$$

2) Compute $I = \iint_D e^{x+y}\, dA$, where D is the region bounded by $y=x$, $y=-x$, $y=1$

AGAIN $f(x,y) = e^{x+y}$ is easy to integrate with respect to both x and y.

$\Rightarrow$ let's see what type gives the nicest description of D.

TYPE I:



lower bound for y changes at $x=0$

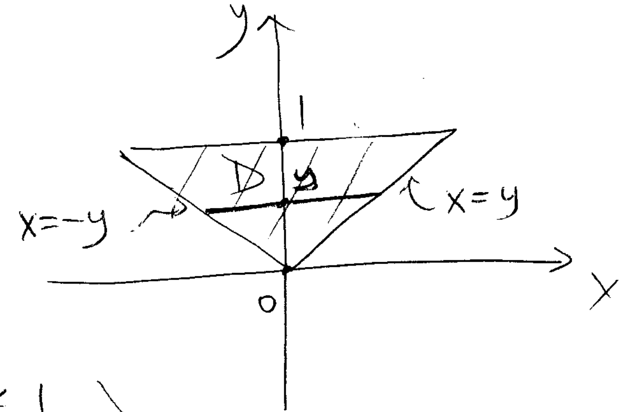
$$D = \left\{ \begin{array}{l} -1 \le x \le 0 \\ -x \le y \le 1 \end{array} \right\} \cup \left\{ \begin{array}{l} 0 \le x \le 1 \\ x \le y \le 1 \end{array} \right\}$$

$$\Rightarrow \quad I = \int_{-1}^{0}\int_{-x}^{1} e^{x+y}\, dy\, dx + \int_{0}^{1}\int_{x}^{1} e^{x+y}\, dy\, dx$$

$\rightarrow$ need 2 integrals!

TRY describing D as a type II region:

TYPE II:



$$D = \left\{ \begin{array}{l} 0 \le y \le 1 \\ -y \le x \le y \end{array} \right\}$$

$$\Rightarrow \quad I = \int_0^1 \left(\int_{-y}^{y} e^{x+y} \, dx \right) dy \quad \text{BETTER!}$$

$$= \int_0^1 \left[e^{x+y} \Big|_{x=-y}^{x=y} \right] dy$$

$$= \int_0^1 (e^{2y} - 1)\, dy$$

$$= \frac{e^{2y}}{2} - y \Big|_0^1 = \frac{e^2}{2} - \frac{3}{2}.$$

3) Evaluate $I = \int_0^1 \int_y^1 \frac{e^x - 1}{x} \, dx \, dy$.

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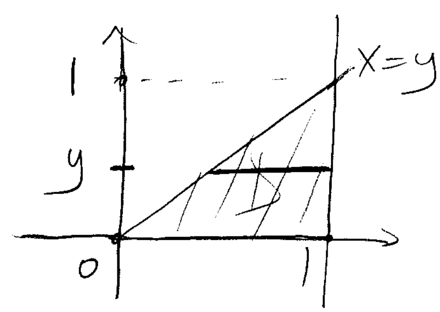
HERE: $f(x,y) = \frac{e^x-1}{x}$ does NOT have an antiderivative with respect to x (since integration by parts won't work because x is in the denominator)

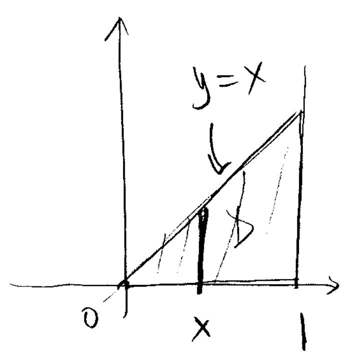
$\Longrightarrow$ Change the order of integration!

NOW, $I = \iint_D \frac{e^x-1}{x} \, dA$, where $D = \left\{ \begin{array}{c} \overset{x=y}{y} \le x \le \overset{x=1}{1} \\ 0 \le y \le 1 \end{array} \right\}$



TYPE II

OR



$D = \left\{ \begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le x \end{array} \right\}$

type I.

$$\Longrightarrow I = \int_0^1 \left(\int_0^x \frac{e^x-1}{x} \, dy \right) dx = \int_0^1 \left(\frac{e^x-1}{x}\right) \left(\int_0^x dy \right) dx$$

$$= \int_0^1 \left(\frac{e^x-1}{\cancel{x}}\right) \cdot \cancel{x} \, dx = \int_0^1 (e^x - 1) \, dx = e^x - x \Big|_0^1 = e-2.$$

4) Evaluate $I = \int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\,dy\,dx$

$\int \cos(y^3)\,dy$

$u = y^3$

$du = 3y^2\,dy$

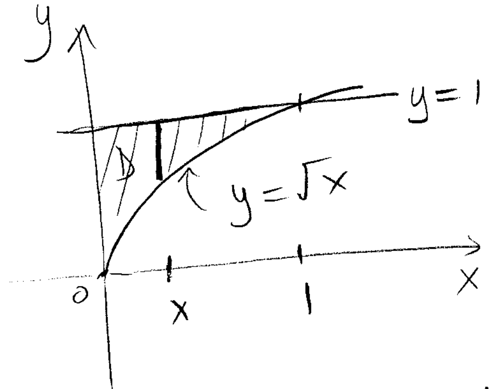
missing!

$\Rightarrow$ cannot integrate $f(x,y) = \cos(y^3)$ with respect to y: SWITCH the order of integration.

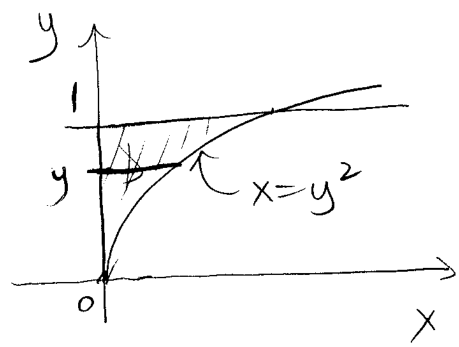
$I = \iint_D \cos(y^3)\,dy\,dx$, where $D = \left\{ \begin{array}{c} 0 \le x \le 1 \\ \sqrt{x} \le y \le 1 \end{array} \right\}$

$y = \sqrt{x}$ $\quad$ $y = 1$

Type I



Now, describe as type II:



$D = \left\{ \begin{array}{c} 0 \le y \le 1 \\ 0 \le x \le y^2 \end{array} \right\}$

$$\Rightarrow \quad I = \int_0^1 \left(\int_0^{y^2} \cos(y^3)\, dx \right) dy$$

$$= \int_0^1 \left(\cos(y^3)\, x \Big|_{x=0}^{x=y^2} \right) dy$$

$$= \int_0^1 \cos(y^3) \cdot y^2\, dy$$

$u = y^3$

$du = 3y^2 dy$

y	u
1	1
0	0

$$= \int_0^1 \cos u \cdot \frac{du}{3} = \frac{\sin u}{3} \Big|_0^1 = \frac{\sin 1}{3}$$