LAGRANGE MULTIPLIER METHOD.

(2-variable case)

Suppose that $f(x,y)$ and $g(x,y)$ are differentiable functions and that the level curve $C: g(x,y) = k$ of g is <u>bounded</u>. Then, if (a,b) is a max./min. of $f(x,y)$ subject to the constraint $g(x,y) = k$, it must satisfy one of the following:

(1) (a,b) is a solution of the system:

$$\begin{cases} \nabla f(a,b) = \lambda \nabla g(a,b) \\ g(a,b) = k \end{cases}$$

(2) $\nabla g(a,b) = (0,0)$

OR

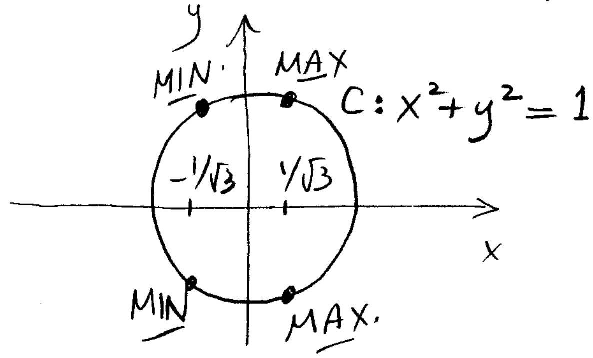
(3) (a,b) is an end-points of $C: g(x,y) = k$.

NOTES: (i) λ = Lagrange multiplier

(ii) Since f is continuous on C (because it is differentiable) and C is bounded, by the Extreme Value Theorem, f HAS a max. and min. on C, which must occur at one of the points you found in (1), (2), or (3).

$\Rightarrow$ if C is bounded, f ALWAYS has a MAX. and MIN. on C.

Ex. 1) Find max./min. of $f(x,y) = xy^2$ subject to $x^2+y^2=1$.



HERE: the constraint curve is $C: g(x,y)=1$ with $g(x,y)= x^2+y^2$.

NOTE that C is bounded.

$\rightarrow$ $\nabla g = (2x, 2y) = (0,0) \Leftrightarrow (x,y) = (0,0) \notin C$.

Since $\nabla g \neq (0,0)$ on C, (2) does NOT apply.

$\rightarrow$ C has NO boundary points $\Rightarrow$ (3) does NOT apply.

$\rightarrow$ Since $\nabla f = (y^2, 2xy)$, the system in (1) is:

$$\left\{ \begin{array}{c} \nabla f = \lambda \nabla g \\ g = 1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} y^2 = \lambda(2x) \\ 2xy = \lambda(2y) \\ x^2+y^2=1 \end{array} \right\}$$

Rewriting the 2^{nd} equ. $\Leftrightarrow$

$$\left\{ \begin{array}{ll} y^2 = \lambda(2x) & (*) \\ y(x-\lambda) = 0 & \rightarrow y=0 \text{ OR } x=\lambda \\ x^2+y^2=1 & \end{array} \right.$$

$y = 0$: $x^2 = 1 \Leftrightarrow x = \pm 1$

$\Rightarrow \boxed{(\pm 1, 0)}$, which does satisfy the 1^{st} equation $(*)$ with $\lambda = 0$.

OR $x = \lambda$ $\to$ replacing in equation $(*)$, we get:

$$y^2 = \lambda(2x) = 2x^2$$

$\rightsquigarrow$ substituting into 3^{rd} equation:

$$x^2 + 2x^2 = 1$$

$\Longrightarrow$ $x^2 = 1/3$ and $y^2 = 2/3$

$\Longrightarrow$ get 4 points:

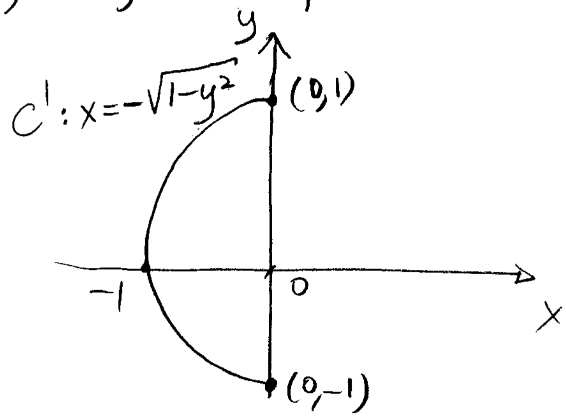
$$\boxed{\left(1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3}\right) \text{ and } \left(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3}\right)}$$

Since C is bounded, the max./min. of f occurs at one of the 6 points found in steps (1) to (3): $(\pm 1, 0)$, $(1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$, $(\pm 1/\sqrt{3}, -\sqrt{2}/\sqrt{3})$.
Comparing values of f at those points:

(x,y)	$(\pm 1, 0)$	$(1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$	$(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$
f	0	$\frac{2}{3\sqrt{3}}$	$-\frac{2}{3\sqrt{3}}$
		$\uparrow$ MAX.	$\uparrow$ MIN.

So: f has a MAX. of $\frac{2}{3\sqrt{3}}$ at $(1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$
AND a MIN. of $-\frac{2}{3\sqrt{3}}$ at $(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$.

2) Find max./min. of $f(x,y) = xy^2$ subject to $x = -\sqrt{1-y^2}$.



HERE: the constraint curve is $C': x = -\sqrt{1-y^2}$, which is the half-circle

$$x^2+y^2=1, -1 \le x \le 0.$$

Set $C': g(x,y) = 1, \ 0 \le x \le 1$, with $g(x,y) = x^2+y^2$.

NOTE that C' is again <u>bounded</u>, so that f will have both a max. and min. on C'.

$\to$ $\nabla g = (2x, 2y) = (0,0) \Leftrightarrow (x,y) = (0,0) \notin C'$

Since $\nabla g \neq (0,0)$ on C', (2) does NOT apply.

$\to$ C' has boundary 2 boundary points: $\boxed{(0,\pm 1)}$

$\to$ Since $\nabla f = (y^2, 2xy)$, the system in (1) is:

$$\left\{ \begin{array}{l} y^2 = \lambda(2x) \\ 2xy = \lambda(2y), \ -1 \le x \le 0 \\ x^2+y^2=1 \end{array} \right\} \Rightarrow$$

This is the same system as in example 1, except that we now have to restrict ourselves to points with $-1 \le x \le 0$.

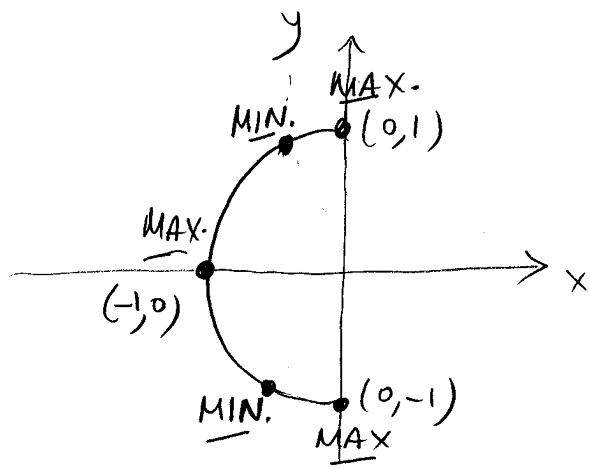
$\Rightarrow$ get 3 points: $\boxed{(-1,0), \left(-1/\sqrt{3}, \pm \sqrt{2}/\sqrt{3}\right)}$

Comparing values at the 5 points found in steps (1) to (3): $(0,\pm 1)$, $(-1,0)$, $\left(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3}\right)$, we get:

(x,y)	$(0,\pm 1)$	$(-1,0)$	$(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$
f	0	0	$-2/3\sqrt{3}$
	MAX.	MAX.	MIN.

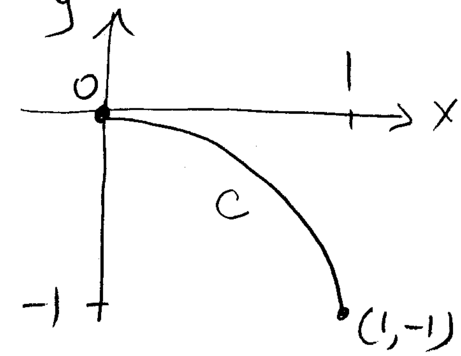
$\Rightarrow$ * The MAX. of f on C' is 0 and occurs at the 2 boundary pts $(0, \pm 1)$ and at $(-1,0)$

* The MIN. of f on C' is $-2/3\sqrt{3}$ and occurs at $(-1/\sqrt{3}, \pm\sqrt{2}/\sqrt{3})$.



NOTE: We see that truncating the circle $x^2+y^2=1$ really changed the problem. In particular, the maximum value of f occurs at different points on the half-circle than it does on the full circle. Moreover, for the half-circle $x=-\sqrt{1-y^2}$, the maximum occurs at the endpts, showing that it is IMPORTANT to consider endpoints

3) Find max./min. of $f(x,y) = x+y$ subject to $y^2 = x^3$, $-1 \le y \le 0$.



HERE: the constraint curve is C: $g(x,y) = 0$, $-1 \le y \le 0$, with $g(x,y) = x^3 - y^2$.

NOTE that C is bounded, so that f will have a max. and min. value on C, and has boundary points $(0,0)$ and $(1,-1)$.

$\rightarrow$ $\nabla g = (3x^2, -2y) = (0,0) \iff (x,y) = (0,0) \in C$

$\Rightarrow$ need to consider $\boxed{(0,0)}$.

$\rightarrow$ C has 2 boundary points: $\boxed{(0,0) \text{ and } (1,-1)}$

$\rightarrow$ Since $\nabla f = (1,1)$, the system (1) becomes:

$$\left\{\begin{array}{l} \nabla f = \lambda \nabla g \\ g = 0,\ -1 \le y \le 0 \end{array}\right\} \iff \left\{\begin{array}{ll} 1 = \lambda(3x^2) & (*) \\ 1 = \lambda(-2y) & (**) \\ y^2 = x^3, & -1 \le y \le 0 \end{array}\right\}$$

NOTE that the first 2 equations $(*)$ & $(**)$ tell us that $x, y, \lambda \neq 0$. Also,

$$\lambda(3x^2) \underset{(*)}{=} 1 \underset{(**)}{=} \lambda(-2y) \iff y = -\tfrac{3}{2}x^2$$

(since $\lambda \neq 0$)

Substituting $y = -\frac{3}{2}x^2$ into the 3rd equation, we get: $\left(-\frac{3}{2}x^2\right)^2 = x^3$, $x \neq 0 \iff \frac{9}{4}x^4 = x^3$, $x \neq 0$

$$\iff x = \frac{4}{9} \Rightarrow x = \frac{4}{9} \text{ and } y = -\frac{3}{2}\left(\frac{4}{9}\right)^2 = \frac{-8}{27}$$

$$\Rightarrow (x,y) = (4/9, -8/27).$$

So, the system has one solution $\boxed{\left(\frac{4}{9}, -\frac{8}{27}\right)}$.

Comparing values at the 3 points found in steps (1) to (3): $(0,0)$, $(1,-1)$, $\left(\frac{4}{9}, -\frac{8}{27}\right)$, we get:

(x,y)	$(0,0)$	$(1,-1)$	$(4/9, -8/27)$
f	0	0	$4/27$
	MIN.	MIN.	MAX.

$\Longrightarrow$ * The MIN. is 0 and occurs at the endpts, one of which is such that $\nabla g(0,0) = 0$.

* The MAX. is $4/27$ and occurs at $(4/9, -8/27)$.

NOTE: (i) Max./min. can occur at a point (a,b) for which $\nabla g(a,b) = (0,0)$.

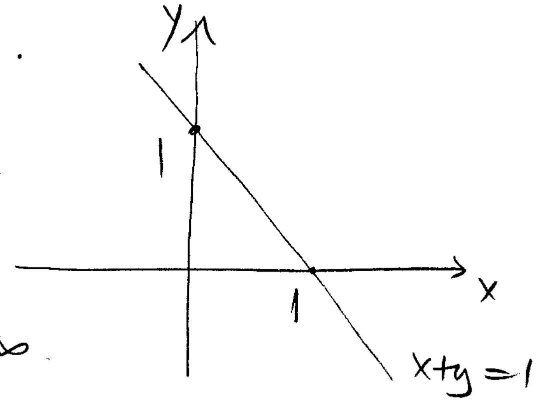
$\Longrightarrow$ need to consider such points!

(ii) When solving system (1), keep track of whether x, y or λ are $\neq 0$, if x and y need to be >0 or <0 $\Longrightarrow$ this will REALLY simplify your solution.

REMARK: What happens if the constraint curve is NOT bounded?

Then, f may not have a max. or min. on the curve.

E.g. 1) $f(x,y) = y$ does not have a max. or min. on the line $x+y=1$.

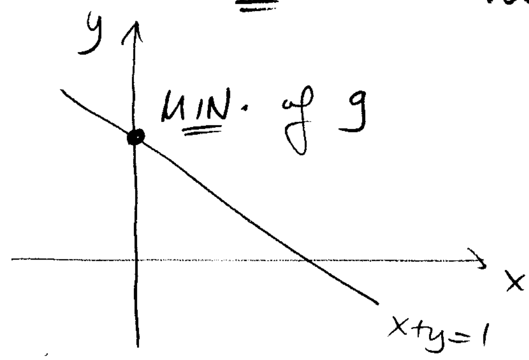


Indeed, the y-coordinate of points on $x+y=1$ can get values that go to $+\infty$ and $-\infty$

$\Rightarrow$ the value of $f(x,y)=y$ will go to $+\infty$ as $y \to +\infty$ and $-\infty$ as $y \to -\infty$

$\Rightarrow$ f has NO max. or min. on $x+y=1$.

2) $g(x,y) = x^2$ has a MIN. but no MAX. on the line $x+y=1$.



This time, $\forall (x,y)$,

$g(x,y) = x^2 \geq 0$

and $g(x,y) = 0 \Leftrightarrow x = 0$.

Now, there is a unique point on $x+y=1$ with $x=0$: $(0,1)$ $\Rightarrow$ g reaches the minimum value 0 at $(0,1)$.

BUT, since x can take values that go to $\pm\infty$ on $x+y=1$, g will go to $+\infty$ as $x \to \pm\infty$

$\Rightarrow$ g has NO max. on $x+y=1$.

So, if the constraint curve $C: g(x,y)=k$ is unbounded, need a fourth step in the Lagrange multiplier method:

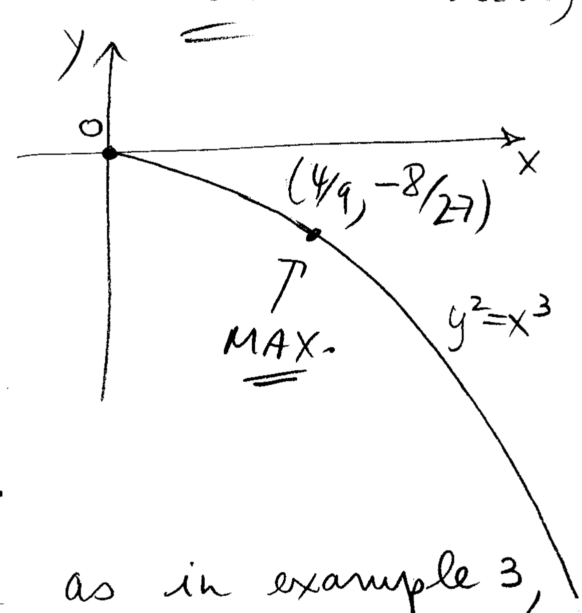
(4) Check $\lim\limits_{\substack{x\to\pm\infty \\ (x,y)\in C}} f(x,y)$ and $\lim\limits_{\substack{y\to\pm\infty \\ (x,y)\in C}} f(x,y)$,

where the limits are taken for point on C where $x\to\pm\infty$ or $y\to\pm\infty$.

IF in (4) you get $+\infty$ AND $-\infty$, then f has no max. or min.. OTHERWISE, f may have max. or min., so compare results of (4) with values of f at points satisfy (1) to (3).

E.g.: $f(x,y) = x+y$ subject to $y^2=x^3$, $y\le 0$.

HERE, the constraint curve is NOT bounded, since y can take values that go to $-\infty$. On $y^2=x^3$, $x=y^{2/3}$, so that



$$\lim_{\substack{y\to-\infty \\ y^2=x^3}} f(x,y) = \lim_{y\to-\infty}\left(y^{2/3}+y\right) = -\infty.$$

$\Rightarrow$ f has no minimum.

HOWEVER, by analysing f as in example 3, the Lagrange multiplier method tells us that f has a max. of $4/27$ at $(4/9, -8/27)$ on $y^2=x^3$, $y\le 0$.

LAGRANGE MULTIPLIER METHOD

(3-variable case)

Suppose that $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions and that the level surface $S: g(x,y,z)=k$ of g is bounded.

THEN, f has a max. and min. on S, and IF (a,b,c) is a max./min. of f on S, it must satisfy one of the following:

(1) (a,b,c) is a solution of: $\begin{cases} \nabla f(a,b,c) = \lambda \nabla g(a,b,c) \\ g(a,b,c) = k \end{cases}$

(2) $\nabla g(a,b,c) = 0$

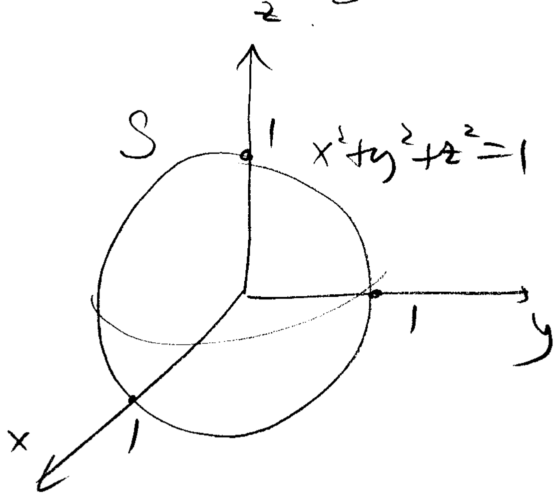
OR

(3) (a,b,c) is a boundary point of S.

BUT, if S is not bounded, you ALSO need to consider the behaviour of f for values of x, y, z that approach $\pm\infty$.

E.g. 1) Find max./min. of $f(x,y,z) = 2x - y + z$ subject to $x^2+y^2+z^2=1$.

HERE, the constraint is the sphere
$$S : g(x,y,z) := x^2+y^2+z^2 = 1,$$
which is bounded



$\Rightarrow$ f has a max. and min. on S.

$\to$ S has NO boundary points: (3) does not hold.

$\to$ $\nabla g = (2x, 2y, 2z) = (0,0,0) \Leftrightarrow (0,0,0) \notin S$
$\Rightarrow$ (2) does not hold.

$\to$ Since $\nabla f = (2,-1,1)$, the system in (1) is:

$$\left\{ \begin{array}{l} \nabla f = \lambda \nabla g \\ g = 1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 2 = \lambda(2x) \\ -1 = \lambda(2y) \\ 1 = \lambda(2z) \\ x^2+y^2+z^2=1 \end{array} \right\} \Rightarrow x, y, z, \lambda \neq 0;$$

So that
$$\left\{ \begin{array}{l} 2 = 2x\lambda \\ -1 = 2y\lambda \\ 1 = \lambda(2z) \end{array} \right. \Rightarrow \lambda = 1/x = \frac{-1}{2y} = \frac{1}{2z}$$
$$\Leftrightarrow x = 2z \text{ and } y = -z$$

$\leadsto$ substituting into $x^2+y^2+z^2=1$, we get
$$(2z)^2 + (-z)^2 + z^2 = 1 \Leftrightarrow z^2 = 1/6 \Leftrightarrow z = \pm 1/\sqrt{6}.$$

THUS: $z = \pm 1/\sqrt{6}$ and $x = \pm 2/\sqrt{6}$ and $y = -(\pm 1/\sqrt{6})$

$\Rightarrow$ get 2 points: $(2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6})$ and
$(-2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6})$.

Comparing the values of f at the 2 points $(2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6})$ and $(-2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6})$ founds in steps (1) to (3), we get.

(x,y,z)	$(2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6})$	$(-2/\sqrt{6}, 1/\sqrt{6}, -1/\sqrt{6})$
f	$6/\sqrt{6}$ ↑ MAX.	$-6/\sqrt{6}$ ↑ MIN.

2) Find the shortest distance from the origin to the surface $xyz^2 = 2$.

If (x,y,z) is a point on $xyz^2 = 2$, then the distance from the origin to (x,y,z) is

$$d = \sqrt{x^2+y^2+z^2}.$$

We are therefore minimizing d subject to the constraint $g(x,y,z) := xyz^2 = 2$.

* Since $\sqrt{\cdot}$ is an increasing function, this is the same as minimizing

$$f(x,y,z) = x^2+y^2+z^2$$

subject to $g(x,y,z) = 2$.

ↄ the min. of d will then be the square root of the minimum of f.

* $S: xyz^2 = 2$ is not bounded since it contains unbounded curves.

E.g. if $z=1$, $xy=2 \Rightarrow S$ contains a hyperbola.

$\Rightarrow f$ may not have a max. or min. on S.

Indeed, for large values of x, y, z, $f = x^2+y^2+z^2$ will go to $+\infty$.

NONETHELESS, since $f = x^2+y^2+z^2 \geq 0$, $\forall (x,y,z)$, the function f is bounded below and we have a minimum on S.

Let's apply the Lagrange multiplier method to $f(x,y,z) = x^2+y^2+z^2$ subject to $xyz^2 = 2$:

→ S has no boundary (since x, y, z can take values that go to $\pm\infty$).

$\Rightarrow$ (3) do <u>not</u> apply.

→ $\nabla g = (yz^2, xz^2, 2xyz) = (0,0,0) \Leftrightarrow (x,y,z) = (0,0,0) \notin S$

$\Rightarrow$ (2) does <u>not</u> apply.

→ Since $\nabla f = (2x, 2y, 2z)$, we need to solve:

$$\begin{cases} 2x = \lambda(yz^2) \\ 2y = \lambda(xz^2) \\ 2z = \lambda(2xyz) \\ xyz^2 = 2 \longrightarrow x,y,z \neq 0 \end{cases} \Rightarrow \lambda \neq 0.$$

<u>So</u>: we may assume that $x, y, z, \lambda \neq 0$.

$x, y, z, \lambda \neq 0$ and

$$\begin{cases} 2x = \lambda(yz^2) & (*) \\ 2y = \lambda(xz^2) & (**) \\ 2z = \lambda(2xyz) \Rightarrow 2xy\lambda = 2 \text{ (since } z \neq 0) \Leftrightarrow \lambda = \dfrac{1}{xy} \\ xyz^2 = 2 \end{cases}$$

(since $x, y \neq 0$)

Substituting $\lambda = \frac{1}{xy}$ in $(*)$ and $(**)$, we get:

$$2x = \frac{1}{x\cancel{y}}(\cancel{y}z^2) \quad \text{and} \quad 2y = \frac{1}{\cancel{x}y}(\cancel{x}z^2)$$

$$\Leftrightarrow \quad z^2 = 2x^2 \quad \text{and} \quad z^2 = 2y^2$$

$$\Leftrightarrow \quad z^2 = 2x^2 \quad \text{and} \quad x^2 = y^2.$$

<u>NOW</u>: $x^2 = y^2 \Leftrightarrow y = \pm x$.

BUT, since $xy = 2/z^2 > 0$, x and y must have the same sign.

$$\Rightarrow \boxed{y = x} \quad \text{and} \quad \boxed{z^2 = 2x^2}$$

Substituting into the last equation, we get: $x(x)(2x^2) = 2 \Leftrightarrow x^6 = 1$

$\Leftrightarrow x = \pm 1$

$\Leftrightarrow y = x = \pm 1$ and $z^2 = 2$

$\Rightarrow$ get 4 points: $\boxed{(1, 1, \sqrt{2}), (1, 1, -\sqrt{2}), (-1, -1, \sqrt{2}), (-1, -1, -\sqrt{2})}$

Compare the value of f at these four points:

$$f(\pm 1, \pm 1, \sqrt{2}) = f(\pm 1, \pm 1, -\sqrt{2}) = 1 + 1 + 2 = 4.$$

$\rightsquigarrow$ So, these four points all give the value 4.

NOTE: if we take any other point on S, say $(2, 1, 1)$,

$$f(2, 1, 1) = 2^2 + 1 + 1 = 6 > 4.$$

$\Longrightarrow$ 4 is indeed the minimum value of f on S.

$\Longrightarrow$ the minimum distance d from the origin to S is therefore

$$d = \sqrt{4} = 2,$$

and it occurs at the points $(\pm 1, \pm 1, \sqrt{2})$ and $(\pm 1, \pm 1, -\sqrt{2})$.