MATH 237 – COMPUTING LIMITS.

A) To show that a limit DNE: we know that if a limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to L, then if one approaches (a,b) along ANY path, the limit EXISTS and is EQUAL to L.

SO, if: * $\exists$ path along which limit DNE

OR

* $\exists$ two $\neq$ paths giving $\neq$ limits,

then: $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE.

E.g: 1) $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$ DNE since, along the path $y=0$:

$$\lim_{\substack{y=0\\x\to 0}} \frac{x}{x^2+0} = \lim_{x\to 0} \frac{1}{x} \text{ DNE.}$$

2) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE since:

* along $x=0$: $\lim_{\substack{x=0\\y\to 0}} \frac{0}{0+y^2} = \lim_{y\to 0} 0 = 0.$

* along $y=0$: $\lim_{\substack{x\to 0\\y=0}} \frac{x^2}{x^2+0} = \lim_{x\to 0} 1 = 1 \neq 0.$

Thus, since the paths $x=0$ and $y=0$ lead to different limits, $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ DNE.

B) TO SHOW THAT A LIMIT EXISTS: Use one of the limit theorems or the Squeeze Theorem.

LIMIT THEOREMS Let f and g be two functions of two variables and assume that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L_1 \quad \underline{\underline{\text{and}}} \quad \lim_{(x,y)\to(a,b)} g(x,y) = L_2$$

exist (!). Then,

(1) $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = L_1 \pm L_2$

(2) $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = L_1 L_2$

(3) $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L_1}{L_2}$, if $L_2 \neq 0$.

(4) If $H(t)$ is continuous at $t = L_1$, then

$$\lim_{(x,y)\to(a,b)} H(f(x,y)) = H(L_1).$$

SQUEEZE THEOREM Let f be a function of two variables. Suppose that there exists a non-negative function $B(x,y)$ such that

$$0 \leq |f(x,y) - L| \leq B(x,y)$$

for ALL (x,y) near (a,b), for some $L \in \mathbb{R}$, <u>AND</u>

$$\lim_{(x,y)\to(a,b)} B(x,y) = 0,$$

THEN, $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L.$

Ex. 1) $\lim_{(x,y)\to(1,1)} 2x^2 + y = 3$ since:

$$\lim_{(x,y)\to(1,1)} x = 1 \quad \text{and} \quad \lim_{(x,y)\to(1,1)} y = 1,$$

so by the limit theorems,

$$\lim_{(x,y)\to(1,1)} 2x^2+y = 2\left(\lim_{(x,y)\to(1,1)} x\right)\left(\lim_{(x,y)\to(1,1)} x\right) + \lim_{(x,y)\to(1,1)} y$$
$$= 2\cdot 1\cdot 1 + 1 = 3.$$

2) Similarly, since $\lim_{(x,y)\to(2,1)} x = 2$ and $\lim_{(x,y)\to(2,1)} y = 1$,

$$\lim_{(x,y)\to(2,1)} x^3 y = \left(\lim_{(x,y)\to(2,1)} x\right)^3 \left(\lim_{(x,y)\to(2,1)} y\right) = 2^3\cdot 1 = 8.$$

IN GENERAL, if $p(x,y) = \begin{pmatrix}\text{polynomial in} \\ x \text{ \& } y\end{pmatrix}$,

$$\lim_{(x,y)\to(a,b)} p(x,y) = p(a,b).$$

3) Since $H(t) = \sin(t)$ is continuous at $\frac{\pi}{2}$ and $\lim_{(x,y)\to(\pi,-1)} (x/2) = \pi/2$, $\lim_{(x,y)\to(\pi,-1)} \sin\left(\frac{x}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

AND

$$\lim_{(x,y)\to(\pi,-1)} 3y^2\sin\left(\frac{x}{2}\right) = 3(-1)^2\cdot\sin\left(\frac{\pi}{2}\right) = 3.$$

4) $\lim_{(x,y)\to(1,0)} \sqrt{9-x^2+3y^4} = \sqrt{9-1+0} = \sqrt{8} = 2\sqrt{2}$, since $H(t) = \sqrt{t}$ is continuous at $t=8$.

Situation gets more complicated with quotients:

5) $$\lim_{(x,y)\to(-1,0)} \frac{\overset{\to 1}{3y^2 - x}}{\underset{\to -2 \neq 0}{2x+y}} = \frac{1}{-2} = -\frac{1}{2} \text{ exists.}$$

6) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}} = 0.$

HERE, we can't use the limit theorems:

$$\lim_{(x,y)\to(0,0)} \frac{\overset{\to 0}{y^2}}{\underset{\to 0}{\sqrt{x^2+y^2}}} \text{ "}= \frac{0}{0}\text{"} \rightsquigarrow \text{INDETERMINATE}$$

$\Rightarrow$ use Squeeze Thm, with $f(x,y) = \dfrac{y^2}{\sqrt{x^2+y^2}}$ and $L = 0$.

NEED to find $B(x,y) \geq 0$ such that

* $|f(x,y) - L| \leq B(x,y)$ near $(0,0)$

and

* $\displaystyle\lim_{(x,y)\to(0,0)} B(x,y) = 0.$

HERE,

$$|f(x,y) - L| = \left| \frac{y^2}{\sqrt{x^2+y^2}} - 0 \right| = \frac{y^2}{\sqrt{x^2+y^2}}$$

since $y^2 \leq x^2+y^2$

$$\leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = B(x,y) \to 0 \text{ as } (x,y)\to(0,0)$$

$\Rightarrow$ By Squeeze Thm, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}} = 0.$

c) COMPUTING LIMITS: $\lim_{(x,y)\to(a,b)} f(x,y)$.

In general, I suggest the following approach to computing limits:

(i) Try using the LIMIT THEOREMS.

If the limit theorems fail, SIMPLIFY the expression of $f(x,y)$ (if necessary) AND

(ii) Try picking a few paths passing through (a,b) to test whether the limit exists.

E.g. If $(a,b)=(0,0)$, try $x=0$, $y=0$, $y=x$, $y=x^2$, $x=y^2$, $y^m=x^n$ (OR possibly $y^m=\alpha x^n$).

BUT, remember to KEEP IT SIMPLE!

→ If you find a path along which the limit DNE or if you find 2 paths that give ≠ limits, then $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE.

→ If you always get limit L, use SQUEEZE THM to show that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

→ If you can't make SQUEEZE THM work, then maybe the limit DNE after all

⟹ try more paths.

ETC....

Ex. 1) $\lim_{(x,y)\to(3,-1)} \dfrac{3y^2 - x}{2x+1} = \dfrac{0}{7} = 0.$

$3y^2 - x \to 0$, $2x+1 \to 7 \neq 0$

2) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^4}{x^2+y^4}$ " $= \dfrac{0}{0}$ " $\leadsto$ indeterminate

↳ can't use limit theorems.

→ pick paths.

$x^2-y^4 \to 0$, $x^2+y^4 \to$

* $x=0$: $\lim_{\substack{x=0\\ y\to 0}} \dfrac{x^2-y^4}{x^2+y^4} = \lim_{y\to 0} \dfrac{-y^4}{y^4} = -1$

* $y=x$: $\lim_{\substack{y=x\\ x\to 0}} \dfrac{x^2-x^4}{x^2+x^4} = \lim_{x\to 0} \dfrac{1-x^2}{1+x^2} = \dfrac{1}{1} = 1 \neq -1.$

$1-x^2 \to 1$, $1+x^2 \to 1$

$\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{x^2-y^4}{x^2+y^4}$ DNE.

3) $\lim_{(x,y)\to(0,0)} \left(\dfrac{4x^2 - y^2(x-2)}{2x^2+y^2} \right) \leadsto \dfrac{0}{0}$: can't use limit thms.

$4x^2 - y^2(x-2) \to 0$, $2x^2+y^2 \to 0$

→ Simplify the expression of $f(x,y)$:

$$\frac{4x^2 - y^2(x-2)}{2x^2+y^2} = 2 - \frac{y^2 x}{2x^2+y^2}$$

→ Check a few paths: here $x=0$, $y=0$, $y=x$, and $y=x^2$, $x=y^2$ give $L=2$ ⇒ use SQUEEZE THM to try to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 2 = L$.

→ $$|f(x,y) - L| = \left| \left(2 - \frac{y^2 x}{2x^2+y^2}\right) - 2 \right| = \frac{y^2|x|}{2x^2+y^2}$$

$$\leq \frac{\cancel{(y^2+2x^2)}|x|}{\cancel{2x^2+y^2}} = |x| = B(x,y) \longrightarrow 0 \text{ as } (x,y)\to(0,0)$$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 2$ by SQUEEZE THM.

D) FINDING $B(x,y)$ when using the SQUEEZE THM.

Suppose that
$$\left(f(x,y) - L\right) = \frac{p(x,y)}{q(x,y)},$$
where p and q are functions that both converge to 0 as $(x,y) \to (a,b)$.

To find $B(x,y)$, try finding an upper bound of $|p(x,y)|$ that's a multiple of $|q(x,y)|$: if
$$|p(x,y)| \leq |q(x,y)| \cdot B(x,y)$$
for all (x,y) near (a,b), then
$$|f(x,y) - L| = \left|\frac{p(x,y)}{q(x,y)}\right| \leq \frac{\cancel{|q(x,y)|} \cdot B(x,y)}{\cancel{|q(x,y)|}} = B(x,y)$$
near (a,b).

BUT, make sure not to over-estimate the upper-bound of $|p(x,y)|$ because you NEED $B(x,y) \to 0$ as $(x,y) \to (a,b)$.

E.g.: If $f(x,y) = \dfrac{x^2y - 2|x| - |y|}{2|x| + |y|}$, $L = -1$, $(a,b) = (0,0)$, then
$$|f(x,y) - L| = \left|\left(\frac{x^2y - 2|x| - |y|}{2|x| + |y|}\right) - (-1)\right|$$
$$= \frac{x^2|y|}{2|x| + |y|} \leq x^2\frac{\cancel{(2|x| + |y|)}}{\cancel{2|x| + |y|}} \quad \left(\begin{array}{l}\text{since} \\ |y| \leq 2|x| + |y|\end{array}\right)$$
$$= x^2 = B(x,y) \to 0 \text{ as } (x,y) \to (0,0).$$

Suppose that $(a,b)=(0,0)$ and

$$\frac{f(x,y)}{g(x,y)} = \frac{|x|^{\alpha}|y|^{\beta}}{|x|^{c}+|y|^{d}}, \quad \alpha,\beta,c,d \geq 0.$$

* If $\alpha \geq c$, try:

$$\frac{|x|^{\alpha}|y|^{\beta}}{|x|^{c}+|y|^{d}} = \frac{|x|^{\alpha-c}|x|^{c}|y|^{\beta}}{|x|^{c}+|y|^{d}}$$

$$\leq \frac{|x|^{\alpha-c}\cancel{(|x|^{c}+|y|^{d})}|y|^{\beta}}{\cancel{|x|^{c}+|y|^{d}}}$$

$$= |x|^{\alpha-c}|y|^{\beta} \longrightarrow 0, \text{ as } (x,y)\to(0,0),$$

if $\alpha - c > 0$ or $\beta > 0$
(since $\alpha - c, \beta \geq 0$).

e.g. $$\frac{x^2|y|^3}{|x|+2y^2} = \frac{|x|\cdot|x|\cdot|y|^3}{|x|+2y^2} \leq \frac{\cancel{(|x|+2y^2)}|x|\cdot|y|^3}{\cancel{|x|+2y^2}} = |x|\cdot|y|^3 \longrightarrow 0$$
as $(x,y)\to(0,0)$.

* If $\beta \geq d$, try:

$$\frac{|x|^{\alpha}|y|^{\beta}}{|x|^{c}+|y|^{d}} = \frac{|x|^{\alpha}|y|^{\beta-d}|y|^{d}}{|x|^{c}+|y|^{d}} \leq |x|^{\alpha}|y|^{\beta-d}\left(\cancel{\frac{|x|^{c}+|y|^{d}}{|x|^{c}+|y|^{d}}}\right)$$

$$= |x|^{\alpha}|y|^{\beta-d} \longrightarrow 0 \text{ as } (x,y)\to(0,0)$$

if $\alpha > 0$ or $\beta - d > 0$
(since $\alpha, \beta - d \geq 0$).

e.g. $$\frac{|x||y|^3}{x^2+2y^2} = \frac{1}{2}\left(\frac{|x||y|\,2y^2}{x^2+2y^2}\right) \leq \frac{1}{2}|x||y|\,\cancel{\frac{(x^2+2y^2)}{(x^2+2y^2)}}$$

$$= \frac{1}{2}|x||y| \longrightarrow 0 \text{ as } (x,y)\to(0,0)$$

* If $\alpha < c$ and $\beta < d$, try:

$$\frac{|x|^{\alpha}|y|^{\beta}}{|x|^c+|y|^d} = \frac{\left(|x|^c\right)^{\alpha/c}\left(|y|^d\right)^{\beta/d}}{|x|^c+|y|^d}$$

$$\leq \frac{\left(|x|^c+|y|^d\right)^{\alpha/c}\left(|x|^c+|y|^d\right)^{\beta/d}}{|x|^c+|y|^d}$$

$$= \left(|x|^c+|y|^d\right)^{\alpha/c+\beta/d-1}$$

etc.....

e.g. $$\frac{x^3|y|}{x^4+y^2} = \frac{\left(x^4\right)^{3/4}\sqrt{y^2}}{x^4+y^2} \leq \frac{\left(x^4+y^2\right)^{3/4}\left(y^2+x^4\right)^{1/2}}{x^4+y^2}$$

$$= (x^4+y^2)^{1/4} \to 0 \text{ as } (x,y)\to(0,0).$$

NOTE: There are sometimes other ways of finding $B(x,y)$, BUT the above suggestions usually work in very FEW steps.

FINALLY, suppose that $p(x,y)$ is a linear combination of powers of x and y.

E.g. $p(x,y) = x^3 + y^3$, $xy - x^{3/2}$, $(x-y)^2$, ...

$\Rightarrow$ use the TRIANGLE INEQUALITY to find an upper bound for $|p(x,y)|$ that's a linear combination of powers of $|x|$ and $|y|$, then proceed as before.

TRIANGLE INEQUALITY: for all $A, B \in \mathbb{R}$,

$$|A \pm B| \leq |A| + |B|.$$

Ex: 1) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^3+y^3}{x^2+y^2}\right) = 0$.

$$\left|\frac{x^3+y^3}{x^2+y^2} - 0\right| = \frac{|x^3+y^3|}{x^2+y^2} \leq \frac{|x^3|+|y^3|}{x^2+y^2}$$

TRIANGLE INEQUALITY for $|x^3+y^3|$

$$= \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2} = \frac{|x|\cdot x^2}{x^2+y^2} + \frac{|y|\cdot y^2}{x^2+y^2}$$

$$\leq |x|\cdot\frac{\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} + |y|\cdot\frac{\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = |x|+|y|$$

$$= B(x,y) \to 0 \text{ as } (x,y)\to(0,0)$$

$\Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$ by Squeeze Theorem.

2) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+2y)^2}{3|x|+6|y|} = 0$.

$$\left|\frac{(x+2y)^2}{3|x|+6|y|}\right| = \frac{|x^2+4xy+4y^2|}{3|x|+6|y|} \leq \frac{x^2+4|x|\cdot|y|+4y^2}{3|x|+6|y|}$$

$$= \frac{\frac{1}{3}|x|\cdot(3|x|) + \frac{4}{6}|x|\cdot(6|y|) + \frac{4}{6}|y|\cdot(6|y|)}{3|x|+6|y|}$$

$$\leq \frac{\frac{1}{3}|x|\cdot\cancel{(3|x|+6|y|)} + \frac{4}{6}|y|\cdot\cancel{(3|x|+6|y|)} + \frac{4}{6}|y|\cdot\cancel{(3|x|+6|y|)}}{\cancel{3|x|+6|y|}}$$

$$= \frac{1}{3}|x| + \frac{4}{6}|y| + \frac{4}{6}|y| = \frac{1}{3}|x| + \frac{4}{3}|y| = B(x,y) \longrightarrow 0$$

as $(x,y)\to(0,0)$.

OR, FASTER:

$$\left|\frac{(x+2y)^2}{3|x|+6|y|}\right| = \frac{|x+2y|^2}{3|x|+6|y|} \leq \frac{(|x|+2|y|)^2}{3|x|+6|y|}$$

$$= \frac{(|x|+2|y|)^{\cancel{2}}}{3\cancel{(|x|+2|y|)}} = \frac{|x|+2|y|}{3}$$

$$= B(x,y) \longrightarrow 0$$

as $(x,y)\to(0,0)$.