Suggestions for solving optimisation problems in $\mathbb{R}^2$ and $\mathbb{R}^3$.

(A) Find the max./min. of $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ over:

* CASE I: A level curve $C: g(x,y) = k$ in $\mathbb{R}^2$:

→ If C can be easily parametrised or if one of variables can easily be expressed in terms of the other on C then use SUBSTITUTION to express the restriction of f to C as a function of one variable $g(t)$, $t \in [a,b]$, AND find max./min. of $g(t)$ on $[a,b]$

OR

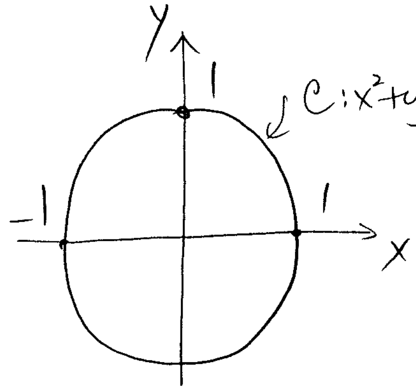
→ Use LAGRANGE MULTIPLIERS.

NOTE: (i) The Method of Lagrange Multipliers will always work, BUT in some cases substitution is a lot simpler.

(ii) C is 1-dimensional, but f is a function of 2 variables.

(iii) If the curve C has endpoints, DON'T forget to check THEM!

E.g.: 1) Find the max./min. of $f(x,y) = xy^2$ on the circle $C: x^2+y^2=1$.



On C, $y^2 = 1-x^2$, $-1 \le x \le 1$. Since the expression of f only involves y^2, we have that

$$f(x,y) = xy^2 = x(1-x^2) = g(x)$$

on C.

$\Rightarrow$ In this case, the easiest thing to do is to find the max./min. of $g(x) = x(1-x^2)$ for $x \in [-1,1]$.

2) Find the max./min. of $f(x,y) = x+y$ on the circle $C: x^2+y^2=1$.

Again, $y^2 = 1-x^2$, $-1 \le x \le 1$, on C. HOWEVER, this time, the expression of f involves y and NOT y^2. So, if one were to use substitution to solve the problem, one would have to consider the two case: $y = \sqrt{1-x^2} \Rightarrow f(x,y) = x + \sqrt{1-x^2} = g_1(x)$

AND

$y = -\sqrt{1-x^2} \Rightarrow f(x,y) = x - \sqrt{1-x^2} = g_2(x)$

(for $x \in [-1,1]$)

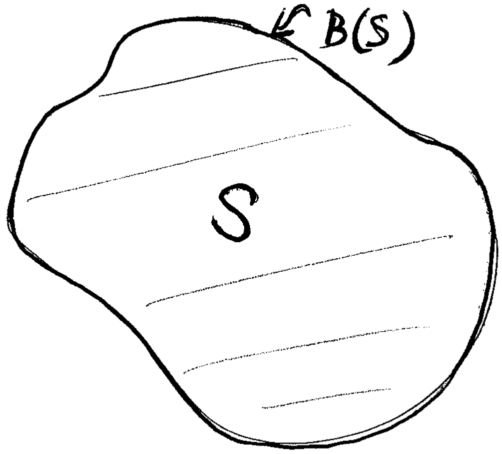
and find max./min. of $g_1(x)$ and $g_2(x)$ on $[-1,1]$

HERE: Faster to use LAGRANGE MULTIPLIERS

3) Find max./min. of $f(x,y) = xy$ on the curve $C: \; xy^2 + x^3 - x + 2y = 0$

In this case, it is difficult to parametrise C or either express x in terms of y or express y in terms of x $\Longrightarrow$ use LAGRANGE.

CASE II: A 2-dimensional subset S of $\mathbb{R}^2$ that is closed and bounded.



1) Find the C.P. of f in S (BUT DON'T test them!).

2) Find max./min. of f on the boundary curve $B(S)$ as in CASE I.

3) Compare the values of f at the points found in 1) and 2).

(B) Find max./min. of $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ on a level surface $g(x,y,z) = k$: USE LAGRANGE.

NOTE: The level surface $g(x,y,z) = k$ is a 2-dimensional subset of $\mathbb{R}^3$ (unlike CASE II in (A), where S is a 2-dim. subset of $\mathbb{R}^2$).