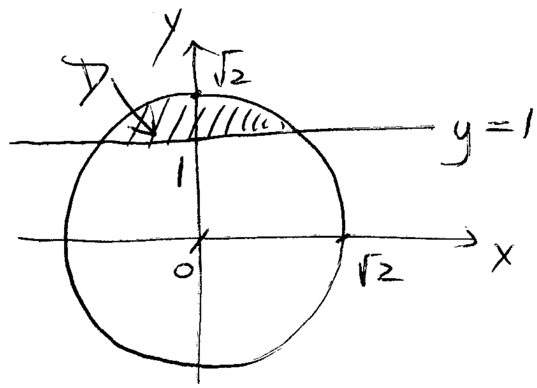
Regions in polar coordinates.

ex. 1) Let D be the region inside $x^2+y^2=2$ above $y=1$.

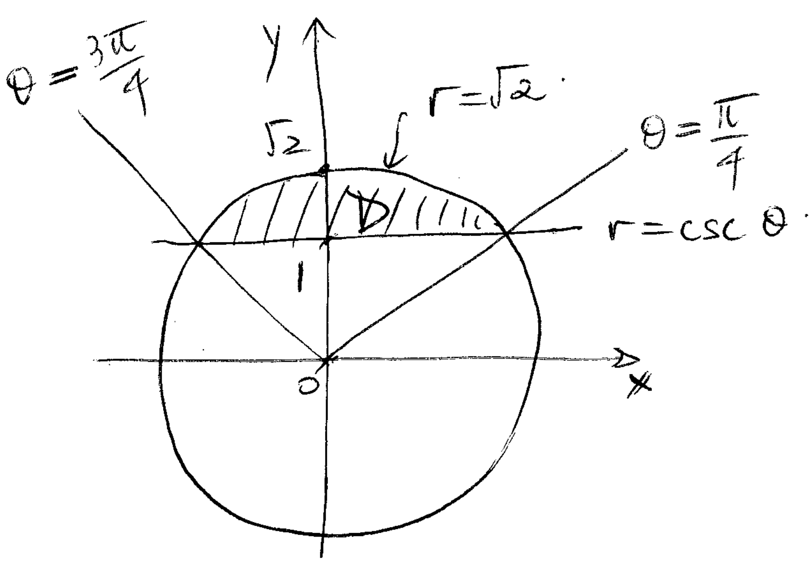


* Convert equations to polar coordinates:

circle: $x^2+y^2=2 \iff r^2=2$
$\iff \boxed{r=\sqrt{2}}$ since $r \geq 0$.

line: $y=1 \iff r\sin\theta=1 \iff \boxed{r=\csc\theta}$

* Bounds for r & θ: take radial strips.



NOTE: The angles θ are bounded above and below by the angle of intersection of the 2 curves

<u>Intersection of the 2 curves:</u>

$r = \sqrt{2}$ and $r = \csc\theta \iff \csc\theta = \sqrt{2}$

$\iff \sin\theta = 1/\sqrt{2}$

$\Rightarrow \boxed{\theta = \frac{\pi}{4}, \frac{3\pi}{4}}$

$\Rightarrow \boxed{\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}}$ in D

Also, $\forall$ fixed $\frac{\pi}{4} \le \theta_0 \le \frac{3\pi}{4}$,

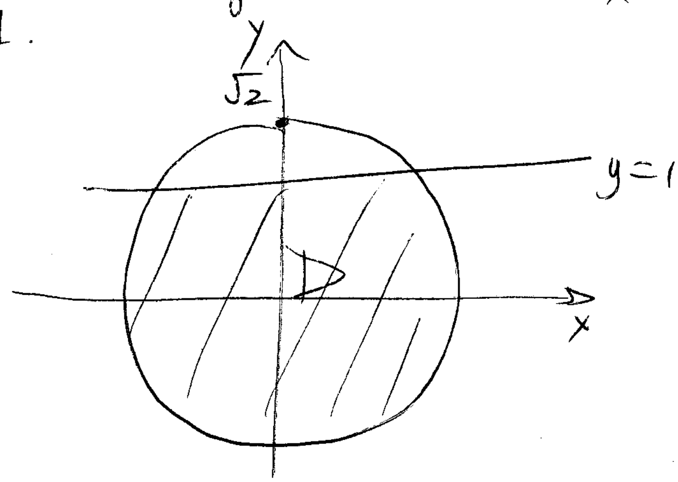
line $\le r \le$ circle

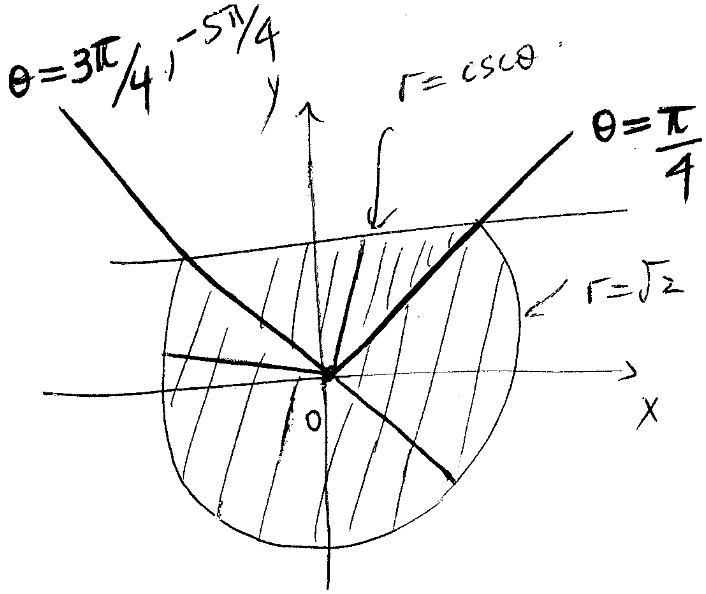
$\iff \boxed{\csc\theta \le r \le \sqrt{2}.}$

Thus,

$$D = \left\{ \begin{array}{l} \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \\ \csc\theta \le r \le \sqrt{2} \end{array} \right\}.$$

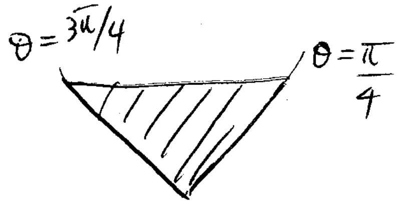
2) Let D be the region inside $x^2 + y^2 = 2$, <u>below</u> $y = 1$.

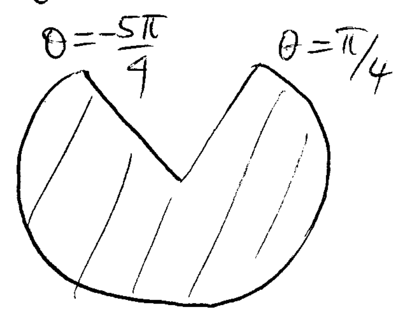




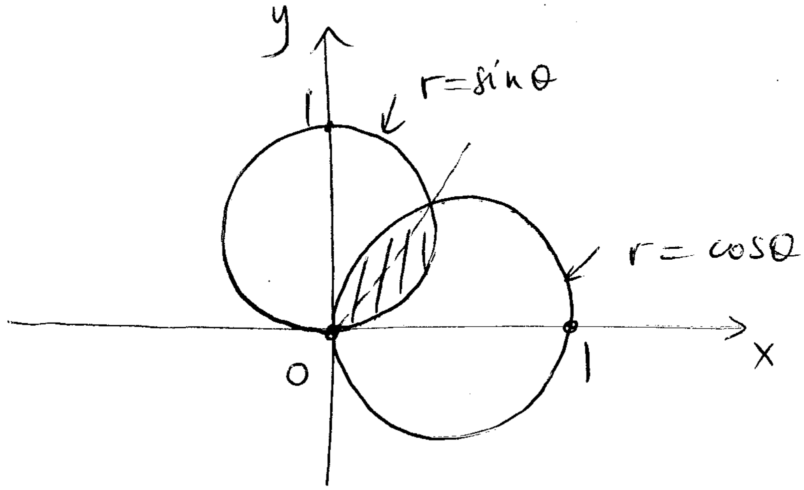
This time, $0 \le \theta \le 2\pi$ in D.
BUT, the upper bound for r changes:

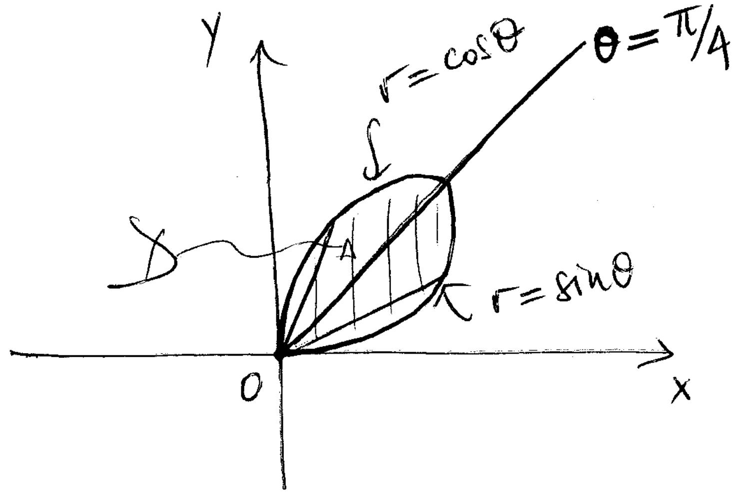
$$D = \left\{ \begin{array}{c} \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \\ 0 \le r \le \csc\theta \end{array} \right\} \cup \left\{ \begin{array}{c} -\frac{5\pi}{4} \le \theta \le \frac{\pi}{4} \\ 0 \le r \le \sqrt{2} \end{array} \right\}$$





3) Let D be the region inside both circles $r = \cos\theta$ and $r = \sin\theta$.





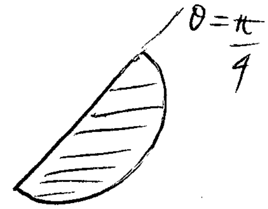
To find the bounds for r, take radial strips: we see that the upper bound for r changes at the angle of intersection of the two curves.

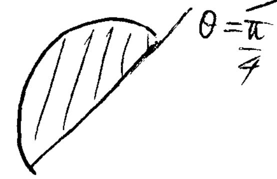
Intersection of the 2 circles:

$$r = \cos\theta \text{ and } r = \sin\theta \iff \cos\theta = \sin\theta$$
$$\iff \tan\theta = 1 \quad \text{with } \theta \in [0, \tfrac{\pi}{2}]$$
$$\iff \boxed{\theta = \frac{\pi}{4}}$$

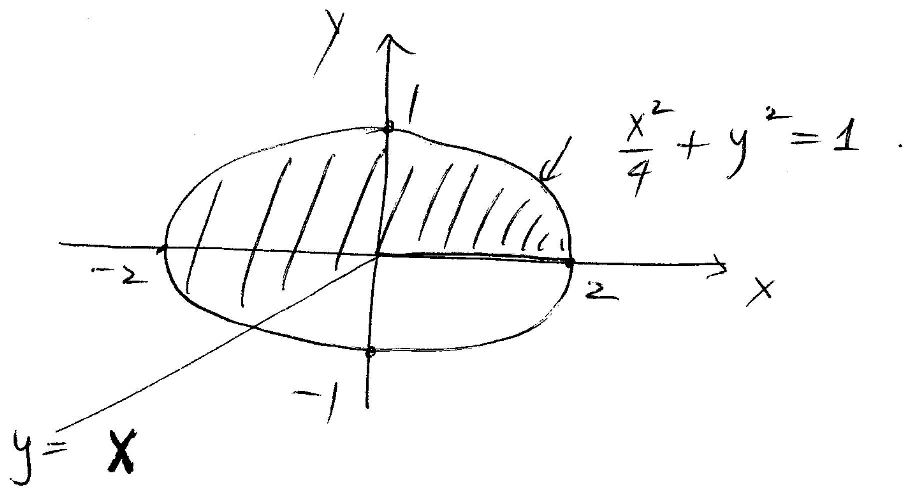
Then,

$$D = \left\{ \begin{array}{c} 0 \le \theta \le \pi/4 \\ 0 \le r \le \sin\theta \end{array} \right\} \cup \left\{ \begin{array}{c} \frac{\pi}{4} \le \theta \le \pi/2 \\ 0 \le r \le \cos\theta \end{array} \right\}$$





4) Let D be the region inside $\frac{x^2}{4}+y^2=1$ and above the positive x-axis and the $\frac{1}{2}$-line $y = x$ for $x \leq 0$.



* Convert equations of curves into polar coordinates:

ellipse: $\frac{x^2}{4}+y^2=1 \iff x^2+4y^2=1.$

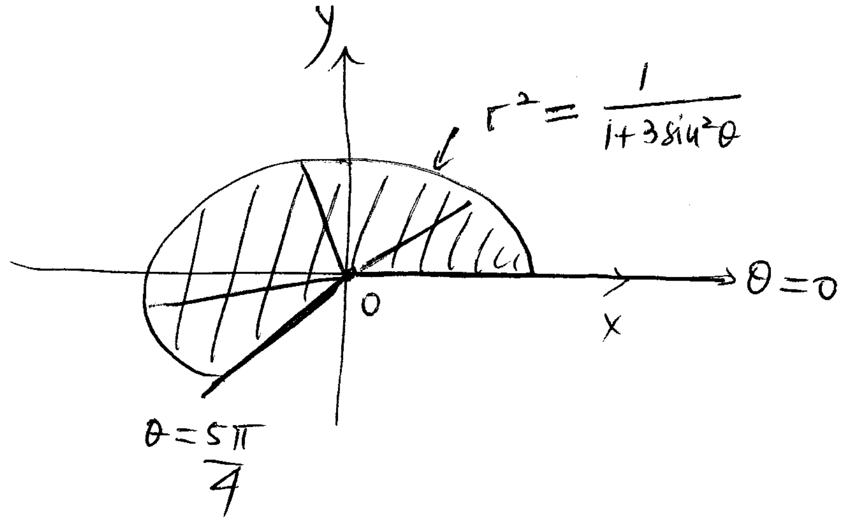
$$\iff \underbrace{(x^2+y^2)}_{r^2} + 3\underbrace{y^2}_{r^2\sin^2\theta} = 1.$$

$$\iff r^2 + 3r^2\sin^2\theta = 1.$$

$$\iff r^2 = \frac{1}{1+3\sin^2\theta}$$

$\frac{1}{2}$ line: $y=-x,\ x\leq 0:\quad \theta = \frac{5\pi}{4}$

positive x-axis: $y=0,\ x\geq 0:\quad \theta=0.$



* Bounds for r & θ: take radial strips.

$$\left\{ \begin{array}{l} 0 \le \theta \le 5\pi/4 \\ 0 \le r^2 \le \dfrac{1}{1+3\sin^2\theta} \end{array} \right\}$$

ellipse