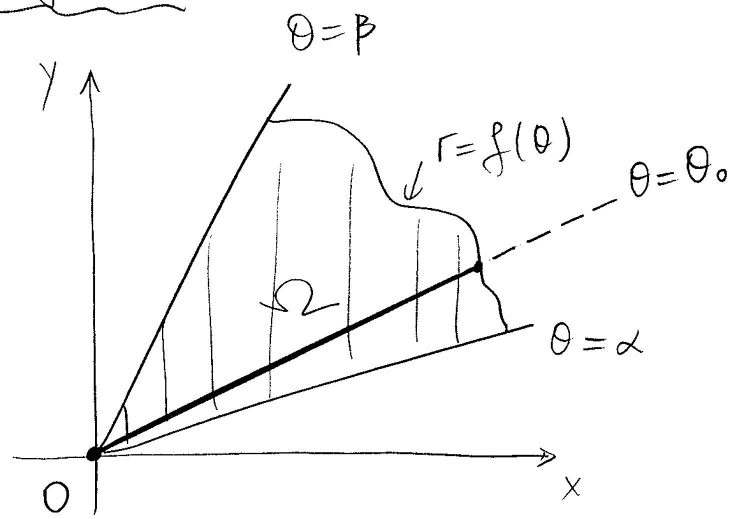
Suggestions for computing areas in polar coordinates.

When computing areas in polar coordinates, the following identities are often useful:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$
$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

HALF-ANGLE FORMULAS

(A) Region bounded by only ONE curve of the form $r = f(\theta)$:



$$\Omega = \left\{ \begin{array}{c} \alpha \le \theta \le \beta \\ 0 \le r \le f(\theta) \end{array} \right\}$$

IMPORTANT

Ω = (region bounded by $r = f(\theta)$ between the rays $\theta = \alpha$ and $\theta = \beta$)

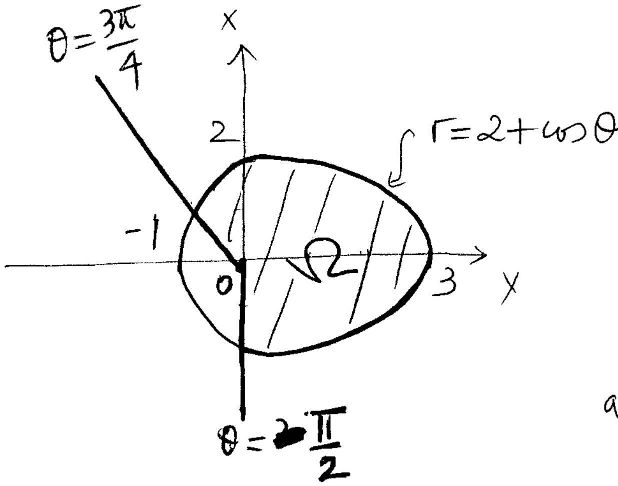
IMPORTANT: If one intersects Ω with ANY ray $\theta = \theta_0$, with $\alpha \le \theta_0 \le \beta$, it must intersect Ω in a line segment whose points are of the form (r, θ_0) with
$$0 \le r \le f(\theta_0).$$

In particular, r takes ALL values from 0 to $f(\theta_0)$, so that the region has NO holes.

For such a region Ω,

$$\boxed{\text{area}(\Omega) = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 \, d\theta .}$$

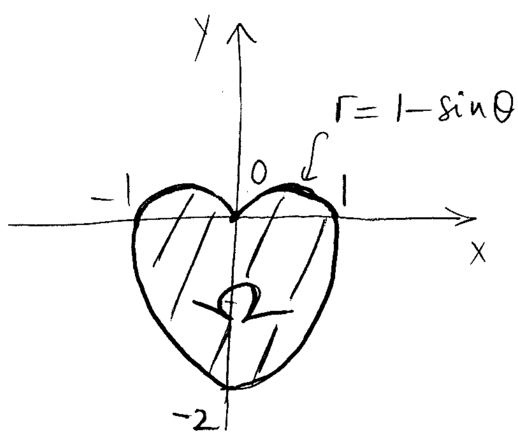
Ex. 1) Express the area of region Ω bounded by $r = 2 + \cos\theta$, $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$, as an integral.



$$\Omega = \left\{ \begin{array}{c} -\frac{\pi}{2} \le \theta \le \frac{3\pi}{4} \\ 0 \le r \le 2 + \cos\theta \end{array} \right\}$$

$$\text{area}(\Omega) = \int_{-\frac{\pi}{2}}^{3\pi/4} \frac{1}{2}(2+\cos\theta)^2 \, d\theta .$$

2) Express the area of the region Ω bounded by $r = 1 - \sin\theta$ as an integral.

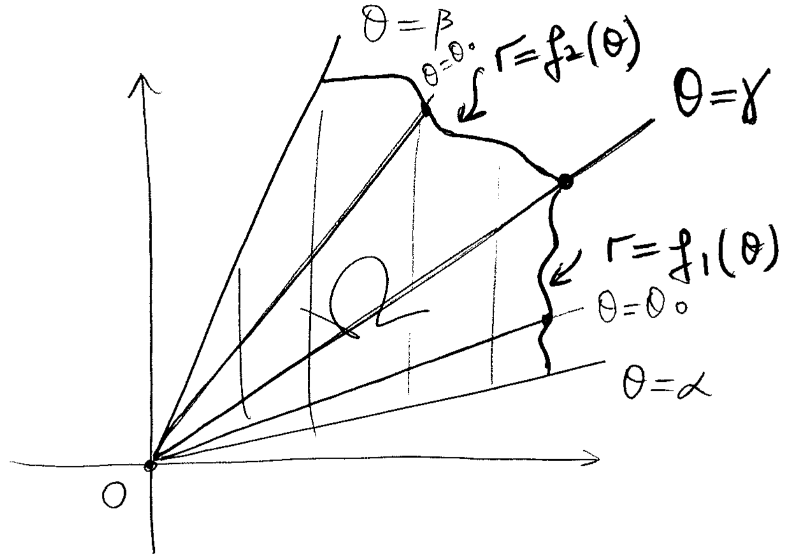


$$\Omega = \left\{ \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 - \sin\theta \end{array} \right\}$$

$$\text{area}(\Omega) = \int_{0}^{2\pi} \frac{1}{2}(1-\sin\theta)^2 \, d\theta$$

(B) Region bounded by more than one curve of the form $r = f(\theta)$:

CASE I: The region has NO holes:



$$\Omega = \left\{ \begin{array}{l} \alpha \le \theta \le \gamma \\ 0 \le r \le f_1(\theta) \end{array} \right\} \cup \left\{ \begin{array}{l} \gamma \le \theta \le \beta \\ 0 \le r \le f_2(\theta) \end{array} \right\}$$

IMPORTANT.

In this case, the upper bound of r changes from $f_1(\theta)$ to $f_2(\theta)$ at $\theta = \gamma$, for points in Ω. NONETHELESS, the region Ω has NO holes since any ray $\theta = \theta_0$, $\alpha \le \theta_0 \le \beta$, intersects Ω in a line segment whose points are of the form (r, θ_0) with

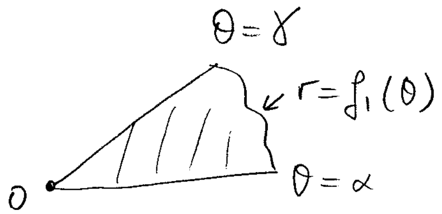
$$0 \le r \le f_1(\theta_0), \text{ if } \alpha \le \theta_0 \le \gamma$$

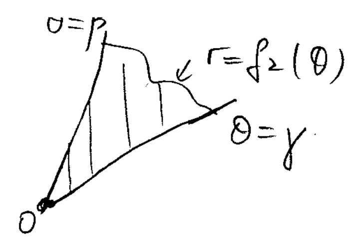
and

$$0 \le r \le f_2(\theta_0), \text{ if } \gamma \le \theta_0 \le \beta,$$

AND r takes ALL values from 0 to $f_1(\theta_0)$ (or $f_2(\theta_0)$).

For such a region Ω,

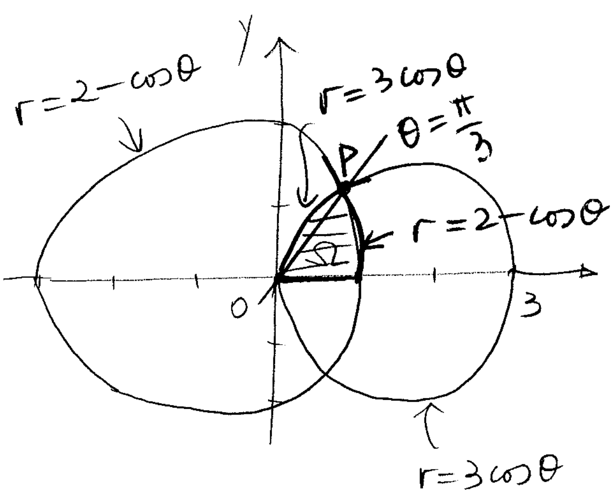
area(Ω) = 

+ 

$$= \int_{\alpha}^{\gamma} \frac{1}{2}\left(f_1(\theta)\right)^2 d\theta + \int_{\gamma}^{\beta} \frac{1}{2}\left(f_2(\theta)\right)^2 d\theta$$

NOTE: This type of region typically happens if one considers a region inside two curves $r = f_1(\theta)$ and $r = f_2(\theta)$.

Ex. Express the area of the region Ω inside BOTH $r = 2 - \cos\theta$ and $r = 3\cos\theta$, in the first quadrant, as an integral.



In this case, the boundary of Ω changes at the intersection point P of the two curves in the first quadrant.

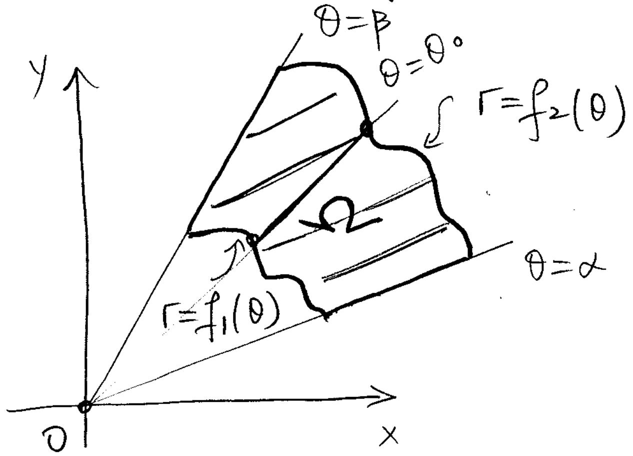
* Intersection point:

$$\left(r = 2 - \cos\theta \text{ and } r = 3\cos\theta,\ \theta \in [0, \tfrac{\pi}{2}]\right)$$
$$\Longleftrightarrow 2 - \cos\theta = 3\cos\theta,\ \theta \in [0, \tfrac{\pi}{2}]$$
$$\Longleftrightarrow \cos\theta = \tfrac{1}{2},\ \theta \in [0, \tfrac{\pi}{2}] \Longleftrightarrow \theta = \frac{\pi}{3}.$$

$$* \ \Omega = \left\{ \begin{array}{l} 0 \le \theta \le \pi/3 \\ 0 \le r \le 2 - \cos\theta \end{array} \right\} \cup \left\{ \begin{array}{l} \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 3\cos\theta \end{array} \right\}$$

$$\Longrightarrow \text{area}(\Omega) = \int_0^{\pi/3} \frac{1}{2}(2 - \cos\theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 d\theta.$$

CASE II: The region HAS holes:



$$\Omega = \left\{ \begin{array}{c} \alpha \leq \theta \leq \beta \\ \\ f_1(\theta) \leq r \leq f_2(\theta) \end{array} \right\}$$

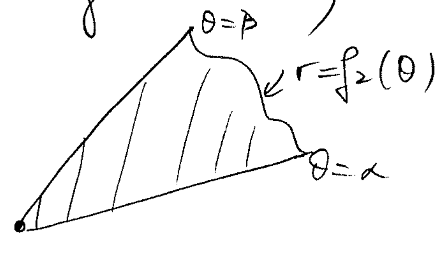
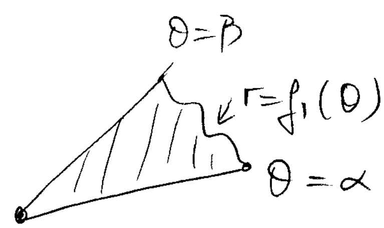
IMPORTANT: the lower bound of r is not always 0.

In such a region, there are rays $\theta=\theta_0$, for $\theta_0 \in [\alpha, \beta]$, that intersect Ω in a line segment whose points (r, θ_0) are such that

$$f_1(\theta_0) \leq r \leq f_2(\theta_0)$$

and $f_1(\theta_0) \neq 0$, so that the lower bound of r is NOT 0.

For such a region Ω,

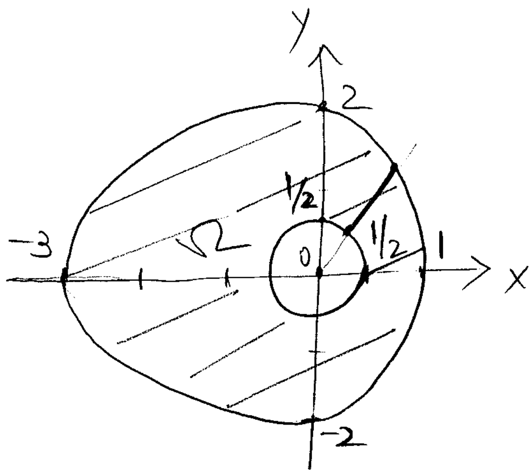
area(Ω) =  − 

$$= \int_\alpha^\beta \frac{1}{2}\left(f_2(\theta)\right)^2 d\theta - \int_\alpha^\beta \frac{1}{2}\left(f_1(\theta)\right)^2 d\theta.$$

(since Ω is the difference of two regions with NO holes that are bounded by only one curve of the form $r=f(\theta)$).

NOTE: This type of region is typically obtained by considering a region inside a curve $r = f_2(\theta)$ and outside another.

Ex. Express the area of the region Ω inside $r = 2 - \cos\theta$ and outside $r = \frac{1}{2}$ as an integral.



$$\Omega = \left\{ \begin{array}{c} 0 \le \theta \le 2\pi \\ \\ \frac{1}{2} \le r \le 2 - \cos\theta \end{array} \right\}$$

$\uparrow$ $\neq 0$!

$$\Longrightarrow \text{area}(\Omega) = \int_0^{2\pi} \frac{1}{2}(2-\cos\theta)^2\, d\theta - \int_0^{2\pi} \frac{1}{2}\left(\frac{1}{2}\right)^2 d\theta .$$

CASE III: More general regions can be described as combinations of the regions described in CASES I & II, and their areas will correspond to sums/differences of integrals.