USING TAYLOR'S THEOREM to ESTIMATE the ERROR of APPROXIMATION of $L_{(a,b)}(x,y)$.

(Taylor's Theorem)

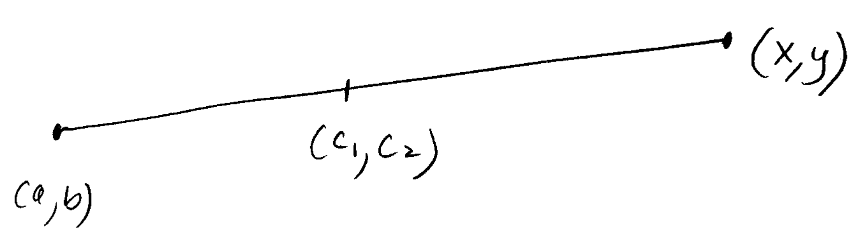
Suppose that f and its first and second order partials are continuous in a neighbourhood of (a,b). Then,

$$f(x,y) = L_{(a,b)}(x,y) + R_{1,(a,b)}(x,y),$$

where

$$R_{1,(a,b)}(x,y) = \frac{1}{2!}\Big[f_{xx}(c_1,c_2)(x-a)^2 + 2f_{xy}(c_1,c_2)(x-a)(y-b) + f_{yy}(c_1,c_2)(y-b)^2\Big],$$

for some point (c_1,c_2) on the line segment joining (a,b) and (x,y).



NOTE: $R_{1,(a,b)}(x,y) = f(x,y) - L_{(a,b)}(x,y)$

$$= \left(\begin{array}{c}\text{error of approximation} \\ \text{of } L_{(a,b)}(x,y)\end{array}\right)$$

Therefore, by Taylor's Theorem,

$$\boxed{\begin{aligned}\text{error} = \frac{1}{2!}\Big[& f_{xx}(c_1,c_2)(x-a)^2 + 2 f_{xy}(c_1,c_2)(x-a)(y-b) \\ & + f_{yy}(c_1,c_2)(y-b)^2\Big]\end{aligned}}$$

for some (c_1, c_2) on the line segment joining (a,b) and (x,y).

In particular, if we try to estimate the error by finding an upper bound for $|\text{error}|$, by the triangle inequality, we have:

$$\boxed{\begin{aligned}|\text{error}| \leq \frac{1}{2!}\Big[& |f_{xx}(c_1,c_2)|\,(x-a)^2 + 2|f_{xy}(c_1,c_2)|\cdot|(x-a)(y-b)| \\ & + |f_{yy}(c_1,c_2)|\,(y-b)^2\Big]\end{aligned}}$$

for (c_1, c_2) on the line segment joining (a,b) and (x,y).

RECAP: To estimate the error, I suggest the following:

By Taylor's Theorem, we know that

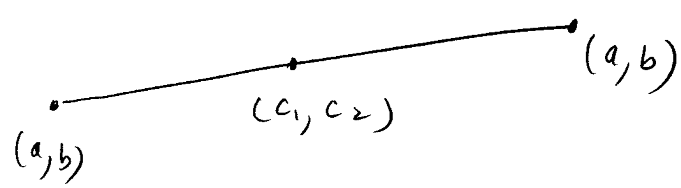
$$|\text{error}| = |\, f(x,y) - L_{(a,b)}(x,y) \,|$$

$$\leq \frac{1}{2!}\Big[|f_{xx}(c_1,c_2)|(x-a)^2 + 2|f_{xy}(c_1,c_2)|\cdot|(x-a)(y-b)| + |f_{yy}(c_1,c_2)|(y-b)^2 \Big]. \quad (*)$$

To find an upper bound for $(*)$

(i) Compute f_{xx}, f_{xy}, f_{yy}.

(ii) Find upper bound for $|f_{xx}(c_1,c_2)|$, $|f_{xy}(c_1,c_2)|$ and $|f_{yy}(c_1,c_2)|$ at points (c_1,c_2) on the line segment joining (a,b) and (x,y).



(iii) Use the following inequality to sharpen the bound you found in (ii):

$$2|x-a|\cdot|y-b| \leq (x-a)^2 + (y-b)^2.$$

Ex. 1) Let $f(x,y) = e^{-3x-y}$ and consider the approximation

$$e^{-3x-y} \simeq 1 - 3x - y$$

for (x,y) sufficiently close to $(0,0)$.

(a) Prove that if $x \geq 0$ and $y \geq 0$, the error satisfies

$$|\text{error}| \leq \frac{13}{2}(x^2 + y^2).$$

(b) Prove that, for all (a,b) and (x,y) in $\mathbb{R}^2$

$$f(x,y) \geq L_{(a,b)}(x,y).$$

(a) FIRST note that $1 - 3x - y$ is a linear polynomial approximation $f(x,y) = e^{-3x-y}$ and so it's probably a linear approximation $L_{(a,b)}(x,y)$ of f. Since we are working near $(0,0)$, try $L_{(0,0)}(x,y)$.

* $f(0,0) = e^0 = 1$

* $f_x = -3e^{-3x-y} \Rightarrow f_x(0,0) = -3$

* $f_y = -e^{-3x-y} \Rightarrow f_y(0,0) = -1$

$$\Rightarrow \boxed{L_{(0,0)}(x,y) = 1 - 3x - y}.$$

and so, by Taylor's Theorem, we know that (5)

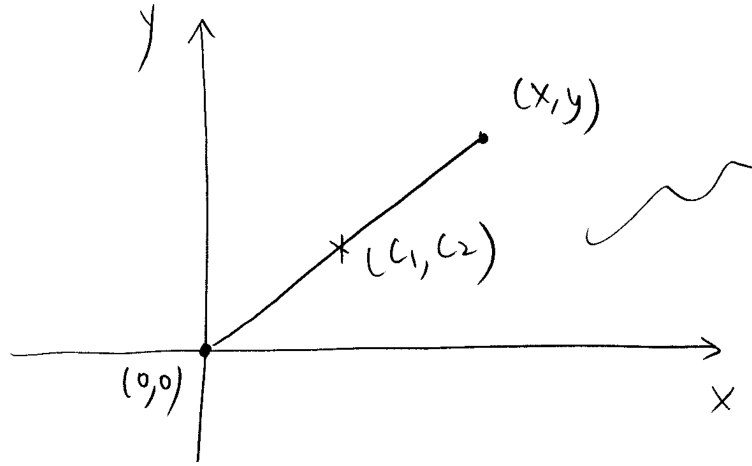
$$e^{-3x-y} \simeq 1 - 3x - y$$

near $(0,0)$ with an error that satisfies

$$|\text{error}| \leq \frac{1}{2!}\Big[|f_{xx}(c_1,c_2)|\,x^2 + 2|f_{xy}(c_1,c_2)|\cdot|xy| + |f_{yy}(c_1,c_2)|\,y^2\Big].$$

for some (c_1, c_2) on the line segment joining $(0,0)$ and (x,y).

If $x, y \geq 0$, (c_1, c_2) must be in the first quadrant, so that $c_1, c_2 \geq 0$:



$$\leadsto \begin{cases} 0 \leq c_1 \leq x \\ 0 \leq c_2 \leq y \end{cases}$$

We must therefore find upper bounds for $|f_{xx}(c_1,c_2)|$, $|f_{xy}(c_1,c_2)|$, and $|f_{yy}(c_1,c_2)|$ for $0 \leq c_1 \leq x$, $0 \leq c_2 \leq y$.

* $f_{xx} = 9e^{-3x-y} \Rightarrow f_{xx}(c_1,c_2) = 9e^{-3c_1-c_1}$, with $c_1, c_2 \geq 0$

BUT, $\boxed{e^{-3c_1-c_2} = e^{-(3c_1+c_2)} \leq 1}$

for $c_1, c_2 \geq 0$ since then $-(3c_1+c_2) \leq 0$.

Therefore, since $c_1, c_2 \geq 0$,

$$|f_{xx}(c_1, c_2)| \leq 9.$$

Similarly,

$$* \; f_{xy} = 3e^{-3x-y} \Rightarrow |f_{xy}(c_1, c_2)| = |3e^{-3c_1-c_2}| \leq 3.$$

$$* \; f_{yy} = e^{-3x-y} \Rightarrow |f_{yy}(c_1, c_2)| = |e^{-3c_1-c_2}| \leq 1.$$

THUS,

$$|\text{error}| \leq \frac{1}{2!}\left[9x^2 + 2(3)|xy| + y^2\right]$$

$$= \frac{1}{2}\left[9x^2 + 3\big(2|x|\cdot|y|\big) + y^2\right]$$

since $2|x|\cdot|y| \leq x^2+y^2$

$$\leq \frac{1}{2}\left[9x^2 + 3(x^2+y^2) + y^2\right]$$

$$= \frac{1}{2}\left[12x^2 + 4y^2\right]$$

$$\leq \frac{1}{2}\left[13x^2 + 13y^2\right] = \frac{13}{2}(x^2+y^2).$$

(b) Let us now show that, for all (x,y) and (a,b) in $\mathbb{R}^2$,

$$f(x,y) \geq L_{(a,b)}(x,y).$$

By Taylor's Theorem, for some (c_1, c_2) on the line segment joining (a,b) and (x,y),

$$f(x,y) - L_{(a,b)}(x,y) = R_{1,(a,b)}(x,y)$$

$$= \frac{1}{2!}\left[f_{xx}(c_1,c_2)(x-a)^2 + 2 f_{xy}(c_1,c_2)(x-a)(y-b) + f_{yy}(c_1,c_2)(y-b)^2 \right]$$

$$= \frac{1}{2}\left[9e^{-3c_1-c_2}(x-a)^2 + 2\left(3e^{-3c_1-c_2}\right)(x-a)(y-b) + e^{-3c_1-c_2}(y-b)^2 \right]$$

$$= \frac{e^{-3c_1-c_2}}{2}\left(9(x-a)^2 + 6(x-a)(y-b) + (y-b)^2 \right)$$

$$= \frac{e^{-3c_1-c_2}}{2}\left(3(x-a) + (y-b) \right)^2 \geqslant 0$$

since $e^{-3c_1-c_2} > 0$ and $\left(3(x-a)+(y-b)\right)^2 \geqslant 0$.

Thus,

$$f(x,y) - L_{(a,b)}(x,y) \geqslant 0$$

proving that

$$f(x,y) \geqslant L_{(a,b)}(x,y)$$

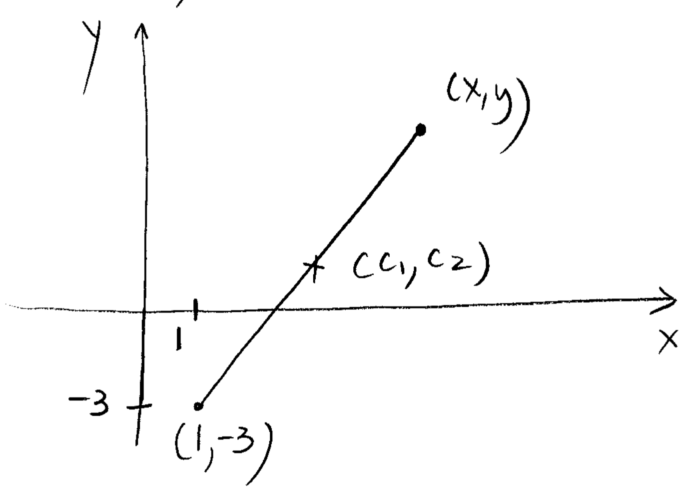
for all (a,b) and (x,y) in $\mathbb{R}^2$. $\boxtimes$

2) Show that the error in the linear approximation $L_{(1,-3)}(x,y)$ of $f(x,y) = \sqrt{4x+y}$ at $(1,-3)$ is at most $\frac{5}{2}\left[(x-1)^2 + (y+3)^2\right]$ for $x \geq 1$ and $y \geq -3$.

We need to show that

$$|\text{error}| \leq \frac{5}{2}\left[(x-1)^2 + (y+3)^2\right]$$

for $x \geq 1$ and $y \geq -3$. By Taylor's Theorem, the error can be expressed in terms of f_{xx}, f_{xy}, and f_{yy} at (c_1, c_2) on the line segment joining $(1,-3)$ and (x,y), with $x \geq 1$ and $y \geq -3$.



This means, in particular, that $c_1 \geq 1$ and $c_2 \geq -3$.
Now, since $f_x = \dfrac{2}{\sqrt{4x+y}}$ and $f_y = \dfrac{1}{2\sqrt{4x+y}}$,

$$* \; f_{xx} = \frac{-4}{(4x+y)^{3/2}} \Rightarrow |f_{xx}(c_1,c_2)| = \frac{4}{(4c_1+c_2)^{3/2}} \leq 4$$

since $4c_1 + c_2 \geq 1$ for $c_1 \geq 1$, $c_2 \geq -3$.

$$* \; f_{xy} = \frac{-1}{(4x+y)^{3/2}} \Longrightarrow |f_{xy}(c_1, c_2)| = \frac{1}{(4c_1+c_2)^{3/2}} \leq 1$$

for $c_1 \geq 1, c_2 \geq -3$.

$$* \; f_{yy} = \frac{-1}{4(4x+y)^{3/2}} \Longrightarrow |f_{yy}(c_1, c_2)| = \frac{1}{4(4c_1+c_2)^{3/2}} \leq \frac{1}{4}$$

for $c_1 \geq 1, c_2 \geq -3$.

THUS, by Taylor's Theorem,

$$|\text{error}| \leq \frac{1}{2!}\Big[|f_{xx}(c_1,c_2)|(x-1)^2 + 2|f_{xy}(c_1,c_2)||(x-1)(y+3)| + |f_{yy}(c_1,c_2)|(y+3)^2\Big]$$

$$\leq \frac{1}{2}\Big[4(x-1)^2 + 2(1)|(x-1)(y+3)| + \frac{1}{4}(y+3)^2\Big]$$

since $2|(x-1)(y+3)| \leq (x-1)^2 + (y+3)^2$

$$\leq \frac{1}{2}\Big[4(x-1)^2 + [(x-1)^2 + (y+3)^2] + \frac{1}{4}(y+3)^2\Big]$$

$$= \frac{1}{2}\Big[5(x-1)^2 + \frac{5}{4}(y+3)^2\Big]$$

$$\leq \frac{1}{2}\Big[5(x-1)^2 + 5(y+3)^2\Big]$$

$$= \frac{5}{2}\Big[(x-1)^2 + (y+3)^2\Big].$$