TOWARDS AN INTERSECTION CHOW COHOMOLOGY THEORY FOR GIT QUOTIENTS

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Abstract. We study the Fulton–MacPherson operational Chow rings of good moduli spaces of properly stable, smooth, Artin stacks. Such spaces are étale locally isomorphic to geometric invariant theory quotients of affine schemes, and are therefore natural extensions of GIT quotients. Our main result is that, with $\mathbb{Q}$-coefficients, every operational class can be represented by a *topologically strong* cycle on the corresponding stack. Moreover, this cycle is unique modulo rational equivalence on the stack. Our methods also allow us to prove that if X is the good moduli space of a properly stable, smooth, Artin stack then the natural map $\mathrm{Pic}(X)_{\mathbb{Q}} \to A^1_{\mathrm{op}}(X)_{\mathbb{Q}}$, $L \mapsto c_1(L)$ is an isomorphism.

1. Introduction

A long standing problem is to extend the classical intersection product to singular varieties. Motivated by topology, one hopes to construct an "intersection Chow cohomology" theory analogous to Goresky and MacPherson's intersection homology. There have been various approaches from this point of view using motivic theories, for example [CH1], [CH2], [FR].

Earlier, Fulton and MacPherson [FM] defined a formal Chow cohomology ring, which we call the operational Chow ring, and proved that it equals the classical intersection ring for smooth varieties. While the operational Chow ring enjoys many natural functorial properties, the product structure, which is given by composition of operations, does not have a natural interpretation in terms of intersecting subvarieties.

In this paper we focus on a class of varieties which we call *reductive quotient varieties*. These are varieties (or more generally algebraic spaces) which are étale locally good quotients of smooth varieties by reductive groups. When a reductive group acts properly the quotient variety has finite quotient singularities, and there is a well-developed geometric intersection theory on such varieties [Vis], [EG].

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However, when the stabilizers are positive-dimensional the singularities are worse and there is no intersection ring. For example, the cone over the quadric surface is the good quotient of $\mathbb{A}^4$ by $\mathbb{G}_m$. More generally, the Cox construction [Cox] shows that all normal toric varieties are good quotients of open sets in affine space. However, only simplicial toric varieties have finite quotient singularities.

If $X = Z/G$ is a quotient variety with Z smooth, then the quotient map $Z \xrightarrow{p} X$ induces a stratification of X coming from the stratification of Z by the dimension of stabilizers. Our main result is that every operational class $c \in A^*_{\mathrm{op}}(X)_{\mathbb{Q}}$ can be represented by a cycle which satisfies certain transversality conditions with respect to this stratification. Moreover, this cycle is unique modulo rational equivalence on the associated quotient stack $[Z/G]$. One consequence of this uniqueness is that the natural map $\mathrm{Pic}(Z/G) \to A^1_{\mathrm{op}}(Z/G)$, $L \mapsto c_1(L)$ is surjective after tensoring with $\mathbb{Q}$. In other words an integer multiple of every codimension-one operational class can be identified with the first Chern class of a line bundle. (See Theorem 1.2 and the remarks thereafter.)

1.1. Statement of results

The point of view of this paper is stack-theoretic. A reductive quotient variety $X = Z/G$ is the good moduli space [Alp] of a smooth stack $\mathcal{X} = [Z/G]$. Given a stack $\mathcal{X}$ with good moduli space X, we say that an integral substack $\mathcal{Y} \subseteq \mathcal{X}$ is *strong* if $\mathcal{Y}$ is saturated with respect to the good moduli space morphism $\pi\colon \mathcal{X} \to X$, i.e., $\mathcal{Y} = \pi^{-1}(\pi(\mathcal{Y}))$. We say $\mathcal{Y}$ is *topologically strong* if $\mathcal{Y} = \pi^{-1}(\pi(\mathcal{Y}))_{\mathrm{red}}$. We prove in Proposition 3.8 that X has a stratification by subspaces with finite quotient singularities, and that if $\mathcal{Y}$ is topologically strong, then the image $Y = \pi(\mathcal{Y})$ is a closed subspace of X satisfying a transversality condition with respect to this stratification.

We define the *relative strong Chow group* $A^k_{\mathrm{st}}(\mathcal{X}/X)$ to be the subgroup of the Chow group $A^k(\mathcal{X})$ generated by classes of strong integral substacks of codimension k and the *relative topologically strong Chow group* $A^k_{\mathrm{tst}}(\mathcal{X}/X)$ to be the subgroup generated by topologically strong cycles. Note that while the ordinary Chow groups of an Artin stack can be non-torsion in arbitrarily high degree, the relative strong and topologically strong Chow groups vanish in degree greater than the dimension of the good moduli space X, and so these Chow groups reflect more of the geometry of X. The strong relative Chow group of an Artin stack was originally defined in [ES]. We show in Example 3.7 that $A^*_{\mathrm{st}}(\mathcal{X}/X)$ need not equal $A^*_{\mathrm{tst}}(\mathcal{X}/X)$ even rationally.

We say that an integral stack $\mathcal{X}$ with good moduli space X is *properly stable* if there is a non-empty open set $\mathcal{U} \subseteq \mathcal{X}$ which is a tame stack in the sense of [AOV] and which is saturated with respect to the good moduli space morphism $\mathcal{X} \to X$.

Theorem 1.1. *Let $\mathcal{X}$ be a smooth connected properly stable Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$. Consider the map $\pi^*\colon A^k_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^k(\mathcal{X})_{\mathbb{Q}}$ defined by $c \mapsto c \cap [\mathcal{X}]$. Then π^* is injective and its image is contained in $A^k_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. In particular every operational class $c \in A^*_{\mathrm{op}}(\mathcal{X})_{\mathbb{Q}}$ determines a topologically strong cycle which is unique modulo rational equivalence in $A^*(\mathcal{X})_{\mathbb{Q}}$.*

As a consequence of Theorem 1.1 we are able to prove the following theorem about Picard groups of properly stable good moduli spaces of smooth Artin stacks.

Theorem 1.2. *Let $\pi\colon \mathcal{X} \to X$ be a properly stable good moduli space morphism with $\mathcal{X}$ smooth. Then the natural map $\mathrm{Pic}(X)_{\mathbb{Q}} \to A^1_{\mathrm{op}}(X)_{\mathbb{Q}}$ is an isomorphism.*

Corollary 1.3. *Let Z be a smooth projective variety with a linearized action of a reductive group G and let $X = Z/\!\!/G$ be the GIT quotient. If $Z^{\mathrm{ps}} \neq \varnothing$ then the natural map $\mathrm{Pic}(X)_{\mathbb{Q}} \to A^1_{\mathrm{op}}(X)_{\mathbb{Q}}$ is an isomorphism.*

(Here Z^{ps} refers to the set of properly stable points for the linearized action of G [MFK, Def. 1.8]. See also Remark 2.6.)

Remark 1.4. For any normal algebraic space the map $\mathrm{Pic}(X) \to A^1_{\mathrm{op}}(X)$ is injective with integer coefficients because the map from $\mathrm{Pic}(X)$ to the divisor class group $A^1(X)$ is injective [EGA4, 21.6.10] and this map factors through the natural map $\mathrm{Pic}(X) \to A^1_{\mathrm{op}}(X)$. Thus to establish Theorem 1.2 we need only prove surjectivity because the good moduli space of any smooth Artin stack is normal [Alp, Thm. 4.16].

Remark 1.5. In characteristic 0, GIT quotients have rational singularities by Boutot's theorem [Bou]. It follows from [KM, Prop. 12.4] that if $k = \mathbb{C}$ then the map $\mathrm{Pic}(X)_{\mathbb{Q}} \to A^1_{\mathrm{op}}(X)_{\mathbb{Q}}$ is an isomorphism (cf. [FMSS, pp. 4–5]).

Remark 1.6. If $k = \mathbb{C}$ and X is spherical (for example if X is a toric variety) then map $\mathrm{Pic}(X) \to A^1_{\mathrm{op}}(X)$ is known to be an isomorphism with integer coefficients. See Totaro's paper [Tot, p22.] where he attributes this fact to Brion.

From Theorem 1.1, we know that the image of π^* is contained in $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. On the other hand, Example 3.7 shows that the image is not generally equal to $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. In Conjecture 7.3, we state a precise conjectural description of the image. The following theorem summarizes some partial results in this direction; see also Theorem 7.4.

Theorem 1.7. *Let $\mathcal{X}$ be a properly stable smooth Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$. Then the following hold:*

(a) *If X is smooth or has finite quotient singularities, then $A^*(X)_{\mathbb{Q}} \simeq A^*_{\mathrm{op}}(X)_{\mathbb{Q}} \simeq A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$, where the second isomorphism is induced by π^*.*

(b) *If X has an ample line bundle, then $\pi^* A^1_{\mathrm{op}}(X) = A^1_{\mathrm{st}}(\mathcal{X}/X)$. In particular, π^* induces an isomorphism $A^1_{\mathrm{op}}(X)_{\mathbb{Q}} \simeq A^1_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}}$.*

(c) *If $\mathcal{Z} \subseteq \mathcal{X}$ is a regularly embedded strong substack of arbitrary codimension k then $[\mathcal{Z}] \in \pi^*(A^k_{\mathrm{op}}(X))_{\mathbb{Q}}$.*

(d) *Assume k equals $\dim X - 1$ or $\dim X$. Then the map $\pi^*\colon A^k_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^k_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$ is an isomorphism. If in addition the maximal saturated tame substack of $\mathcal{X}$ is representable, then $A^k_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}} = A^k_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$.*

Corollary 1.8. *If $\dim X \leq 3$, the maximal saturated tame substack of $\mathcal{X}$ is representable, and X has an ample line bundle, then the map $A^*_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^*_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}}$ is an isomorphism.*

Remark 1.9. Theorem 1.7(a) implies that if the good moduli space X is smooth or has finite quotient singularities, then $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}} \subseteq A^*(\mathcal{X})_{\mathbb{Q}}$ is a subring, i.e., it is closed under intersection products. Furthermore, it shows that $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$ is independent of the stack $\mathcal{X}$. In other words, any two stacks $\mathcal{X}'$ and $\mathcal{X}$ with good moduli space X have rationally isomorphic topologically strong Chow groups. For Deligne–Mumford or tame stacks with coarse moduli space X this is well known. Our result shows that even if the stack $\mathcal{X}$ is not tame, there is still a canonical subring $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}} \subseteq A^*(\mathcal{X})_{\mathbb{Q}}$ which captures the rational Chow ring of the moduli space X.

2. Background results

Throughout this paper, all schemes and stacks are assumed to be of finite type over an algebraically closed field. All algebraic stacks are assumed to have affine diagonal. An Artin stack is *tame* if it has finite inertia and satisfies any (and hence all) of the equivalent conditions of [AOV, Thm. 3.2]. In characteristic 0, all Deligne–Mumford stacks with finite inertia are tame.

2.1. Intersection theory on schemes and stacks

Let X be a scheme or algebraic space. We denote by $A_n(X)$ the Chow group of n-dimensional cycles modulo rational equivalence and we denote by $A_*(X)$ the direct sum of all Chow groups. Unless otherwise stated, we assume that a scheme is equidimensional and use the notation $A^k(X)$ to denote the Chow group of codimension-k cycles modulo rational equivalence. We denote by $A^k_{\mathrm{op}}(X)$ the codimension-k operational Chow group of X as defined in [Ful, Chap. 17]. By definition, an element $c \in A^k_{\mathrm{op}}(X)$ is an assignment, for every morphism $T \to X$ a homomorphism $c_T\colon A_*(T) \to A_{*-k}(T)$ which is compatible with the basic operations of Chow groups (flat and lci pullbacks, as well as proper pushforward). If $\alpha \in A_*(T)$ we write $c \cap \alpha$ for $c_T(\alpha)$. The group $\bigoplus A^k_{\mathrm{op}}(X)$ is a graded ring with multiplication given by composition. When X is smooth, the Poincaré duality map $A^*_{\mathrm{op}}(X) \to A^*(X)$, $c \mapsto c \cap [X]$ is an isomorphism of rings [Ful, Cor. 17.4] where the product on $A^*(X)$ is the intersection product defined in [Ful, Chap. 8].

There is also an intersection theory for stacks which was developed in [EG] and [Kre] building on earlier work of Gillet [Gil] and Vistoli [Vis]. If $\mathcal{X} = [Z/G]$ is a quotient stack then we can identify the Chow group $A^k(\mathcal{X})$ with the equivariant Chow group $A^k_G(X)$ defined in [EG]. For any stack we can define an operational Chow ring in a manner similar to the definition for schemes. Precisely, an element $c \in A^k_{\mathrm{op}}(\mathcal{X})$ is an assignment to every morphism from a scheme $T \to \mathcal{X}$ an operation $c_T\colon A^*(T) \to A^{*-k}(T)$ with the usual compatibilities. When $\mathcal{X} = [Z/G]$ is a quotient we can identify the operational Chow ring $A^*_{\mathrm{op}}(\mathcal{X})$ with the operational equivariant Chow ring of $A^*_{\mathrm{op},G}(Z)$ defined in [EG]. Again if $\mathcal{X}$ is smooth, there is a Poincaré duality isomorphism $A^*_{\mathrm{op}}(\mathcal{X}) \to A^*(\mathcal{X})$, $c \mapsto c \cap [\mathcal{X}]$ [Ed13].

Let $\mathcal{X}$ be an integral Artin stack which can be stratified by quotient stacks[2] and let $\pi\colon \mathcal{X} \to X$ be any morphism to an algebraic space. The following proposition gives the construction of an evaluation map $A^*_{\mathrm{op}}(X) \to A^*(\mathcal{X})$, $c \mapsto c \cap [\mathcal{X}]$.

Proposition 2.1. *There exists an evaluation map $A^*_{\mathrm{op}}(X) \to A^*(\mathcal{X})_{\mathbb{Q}}$, $c \mapsto c \cap [\mathcal{X}]_{\mathbb{Q}}$. Moreover, if $\mathcal{X}$ is a quotient stack or if the characteristic of the ground field is 0 then this map can be defined with $\mathbb{Z}$-coefficients.*

[2] Any stack with a good moduli space necessarily has affine stabilizers and so can be stratified by quotient stacks by [Kre, Prop. 3.5.9].

Proof. For general $\mathcal{X} \to X$, we can apply Chow's lemma and deJong's alteration theorem to construct a generically finite morphism of degree d, $g\colon X' \to X$ where X' is smooth and has an ample line bundle.

Since X' is smooth and has an ample line bundle, we can, by the Riemann–Roch theorem, express the image of g^*c in $A^*(\mathcal{X}')_{\mathbb{Q}}$ as $p(E_1, \ldots, E_r)$ where p is a polynomial in the Chern classes of vector bundles $E_1, \ldots, E_r$ on X'.

Let $\mathcal{X}' = X' \times_X \mathcal{X}$ be the stack obtained by base change, so we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array} .$$

By [Kre, Thm. 2.1.12(vii)] Chern class operations are defined on the Chow groups of any Artin stack. Thus we define $c \cap [\mathcal{X}]$ by the formula

$$\frac{1}{d} f_* \left(p((\pi')^* E_1, \ldots, (\pi')^* E_r)) \cap [\mathcal{X}']\right).$$

Standard arguments using the projection formula show that this is independent of choice of alteration X'.

If we work over a field of characteristic 0 then using resolution of singularities we can assume that $X' \to X$ is birational. Let $h\colon \tilde{\mathcal{X}}' \to \mathcal{X}'$ be a resolution of singularities[3] of the stack $\mathcal{X}'$ obtained by base change along the morphism $g\colon X' \to X$. Let $p = \pi' \circ h$ and $\tilde{f} = f \circ h$. Then we have a commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{X}}' & \xrightarrow{\tilde{f}} & \mathcal{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array} .$$

Both $\tilde{\mathcal{X}}'$ and X' are smooth so the morphism p is l.c.i. Also the stack $\tilde{\mathcal{X}}'$ can be stratified by quotient stacks since it admits a representable morphism to the stack $\mathcal{X}$ which has the same property.

By [Kre, Thm. 2.1.12(xi)] there is an l.c.i. pullback $p^*\colon A^*(X') = A^*_{\mathrm{op}}(X') \to A^*(\tilde{\mathcal{X}}')$. We define

$$c \cap [\mathcal{X}] = \tilde{f}_*\big(p^* g^* c \cap [\tilde{\mathcal{X}}']\big).$$

Again this formula is independent of choices of resolutions.

Finally if $c \in A^k_{\mathrm{op}}(X)$ and $\mathcal{X} = [Z/G]$ is a quotient stack, then $A^k(\mathcal{X})$ is identified with $A^k((Z \times U)/G)$ for some open set U in a representation of G. In this case we identify $c \cap [\mathcal{X}]$ with $c \cap [(Z \times U)/G] \in A^k((Z \times U)/G)$. $\square$

[3]The existence of resolution of singularities for Artin stacks follows from functorial resolution of singularities for schemes. Since resolutions are functorial for smooth morphisms they can be constructed for Artin stacks which are locally schemes in the smooth topology. The functoriality also implies that the resolution of singularities morphism is representable.

2.2. Properly stable good moduli spaces and Reichstein transforms

We briefly review the material on properly stable good moduli spaces and Reichstein transforms from [ER], [ES].

Definition 2.2 ([Alp, Def. 4.1])**.** Let $\mathcal{X}$ be an algebraic stack and let X be an algebraic space. We say that X is a *good moduli space of* $\mathcal{X}$ if there is a morphism $\pi\colon \mathcal{X} \to X$ such that

(1) π is *cohomologically affine* meaning that the pushforward functor π_* on the category of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules is exact;
(2) π is *Stein* meaning that the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism.

More generally, a morphism $\pi\colon \mathcal{X} \to \mathcal{Y}$ of algebraic stacks is a *good moduli space morphism* if it satisfies conditions (1) and (2) above.

Remark 2.3. By [Alp, Thm. 6.6], a good moduli space morphism $\pi\colon \mathcal{X} \to X$ is the universal morphism from $\mathcal{X}$ to an algebraic space. That is, if X' is an algebraic space then any morphism $\mathcal{X} \to X'$ factors through a morphism $X \to X'$. Consequently X is unique up to unique isomorphism, so we will refer to X as *the* good moduli space of $\mathcal{X}$.

Remark 2.4. If $\mathcal{X} = [Z/G]$ where G is a linearly reductive algebraic group then the statement that X is a good moduli space for $\mathcal{X}$ is equivalent to the statement that X is the good quotient of Z by G.

Definition 2.5 ([ER])**.** Let $\mathcal{X}$ be an Artin with good moduli space X and let $\pi\colon \mathcal{X} \to X$ be the good moduli space morphism. We say that a closed point of $\mathcal{X}$ is *stable* if $\pi^{-1}(\pi(x)) = x$ under the induced map of topological spaces $|\mathcal{X}| \to |X|$. A point x of $\mathcal{X}$ is *properly stable* if it is stable and the stabilizer of x is finite.

We say $\mathcal{X}$ is stable (resp. properly stable) if there is a good moduli space $\pi\colon \mathcal{X} \to X$ and the set of stable (resp. properly stable) points is non-empty. Likewise we say that π is a stable (resp. properly stable) good moduli space morphism.

Remark 2.6. This definition is modeled on GIT. If G is a linearly reductive group and X^{ss} is the set of semistable points for a linearization of the action of G on a projective variety X, then a (properly) stable point of $[X^{\mathrm{ss}}/G]$ corresponds to a (properly) stable orbit in the sense of GIT. The stack $[X^{\mathrm{ss}}/G]$ is stable if and only if $X^{\mathrm{s}} \neq \varnothing$. Likewise $[X^{\mathrm{ss}}/G]$ is properly stable if and only if $X^{\mathrm{ps}} \neq \varnothing$. As is the case for GIT quotients, the set of stable (resp. properly stable points) is open [ER].

We denote by $\mathcal{X}^{\mathrm{s}}$ (resp. $\mathcal{X}^{\mathrm{ps}}$) the open substack of $\mathcal{X}$ consisting of stable (resp. properly stable) points. The stack $\mathcal{X}^{\mathrm{ps}}$ is the maximal tame substack of $\mathcal{X}$ which is saturated with respect to the good moduli space morphism $\pi\colon \mathcal{X} \to X$. In particular, a stack $\mathcal{X}$ with good moduli space X is properly stable if and only if it contains a non-empty *saturated* tame open substack.

The following definition is a straightforward extension of the one originally made in [EM1].

Definition 2.7. Let $\pi\colon \mathcal{X} \to X$ be a good moduli space morphism and let $\mathcal{C} \subseteq \mathcal{X}$ be a closed substack. Let $f\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up along $\mathcal{C}$. The *Reichstein* *transform* with center $\mathcal{C}$, is the stack $R(\mathcal{X},\mathcal{C})$ obtained by deleting the strict transform of the saturation $\pi^{-1}(\pi(\mathcal{C}))$ in the blow-up of $\mathcal{X}$ along $\mathcal{C}$.

Theorem 2.8 ([ER, Props. 3.4, 4.5])**.** *Let $\pi\colon \mathcal{X} \to X$ be a good moduli space morphism with $\mathcal{X}$ smooth and let $\mathcal{C} \subseteq \mathcal{X}$ be a smooth closed substack with sheaf of ideals $\mathcal{I}$. Then $R(\mathcal{X},\mathcal{C}) \to \mathbb{P}(\bigoplus \pi_*(\mathcal{I}^n))$ is a good moduli space morphism.*

Let $\pi\colon \mathcal{X} \to X$ be a stable good moduli space morphism with $\mathcal{X}$ smooth and irreducible. Let $\mathcal{Y} \subseteq \mathcal{X}$ be the locus of points with maximal dimensional stabilizer n. By [ER], $\mathcal{Y}$ is a closed smooth substack of $\mathcal{X}$. Let $f\colon \mathcal{X}' \to \mathcal{X}$ be the Reichstein transform along $\mathcal{Y}$ and let $g\colon X' \to X$ be the induced morphism of good moduli spaces. If $\mathcal{Y}$ is a proper closed substack then by [ER, Prop. 5.3], g is a projective birational morphism and the stabilizer of every point of $\mathcal{X}'$ is strictly less than n. As a consequence, after a finite sequence of Reichstein transforms we may obtain a stack $\mathcal{X}''$ where the dimension of the stabilizer is constant. If $\mathcal{X}$ is properly stable then $\mathcal{X}''$ is a tame stack; otherwise $\mathcal{X}''$ is a gerbe over a tame stack.

3. Strong and topologically strong Chow groups

We begin with the two central definitions of the paper, the first of which having previously been introduced in [ES]. Note first that if $\pi\colon \mathcal{X} \to X$ is a good moduli space morphism and $\mathcal{Z} \subseteq \mathcal{X}$ is a closed substack, then $\pi(\mathcal{Z}) \subseteq X$ inherits a natural subscheme structure. Indeed, if $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is the coherent sheaf of ideals defining $\mathcal{Z}$, then since π_* is Stein and cohomologically affine, we see $\pi_*\mathcal{I} \subseteq \pi_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ is a coherent sheaf of ideals and set-theoretically cuts out $\pi(\mathcal{Z})$, thereby giving $\pi(\mathcal{Z})$ a natural scheme structure.

Definition 3.1. Let $\mathcal{X}$ be an irreducible Artin stack with stable good moduli space $\pi\colon \mathcal{X} \to X$. A closed integral substack $\mathcal{Z} \subseteq \mathcal{X}$ is *strong* if $\operatorname{codim}_{\mathcal{X}} \mathcal{Z} = \operatorname{codim}_X \pi(\mathcal{Z})$ and $\mathcal{Z}$ is saturated with respect to π, i.e., $\pi^{-1}(\pi(\mathcal{Z})) = \mathcal{Z}$ as stacks. We say $\mathcal{Z}$ is *topologically strong* if $\operatorname{codim}_{\mathcal{X}} \mathcal{Z} = \operatorname{codim}_X \pi(\mathcal{Z})$ and $\pi^{-1}(\pi(\mathcal{Z}))_{\mathrm{red}} = \mathcal{Z}$.

Remark 3.2. Note that if $\mathcal{X}$ is tame then $\mathcal{X}$ has a coarse moduli space X which is also the good moduli space [Alp, Example 8.1]. If $\mathcal{X} \to X$ is the coarse/good moduli space morphism, then any integral substack $\mathcal{Z} \subseteq \mathcal{X}$ is topologically strong. If $\mathcal{X} \to X$ is properly stable then $\dim \mathcal{X} = \dim X$, so $\dim \mathcal{Z} = \dim \pi(Z)$ if $\mathcal{Z}$ is strong or topologically strong.

Lemma 3.3. *Let $\pi\colon \mathcal{X} \to X$ be a stable good moduli space morphism and $\mathcal{Z} \subseteq \mathcal{X}$ an irreducible closed substack. Consider the cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array}$$

with g an étale cover. Let $\mathcal{Z}' = f^{-1}(\mathcal{Z})$ and $Z' = \pi'(\mathcal{Z}')$. Then $\mathcal{Z}$ is strong (resp. topologically strong) with respect to π if and only if $\mathcal{Z}'$ is reduced, $\operatorname{codim}_{\mathcal{X}'} \mathcal{Z}' = \operatorname{codim}_{X'} Z'$, *and $\mathcal{Z}' = \pi'^{-1}(Z')$ (resp. $\mathcal{Z}' = (\pi'^{-1}(Z'))_{\mathrm{red}}$).*

Proof. Notice that $\mathcal{Z}$ is integral if and only if $\mathcal{Z}$ is reduced, and since f is an étale cover, this holds if and only if $\mathcal{Z}'$ is reduced.

Next, we show that $Z' = g^{-1}(\pi(\mathcal{Z}))$. The diagram in the statement of the lemma is cartesian, and so

$$\begin{array}{ccc} \mathcal{Z}' & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & X \end{array}$$

is as well. This factors into a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}' & \longrightarrow & \mathcal{Z} \\ \downarrow & & \downarrow \\ g^{-1}(\pi(\mathcal{Z})) & \longrightarrow & \pi(\mathcal{Z}) \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & X \end{array}$$

and the bottom square is cartesian, so the top square is as well. Now $\mathcal{Z} \to \pi(\mathcal{Z})$ is surjective, so $\mathcal{Z}' \to g^{-1}(\pi(\mathcal{Z}))$ is as well, showing $Z' = g^{-1}(\pi(\mathcal{Z}))$.

Since f and g are étale, $\dim \mathcal{Z} = \dim \mathcal{Z}'$ and $\dim \pi(\mathcal{Z}) = \dim g^{-1}\pi(\mathcal{Z}) = \dim Z'$. Therefore, $\operatorname{codim}_{\mathcal{X}} \mathcal{Z} = \operatorname{codim}_X \pi(\mathcal{Z})$ if and only if $\operatorname{codim}_{\mathcal{X}'} \mathcal{Z}' = \operatorname{codim}_{X'} Z'$.

Lastly, since f is an étale cover, the canonical map $\mathcal{Z} \to \pi^{-1}\pi(\mathcal{Z})$ is an isomorphism if and only if $\mathcal{Z}' \to f^{-1}\pi^{-1}\pi(\mathcal{Z}) = \pi'^{-1}Z'$ is an isomorphism. Similarly, we show $\mathcal{Z} = (\pi^{-1}\pi(\mathcal{Z}))_{\mathrm{red}}$ if and only if $\mathcal{Z}' = (\pi'^{-1}(Z'))_{\mathrm{red}}$ under the presence of the equivalent hypotheses: $\mathcal{Z}$ is integral and $\mathcal{Z}'$ is reduced. To see this, note that the canonical map $\mathcal{Z} \to (\pi^{-1}\pi(\mathcal{Z}))_{\mathrm{red}}$ is an isomorphism if and only if $\mathcal{Z}' \to f^{-1}((\pi^{-1}\pi(\mathcal{Z}))_{\mathrm{red}}) = (f^{-1}\pi^{-1}\pi(\mathcal{Z}))_{\mathrm{red}} = (\pi'^{-1}Z')_{\mathrm{red}}$ is an isomorphism. □

Remark 3.4 (Local structure of strong and topologically strong substacks). Let $\pi\colon \mathcal{X} \to X$ be a good moduli space morphism. By the local structure theorem of [AHR], for any point x of $\mathcal{X}$, letting G_x denote the stabilizer of x, there is an étale neighborhood of x isomorphic to $[U/G_x]$ such that the diagram

$$\begin{array}{ccc} [U/G_x] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ U/G_x & \longrightarrow & X \end{array}$$

is cartesian and the horizontal maps are étale. Shrinking U is necessary; we may assume it is affine.

By Lemma 3.3, we can check if $\mathcal{Z} \subseteq \mathcal{X}$ is strong (resp. topologically strong) étale locally. Thus, we need only understand the structure of those $\mathcal{Z} \subseteq \mathcal{X} = [U/G]$ with $U = \operatorname{Spec} A$, where $\mathcal{Z}$ satisfies the conclusion of the lemma. Let $\mathcal{Z}$ be a reduced substack defined by an ideal $I \subseteq A$, let $\pi\colon \mathcal{X} \to U/G$ be the good moduli space map, and let $Z = \pi(\mathcal{Z})$. The condition that $\operatorname{codim}_{\mathcal{X}} \mathcal{Z} = \operatorname{codim}_{\mathcal{X}} Z$ is equivalent to $\operatorname{ht}_A I = \operatorname{ht}_{A^G} I^G = \operatorname{codim}_{\mathcal{X}} \mathcal{Z}$. We have $\mathcal{Z} = \pi^{-1}(Z)$ if and only if $I^G A = I$, or equivalently $I = (f_1, \dots, f_r)$ for some $f_i \in A^G$. Lastly, $\mathcal{Z} = (\pi^{-1}(Z))_{\mathrm{red}}$ if and only if there is an auxiliary ideal $J = (f_1, \dots, f_r)$ with $f_i \in A^G$ and $\sqrt{J} = I$.

Using the notions of strong and topologically strong substacks, we obtain corresponding subgroups of Chow. The relative strong Chow groups were introduced in [ES].

Definition 3.5. Let $\mathcal{X} \to X$ be a stable good moduli space morphism and assume that $\mathcal{X}$ has pure dimension. Define the relative strong Chow group $A^k_{\mathrm{st}}(\mathcal{X}/X)$ to be the subgroup of $A^k(\mathcal{X})$ generated by the fundamental classes of strong integral substacks of codimension k. Likewise let $A^k_{\mathrm{tst}}(\mathcal{X}/X)$ be the subgroup generated by topologically strong integral substacks of codimension k.

Remark 3.6. Unlike the Chow groups $A^k(\mathcal{X})$, we see from the definition that both $A^k_{\mathrm{st}}(\mathcal{X}/X)$ and $A^k_{\mathrm{tst}}(\mathcal{X}/X)$ vanish for $k > \dim \mathcal{X}$. Thus, the relative strong Chow group reflects some of the geometry of the good moduli space X. Moreover, if $\mathcal{X}$ is tame we also know that $A^k(\mathcal{X})_{\mathbb{Q}}$ is generated by fundamental classes of codimension-k integral substacks of $\mathcal{X}$, as opposed to integral substacks on vector bundles over $\mathcal{X}$. Hence, $A^k_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}} = A^k(\mathcal{X})_{\mathbb{Q}} \simeq A^k(X)_{\mathbb{Q}}$.

However, the following example shows that the group $A^k_{\mathrm{st}}(\mathcal{X}/X)$ need not equal $A^k_{\mathrm{tst}}(\mathcal{X}/X)$ even after tensoring with $\mathbb{Q}$.

Example 3.7 ($A^k_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}}$ and $A^k_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$ can differ). Consider $\mathbb{G}_m$ acting on $\mathbb{A}^4$ with weights $(1, -1, 1, -1)$ and denote by x_1, x_2, x_3, x_4 the coordinate functions on $\mathbb{A}^4$. Let $\mathcal{X} = [\mathbb{A}^4/\mathbb{G}_m]$. Then the good moduli space is

$$X = \operatorname{Spec} k[x_1x_2, x_1x_4, x_2x_3, x_3x_4].$$

If χ is the defining character of $\mathbb{G}_m$ then the Chow group $A^1(\mathcal{X}) = \mathbb{Z}[t]$ where $t = c_1(\chi)$. If $\mathcal{D} \subseteq \mathcal{X}$ is an integral divisor, then since $\mathcal{X}$ is smooth, $\mathcal{D}$ is Cartier, hence of the form $[V(f)/\mathbb{G}_m]$. Then by Remark 3.4, $\mathcal{D}$ is strong if and only if f is a $\mathbb{G}_m$-fixed polynomial. But such a polynomial has $\mathbb{G}_m$-weight 0 so its Chow class is 0. Hence $A^1_{\mathrm{st}}(\mathcal{X}/X) = 0$. On the other hand, the substack $\mathcal{D}' = [V(g)/\mathbb{G}_m]$ with $g = x_1^2x_2 + x_3^2x_4$ is topologically strong and has non-trivial Chow class. To see that $\mathcal{D}'$ is topologically strong, note that if $I = (g)$ then $I^{\mathbb{G}_m} k[x_1, x_2, x_3, x_4] = (x_4g, x_2g)$ and $V(x_4g, x_2g)$ is supported on $V(g)$ with an embedded component $V(x_2^2, x_4)$. Since the $\mathbb{G}_m$-weight of g is 1, we see $[\mathcal{D}'] = t \neq 0$. Hence $A^1_{\mathrm{tst}}(\mathcal{X}/X) = \mathbb{Z}t = A^1(\mathcal{X})$.

3.1. Strong cycles and stratification by stabilizer

Let $\mathcal{X}$ be a smooth connected Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$. We show that π induces a stratification of X where each open stratum has finite quotient singularities. Let $\mathcal{X}_0$ be the locus of maximal dimensional stabilizer in $\mathcal{X}$ and let $X_0 \subseteq X$ be its image under π. By [ER], $\mathcal{X}_0$ is closed in $\mathcal{X}$ and thus $X_0 \subseteq X$ is as well. By [Alp], the restriction $\pi|_{\mathcal{X}_0}\colon \mathcal{X}_0 \to X_0$ is a good moduli space morphism. Now $\mathcal{X}_0$ is a smooth stack where the stabilizer dimension is constant. By [ER, Prop. A.2] its rigidification is a smooth tame stack $\mathcal{X}_0^{rig}$ and the good moduli space morphism factors as $\mathcal{X}_0 \to \mathcal{X}_0^{rig} \stackrel{\eta}{\to} X_0$, where η is a coarse space map. Hence X_0 has finite quotient singularities. Next, let $\mathcal{Y}_1$ be the locus of maximal dimensional stabilizer in $\mathcal{X} \smallsetminus \pi^{-1}(X_0)$. Again $\mathcal{Y}_1$ is a gerbe over a tame stack so its image $Y_1 \subseteq X$ has finite quotient singularities. We let $\mathcal{X}_1 = \overline{\mathcal{Y}}_1 \cup \mathcal{X}_0$ which is closed, and let $X_1 = \pi(\mathcal{X}_1) \subseteq X$ which is then closed as well. By upper semicontinuity of the dimension of stabilizers, we see $\overline{\mathcal{Y}}_1 \smallsetminus \mathcal{Y}_1 \subseteq \mathcal{X}_0$, and so $\mathcal{X}_1 \smallsetminus \pi^{-1}(X_0) = \overline{\mathcal{Y}}_1 \smallsetminus \pi^{-1}(X_0) = \mathcal{Y}_1$. As a result, $X_1 \smallsetminus X_0 = \pi(\mathcal{X}_1 \smallsetminus \pi^{-1}(X_0)) = \pi(\mathcal{Y}_1) = Y_1$, which has finite quotient singularities. Continuing this process inductively we obtain a stratification $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$ where $X_{k+1} \smallsetminus X_k$ has finite quotient singularities. Note that $X \smallsetminus X_{n-1}$ is the image of $\mathcal{X}^{\mathrm{s}}$ under the good moduli space morphism.

The next proposition shows that topologically strong substacks of $\mathcal{X}$ must satisfy a transversality condition with respect to the above stratification.

Proposition 3.8. *Let $\pi\colon \mathcal{X} \to X$ be a properly stable good moduli space morphism and let $\mathcal{Z}$ be a topologically strong k-dimensional integral substack with image $Z \subseteq X$. Then Z satisfies the following transversality conditions with respect to the stratification above:*

1. $Z \cap (X \smallsetminus X_{n-1}) \neq \varnothing$.
2. *For $k \leq n-1$, $\dim \pi^{-1}(Z \cap Y) < \dim Z$ for every connected component Y of X_k.*

Conversely, if $Z \subseteq X$ is an integral closed subspace such that $Z \cap (X \smallsetminus X_{n-1}) \neq \varnothing$, then let $\mathcal{Z}$ be the closure of $\pi^{-1}(Z \cap (X \smallsetminus X_{n-1}))$. If $\mathcal{Z} = \pi^{-1}(Z)_{\mathrm{red}}$ then $\mathcal{Z}$ is topologically strong.

Proof. Since $\mathcal{Z}$ is saturated with respect to the good moduli space morphism it must intersect the open substack $\mathcal{X}^{\mathrm{s}}$ because otherwise $\dim \pi(\mathcal{Z}) < \dim \mathcal{Z}$. Likewise if Y is a connected component of X_k and $\dim \pi^{-1}(Z \cap Y) \geq \dim Z$ then $\pi^{-1}(Z)$ would have an additional irreducible component which does not dominate the image Z.

To prove the converse note that the good moduli space morphism π is a homeomorphism over $X \smallsetminus X_{n-1}$, so $\pi^{-1}(Z \cap (X \smallsetminus X_{n-1}))$ is irreducible of the same dimension as Z, and so its closure $\mathcal{Z}$ has the same property. By assumption, $\mathcal{Z}$ is reduced so it is integral. Lastly, $\mathcal{Z} = \pi^{-1}(Z)_{\mathrm{red}}$ by assumption, so $\mathcal{Z}$ is topologically strong. □

Remark 3.9. The statement of Proposition 3.8 is easily modified when $\mathcal{X} \to X$ is stable but not properly stable by adjusting our dimension counts by the dimension of the generic stabilizer of $\mathcal{X}$.

Example 3.10. Note that conditions (3.8) and (3.8) of Proposition 3.8 are not sufficient for an integral subspace $Z \subseteq X$ to be the image of a topologically strong cycle. For example, consider $\mathcal{X} = [\mathbb{A}^4/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weights $(1,-1,1,-1)$ on $\mathbb{A}^4$ whose coordinates are (x_1, x_2, x_3, x_4). In this case the good moduli space of $\mathcal{X}$ is given by $X = \operatorname{Spec} k[x_1x_2, x_1x_4, x_2x_3, x_3x_4]$ and the stratification is $\{O\} \subseteq X$ where O is the origin corresponding to all coordinates equal to 0. In this case the integral Weil divisor $Z = V(x_1x_2, x_1x_4) \subseteq X$ satisfies the two conditions of the proposition. However, the closure of $\pi^{-1}(Z \cap (X \smallsetminus \{O\}))$ is $[V(x_1)/\mathbb{G}_m]$ while $\pi^{-1}(Z) = [V(x_1)/\mathbb{G}_m] \cup [V(x_2, x_4)/\mathbb{G}_m]$.

On the other hand, for a curve or point $Z \subseteq X$ to be the image of a topologically strong cycle then Z must not contain the origin O, because the fiber of the good moduli space map $[\mathbb{A}^4/\mathbb{G}_m] \to X$ over O is the union of two one-dimensional stacks $[V(x_1, x_3)/\mathbb{G}_m] \cup [V(x_2, x_4)/\mathbb{G}_m]$. Hence Z must be contained in $X \smallsetminus \{O\}$. This condition is sufficient for Z to be the image of a topologically strong substack because the good moduli space map $[\mathbb{A}^4 \smallsetminus \{(0,0,0,0)\}/\mathbb{G}_m] \to X \smallsetminus \{O\}$ is an isomorphism.

The reasoning at the end of Example 3.10 can be extended to any stable stack:

Corollary 3.11. *With the notation as in Proposition* 3.8, *if* $\dim Z = 0, 1$ *then* Z *is the image of a topologically strong cycle if and only if* $Z \subseteq X \smallsetminus X_{n-1} = \pi(\mathcal{X}^{\mathrm{s}})$.

We end with a lemma that will be used in the following section. Before we state the result, we establish some notation. Suppose $\mathcal{X}$ is an Artin stack with stable good moduli space $\pi\colon \mathcal{X} \to X$ and $\mathcal{Z} \subseteq \mathcal{X}$ is a strong (resp. topologically) strong substack. Let $Z = \pi(\mathcal{Z})$. By [Alp, Lem. 4.14] the restriction of π to a morphism $\pi_{\mathcal{Z}}\colon \mathcal{Z} \to Z$ is a good moduli space morphism. Moreover, by Proposition 3.8 we know $\mathcal{Z} \cap \mathcal{X}^{\mathrm{s}} \neq \varnothing$, so $\pi_{\mathcal{Z}}\colon \mathcal{Z} \to Z$ is also stable.

Lemma 3.12. *Let* $\pi\colon \mathcal{X} \to X$ *be an Artin stack with stable good moduli space and let* $\mathcal{Z}$ *be a strong* (*resp. topologically strong*) *substack of* $\mathcal{X}$. *Let* $\mathcal{W} \subseteq \mathcal{Z}$ *be a strong* (*resp. topologically strong*) *substack of* $\mathcal{Z}$. *Then* $\mathcal{W}$ *is a strong* (*resp. topologically strong*) *substack of* $\mathcal{X}$.

Proof. First suppose that $\mathcal{Z}$ is strong in $\mathcal{X}$ and that $\mathcal{W}$ is strong in $\mathcal{Z}$. Let $Z = \pi(\mathcal{Z})$, $W = \pi_{|\mathcal{Z}}(\mathcal{W})$. By definition $\operatorname{codim}_{\mathcal{Z}} \mathcal{X} = \operatorname{codim}_Z X$, $\operatorname{codim}_{\mathcal{W}} \mathcal{Z} = \operatorname{codim}_W Z$ and $\mathcal{Z} = Z \times_X \mathcal{X}$, and $\mathcal{W} = W \times_Z \mathcal{Z}$. Therefore $\operatorname{codim}_{\mathcal{W}} \mathcal{X} = \operatorname{codim}_W X$ and $\mathcal{W} = W \times_X \mathcal{X}$, so $\mathcal{W}$ is strong in $\mathcal{X}$.

Now suppose $\mathcal{Z}$ is only topologically strong in $\mathcal{X}$ and that $\mathcal{W}$ is topologically strong in $\mathcal{Z}$. To show that $\mathcal{W}$ is topologically strong in $\mathcal{X}$ we need to show that $(W \times_X \mathcal{X})_{\mathrm{red}} = \mathcal{W}$. If we denote the fiber product $Z \times_X \mathcal{X}$ by $\mathcal{Z}'$, this is equivalent to showing that $(W \times_Z \mathcal{Z}')_{\mathrm{red}} = \mathcal{W}$.

Since $\mathcal{Z}$ is topologically strong in $\mathcal{X}$, we know that $\mathcal{Z}'_{\mathrm{red}} = \mathcal{Z}$. We also know that $(W \times_Z \mathcal{Z})_{\mathrm{red}} = \mathcal{W}$ because $\mathcal{W}$ is topologically strong in $\mathcal{Z}$. Since the closed immersion of stacks $\mathcal{Z} \subseteq \mathcal{Z}'$ is a homeomorphism, the map obtained by base change $(W \times_Z \mathcal{Z}) \to (W \times_Z \mathcal{Z}')$ is a closed immersion of stacks which is also a homeomorphism. It follows that they have the same reduced induced substack structure in $\mathcal{X}$. Therefore $\mathcal{W}$ is topologically strong in $\mathcal{X}$. $\square$

4. The pullback of an operational class is topologically strong

In this section we prove part of Theorem 1.1, namely that the pullback of operational classes are topologically strong. For this theorem we only require that $\pi\colon \mathcal{X} \to X$ is a stable good moduli space morphism.

Theorem 4.1. *Let $\pi\colon \mathcal{X} \to X$ be a stable good moduli space morphism with $\mathcal{X}$ irreducible. If $c \in A^k_{\mathrm{op}}(X)_{\mathbb{Q}}$ then $\pi^* c \cap [\mathcal{X}]$ is represented by a topologically strong cycle.*

Remark 4.2. Note that Theorem 4.1 holds without the assumption that the stack $\mathcal{X}$ is smooth.

We begin with a series of lemmas, keeping the notation of the theorem throughout this section.

Lemma 4.3. *Let $\mathcal{X}$ be an integral stack and let $\pi\colon \mathcal{X} \to X$ be a stable good moduli space morphism. If X has an ample line bundle and L is any line bundle on X, then $c_1(\pi^* L) \cap [\mathcal{X}]$ is a strong cycle.*

Proof. Since X has an ample line bundle we can write $L = L_1 \otimes L_2^{-1}$ with L_1 and L_2 very ample, see, e.g., [Har, Exercise II.7.5]. Thus, it suffices to prove that $c_1(\pi^* L) \cap [\mathcal{X}]$ is strong when L is very ample. Let $\mathcal{X}^{\mathrm{s}}$ be the open set of stable points and let $X^{\mathrm{s}} = \pi(\mathcal{X}^{\mathrm{s}})$ be its good moduli space. The dimension of the stabilizers is constant on $\mathcal{X}^{\mathrm{s}}$ and so it is a gerbe over a tame stack $(\mathcal{X})^{\mathrm{s}}_{\mathrm{tame}}$ whose coarse space is X^{s}. Since both a gerbe morphism and a coarse moduli space morphism are homeomorphisms of Zariski spaces the good moduli space morphism $\pi_{|\mathcal{X}^{\mathrm{s}}}$ induces a homeomorphism $|\mathcal{X}^{\mathrm{s}}| \to |X^{\mathrm{s}}|$.

If $\dim X = 1$, then we can choose a divisor $D = P_1 + \cdots + P_d$ where $P_1, \ldots, P_d$ are distinct points contained in the intersection $X^{\mathrm{s}} \cap X^{\mathrm{sm}}$ such that $L = L(D)$; here X^{sm} denotes the smooth locus of X, which is non-empty because X is reduced and we work over an algebraically closed field. In this case the inverse image of each P_i is a strong stacky point $\mathcal{P}_i$ in $\mathcal{X}$ and so $c_1(\pi^* L) \cap [\mathcal{X}] = [\mathcal{P}_1] + \cdots + [\mathcal{P}_d]$ is strong.

Next suppose $\dim X > 1$. Since $\mathcal{X}^{\mathrm{s}}$ is dense in $\mathcal{X}$, its complement contains finitely many (possibly zero) divisors $\mathcal{D}_1, \ldots, \mathcal{D}_r$ of $\mathcal{X}$. Let $W_i = \pi(\mathcal{D}_i)$. Since L is very ample we can choose a Cartier divisor D with $|D|$ integral such that $L = L(D)$. Moreover, we can choose D so that $D \cap X^{\mathrm{s}} \neq \varnothing$ and D does not contain any of the W_i. The class $c_1(\pi^* L) \cap [\mathcal{X}]$ is represented by the Cartier divisor $\pi^{-1}(D)$. Let $\mathcal{D}$ be the closure of $\pi^{-1}(D \cap X^{\mathrm{s}})$. Since the map $|\mathcal{X}^{\mathrm{s}}| \to |X|$ a homeomorphism, $\mathcal{D} = \overline{\pi^{-1}(D \cap X^{\mathrm{s}})}$ is irreducible because $\pi^{-1}(D \cap X^{\mathrm{s}})$ is irreducible.

We claim that $\mathcal{D} = \pi^{-1}(D)$. Upon showing this, we are done because $\pi^{-1}(D \cap X^{\mathrm{s}})$ is integral and has the same dimension as D, so the same is true for $\mathcal{D}$. It follows that $\pi^{-1}(D) = \mathcal{D}$ is strong, so $c_1(\pi^* L) \cap [\mathcal{X}]$ is as well.

It remains to prove the claim that $\mathcal{D} = \pi^{-1}(D)$. First note that $\mathcal{D} \subseteq \pi^{-1}(D)$, and moreover $\mathcal{D} \cap \mathcal{X}^{\mathrm{s}} = \pi^{-1}(D) \cap \mathcal{X}^{\mathrm{s}}$. So if $\mathcal{D}$ and $\pi^{-1}(D)$ do not have the same support, $\pi^{-1}(D)$ must contain an irreducible component in $\mathcal{X} \smallsetminus \mathcal{X}^{\mathrm{s}}$. Since $\pi^{-1}(D)$ is a Cartier divisor, it would have to contain a divisorial component of $\mathcal{X} \smallsetminus \mathcal{X}^{\mathrm{s}}$, but this is not possible since D does not contain any of the W_i. As a result, $\mathcal{D}_{\mathrm{red}} = \pi^{-1}(D)_{\mathrm{red}}$. As $\pi^{-1}(D)$ is Cartier, it has no embedded components. Thus $\pi^{-1}(D)$ agrees with $\mathcal{D}$ up to multiplicity since they both have the same support and this support is irreducible. Since $\mathcal{D}$ is reduced and $\pi^{-1}(D) \cap \mathcal{X}^{\mathrm{s}} = \mathcal{D} \cap \mathcal{X}^{\mathrm{s}}$, these multiplicities agree and are equal to 1. □

Remark 4.4. If $\mathcal{X}$ is not reduced, the proof of Lemma 4.3 still shows that $c_1(\pi^*L)\cap[\mathcal{X}]$ is topologically strong.

Proposition 4.5. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ X' & \xrightarrow{g} & X \end{array} \tag{4.6}$$

be a cartesian diagram. We assume π and π' are stable good moduli space morphisms, $\mathcal{X}'$ and $\mathcal{X}$ are irreducible, and f and g are proper. Then for all $\alpha \in A^*_{\mathrm{tst}}(\mathcal{X}'/X')$, the proper pushforward $f_*\alpha$ is contained in $A^*_{\mathrm{tst}}(\mathcal{X}/X)$.

Proof. We can assume that $\alpha = [\mathcal{Z}']$ where $\mathcal{Z}'$ is a topologically strong integral substack. Let $\mathcal{Z} = f(\mathcal{Z}')$ with its reduced substack structure. We may also assume that $\dim\mathcal{Z} = \dim\mathcal{Z}'$ since otherwise $f_*[\mathcal{Z}'] = 0$.

Since $\mathcal{Z}'$ is topologically strong the same argument used in the proof of Proposition 3.8 in the properly stable case shows that $\mathcal{Z}'\cap(\mathcal{X}')^{\mathrm{s}} \neq \varnothing$, where $(\mathcal{X}')^{\mathrm{s}}$.

Claim:

$$f^{-1}(\mathcal{X}^{\mathrm{s}}) = (\mathcal{X}')^{\mathrm{s}}.$$

If x' is any closed point of $\mathcal{X}$ set $\mathrm{x} = f(\mathrm{x}')$, $x' = \pi'(\mathrm{x}')$, and $x = \pi(\mathrm{x})$. Then we have the following diagram of stacks and good moduli spaces where all squares are cartesian:

$$\begin{array}{ccccc} \pi'^{-1}(x') & \hookrightarrow & \pi'^{-1}(f^{-1}(\mathrm{x})) & \xrightarrow{f} & \pi^{-1}(x) \\ \downarrow & & \downarrow & & \downarrow \\ x' & \hookrightarrow & g^{-1}(x) & \xrightarrow{g} & x \end{array}\ . \tag{4.7}$$

Now if $\mathrm{x}' \in f^{-1}(\mathcal{X}^{\mathrm{s}})$ then $|\pi^{-1}(x)| = \mathrm{x}$. By base change along the finite morphism of $x' \to x$, we know that $\pi'^{-1}(x') \to \pi^{-1}(x)$ is a finite morphism. Since $|\pi^{-1}(x)|$ is a single point, it follows that $|\pi'^{-1}(x')|$ is discrete. However, we also know that $|\pi'^{-1}(x')|$ has a unique closed point, so it must be a singleton. Hence x' is saturated.

Now suppose $\mathrm{x}' \in (\mathcal{X}')^{\mathrm{s}}$. We wish to show that $\mathrm{x} = f(\mathrm{x}')$ is also stable; i.e., that $|\pi^{-1}(\pi(\mathrm{x}))|$ consists of the single point x.

Again we refer to diagram (4.7). The map of points $x' \to x$ is surjective, so $|\pi^{-1}(x)| \to |\pi'^{-1}(x')|$ is surjective as well. By assumption that x' is stable, $\pi^{-1}(x')$ consists of a single point, so therefore $|\pi^{-1}(x)|$ does as well. Hence, $f((\mathcal{X}')^{\mathrm{s}}) \subset \mathcal{X}^{\mathrm{s}}$. This proves our claim. □

Given the claim it follows that $\mathcal{Z} = f(\mathcal{Z}')$ has non-empty intersection with the open substack $\mathcal{X}^{\mathrm{s}}$. Thus $\operatorname{codim}_X \pi(\mathcal{Z}) = \operatorname{codim}_{\mathcal{X}} \pi(\mathcal{Z}\cap\mathcal{X}^{\mathrm{s}}) = \operatorname{codim}_{\mathcal{X}} \mathcal{Z}$.

Let $Z = \pi(\mathcal{Z})$ and let $X^{\mathrm{s}} = \pi(\mathcal{X}^{\mathrm{s}})$. Since $\mathcal{X}^{\mathrm{s}} \to X^{\mathrm{s}}$ factors as gerbe morphism followed by a coarse moduli space map for a tame stack, we know that $\pi^{-1}(Z\cap X^{\mathrm{s}})_{\mathrm{red}} = \mathcal{Z}\cap\mathcal{X}^{\mathrm{s}}$. Thus, to prove that $\pi^{-1}(Z)_{\mathrm{red}} = \mathcal{Z}$ it suffices to show that $\pi^{-1}(Z)$ is irreducible.

Since $\mathcal{Z}' = {\pi'}^{-1}(Z')_{\mathrm{red}}$ we see $\mathcal{Z}'$ and ${\pi'}^{-1}(Z')$ have the same image under the morphism f. Hence, we can replace X' by Z' and $\mathcal{X}'$ by $\mathcal{Z}'$. That is, we may assume that f, g are surjective and prove that $\pi^{-1}(g(X'))$ is irreducible.

Since the diagram is cartesian (and thus commutative) the map $f\colon \mathcal{X}' \to \mathcal{X}$ factors through the closed substack $\pi^{-1}(g(X')) \subseteq \mathcal{X}$. Thus, by the universal property of scheme-theoretic images, we have a closed immersion $i\colon f(\mathcal{X}') \to \pi^{-1}(g(X'))$. We therefore have a commutative diagram

$$\begin{array}{ccccccc} \mathcal{X}' & \overset{f}{\twoheadrightarrow} & f(\mathcal{X}') & \overset{i}{\hookrightarrow} & \pi^{-1}(g(X')) & \hookrightarrow & \mathcal{X} \\ {\scriptstyle h}\downarrow & & & & \downarrow & & \downarrow{\scriptstyle \pi} \\ X' & & \overset{g}{\twoheadrightarrow} & & g(X') & \hookrightarrow & X \end{array} .$$

Since the right-hand square is cartesian, and the outer rectangle is cartesian, we see the left-hand rectangle is cartesian as well. Therefore, the composite $i \circ f$ is surjective by base change. Hence i is a surjective closed embedding. Hence $f(\mathcal{X}') = \pi^{-1}(g(X'))$ as closed subsets of $\mathcal{X}$. Therefore, $\pi^{-1}(g(X'))$ is irreducible. $\square$

Proposition 4.8. *Let $\pi\colon \mathcal{X} \to X$ be a stable good moduli space morphism with X a separated algebraic space and $\mathcal{X}$ irreducible. If E is a vector bundle on X, then $c_i(\pi^* E) \cap [\mathcal{X}]$ is a topologically strong cycle. More generally, if $q(E_1, \ldots, E_r)$ is a polynomial in the Chern classes of vector bundles $E_1, \ldots, E_r$ then $\pi^* q(E_1, \ldots, E_r) \cap [\mathcal{X}]$ is a topologically strong cycle.*

Proof. We first use induction to prove the special case that $c_1(\pi^* L)^m \cap [\mathcal{X}]$ is topologically strong for all line bundles L, all exponents m, and all stable good moduli space morphisms $\pi\colon \mathcal{X} \to X$ with X a separated algebraic space. We begin with the base case $m = 1$. By Chow's lemma [Stacks, Tag 089K], there is a scheme X' which has an ample line bundle and a proper morphism $g\colon X' \to X$ which is an isomorphism over a dense open set. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \overset{f}{\longrightarrow} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \overset{g}{\longrightarrow} & X \end{array} .$$

If L is any line bundle on X, then Lemma 4.3 shows that $c_1({\pi'}^* L') \cap [\mathcal{X}']$ is represented by a strong cycle on $\mathcal{X}'$ where $L' = g^* L$. Then by Proposition 4.5, we know that $f_*(c_1({\pi'}^* L') \cap [\mathcal{X}'])$ is topologically strong. Since ${\pi'}^* L' = f^* \pi^* L$, by the projection formula $f_*(c_1({\pi'}^* L') \cap [\mathcal{X}']) = c_1(\pi^* L) \cap f_*[\mathcal{X}'] = c_1(\pi^* L) \cap [\mathcal{X}]$.

We now show $c_1(\pi^* L)^{m+1} \cap [\mathcal{X}]$ is topologically strong assuming the result for m. By assumption, we can write $c_1(\pi^* L)^m \cap [\mathcal{X}] = \sum a_i [\mathcal{Z}_i]$ where the $\mathcal{Z}_i$ are topologically strong substacks of $\mathcal{X}$. Let $Z_i = \pi(\mathcal{Z}_i)$ and $\pi_i = \pi|_{\mathcal{Z}_i}$. By [Alp, Lemma 4.14] $\pi_i\colon \mathcal{Z}_i \to Z_i$ is a good moduli space. Since $\mathcal{Z}_i \cap \mathcal{X}^{\mathrm{s}} \neq \varnothing$ we know that π_i is also a stable good moduli space morphism. By the case $m = 1$ of our statement we know that, $\pi_i^* L|_{Z_i} \cap [\mathcal{Z}_i]$ is a represented by a topologically strong cycle on $\mathcal{Z}_i$.

Now note that

$$c_1(\pi^*L)^{m+1} \cap [\mathcal{X}] = \sum_i a_i \, c_1(\pi^*L) \cap [\mathcal{Z}_i] = \sum_i a_i \, c_1(\pi_i^*L|_{Z_i}) \cap [\mathcal{Z}_i].$$

Since $c_1(\pi_i^*L|_{Z_i}) \cap [\mathcal{Z}_i]$ is topologically strong on $\mathcal{Z}_i$, it is also topologically strong on $\mathcal{X}$ by Lemma 3.12. This proves the result for $c_1(\pi^*L)^{m+1} \cap [\mathcal{X}]$.

Now let E be a vector bundle of rank r on X. We show that $c_i(\pi^*E) \cap [\mathcal{X}]$ is topologically strong using the construction of Chern classes via Segre classes, see [Ful, Chap. 3].[4] Consider the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}(\pi^*E) & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}(E) & \longrightarrow & X \end{array} .$$

Since we have already established the proposition for powers of the first Chern class for all good moduli space morphisms, applying this to the morphism η shows that $c_1(\eta^*\mathcal{O}_{\mathbb{P}(E)(1)})^{i+r-1} \cap [\mathbb{P}(\pi^*E)]$ is topologically strong. Then by Proposition 4.5, the Segre classes

$$s_i(\pi^*E) \cap [\mathcal{X}] = f_*(c_1(\eta^*\mathcal{O}_{\mathbb{P}(E)}(1))^{i+r-1} \cap [\mathbb{P}(\pi^*E)])$$

are topologically strong. Since the Chern classes of E are polynomials in the Segre classes, the proposition follows by induction on the degree of the polynomial in the same manner as the argument for powers of the first Chern class. □

Proof of Theorem 4.1. By Chow's lemma and resolution of singularities[5] there is a smooth scheme X' with ample line bundle and a proper morphism $g\colon X' \to X$ which is an isomorphism on a dense open set. Consider the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ X' & \xrightarrow{g} & X \end{array} .$$

Let $c \in A^*_{\mathrm{op}}(X)_{\mathbb{Q}}$. By the Riemann–Roch theorem for smooth schemes, $A^*_{\mathrm{op}}(X')_{\mathbb{Q}}$ is generated by Chern classes of vector bundles. Then g^*c is a rational polynomial in Chern classes of vector bundles on X', so by Proposition 4.8, $\pi'^*g^*c \cap [\mathcal{X}']$ is topologically strong. Then Proposition 4.5 says $f_*(\pi'^*g^*c \cap [\mathcal{X}'])$ is topologically strong. Lastly, notice by the projection formula that $f_*(\pi'^*g^*c \cap [\mathcal{X}']) = f_*(f^*\pi^*c \cap [\mathcal{X}']) = \pi^*c \cap [\mathcal{X}]$. □

[4]One may alternatively proceed by using the splitting principle.

[5]This proof also works in positive characteristic if we use a smooth alteration instead of a resolution of singularities.

5. Injectivity of the pullback and applications

5.1. Proof of Theorem 1.1

To complete the proof of the theorem we need to show that π^* is injective when $\pi\colon \mathcal{X} \to X$ is properly stable.

Proof of the injectivity of π^*. We use induction on n which is the maximum dimension of a stabilizer of a point of $\mathcal{X}$. When $n = 0$ then $\mathcal{X}$ is tame and the pullback map $\pi^*\colon A^*_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^*(\mathcal{X})_{\mathbb{Q}}$ is an isomorphism.

Next, suppose the theorem holds for stacks with maximum dimension of a stabilizer of a point less than or equal to $n-1$. Let $\mathcal{Y} \subseteq \mathcal{X}$ be the locus with stabilizer dimension n and let $f\colon \mathcal{X}' \to X$ be the Reichstein transformation along $\mathcal{Y}$. By [ER] we know that the maximum dimension of the stabilizer of a point of $\mathcal{X}'$ is less than n. Let $\pi'\colon \mathcal{X}' \to X'$ be a good moduli space morphism and consider the induced commutative diagram:

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array} .$$

Since $g\colon X' \to X$ is proper and surjective, the pullback $g^*\colon A^*_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^*_{\mathrm{op}}(X')_{\mathbb{Q}}$ is injective by [Kim, Lem. 2.1]. Also by our induction hypothesis we know that $(\pi')^*\colon A^*_{\mathrm{op}}(X')_{\mathbb{Q}} \to A^*(\mathcal{X}')_{\mathbb{Q}}$ is injective. By commutativity of the diagram we conclude that $f^*\pi^*$ is injective, hence π^* is injective. □

5.2. Applications to Picard groups of good moduli spaces: proof of Theorem 1.2

The proof of Theorem 1.2 makes essential use of the following result.

Theorem 5.1. *Let $\pi\colon \mathcal{X} \to X$ be a properly stable good moduli space morphism with $\mathcal{X}$ smooth, and consider the commutative diagram*

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{X}) & \xrightarrow{\simeq} & A^1(\mathcal{X}) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ \mathrm{Pic}(X) & \longrightarrow & A^1_{\mathrm{op}}(X) \end{array} .$$

Then the images of $\mathrm{Pic}(X)$ *and* $A^1_{\mathrm{op}}(X)$ *in* $A^1(\mathcal{X})$ *agree.*

Proof. Let $c \in A^1_{\mathrm{op}}(X)$. Then $\pi^*c \in A^1(\mathcal{X}) \simeq \mathrm{Pic}(\mathcal{X})$ so it is represented by a line bundle Ł on $\mathcal{X}$. Showing that Ł is the pullback of a line bundle from X is equivalent to showing that the stabilizer actions on Ł are trivial. Let x be a point of $\mathcal{X}$ with stabilizer G_x. The map from the residual gerbe $i\colon BG_x \to \mathcal{X}$ induces a map of good moduli spaces making the diagram

$$\begin{array}{ccc} BG_x & \xrightarrow{i} & \mathcal{X} \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}\, k & \xrightarrow{j} & X \end{array}$$

commute. Then $i^*\text{Ł}$ is a line bundle on BG_x corresponding to a character χ of G_x. The stabilizer action of G_x on Ł is trivial if and only if χ is the trivial character. Note that $j^*c \in A^1_{\mathrm{op}}(\operatorname{Spec} k) = 0$ and so j^*c is represented by the trivial line bundle. Since $i^*\text{Ł} = \eta^* j^* c$, we see $i^*\text{Ł}$ is also the trivial line bundle, i.e., it corresponds to the trivial representation of G_x, and hence the character χ is trivial. □

Proof of Theorem 1.2. Let $\alpha\colon \operatorname{Pic}(X) \to A^1_{\mathrm{op}}(X)$ denote the natural map. Given an element $c \in A^1_{\mathrm{op}}(X)$, by Theorem 5.1 there exists $L \in \operatorname{Pic}(X)$ such that $\pi^*L = \pi^*c$ in $A^1(\mathcal{X})$. That is, $\pi^*\alpha(L) = \pi^*c$. Since $\pi^*\colon A^1_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^1(\mathcal{X})_{\mathbb{Q}}$ is injective by Theorem 1.1, we see $\alpha(L) = c$, and so α is surjective. Injectivity follows from Remark 1.4. □

6. Surjectivity results for the pullback map

6.1. The case of a moduli space with finite quotient singularities: proof of Theorem 1.7(a)

Let $\mathcal{Z}$ be a topologically strong integral substack of codimension-k in $\mathcal{X}$ with image $Z \subseteq X$. Since X has finite quotient singularities, $A^*(X)_{\mathbb{Q}} = A^*_{\mathrm{op}}(X)_{\mathbb{Q}}$. Let c_Z be the bivariant class in $A^k(Z \to X)$ corresponding to the refined intersection with Z. In other words, given a map $T \to X$ and $\alpha \in A_*(T)$ we set $c_Z \cap \alpha = [Z] \cdot \alpha \in A_{*-k}(Z \times_X T)$. Since $Z \to X$ is proper there is a pushforward $A^k(Z \to X)_{\mathbb{Q}} \to A^k_{\mathrm{op}}(X)_{\mathbb{Q}} = A^k(X)_{\mathbb{Q}}$ [Ful, 17.2 P2] and the image is the fundamental class of Z.

To compute the pullback of $[Z]$ we can compute $j_*(c_Z \cap [\mathcal{X}])$ where $j\colon \mathcal{Z} \to \mathcal{X}$ is the inclusion morphism. Now $c_Z \cap [\mathcal{X}]$ is a cycle of dimension equal to $\dim \mathcal{X} - k = \dim X - k = \dim Z$ supported on the fiber product $Z \times_X \mathcal{X}$. Since $\mathcal{Z}$ is assumed to be topologically strong, $(Z \times_X \mathcal{X})_{\mathrm{red}} = \mathcal{Z}$. Hence, $c_Z \cap [\mathcal{X}]$ is a multiple of the fundamental class in $A^0(\mathcal{Z})$. This multiple is necessarily non-zero because it is non-zero when we restrict to the non-empty open subset $\mathcal{X}^{\mathrm{ps}} \cap Z$. Hence the pullback of $[Z]$ in $A^*(\mathcal{X})$ is a non-zero multiple of $[\mathcal{Z}]$. In other words $[\mathcal{Z}]$ is in the image of $A^k_{\mathrm{op}}(X)_{\mathbb{Q}} = A^k(X)_{\mathbb{Q}}$.

6.2. The case of codimension-one cycles: proof of Theorem 1.7(b)

By Theorem 5.1 we know that the image of $A^1_{\mathrm{op}}(X)$ in $A^1(\mathcal{X})$ is the same as the image of $\operatorname{Pic}(X)$ under the map $L \mapsto c_1(\pi^*L) \cap [\mathcal{X}]$. Since X has an ample line bundle we know by Lemma 4.3 that $c_1(\pi^*L) \cap [\mathcal{X}]$ is strong. Hence the image of $A^1_{\mathrm{op}}(X)$ is contained in $A^1_{\mathrm{st}}(\mathcal{X}/X)$. Conversely, if $\mathcal{Z} \subseteq \mathcal{X}$ is a strong codimension-one substack, then $[\mathcal{Z}]$ is an effective Cartier divisor whose local equation in a neighborhood of every point x is G_x-fixed, where G_x denotes the stabilizer of x. Hence, the restriction of $\mathcal{O}_{\mathcal{X}}(\mathcal{Z})$ to BG_x is the trivial character. Thus, $\mathcal{O}_{\mathcal{X}}(\mathcal{Z})$ is the pullback of a line bundle on X, so $\pi^*A^1_{\mathrm{op}}(X) = A^1_{\mathrm{st}}(\mathcal{X}/X)$.

6.3. Strong regular cycles: proof of Theorem 1.7(c)

Let $\mathcal{Z} \subseteq \mathcal{X}$ be strong and regularly embedded. By [ES, Prop. 1.14], we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

and $Z \subseteq X$ is regularly embedded. Letting $K_\bullet(I_Z/I_Z^2)$ be the Koszul resolution of the conormal bundle, we have an equality $[\mathcal{O}_Z] = [K_\bullet(I_Z/I_Z^2)]$ in K-theory. Thus $\mathcal{O}_Z$ has a Chern character and $[Z] = \mathrm{ch}(\mathcal{O}_Z) \cap [X]$. Similarly, $\mathcal{O}_{\mathcal{Z}}$ has a Chern character and $[\mathcal{Z}] = \mathrm{ch}(\mathcal{O}_{\mathcal{Z}}) \cap [\mathcal{X}]$. To finish the proof notice that $\mathrm{ch}(\mathcal{O}_{\mathcal{Z}}) \cap [\mathcal{X}] = \mathrm{ch}(\pi^*\mathcal{O}_Z) \cap [\mathcal{X}]$ which is the pullback of an operational class. □

6.4. The case of small dimension: proof of Theorem 1.7(d) and Corollary 1.8

Let $\mathcal{Z} \subseteq \mathcal{X}$ be a topologically strong integral substack of dimension 0 or 1 and let $Z \subseteq X$ be its image in X. By Corollary 3.11 we know that $Z \subseteq X^{\mathrm{ps}}$ where X^{ps} is the image of the maximal saturated tame stack $\mathcal{X}^{\mathrm{s}}$ and moreover, $A^*_{\mathrm{op}}(X^{\mathrm{ps}})_{\mathbb{Q}} = A^*(X^{\mathrm{ps}})_{\mathbb{Q}} = A^*(\mathcal{X}^{\mathrm{ps}})_{\mathbb{Q}}$. Let $c_Z \in A^*(Z \to X^{\mathrm{ps}})_{\mathbb{Q}}$ be the bivariant class corresponding to the refined intersection with Z in X^{s}. In other words, given a map $T \to X^{\mathrm{s}}$ and $\alpha \in A^*(T)$, we define $c_Z \cap \alpha = [Z] \cdot \alpha \in A_*(Z \times_{X^{\mathrm{s}}} T)$. As was the case in the proof of Theorem 1.7(a), $c_Z \cap [\mathcal{X}^{\mathrm{ps}}]$ is necessarily a non-zero multiple of the fundamental class of $\mathcal{Z} = \pi^{-1}(Z)_{\mathrm{red}}$.

Since $Z \subseteq X$ is closed, c_Z has a canonical lift to a bivariant class $A^*_{\mathrm{op}}(Z \to X)$ defined by the formula $\alpha \mapsto Z \cdot \alpha|_{X^{\mathrm{s}} \times_X T}$ and we let c be the direct image in $A^k_{\mathrm{op}}(X)$. Since $c_Z \cap [\mathcal{X}^{\mathrm{ps}}]$ is a non-zero multiple of $[\mathcal{Z}]$ in $A^k(\mathcal{X}^{\mathrm{ps}})$, it follows that $c \cap [\mathcal{X}]$ is a non-zero multiple of $[\mathcal{Z}]$ in $A^k(\mathcal{X})$. Hence, $[\mathcal{Z}]$ is in the image of $A^k_{\mathrm{op}}(X)_{\mathbb{Q}}$. Finally if $\mathcal{X}^{\mathrm{ps}}$ is representable then $\mathcal{X}^{\mathrm{ps}} = X^{\mathrm{ps}}$ so $\mathcal{Z} = Z$ is automatically strong.

Corollary 1.8 is now a consequence of what has already been proved. Clearly $A^0(\mathcal{X})_{\mathbb{Q}} = A^0_{\mathrm{st}}(\mathcal{X}/X) = A^0(X) = \mathbb{Q}$ for any integral stack $\mathcal{X}$. If X has an ample line bundle then $A^1_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}} = A^1_{\mathrm{op}}(X)_{\mathbb{Q}}$ by Theorem 1.7(b). Since $\dim X \le 3$, $A^2_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}} = A^2_{\mathrm{op}}(X)_{\mathbb{Q}}$ and $A^3_{\mathrm{st}}(\mathcal{X}/X)_{\mathbb{Q}} = A^3_{\mathrm{op}}(X)_{\mathbb{Q}}$ by Theorem 1.7(d).

Example 6.1. (The strong Chow ring of the toric stack associated to the projectivized quadric cone) We consider the strong Chow ring of the toric stack $\mathcal{X}$ associated to the projectivized quadric cone which was studied in [ES, Example 3.12]. There we computed the integral strong Chow ring modulo a conjecture about $A^2_{\mathrm{st}}(\mathcal{X}/X)$. Using Corollary 1.8 we can verify this conjecture without qualification.

Precisely, we let $\mathcal{X} = [U/\mathbb{G}_m^2]$ where $U = \mathbb{A}^5 \smallsetminus V(x_1x_2, x_1x_4, x_2x_3, x_3x_4, v)$ and $\mathbb{G}_m^2$ acts diagonally on $\mathbb{A}^5$ with weight matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Here we have coordinates (x_1, x_2, x_3, x_4, v) on $\mathbb{A}^5$. Then the good moduli space of $\mathcal{X}$ is given by $X = \operatorname{Proj} k[x_1x_2, x_1x_4, x_2x_3, x_3x_4, v]$ which is the projective closure in $\mathbb{P}^4$ of the cone over the quadric surface in $\mathbb{P}^3$. In [ES, Example 3.12] we showed that $A^*(\mathcal{X}) = \mathbb{Z}[s,t]/(s^2(s+t), t^2(s+t))$ where s, t are the first Chern classes of the characters of $\mathbb{G}_m^2$ corresponding to projection onto the first or second factors. We also showed that $A^1_{\mathrm{st}}(\mathcal{X}/X)$ is the free abelian group generated by $s+t$, and that $A^3_{\mathrm{st}}(\mathcal{X}/X)$ is the free abelian group generated by $st(s+t)$. We showed that $A^2_{\mathrm{st}}(\mathcal{X}/X)$ contains the free group of rank two generated by $s(s+t)$ and $t(s+t)$ and conjectured that this was the entire relative strong Chow group in codimension 2. Since X is projective, by Corollary 1.8 we know that $\operatorname{rk} A^2_{\mathrm{op}}(X) = \operatorname{rk} A^2_{\mathrm{st}}(\mathcal{X}/X)$.

On the other hand, X is a toric variety so by [FS] we know that $A^2_{\mathrm{op}}(X) = \hom(A_2(X), \mathbb{Z})$. Since X is toric, it is easy to calculate by hand that the group $A_2(X)$ has rank 2. Thus $\operatorname{rk} A^2_{\mathrm{st}}(\mathcal{X}/X) = 2$. Moreover, the lattice $\langle s(s+t), t(s+t)\rangle$ is primitive in the rank 3 free abelian group $A^2(\mathcal{X})$, so we conclude $A^2_{\mathrm{st}}(\mathcal{X}/X) = \langle s(s+t), t(s+t)\rangle$.

Note that the classes $s(s+t)$, and $t(s+t)$ which correspond to distinct operational classes are represented by the cycles $[V(x_1, v)/\mathbb{G}_m]$ and $[V(x_2, v)]/\mathbb{G}_m$. These cycles have rationally equivalent images in $A^2(X)$. This illustrates the power of viewing the singular toric variety X together with the smooth quotient stack $\mathcal{X}$.

7. Conjectures and speculations

We end this paper with some questions and conjectures about the image of $A^*_{\mathrm{op}}(X)$ in $A^*(\mathcal{X})$.

Theorem 1.1 states that $A^*_{\mathrm{op}}(X)_{\mathbb{Q}}$ injects into $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. However, we know that the image need not equal $A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. This follows from Example 3.7 where we constructed a topologically strong Cartier divisor that is not strong. Since the image of $A^1_{\mathrm{op}}(X)$ equals $A^1_{\mathrm{st}}(\mathcal{X}/X)$, (Theorem 1.7(b)) we see that not every topologically strong cycle is a pullback of an operational class.

This of course raises the question: what is the image of π^*? We state a conjectural answer after first introducing some definitions.

Definition 7.1. Let $\mathcal{X}$ be an Artin stack and $\pi\colon \mathcal{X} \to X$ a stable good moduli space. We say an integral substack $\mathcal{Z} \subseteq \mathcal{X}$ is *primarily strong* if $\mathcal{Z}$ is topologically strong and $\pi^{-1}(\mathcal{Z})$ has no embedded components. If $\mathcal{X}$ has pure dimension, then we define $A^k_{\mathrm{pst}}(\mathcal{X}/X)$ to be the subgroup of $A^k(\mathcal{X})$ generated by primarily strong integral substacks of codimension k.

Remark 7.2. The proof of Lemma 3.3 also gives an étale local characterization of its primarily strong substacks. As in Remark 3.4, we see that primarily strong substacks are those that étale locally have the following form: $\mathcal{Z} \subseteq \mathcal{X} = [U/G]$ with G the stabilizer of a point, $U = \operatorname{Spec} A$, and $\mathcal{Z}$ is defined by an ideal $I \subseteq A$ with $\dim A/I = \dim A^G/I^G$ and such that there is an I-primary ideal J satisfying $J = (f_1, \ldots, f_r)$ with each $f_i \in A^G$.

Conjecture 7.3. *Let $\mathcal{X}$ be a smooth connected properly stable Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$. Then the injection $\pi^*\colon A^*_{\mathrm{op}}(X)_{\mathbb{Q}} \to A^*_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$ induces an isomorphism $A^*_{\mathrm{op}}(X)_{\mathbb{Q}} \simeq A^*_{\mathrm{pst}}(\mathcal{X}/X)_{\mathbb{Q}}$.*

We prove Conjecture 7.3 in several cases by giving a minor strengthening of Theorem 1.7.

Theorem 7.4. *Let $\mathcal{X}$ be a properly stable smooth Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$. Then the following hold:*

(a) *Conjecture* 7.3 *holds if X is smooth.*
(b) *If X has an ample line bundle, then Conjecture* 7.3 *holds in codimension* 1.
(c) *Assume k equals* $\dim X - 1$ *or* $\dim X$. *If the maximal saturated tame substack of $\mathcal{X}$ is representable, then Conjecture* 7.3 *holds in codimension k.*

Proof. We begin with case (a). By Theorem 1.7(a), it suffices to prove that we have $A^*_{\text{tst}}(\mathcal{X}/X)_{\mathbb{Q}} = A^*_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}}$. Since $\mathcal{X}$ and X are smooth, π is an lci morphism. Furthermore, it has relative dimension 0 since the maximal saturated tame substack $\mathcal{U} \subseteq \mathcal{X}$ is non-empty and $\mathcal{U} \to \pi(\mathcal{U})$ has relative dimension 0. Thus, π locally factors as $\mathcal{X} \xrightarrow{j} \mathcal{Y} \xrightarrow{p} X$ with j a regular embedding of codimension e and p a smooth morphism of relative dimension e. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a topologically strong substack of dimension d and let $Z = \pi(\mathcal{Z})$. By hypothesis, $\dim Z = \dim \mathcal{Z} = d$. Then $p^{-1}(Z)$ has dimension $d+e$ and $\pi^{-1}(Z) = j^{-1}p^{-1}(Z)$ is cut out by e equations within $p^{-1}(Z)$. As a result, each component (irreducible or embedded) of $\pi^{-1}(Z)$ has dimension at least d. Since $\mathcal{Z}$ is topologically strong, $\mathcal{Z} = \pi^{-1}(Z)_{\text{red}}$ and so $\pi^{-1}(Z)$ has exactly one irreducible component of dimension d, As a result, $\pi^{-1}(Z)$ has no embedded components since such components would have dimension less than d.

To handle case (b), it suffices by Theorem 1.7(b) to show that $A^1_{\text{st}}(\mathcal{X}/X)_{\mathbb{Q}} = A^1_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}}$. Let $\mathcal{Z} \subseteq \mathcal{X}$ be a primarily strong divisor. Since $\mathcal{X}$ is smooth, $\mathcal{Z}$ is Cartier, and so some multiple of its local equation is invariant under the stabilizer actions. As a result, $n[\mathcal{Z}] = c_1(\pi^*\text{Ł}) \cap [\mathcal{X}]$ for some line bundle Ł on X. Then Lemma 4.3 tells us that $n[\mathcal{Z}]$ is strong.

Lastly, we turn to case (c). By the first assertion of Theorem 1.7(d), we need only show $A^k_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}} = A^k_{\text{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. Since $A^k_{\text{st}}(\mathcal{X}/X)_{\mathbb{Q}} \subseteq A^k_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}} \subseteq A^k_{\text{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$, the second assertion of Theorem 1.7(d) shows that $A^k_{\text{st}}(\mathcal{X}/X)_{\mathbb{Q}} = A^k_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}} = A^k_{\text{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$. □

Remark 7.5. One way to partially verify Conjecture 7.3 is to generalize Proposition 4.5 by proving that if

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

is a cartesian diagram of properly stable stacks and good moduli spaces, then the image of a primarily strong cycle is primarily strong. Using the methods of Section 4 this would imply that the image of $A^*_{\text{op}}(X)$ is contained in $A^*_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}}$.

Example 7.6. We note it is not obvious how to generalize Theorem 7.4(a) from the case where X is smooth to the case where it has finite quotient singularities. The reason for this is that even when the stack $\mathcal{X}$ is smooth and tame, topologically strong substacks $\mathcal{Z} \subseteq \mathcal{X}$ need not be primarily strong. For example, let $\mathcal{X} = [\mathbb{A}^2/(\mathbb{Z}/2)]$ with $\mathbb{Z}/2$ acting on $\mathbb{A}^2$ by $(x,y) \mapsto (-x,-y)$. Then $X = \operatorname{Spec} k[x^2, xy, y^2]$. If $\mathcal{Z} = V(x)$, then $\pi^{-1}(\pi(Z)) = V(x^2, xy)$. Notice that $\mathcal{Z}$ is topologically strong since the reduction of $V(x^2, xy)$ is $V(x)$; however, $V(x^2, xy)$ has an embedded point so $\mathcal{Z}$ is not primarily strong.

We end the paper with two questions:

Question 7.7. *If $\mathcal{X}$ is a smooth connected properly stable Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$, then are $A^*_{\text{st}}(\mathcal{X}/X)_{\mathbb{Q}}$ and $A^*_{\text{pst}}(\mathcal{X}/X)_{\mathbb{Q}}$ equal?*

In general, if $k > 0$ and $\alpha \in A^k(\mathcal{X})$ is the image of an operational class $c \in A^k_{\text{op}}(X)$ then α must, in addition to being represented by a topologically strong cycle, satisfy $\alpha \cdot [BG_{\mathrm{x}}] = 0$ for any residual gerbe $BG_{\mathrm{x}} \subseteq \mathcal{X}$ where x is closed point of $\mathcal{X}$. The reason for this is as follows: let x be the image of x in X and let p be the restriction of π to a morphism $\mathrm{x} \to x$. Functorial properties of operational Chow groups imply that $\alpha \cap [BG_{\mathrm{x}}] = p^*(c \cap [x])$ where p^* is the flat pullback. However, $c \cap [x] = 0$ as $c \in A^k_{\mathrm{op}}(X)$ with $k > 0$ and $[x]$ is a zero-cycle.

We ask whether this necessary condition is sufficient.

Question 7.8. *Let $k > 0$ and $\mathcal{X}$ be a smooth connected properly stable Artin stack with good moduli space $\pi\colon \mathcal{X} \to X$. Suppose $\alpha \in A^k_{\mathrm{tst}}(\mathcal{X}/X)_{\mathbb{Q}}$ and $\alpha \cdot BG_{\mathrm{x}} = 0$ for all residual gerbes $BG_{\mathrm{x}} \subseteq \mathcal{X}$. Then do we have $\alpha = \pi^* c$ for some $c \in A^k_{\mathrm{op}}(X)_{\mathbb{Q}}$? Do we have $\alpha \in A^k_{\mathrm{pst}}(\mathcal{X}/X)_{\mathbb{Q}}$?*

A natural testing ground for the conjectures and questions raised in this section is the theory of toric Artin stacks [GS1], [GS2].

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