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Toric Geometry

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Abstract. Toric geometry is a subfield of algebraic geometry with deep intersections with combinatorics. This workshop brought together researchers working in toric geometry, applying toric geometry elsewhere in algebraic geometry, and applying toric geometry elsewhere inside and outside mathematics.

Mathematics Subject Classification (2010): 14M25.

Introduction by the Organisers

Toric geometry is a subfield of algebraic geometry with deep intersections with combinatorics. A toric variety X is a partial compactification of the algebraic torus $T \cong (\mathbb{C}^*)^n$ with an action of T that extends the action of T on itself. Behind this simple definition, however, is a striking combinatorial dictionary that relates algebro-geometric invariants of the variety X to geometric-combinatorial invariants of an associated lattice polytope or polyhedral fan. This bridge between the two fields has made toric geometry to an important source of examples and counterexamples in algebraic geometry.

Toric techniques also have applications in other areas, both inside algebraic geometry, in other areas of mathematics, and outside mathematics. Examples inside algebraic geometry include the study of Mori Dream Spaces, varieties with torus actions, Newton-Okounkov bodies, tropical geometry, and degenerations to toric varieties. There are also strong connections to string theory and symplectic geometry, and increasing ties to arithmetic geometry and commutative algebra. Finally, toric varieties also have applications outside mathematics, in areas as diverse as statistics, coding theory, computer modelling, and chemistry.

This workshop brought together researchers working in all aspects of the subject. The talks presented current developments and recent results in "classical" toric geometry, toric-inspired topics, and the use of toric tools in other fields ranging from algebraic geometry via commutative algebra, topology and arithmetic geometry to applications.

Some of the broad themes covered were:

(1) Applications to combinatorics (Huh, Katz, Lasoń)
(2) Connections to number theory and topology (Gubler, De Cataldo)
(3) Applications outside mathematics: Dickenstein (biochemistry), He (physics), Michalek (statistics)
(4) Toric inspired algebraic geometry (Brion, Karu, Laface, Satriano)
(5) Algebraic aspects (Hering, Kaveh, Smith)
(6) Classical toric questions (Altmann, Arzhantsev, Brown, Di Rocco, Grassi, Ilten, Mustaţă, Teissier)

One aspect that we would like to highlight was an evening session on Tuesday of five minute talks by largely junior participants. The session was lively, and began with a five minute talk by Sturmfels, and finished with one by Batyrev. As with most Oberwolfach workshops, the informal conversations that followed the talks, over meals, during coffee breaks, and in the evening, also contributed to a rich scientific week, and we are grateful to Oberwolfach for facilitating that.

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Workshop: Toric Geometry

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Abstracts

The *F*-splitting ratio of a seminormal affine toric variety

Milena Hering
(joint work with Kevin Tucker)

The F-signature, or its more refined cousin, the F-splitting ratio, are measures of the singularities of a ring of characteristic p defined using the Frobenius endomorphism. We review the computation of the F-signature of a normal semigroup ring, and compute the F-splitting ratio of a seminormal semigroup ring.

Let R be a reduced Noetherian local or graded ring of prime characteristic p with perfect residue field. The powers of Frobenius act on such a ring, and we let $F^e_* R$ be the R-module whose underlying set is R with module structure given by $r \star s = r^{p^e} s$. We assume that R is F-finite, i.e., that $F^e_* R$ is module finite over R. Then $R = R^{a_e} \oplus M_e$, where M_e has no free direct summands.

Tucker proved in [6] that the limit

$$s(R) := \lim_{e\to\infty} \frac{a_e}{p^{ed}}$$

exists. It is called the *F-signature* of R and was originally defined by Huneke and Leuschke. It is a measure of the singularities of R. For example $s(R) = 1$ if and only if R is regular. And if $s(R) > 0$, then R is normal and Cohen-Macaulay. Moreover, if R is the invariant ring of a finite group G acting on a regular local ring, then $s(R) = \frac{1}{|G|}$.

The F-signature of an affine toric ring was computed by Bruns, Singh, and v. Korff [3, 5, 4]. To see how it works, we need to introduce a bit of notation. Let $M \cong \mathbb{Z}^n$ be a lattice and let $C \subset M_{\mathbb{R}}$ be a rational polyhedral cone. Then $S = M \cap C$ is a finitely generated normal semigroup. We denote by $k[S]$ the associated semigroup ring. It is a normal ring. Let $\sigma \subset N_{\mathbb{R}}$ be the dual cone to C. Then $\sigma = \langle v_1, \dots, v_r \rangle$, where v_i are the primitive generators of the rays of σ. To S we associate a polytope $P_S := \{u \in M \mid 0 \le \langle u, v_i \rangle \le 1 \text{ for } 1 \le i \le r\}$. Then $s(k[S])$ is the lattice volume of P_S.

When $s(R) = 0$, we can define a refined version of the F-signature, called the F-splitting ratio. Indeed, Tucker, based on work of Aberbach and Enescu [1], shows that for R as above with R F-split, (i.e., $a_1 \ge 1$), there exists a positive integer δ, called the *splitting dimension*, such that the limit

$$r(R) := \lim_{e\to\infty} \frac{a_e}{p^{e\delta}}$$

exists. Moreover, this limit is positive [2].

We now consider a semigroup ring that is not necessarily normal. If $k[S]$ is F-split, then $k[S]$ must be a seminormal ring. So for our purposes it suffices to consider seminormal rings. There is a combinatorial condition on S that determines whether $k[S]$ is seminormal. Let $C = \mathbb{R}_{\ge 0} S$, be the cone generated by S. Then S is seminormal if for every face D of C, there is a sublattice M_D of M such that $S \cap \mathrm{int}(D) = M_D \cap \mathrm{int}(D)$. One can then show that $k[S]$ is seminormal if and only if S is seminormal. Moreover, $k[S]$ is F-split if and only if, for every face D of C, p does not divide the index $[M\colon M_D]$. We call a face D *relatively unsaturated* (RUF) if

$$M_D \subsetneq \bigcap_{D \subsetneq D'} M_{D'}.$$

We show that for a F-split seminormal affine semigroup ring $k[S]$, the splitting dimension is given by $\delta = \dim \cap_{D\mathrm{RUF}} D$ and the F-splitting ratio by

$$r(k[S]) = \mathrm{Vol}\left(P_{\overline{S}} \cap \left(\bigcap_{D\mathrm{RUF}} D\right)\right),$$

where the volume is taken with respect to the lattice $\bigcap_{D\mathrm{RUF}} M_D$.

We also compute $\mathrm{Hom}(R, F^e_* R)$, even in the case when $k[S]$ is not F-split, and we use this to compute the test ideal.

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Toric degenerations and symplectic geometry of projective varieties

Kiumars Kaveh

We talked about some general results on symplectic geometry of smooth projective varieties from the recent preprint [5]. This approach relies on a construction of toric degenerations motivated by commutative algebra and the theory of Newton-Okounkov bodies ([9, 7, 6, 10]). At the end we briefly discussed some applications to symplectic topology, in particular obtaining lower bounds for the Gromov width and symplectic ball packings of smooth projective varieties.

Let us start with the toric degeneration result. Let X be a smooth complex projective variety of dimension n embedded in a projective space $\mathbb{CP}^N$. We construct a smooth family $\pi : \mathcal{X} \to \mathbb{C}$ together with an embedding into $\mathbb{CP}^N \times \mathbb{C}$ such that the general fiber of the family is X and the special fiber is the algebraic torus $(\mathbb{C}^*)^n$ embedded in $\mathbb{CP}^N$ via a monomial embedding (throughout $\mathbb{C}^*$ denotes $\mathbb{C} \setminus \{0\}$, the multiplicative group of nonzero complex numbers). Notice that the general fiber of the family is X and hence a projective variety, while the special fiber is $(\mathbb{C}^*)^n$ and not projective. In fact if we take the closure $\overline{\mathcal{X}}$ of the family in $\mathbb{CP}^N \times \mathbb{C}$, the general fiber is still X but the special fiber may become reducible while at least one of its irreducible components is an n-dimensional toric variety.

The construction of the family $\mathcal{X}$ is a generalization of the deformation to the normal cone in algebraic geometry. The construction of the embedding of $\mathcal{X}$ in $\mathbb{CP}^N \times \mathbb{C}$ depends on the choice of a $\mathbb{Z}^n$-valued valuation v on the field of rational functions on X.

Our construction of a degeneration in this general setting is motivated by the works of D. Anderson [1] and B. Teissier [10], as well as [3]. Interestingly the author learned about the latter work from Bernard Teissier in a previous Oberwolfach workshop.

We use the family $\mathcal{X}$ and its embedding in $\mathbb{CP}^N \times \mathbb{C}$ to obtain results about the symplectic geometry of X. Specifically we prove the following: let ω be a Kähler form on X which is integral (i.e. its cohomology class lies in $H^2(X, \mathbb{Z})$). Then for any $\epsilon > 0$ there exists an open subset $U \subset X$ (in the usual classical topology) such that $vol(X \setminus U) < \epsilon$ and (U, ω) is symplectomorphic to $(\mathbb{C}^*)^n$ equipped with a toric Kähler form (Theorem 2 below).

Let us explain the above more precisely. Fix a finite set $\mathcal{A} = \{\beta_1, \ldots, \beta_r\} \subset \mathbb{Z}^n$ and a point $c = (c_1, \ldots, c_r) \in (\mathbb{C}^*)^r$. We assume that the set of differences of elements in $\mathcal{A}$ generates the lattice $\mathbb{Z}^n$ and hence the orbit map:

$$\psi_{\mathcal{A},c} : u \mapsto (u^{\beta_1} c_1 : \cdots : u^{\beta_r} c_r), \tag{1}$$

is an isomorphism of varieties from $(\mathbb{C}^*)^n$ to its image $O_{\mathcal{A},c} \subset \mathbb{CP}^{r-1}$. Here $u = (u_1, \ldots, u_n) \in (\mathbb{C}^*)^n$ and u^α is shorthand for $u_1^{a_1} \cdots u_n^{a_n}$, where $\alpha = (a_1, \ldots, a_n)$. The closure of $O_{\mathcal{A},c}$ is a (not necessarily normal) projective toric variety. The map $\psi_{\mathcal{A},c}$ also induces a Kähler form on $(\mathbb{C}^*)^n$ as follows: Consider the standard Hermitian product on $\mathbb{C}^r$. Let Ω be the associated Fubini-Study Kähler form on $\mathbb{CP}^{r-1}$ and let $\omega_{\mathcal{A},c}$ be the pull-back of Ω to $(\mathbb{C}^*)^n$ under the map $\psi_{\mathcal{A},c}$. The symplectic manifold $((\mathbb{C}^*)^n, \omega_{\mathcal{A},c})$ is a Hamiltonian space with respect to the natural action of the compact torus $(S^1)^n$ on $(\mathbb{C}^*)^n$ by multiplication. The image of its moment map is the interior of the convex hull of $\mathcal{A}$.

Now let $X \subset \mathbb{CP}^{r-1}$ be a smooth projective variety embedded in some projective space $\mathbb{CP}^{r-1}$. We construct a complex manifold $\mathcal{X}$ together with a holomorphic function $\pi : \mathcal{X} \to \mathbb{C}$, as well as a an embedding $\mathcal{X} \hookrightarrow \mathbb{CP}^{r-1} \times \mathbb{C}$ such that:

(a) The family $\mathcal{X}$ is trivial over $\mathbb{C}^*$ i.e. $\pi^{-1}(\mathbb{C}^*) \cong X \times \mathbb{C}^*$. In particular for each $t \neq 0$ we have $X_t := \pi^{-1}(t)$ is biholomorphic to X. Moreover, $X_1 \hookrightarrow \mathbb{CP}^{r-1} \times \{1\}$ coincides with the original embedding $X \hookrightarrow \mathbb{CP}^{r-1}$.

(b) The fiber $X_0 = \pi^{-1}(0)$ is the algebraic torus $(\mathbb{C}^*)^n$ embedded in $\mathbb{CP}^{r-1} \times \{0\}$ via a monomial map $\psi_{\mathcal{A},c}$ for some finite set $\mathcal{A} \subset \mathbb{Z}^n$ and $c \in (\mathbb{C}^*)^r$ as above.

(c) The map $\pi : \mathcal{X} \to \mathbb{C}$ has no critical points, i.e. $d\pi$ is nonzero at every point in $\mathcal{X}$.

Next consider a smooth projective variety X equipped with a very ample line bundle L. The line bundle L gives rise to the Kodaira embedding $X \hookrightarrow \mathbb{P}(H^0(X, L)^*)$. Let ω be a Kähler form in the class $c_1(L)$. We note that by

Moser's trick any two Kähler forms in $c_1(L)$ are symplectomorphic. In particular ω is symplectomorphic to the pull-back of a Fubini-Study Kähler form on the projective space $\mathbb{P}(H^0(X,L)^*)$ to X.

Using the family $\mathcal{X}$ above we prove the following:

Theorem 1. *There exists an open subset $U \subset X$ (in the usual classical topology) such that (U,ω) is symplectomorphic to $((\mathbb{C}^*)^n, \omega_{\mathcal{A},c})$, for some $\mathcal{A} \subset \mathbb{Z}^n$ and $c \in (\mathbb{C}^*)^r$ as above (i.e. $\mathcal{A}$ is a finite subset such that the differences of elements in $\mathcal{A}$ generate the lattice $\mathbb{Z}^n$).*

Consider a Kähler form on $(\mathbb{C}^*)^n$ of the form $\frac{1}{m}\omega_{\mathcal{A},c}$, where m is a positive integer. We call such a form a *rational toric Kähler form.* The following is our main result about the symplectic geometry of smooth projective varieties. It states that we can enlarge the open subset U in Theorem 1 as much as we wish provided that we consider rational toric Kähler forms on $(\mathbb{C}^*)^n$.

Let X be a smooth projective variety and let ω be a Kähler form on X which is integral (i.e. its cohomology class lies in $H^2(X,\mathbb{Z})$). We recall that by the Lefschetz theorem on $(1,1)$-classes any integral Kähler form is in $c_1(L)$ for an ample line bundle L.

Theorem 2. *For any $\epsilon > 0$ we can find an open subset $U \subset X$ such that $vol(X \setminus U) < \epsilon$ and (U,ω) is symplectomorphic to $(\mathbb{C}^*)^n$ equipped with a rational toric Kähler form.*

Roughly speaking, this result claims that *in symplectic category and over arbitrarily large open subsets, any smooth projective variety looks like a toric variety equipped with a toric Kähler form.*

Finally we apply the above to problems in symplectic geometry i.e. Gromov width and symplectic packing problems ([8, 2]). We get the following:

- We give lower bounds for the Gromov width of (X,ω) in terms of associated convex bodies $\Delta \subset \mathbb{R}^n$, namely its Newton-Okounkov bodies. More precisely the Gromov width is at least the supremum of the sizes of simplices that lie in the interior of a Newton-Okounkov body of (X,L) (this is related to the results in [4]). In particular, this readily implies that when L is very ample the Gromov width of (X,ω) is at least 1.
- Moreover, we show that when L is very ample the symplectic manifold (X,ω) has a full symplectic packing by d equal balls, where d is the degree of the line bundle L (i.e. the self-intersection number of its divisor class).

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Quivers, dessins and toric Calabi-Yau

Yang-Hui He

In this talk we give an overview of the joint work with many collaborators over the last decade or so, notably Amihay Hanany, Bo Feng, Sebastian Franco, Sanjaye Ramgoolam, Vishnu Jejjala, Diego Rodriguez-Gomez and Cumrun Vafa et al., from whose friendship I have much benefited. The audience is referred to the brief reviews in [1] and citations therein.

Let Q be a quiver, i.e., a labeled directed multigraph allowing for cycles and loops, with N_G nodes and N_F arrows. For simplicity let the dimension vector be $(1, 1, \ldots, 1)$ so that all arrows X_i correspond to complex numbers (as elements in $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$). We will also impose relations coming from the Jacobian of a polynomial $W(X_i)$, i.e., $\partial_{X_i} W(X_i) = 0$. In physics, this corresponds to a $U(1)^{N_G}$ supersymmetric gauge theory in $3+1$-dimensions with N_F fields X_i and superpotential W.

The representation variety $\mathcal{M}$ of Q is the GIT quotient of this Jacobian by appropriate group action. Computationally, let $GIO_{i=1,\ldots,k}$ be the generating set, in the sense of arrow composition, of directed minimal cycles in Q (each of which in physics is called a gauge invariant operator) and consider the polynomial rings $\mathbb{C}[X_i]$ and $\mathbb{C}[z_1, \ldots, z_k]$. Then $\mathcal{M}$ is the affine variety corresponding to the image of the map $GIO_j : \mathbb{C}[X_i]/\langle \partial_{X_i} W(X_i)\rangle \to \mathbb{C}[z_j]$.

In general, $\mathcal{M}$ could be any affine variety of arbitrary dimension. However, when it is an affine Calabi-Yau threefold, the theory is of particular interest in that there is a natural embedding into string theory and the above setup can be thought of as the algebro-geometric realization of the so-called AdS/CFT Correspondence which had since Maldacena become part of the canon of modern physics. Consider the simplest example: a clover-like Q with a single node and 3 arrows $X_{i=1,2,3}$ thereon forming 3 loops. In general, take the label to be N for the node, rendering each of X_i to be an element of $U(N)$. The standard superpotential one imposes is $W = \mathrm{Tr}(X_1X_2X_3 - X_1X_3X_2)$ and this quiver theory is known as $\mathcal{N} = 4$ super-Yang-Mills theory. As aforementioned, take $N = 1$ for convenience so that $W = 0$ and no relations are placed on the X_i. Clearly, the minimal cycles here are simply the three X_i each of which is a complex number. Therefore, $\mathcal{M}$ is here $\mathrm{Im}\,(z_j = X_j : \mathbb{C}[X_1, X_2, X_3] \to \mathbb{C}[z_1, z_2, z_3]) \simeq \mathbb{C}^3$, the simplest (trivial) affine Calabi-Yau threefold.

Of special note should be the fact that in the above W, with a total of $N_W = 2$ monomial terms, is of a specific form: each field appears exactly twice with opposite sign. Moreover, we have that $N_G - N_F + N_W = 1 - 3 + 2 = 0$. That these two conditions are highly suggestive (the first, due to the its making $\langle \partial W \rangle$ a binomial ideal, should evoke toric varieties, and the second, the Euler relation for a torus) has inspired much research and has by now become a captivating story (cf. [2]): *when $\mathcal{M}$ is a toric Calabi-Yau threefold, (Q, W) can be recast into a bipartite graph (variously called a dimer-model or a brane-tiling) drawn on a T^2.*

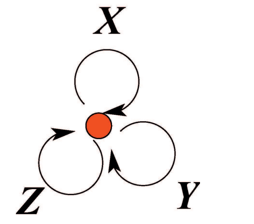
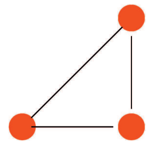
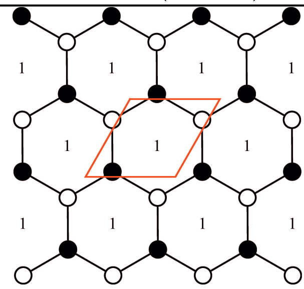
To see this, one associates a, say, black node to each term in W with a plus sign, and a white node, that with a minus sign, so that around a black (respectively white) node we write one monomial contribution to W by writing the variables X_i clockwise (respectively counter-clockwise); thus in our $\mathbb{C}^3$ example, $X_1X_2X_3$ is assigned a black node and $-X_1X_3X_2$, a white node. One readily concludes that the subsequent graph G is bipartite (no links exist between nodes of the same colour) and there is no further need of arrows since our choice of (counter-)clockwise orientation encodes the directed arrows in Q. Indeed, G is precisely the graph dual of Q in that each node/cycle of Q is now a face/node in G. Moreover, G is a bipartite tiling of the doubly-periodic plane (i.e., T^2), which for our running example is the hexagonal tiling with alternating black-white nodes so that in the fundamental region there is exactly one pair of black/white nodes, each of valency 3, as well as exactly three edges.

As we enter the realm of bipartite graphs on Riemann surfaces, we are inevitably lead to the subject of *dessin d'enfant* [3]. Briefly, we recall Belyi's theorem, which states that a compact smooth Riemann surface Σ has a model over $\overline{\mathbb{Q}}$ iff there exists a sujective map $\beta : \Sigma \to \mathbb{P}^1$ ramified at only three points, which can be taken to be $(0, 1, \infty)$ by $SL(2;\mathbb{C})$. Calling each preimage of 0 "white" and each preimage of 1 "black", the preimage of any continuous path from 0 to 1 on $\mathbb{P}^1$ is thus a bipartite graph on Σ with each face corresponding to a preimage of ∞: this graph reminded Grothendieck of children's drawings, hence the name.

It is thus expedient to think of our bipartite graph G on T^2 as a dessin d'enfant. For our example, one can check that the algebraic model for the T^2 is the elliptic curve $E : y^2 = x^3 + 1$ on which the rational map $\beta(x, y) = \frac{1}{2}(y + 1)$ is ramified only at 3 points with $\beta^{-1}(0) = (0, -1) \in E$ and $\beta(1) = (0, 1) \in E$, each with ramification index 3, and $\beta^{-1}(\infty) = (\infty, \infty) \in E$, also of ramification index 3, producing for us the trivalent hexagonal bipartite graph G on E.

The foregoing discussions in our running example can be diagrammatically summarized as follows (we have renamed, for convenience, the arrows $X_{i=1,2,3}$

as (X, Y, Z)):

Quiver	Toric Diagram	Belyĭ Pair	Dessin (dimer)
 $W = \mathrm{Tr}(X[Y,Z])$	 $\mathcal{M} \simeq \mathbb{C}^3$	$y^2 = x^3 + 1$ $\beta(x,y) = \frac{y+1}{2}$	

On this tapestry of quivers, gauge theories, dessins and Calabi-Yau varieties one can weave an intricate web of mathematics and physics. For example, to enumerate perfect matchings in the dimer, one traditionally computes, as dictated by statistical mechanics, the Kasteleyn matrix K which is a weighted adjacency matrix. It turns out that $\det(K)$ is the bi-variate Newton polynomial $P(z, w)$ of the toric diagram (note that for affine toric Calabi-Yau varieties, the end points of the extremal vector of the toric cone is actually co-hyperplanar by the vanishing of the first Chern class, hence all toric Calabi-Yau threefolds can be represented by a planar convex lattice polygon D). For the case where there is a single or no interior point in D, the tropicalization of $P(z, w)$ in the sense of taking the spine of the amœba-projection of $P(z, w)$ is the dual diagram to D.

Furthermore, the local-mirror to $\mathcal{M}$, by Strominger-Yau-Zaslow and Hori-Vafa, is given explicitly as the double-hypersuface $\zeta = uv = P(z, w)$ in $\mathbb{C}[u, v, z, w, \zeta]$ which is another (not necessarily toric) affine Calabi-Yau threefold $\mathcal{W}$. One can retrieve the quiver Q by associating its adjacency matrix as the intersection matrix of 3-cycles in $\mathcal{W}$: this is an archetypal consequence of (homological) mirror symmetry that there is a categorical equivalence between $D^\flat(\mathcal{M})$, the bounded derived category of coherent sheaves on $\mathcal{M}$ and the Fukaya category $Fuk(\mathcal{W})$ of special-Lagrangian 3-cycles in the mirror $\mathcal{W}$.

As another enticing example, let us consider the most important quantum field theoretic duality for these gauge theories, viz., Seiberg's strong-weak duality. It turns out [4] that this is none other than cluster mutation of the quiver [5]. It is curious that mathematicians and physicists, completely unbeknownst to each other, arrived at same quiver transformation around the same time. From the mirror perspective, the cluster mutation is Picard-Lefschetz monodromy in the 3-cycles and from the dessin point of view, it is a different choice of Belyi maps on the same elliptic curve.

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Splendid complexes on products of projective space

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(joint work with Christine Berkesch Zamaere, Daniel Erman)

There are two distinct constructions for locally-free resolutions of a coherent sheaf on a smooth complete toric variety X. The geometric procedure, which appears in Section 4 of [1] and recursively expresses a coherent sheaf as the quotient of locally-free sheaf of finite rank, terminates with a vector bundle after at most $\dim(X)$ steps. The algebraic approach, which is described in Section 7 in [3] and sheafifies the minimal free resolution of the corresponding saturated module over the Cox ring S, has length at most $\dim(S) - 1$. In both cases, the finite length of the resolution is derived from Hilbert's Syzygy Theorem; the geometric procedure uses a local version and the algebraic approach uses a global version for polynomial rings. Nevertheless, the geometric resolution is shorter than the algebraic one whenever $X \neq \mathbb{P}^n$. On the other hand, all of the vector bundles appearing in the algebraic resolution are simply direct sums of line bundles, unlike the geometric resolution. The goal of this presentation is to indicate how to effectively construct locally-free resolutions that are simultaneously short and simple.

To be more precise, fix $r \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{N}^r$. Consider the smooth projective toric variety $X := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ and let $S := \mathbb{C}[x_{j,k} : 1 \leq j \leq r,\, 0 \leq k \leq n_j]$ be its Cox ring. Since $r = \dim(S) - \dim(X)$, we see that the difference in length between the algebraic and geometric resolutions can be arbitrarily large. The polynomial ring S has the $\mathbb{Z}^r$-grading induced by setting $\deg(x_{j,k}) := \mathbf{e}_j \in \mathbb{Z}^r$ and the irrelevant ideal for X is $B := \bigcap_{j=1}^{r} \langle x_{j,0}, x_{j,1}, \ldots, x_{j,n_j} \rangle$. As better homological objects and replacements for minimal free resolutions over S, we focus on a special collection of complexes. A $\mathbb{Z}^r$-graded free complex F of S-modules is called a *splendid complex* of a $\mathbb{Z}^r$-graded S-module M (or the sheaf $\widetilde{M}$) if the complex $\widetilde{F}$ of $\mathcal{O}_X$-modules is a resolution of $\widetilde{M}$. Corollary 3.6 in [2] shows that a $\mathbb{Z}^r$-graded free complex F is splendid if and only if, for all $i \neq 0$, the homology $\mathrm{H}_i(F)$ is a B-torsion module. As a proof of concept, we demostrate that every $\mathcal{O}_X$-module has a splendid complex of length at most $\dim(X)$, by exploit a resolution of the diagonal $X \hookrightarrow X \times X$.

Punctual subschemes in X provide more compelling examples of splendid complexes. For instance, the minimal free resolution of the quotient S/I_Z, where I_Z

is the B-saturated ideal defining two generic points in $\mathbb{P}^1 \times \mathbb{P}^1$, has the form

$$S^1 \longleftarrow \begin{matrix} S(-2,0)^1 \\ \oplus \\ S(-1,-1)^2 \\ \oplus \\ S(0,-2)^1 \end{matrix} \longleftarrow \begin{matrix} S(-2,-1)^2 \\ \oplus \\ S(-1,-2)^2 \end{matrix} \longleftarrow S(-2,-2)^1 \,,$$

but there is a splendid complex of the form $S^1 \longleftarrow S(-1,-1)^2 \longleftarrow S(-2,-2)^1$. For $\mathbf{v} \in \mathbb{N}^r$, we set $B^{\mathbf{v}} := \bigcap_{j=1}^r \langle x_{j,0}, x_{j,1}, \ldots, x_{j,n_j} \rangle^{v_j}$. If $Z \subset X$ with $\dim(Z) = 0$ and B-saturated S-ideal I_Z, then we show that there exists $\mathbf{v} \in \mathbb{N}^r$ with $v_r = 0$ such that the minimal free resolution of $I_Z \cap B^{\mathbf{v}}$ is a splendid complex of $\mathcal{O}_Z$ with length $\dim(X)$. In particular, every punctual subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ has a Cohen–Macaulay representation of the form $S/(I_Z \cap B^{\mathbf{v}})$ and the Hilbert–Burch Theorem describes the corresponding splendid complex.

More generally, intriguing splendid complexes can be extracted from minimal free resolutions. Given $\mathbf{m} \in \mathbb{Z}^r$, a B-saturated $\mathbb{Z}^r$-graded S-module M is $\mathbf{m}$-regular if, for all $i > 0$ and for all $\mathbf{u} \in \mathbb{N}^r$ with $u_1 + u_2 + \cdots + u_r = i - 1$, we have $H^i_B(M)_{\mathbf{m}-\mathbf{u}} = 0$; compare with Definition 1.1 in [3]. If M is an $\mathbf{m}$-regular $\mathbb{Z}^r$-graded B-saturated S-module, then we prove that the subcomplex of its minimal free resolution consisting of all summands of degree at most $\mathbf{m}+\mathbf{n}$ gives a splendid complex of M. For example, the minimal free resolution of six generic points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ has the form $S^1 \longleftarrow S^{37} \longleftarrow S^{120} \longleftarrow S^{116} \longleftarrow S^{120} \longleftarrow S^{45} \longleftarrow S^7$. However, the module S/I_Z is $(0,0,2)$-regular and the associated splendid complex has the form $S^1 \longleftarrow S^{22} \longleftarrow S^{51} \longleftarrow S^{42} \longleftarrow S^{12}$. Although splendid complexes are not unique, these results suggest that they are the right algebraic mechanism for capturing the underlying intrinsic geometry.

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Local heights of toric varieties over non-archimedean fields

Walter Gubler
(joint work with Julius Hertel)

The main reference for the talk is [5].

1. Introduction

Let X_Σ be a proper toric variety of dimension n over the field K corresponding to the fan Σ in $N_\mathbb{R}$. A base-point-free toric Cartier divisor D is given by a concave piecewise linear function Ψ on Σ. Let Δ_Ψ be the associated polytope in the dual space $M_\mathbb{R}$ of $N_\mathbb{R}$. Then we have the following famous formula in toric geometry:

$$\deg_L(X_\Sigma) = n! \cdot \mathrm{vol}_M(\Delta_\Psi).$$

We will give an arithmetic version of this formula, where the height replaces the degree, generalizing results of Burgos, Philippon and Sombra in [1].

2. Metrized line bundles

From now on, K is endowed with a non-trivial non-archimedean complete absolute value $|\ |$ (e.g. $\mathbb{Q}_p$). We also assume that K is algebraically closed which may always be achieved by passing to the completion of the algebraic closure. Let K° be the valuation ring of K.

Let X be a variety of dimension n and let X^{an} be the associated Berkovich analytic space. This analytification has similar properties as its complex analytic analogue. We will always assume that X is proper over K which means that X^{an} is compact. A metric $\|\ \|$ on a line bundle L over X means a continuously varying family of norms on the fibres of L^{an} over X^{an}.

Let $(\mathcal{X}, \mathcal{L})$ be a K°-model of (X, L) which means that $\mathcal{X}$ is a proper variety over K° with $\mathcal{X} \otimes_{K^\circ} K = X$ and $\mathcal{L}$ is a line bundle on $\mathcal{X}$ with $\mathcal{L}|_X = L$. Zhang has noticed that such a model induces a metric $\|\ \|_\mathcal{L}$ on L which we call the *algebraic metric* associated to $\mathcal{L}$. This metric is called *semipositive* if the restriction of $\mathcal{L}$ to the special fibre of $\mathcal{X}$ is nef.

More generally, Zhang calls a metric $\|\ \|$ on L *semipositive* if there is a sequence $m_k \geq 1$ and models $\mathcal{L}_{m_k}$ of $L^{\otimes m_k}$ with semipositive algebraic metrics $\|\ \|_{\mathcal{L}_{m_k}}$ such that $\|\ \|$ is the uniform limit of the metrics $\|\ \|_{\mathcal{L}_{m_k}}^{1/m_k}$ on L^{an}. Such metrics are important in the study of arithmetic dynamical systems.

Chambert–Loir has introduced a positive Radon measure $c_1(L_1, \|\ \|_1) \wedge \cdots \wedge c_1(L_n, \|\ \|_n)$ on X^{an} for line bundles $(L_1, \|\ \|_1), \ldots, (L_n, \|\ \|_n)$ on X endowed with semipositive metrics. This is a non-archimedean analogue of the Monge–Ampère measure in complex analysis. We refer to [5] for more details and references.

3. Local heights

Let $\hat{D} = (D, \| \ \|)$ be a metrized Cartier divisor on X which means a metric $\| \ \|$ on the underlying line bundle $O(D)$ of the Cartier divisor D on X. Following Weil and Néron, we consider the local height function

$$\lambda_{\hat{D}} : X^{\mathrm{an}} \setminus D^{\mathrm{an}} \longrightarrow \mathbb{R}, \quad x \mapsto -\log \|s_D(x)\|,$$

where s_D is the canonical meromorphic section of $O(D)$.

We have the following generalization to higher dimensions. Recall that n is the dimension of the proper variety X. Let $\hat{D}_0, \ldots, \hat{D}_n$ be semipositively metrized Cartier divisors on X satisfying

$$\operatorname{supp}(D_0) \cap \cdots \cap \operatorname{supp}(D_n) = \emptyset. \tag{1}$$

Then the *local height* $\lambda_{\hat{D}_0,\ldots,\hat{D}_n}(X)$ *of* X *with respect to* $\hat{D}_0, \ldots, \hat{D}_n$ is characterized by the following properties:

(i) If the metrics of $\hat{D}_0, \ldots, \hat{D}_n$ are algebraic metrics induced by line bundles $\mathcal{L}_0, \ldots, \mathcal{L}_n$ on a common K°-model $\mathcal{X}$ of X, then

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_n}(X) = \deg(\operatorname{div}(s_0) \ldots \operatorname{div}(s_n).\mathcal{X})$$

where s_i is the canonical meromorphic section of $\mathcal{L}_i$ induced by D_i. Note that this intersection number on $\mathcal{X}$ makes sense by (1) using [3].

(ii) The local height $\lambda_{\hat{D}_0,\ldots,\hat{D}_n}(X)$ is continuous with respect to uniform convergence of the metrics.

Theorem 1. *Under the hypothesis* (1)*, we have the following induction formula:*

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_n}(X) = \lambda_{\hat{D}_0,\ldots,\hat{D}_{n-1}}(\operatorname{div}(s_{D_n})) - \int_{X^{\mathrm{an}}} \log \|s_{D_n}\|_n \, \mu$$

where $\mu = c_1(O(D_0), \| \ \|_0) \wedge \cdots \wedge c_1(O(D_{n-1}), \| \ \|_{n-1})$.

The induction formula follows easily from the definitions if all metrics are algebraic. For arbitrary semipositive metrics, the induction formula is due to Chambert-Loir and Thuillier [2] in case of a discrete absolute value on K. The general case is done in [5].

4. Toric local heights

Now we come back to the toric setting of the introduction, but assuming that K is a non-archimedean field as above. Then the base-point-free toric Cartier divisor D on the toric variety X_Σ with associated concave piecewise linear function Ψ on the fan Σ leads to a toric line bundle $L = O(D)$ over X_Σ together with a toric section s_D of L.

Theorem 2. *There are bijective correspondences between the sets of*

(i) *semipositive toric metrics on* L*;*
(ii) *concave functions* ψ *on* $N_\mathbb{R}$ *such that the function* $|\psi - \Psi|$ *is bounded;*
(iii) *continuous concave functions* ϑ *on* Δ_Ψ.

For the bijection $(i) \leftrightarrow (ii)$, one associates to the toric metric $\| \ \|$ the function ψ on $N_{\mathbb{R}}$ given by $\psi(u) = \log \|s_D \circ \mathrm{trop}^{-1}(u)\|$, where trop is the tropicalization map from the dense torus to $N_{\mathbb{R}}$. For the bijection $(ii) \leftrightarrow (iii)$, ϑ is the Legendre-Fenchel transform of ψ.

Theorem 2 is due to Burgos, Philippon and Sombra [1] in case of a discrete absolute value on K. The general case is done in [5] based on the study of toric schemes over the valuation ring K° from [4].

Note that if we choose $\psi := \Psi$, then we get a canonical semipositive toric metric on L which we denote by $\| \ \|_{\mathrm{can}}$.

For any semipositive toric metric $\| \ \|$ on L, we define the *toric local height of* X_Σ by choosing Cartier divisors $D_0, \ldots, D_n$ on X_Σ with $O(D_i) = L$ satisfying (1) and then we set

$$\lambda^{\mathrm{tor}}_{(L,\| \ \|)}(X_\Sigma) = \lambda_{\hat{D}_0,\ldots,\hat{D}_n}(X_\Sigma) - \lambda_{\hat{D}_0^{\mathrm{can}},\ldots,\hat{D}_n^{\mathrm{can}}}(X_\Sigma),$$

where we always use the given metric $\| \ \|$ on $L = O(D_i)$ for all metrized Cartier divisors $\hat{D}_i$ and where $\hat{D}_i^{\mathrm{can}} = (D_i, \| \ \|_{\mathrm{can}})$. It is easy to see that the toric local height is independent of the choice of $D_0, \ldots, D_n$.

We have the following analogue of the degree formula from toric geometry:

Theorem 3. *Let $\| \ \|$ be a semipositive toric metric on L. Then we have*

$$\lambda^{\mathrm{tor}}_{(L,\| \ \|)}(X_\Sigma) = (n+1)! \int_{\Delta_\Psi} \vartheta \, \mathrm{dvol}_M,$$

where ϑ is corresponding to $\| \ \|$ as in Theorem 2.

This result is due to Burgos, Philippon and Sombra [1] in case of a discrete absolute value on K and is generalized in [5]. The proof uses Theorem 1 and Theorem 2.

In case of a global field, this leads to a similar formula for global heights. In [5], an application is presented where non-discrete non-archimedean absolute values really matter to compute natural global heights for fibrations of projective varieties over $\mathbb{Q}$ which are generically toric.

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The Noether-Lefschetz locus of surfaces in toric threefolds

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(joint work with Ugo Bruzzo)

The classical Noether-Lefschetz theorem states that any curve in a very general surface X in $\mathbb{P}^3$ of degree $d \geq 4$ is a restriction of a surface in the ambient space, namely the Picard number of X is 1 (a point is very general if it lies outside a countable union of closed subschemes of positive codimension). The Noether-Lefschetz locus is the locus where the Picard number is greater than 1.

Consider a projective toric variety $\mathbb{P}_\Sigma$ with orbifold singularities; $\mathbb{P}_\Sigma$ is associated with a 3-dimensional complete simplicial fan Σ and is $\mathbb{Q}$-factorial . Let β be a nef (numerically effective) class in the class group $\mathrm{Cl}(\mathbb{P}_\Sigma)$ of Weil divisors modulo rational equivalence and consider a surface X in $\mathbb{P}_\Sigma$ whose class (degree) in $\mathrm{Cl}(\mathbb{P}_\Sigma)$ is β. If X is general, it is quasi-smooth, that is, its only singularities are those inherited from $\mathbb{P}_\Sigma$.

Let $\mathcal{M}_\beta$ be the moduli space of surfaces in $\mathbb{P}_\Sigma$ of degree β modulo automorphisms of $\mathbb{P}_\Sigma$. In [1], we proved that for β ample and $-\beta_0$ the canonical class of $\mathbb{P}_\Sigma$, if the multiplication morphism

$$R(f)_\beta \otimes R(f)_{\beta-\beta_0} \to R(f)_{2\beta-\beta_0} \tag{1}$$

is surjective, very general points of $\mathcal{M}_\beta$ correspond to surfaces whose Picard number equals the Picard number of $\mathbb{P}_\Sigma$; here $R(f)$ is the Jacobian ring. If $\mathbb{P}_\Sigma = \mathbb{P}^3$ and $\beta = d \geq 4$, or equivalently $\beta - \beta_0$ is nef, the morphism in (1) is always surjective. Also, the morphism is surjective whenever $\beta - \beta_0$ is trivial, that is, X is a K3 surface in a Fano threefold $\mathbb{P}_\Sigma$. If the sum of two polytopes associated with a nef and an ample divisor is equal to their Minkowski sum, the multiplication map in (1) is always surjective. We refer to these varieties as "Oda varieties".

If we write $\beta - \beta_0 = n\eta$ for an ample Cartier primitive class η, the condition $n \geq 0$ generalizes the classical condition $d \geq 4$. We define the Noether-Lefschetz locus with respect to β to be the closed subscheme $U_\eta(n)$ of $\mathcal{M}_\beta$ corresponding to quasi smooth surfaces whose Picard number is strictly larger than that of $\mathbb{P}_\Sigma$. In particular we have an upper bound on the codimension of any irreducible component on the Noether-Lefschetz locus:

Proposition 1.

$$\operatorname{codim} U_\eta(n) \leq h^{2,0}(S) = h^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(n\eta)), \tag{2}$$

where S is a quasi-smooth surface in the linear system β.

The classical proof of the codimension of the Noether-Lefschetz locus for $\mathbb{P}_\Sigma = \mathbb{P}^3$ relies implicitly on the fact that $\eta = \mathcal{O}_{\mathbb{P}^3}(1)$ is (-1)-regular and any line bundle of degree $d \geq 4$ is 0-regular. However, we show that $\mathbb{P}^n$ is the only simplicial toric n-fold with an ample (-1)-regular line bundle. So we consider toric varieties with a 0-regular ample line bundle. Our proof generalizes the arguments of Green in [4, 3] and relies on vanishing theorems and dualities for toric varieties, as well as the Castelnuovo-Mumford regularity of certain bundles. Recall that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is m-regular with respect to the very ample line bundle η if

$$H^q(X, \mathcal{F} \otimes \eta^{m-q}) = 0$$

for all $q > 0$.) Then we bound the codimension of the Noether-Leschetz locus. We prove

Theorem 2. *Let $\mathbb{P}_\Sigma$ be a simplicial toric variety, η an ample primitive Cartier class, $\beta_0 = -K_{\mathbb{P}_\Sigma}$, $\beta \in \operatorname{Pic}(\mathbb{P}_\Sigma)$ an ample Cartier class that satisfies $\beta - \beta_0 = n\eta$ for some $n \geq 0$. Assume that β is 0-regular with respect to η. If η is (-1)-regular, then*

$$\operatorname{codim} U_\eta(n) \geq n + 1. \tag{3}$$

If η is 0-regular, then

$$\operatorname{codim} U_\eta(n) \geq n. \tag{4}$$

Corollary 3. *Let $\mathbb{P}_\Sigma$ be a simplicial Fano toric variety, η a primitive nef divisor, $\beta_0 = -K_{\mathbb{P}_\Sigma}$, $\beta \in \operatorname{Pic}(\mathbb{P}_\Sigma)$ an ample Cartier class that satisfies $\beta - \beta_0 = n\eta$ for some $n \geq 3$.*
If η is (-1)-regular then

$$\operatorname{codim} U_\eta(n) \geq n + 1. \tag{5}$$

If η is 0-regular, then

$$\operatorname{codim} U_\eta(n) \geq n. \tag{6}$$

We prove a similar result for Oda varieties. We consider various examples and study the components of the loci $U_\eta(n)$ which contain a line, defined as a rational curve that is "minimal" in a suitable sense (i.e., its intersection with the ample class η is 1). We show that the codimension of these components is $n + 1$, as in the classical case.

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Infinitesimal qG-deformations of cyclic quotient singularities

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(joint work with János Kollár)

The deformation theory of a two-dimensional toric singularity, i.e. a quotient singularity $S_{n,q} = \mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ (acting via (ξ, ξ^q), $q \in (\mathbb{Z}/n\mathbb{Z})^*$, $q \neq -1$), was studied a lot. Nowadays, there is a clear understanding of its infinitesimal deformations, obstructions, and of the component structure of the versal deformation and its relations to so-called P-resolutions.

However, not every flat deformation should be allowed in moduli theory. There, it becomes important that several (or all) reflexive powers $\omega_S^{[g]}$ fit into the deformation as well, i.e. for a deformation $f : X \to B$ of S not only X, but also the reflexive powers $\omega_{X|B}^{[g]}$ should become flat over the base space B. We will denote this property by $(*)_g$ and study it in dependence on $g \in \mathbb{Z}$. Three variants of this compatibility have been established in the past – they differ by the set of exponents g one asks the compatibility for. Note that $g = 0$ just encodes the flatness of the family itself.

Definition 1. The deformation f is called a V-, W-, or a qG-deformation if $(*)_g$ is satisfied for $g = \operatorname{index}(\omega_S)$ (implying it for all multiples, too), for $g = -1$, or for all $g \in \mathbb{Z}$, respectively. Deformations being both V- and W-deformations are called VW-deformations.

While in characteristic zero and over reduced base spaces B the concepts V, VW and qG coincide, cf. [2], we will now focus on the infinitesimal theory, that is $B = \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$. We will use the standard toric language, i.e. $M \cong \mathbb{Z}^2$ and $N \cong \mathbb{Z}^2$ denote the mutually dual lattices, and S is given by a two-dimensional, rational polyhedral cone $\sigma = \langle \alpha, \beta \rangle \subseteq N_{\mathbb{R}}$ with $\det(\alpha, \beta) = n$ and $n | \alpha q + \beta$. Then, the vector space T_S^1 of infinitesimal deformations becomes a subquotient of $\mathbb{C}[M] \otimes_{\mathbb{Z}} N$, i.e. its homogeneous elements of degree $-R$ can essentially be written as $\xi = x^{-R}\partial_a$ with $\partial_a : x^r \mapsto \langle a, r \rangle \cdot x^r$ for $a \in N$.

It is well-known, cf. [3], that $T_S^1(-R)$ vanishes unless in a few cases. To name them, we denote by $E = \{w^1, \ldots, w^e\} \subset \sigma^\vee \cap M$ the Hilbert basis (with $w^1 \in \alpha^\perp$ and $w^e \in \beta^\perp$ being the rays of $\sigma^\vee$). Then

(i) $R = w^2$ or $R = w^{e-1}$: $\dim_{\mathbb{C}} T_S^1(-R) = 1$, namely $N_{\mathbb{C}}/\,\mathbb{C}\cdot\alpha$ and $N_{\mathbb{C}}/\,\mathbb{C}\cdot\beta$, respectively,

(ii) $R = w^i$ for $i = 3, \ldots, e-2$: $T_S^1(-R) = N_{\mathbb{C}}$ is two-dimensional, and

(iii) $R = k \cdot w^i$ for $i = 2, \ldots, e-1$ with $2 \leq k \leq a_i - 1$: $T_S^1(-R) = (w^i)^\perp \subseteq N_{\mathbb{C}}$ is again one-dimensional.

Here we have denoted by $a_i \in \mathbb{Z}_{\geq 2}$ the numbers defined by $w^{i-1} + w^{i+1} = a_i \cdot w^i$. They appear also in the continued fraction expansion $\frac{n}{n-q} = [a_2, \ldots, a_{e-1}]$. To describe the V-deformations, we denote $b := \gcd(n, q+1)$, $m := n/b$, and $\overline{R} = (w^1 + w^e)/b \in M$. Note that $m = \operatorname{index}(\omega_S)$.

Theorem 2. *The homogeneous* $\xi = x^{-R}\,\partial_a \in T^1_S(-R)$ *is a V-deformation iff* $a \in (\overline{R} - mR)^\perp$.

This implies that the multidegrees of (i) do not provide V-deformations at all. From the list in (ii) we obtain $e-4$ dimensions (one dimension within each $T^1_S(-w^i)$), and the degrees $R = k \cdot w^i$ of (iii) ask for $(w^i, \overline{R} - mw^i)^\perp = (w^i, w^1 + w^e)^\perp$ within the 2-dimensional $N_{\mathbb{C}}$. This leads to

Definition 3. The singularity S (or the cone σ) is called grounded if $\overline{R}$ belongs to the Hilbert basis E, i.e. if it is one of the w^i of the list in (ii); we call it w^ν then.

Using this terminology, we can identify the remaining V-deformations as exactly those of the list in (iii) with $i = \nu$; it is the only index making w^i and $w^1 + w^e$ linearly dependent. In particular, this requires that S is grounded.

We may use the primitive $\overline{R} \in M$ also to introduce an alternative way of describing cyclic quotient singularities. Associating $\sigma \mapsto I := \sigma \cap [\overline{R} = 1]$ and interpreting the affine line $[\overline{R} = 1] \subseteq N_{\mathbb{R}}$ as an ordinary line $\mathbb{R}^1$ with a well-defined lattice structure, we obtain a one-one correspondence between the following sets:

$$\big\{\text{cones } \sigma\big\}/\mathrm{Sl}(2,\mathbb{Z})$$

$$\big\{\text{rational intervals } I \subseteq \mathbb{R} \text{ with uniform denominators}\big\}/\{\mathbb{Z}\text{-shifts}\}$$

where we call I to have "*uniform denominators*" (at the end points) if both become equal in the reduced forms.

Under this correspondence, the groundedness of cones translates into the property that I contains interior integers. When this is the case, then we may, w.l.o.g., suppose that $0 \in \mathrm{int} I$, i.e. that

$$I = [-A, B] \subseteq \mathbb{R} \quad \text{with} \quad A, B \in \mathbb{Q}_{>0}$$

having the same denominator m in their reduced form. This language allows to express the ν-th element a_ν of the continued fraction $[a_2, \ldots, a_{e-1}]$ as

$$a_\nu - 2 = \lfloor A \rfloor + \lfloor B \rfloor \le \lfloor A + B \rfloor \le A + B = |I|.$$

Theorem 4. *S has neither qG- or VW-deformations unless it is grounded. If this is the case, and if S is given by $I = [-A, B]$ with $\overline{R} = w^\nu$, then the homogeneous qG-deformations are formed by the one-dimensional subspaces $\overline{R}^\perp \subseteq T^1(-k \cdot w^\nu) \subseteq N_{\mathbb{C}}$ with $k = 1, \ldots, \lfloor A + B \rfloor$.*

Note that the left and right inclusions become equalities if $k \ge 2$ or $k = 1$, respectively. Since $a_\nu - 2 \le \lfloor A + B \rfloor \le a_\nu - 1$, it follows that we have the qG- (and hence the VW-) property for at least $k = 1, \ldots, a_\nu - 2$. Thus, to compare the qG- and the VW-notion, it remains to analyze the "last deformation", i.e. the one-dimensional $\overline{R}^\perp \subseteq T^1_S\big(-(a_\nu - 1) \cdot w^\nu\big)$.

Theorem 5. *Assume that S is grounded and given by $I = [-A, B]$; denote by m the denominator of the end points. Then,*

(a) *the last deformation is qG* $\iff$ $\{A\} + \{B\} \ge 1$.
(b) *If* $\{A\}, \{B\} \ne \frac{1}{m}$, *then the last deformation is VW.*

(c) *Otherwise, if either* $\{A\} = \frac{1}{m}$ *or* $\{B\} = \frac{1}{m}$*, then the last deformation is VW iff it is qG.*

Thus, while the spaces of infinitesimal V- and qG-deformations differ a lot (the difference of their respecive dimensions is $e-4$ or $e-5$), the spaces of infinitesimal VW- and qG-deformations do differ by at most one dimension. Nevertheless, we have the following

Example 6. If $I = [-\frac{2}{5}, \frac{2}{5}]$, i.e. $n = 20$, $q = 11$, $e = 7$, $\frac{20}{9} = [a_2, \ldots, a_6] = [3, 2, 2, 2, 3]$, and $\overline{R} = w^4$ with $a_4 = 2$, then there is no qG-deformation, but the spaces of infinitesimal V- and VW-deformations are 3- and 1-dimensional, respectively.

The main step in the proof of these theorems is the translation of the property $(*)_g$ into combinatorics. This is done by the characterization

$$x^{-R}\,\partial_a \text{ satisfies } (*)_g \iff \left(\tfrac{g}{m}\,\overline{R} + Z_R\right) \cap M \subseteq a^{\perp} + g \cdot R$$

where Z_R denotes the bounded region $Z_R := \sigma^{\vee} \cap (R - \mathrm{int}\sigma^{\vee}) \subset M_{\mathbb{R}}$.

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Blowups of toric varieties with non-finitely generated Cox rings

Kalle Karu

(joint work with José Luis González)

Cox rings of a normal projective variety X were defined by Hu and Keel [4] as

$$\mathrm{Cox}(X) = \sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} \mathcal{O}_X(a_1D_1 + a_2D_2 + \cdots a_nD_n),$$

where $d_1, \ldots, d_n$ are Weil divisors that span the class group $\mathrm{Cl}(X)_{\mathbb{Q}}$. Cox rings generalize the homogeneous coordinate rings of toric varieties studied by Cox [1]. Varieties with finitely generated Cox rings are called Mori Dream Spaces (MDS). All toric varieties are MDS.

This talk describes joint work with José Luis González in which we construct examples of toric varieties blown up at a point e in the torus that are not MDS. The motivation for this comes from the theorem by Castravet and Tevelev [2] that the moduli spaces $\overline{M}_{0,n}$ are not MDS for n large. They reduced the problem to an earlier result by Goto, Nishida and Watanabe [3] who gave an infinite family of weighted projective planes blown up at a point that are not MDS.

Our examples of toric varieties are defined by rational convex polytopes Δ that satisfy a combinatorial condition. The normal fan of the polytope then defines the toric variety whose blowup at a point is not a MDS.

In the two-dimensional case the polytope is a 4-gon with vertices

$$(0,0), \quad (0,1), \quad (x_1, y_1), \quad (x_2, y_2),$$

where $x_1 < 0$ and $x_2 > 0$. The polytopes Δ must satisfy the conditions:

(1) $x_2 - x_1 \leq 1$.
(2) If $m\Delta$ has integral vertices for some $m > 0$, then there can be at most one lattice point in $m\Delta$ with first coordinate $mx_1 + 1$.

Then the blowup of the toric variety corresponding to Δ is not a MDS. The second condition on the number of lattice points can be relaxed further.

An analogous construction in dimension 3 gives a rational polytope Δ with vertices

$$(0,0,0), \quad (0,1,0), \quad (0,0,1), \quad (x_1, y_1, z_1), \quad (x_2, y_2, z_2),$$

where $x_1 < 0$, $x_2 > 0$. The same combinatorial condition as above applied to Δ ensures that the blowup of the toric variety defined by Δ is not a MDS. This construction can be generalized to arbitrary dimension.

When the polytope Δ degenerates to a simplex then the toric variety will have Picard number 1. The corresponding toric varieties blown up at a point are again not MDS if one adds another combinatorial condition. In the examples described above it amounts to the condition that the single lattice point in $m\Delta$ with first coordinate $x_1 + 1$ should not lie on the edge of $m\Delta$ connecting the two vertices with nonzero x-coordinates.

Among the toric varieties defined by simplices are the weighted projective spaces $\mathbb{P}(a, b, c, \ldots)$. We have done a computer search to find weighted projective spaces that satisfy the combinatorial conditions and hence are not MDS when blown up at a point. In an earlier article [5] we reported a long list of such weighted projective planes, for example

$$\mathbb{P}(7, 15, 26), \quad \mathbb{P}(7, 17, 22), \quad \mathbb{P}(12, 13, 17).$$

We can now add to this a list of weighted projective 3-spaces, such as

$$\mathbb{P}(7, 18, 27, 47), \quad \mathbb{P}(7, 17, 22, 51), \quad \mathbb{P}(15, 19, 20, 41), \quad \mathbb{P}(17, 18, 20, 27).$$

The numbers (a, b, c, d) appearing in the 3-dimensional case tend to be larger than in dimension 2.

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Linear subspaces of projective toric varieties

Nathan Ilten
(joint work with Sasha Zotine)

Given an embedded projective variety $X \subset \mathbb{P}^n$ over an algebraically closed field $\mathbb{K}$, its kth Fano scheme $\mathbf{F}_k(X)$ is the fine moduli space parametrizing k-dimensional projective linear subspaces of X. In this report we describe work in progress studying the Fano scheme of an embedded projective toric variety. To fix notation, let $\mathcal{A} \subset \mathbb{Z}^n$ be a finite set of lattice points, and $S_\mathcal{A}$ the semigroup generated by elements of the form $(u, 1) \in \mathcal{A} \times \mathbb{Z}$. By $X_\mathcal{A}$ we denote the projective toric variety whose homogeneous coordinate ring is the semigroup algebra of $S_\mathcal{A}$, that is,

$$X_\mathcal{A} = \operatorname{Proj} \mathbb{K}[S_\mathcal{A}]$$

with $\mathbb{Z}$-grading given by projection to the final $\mathbb{Z}$-factor.

The most important concept for understanding the Fano scheme $\mathbf{F}_k(X_\mathcal{A})$ is that of a *Cayley structure*. Consider a face τ of $\mathcal{A}$, that is, the intersection of $\mathcal{A}$ with a face of its convex hull. A Cayley structure on τ is a map $\pi\colon \tau \to \Delta_l$ whose image consists of the vertices of Δ_l and which preserves affine relations. Here, Δ_l denotes the standard l-simplex. We will be interested in the set of all Cayley structures where τ ranges over faces of $\mathcal{A}$, and $l \geq k$. This set comes with a natural partial order: for Cayley structures $\pi\colon \tau \to \Delta_l$ and $\pi'\colon \tau' \to \Delta_{l'}$, $\pi \leq \pi'$ if τ is contained in τ', and there is a map $\phi\colon \Delta_{l'} \to \Delta_l$ such that $\phi \circ \pi'$ restricts to π on τ. Note that we do not differentiate between two Cayley structures if they differ by an automorphism of the standard simplex.

Our main result is the following theorem.

Theorem 1. *There is a bijection between irreducible components of $\mathbf{F}_k(X_\mathcal{A})$ and maximal Cayley structures $\pi\colon \tau \to \Delta_l$ where τ is a face of $\mathcal{A}$ and $l \geq k$.*

A similar result has recently been obtained independently in work in progress by K. Furukawa and A. Ito.

The bijection in the above theorem can be described quite explicitly. The action of the dense torus T of $X_\mathcal{A}$ preserves linear subspaces, and thus induces an action on $\mathbf{F}_k(X_\mathcal{A})$. Given a maximal Cayley structure $\pi\colon \tau \to \Delta_l$ for τ a face of $\mathcal{A}$ and $l \geq k$, we may consider the l-plane L_π of $X_\mathcal{A}$ corresponding to the unique semigroup homomorphism $S_\mathcal{A} \to \mathbb{N}^{l+1}$ sending $(u, 1)$ to $\pi(u)$ for $u \in \mathcal{A}$. Here we have identified the vertices of Δ_l with the standard basis vectors of $\mathbb{Z}^{l+1}$. The irreducible component of $Z_{\pi,k}$ of $\mathbf{F}_k(X_\mathcal{A})$ corresponding to $\mathcal{A}$ is obtained by taking the closure in $\mathbf{F}_k(X_\mathcal{A})$ of all T-orbits of points corresponding to k-planes $L \subset L_\pi$. These components have a natural affine cover for which each chart is itself isomorphic to a toric variety. In fact, if $l = k$, then $Z_{\pi,k}$ is globally a toric variety.

The key step in proving the above theorem is to show that each k-plane $L \subset X_{\mathcal{A}}$ corresponds to a point of $Z_{\pi,k}$ for some Cayley structure π. Suppose that L is the rowspan of a $(k+1) \times \#\mathcal{A}$ matrix (α_{iu}) with columns indexed by $u \in \mathcal{A}$. For each $u \in \mathcal{A}$, let y_u be the linear form

$$y_u = \sum_i \alpha_{iu} y_i \in \mathbb{K}[y_0, \ldots y_k].$$

We set $\tau = \{u \in \mathcal{A} \mid y_u \neq 0\}$; this turns out to be a face of $\mathcal{A}$. Taking V to be the vector space of linear forms in the variables y_i, we have a natural map $\pi\colon \tau \to \mathbb{P}(V)$. The image of π contains at least $k+1$ elements, since (α_{iu}) has rank $k+1$. Identifying the elements of $\pi(\mathcal{A})$ with the vertices of Δ_l for some $l \geq k$ leads to a Cayley structure $\pi\colon \tau \to \Delta_l$. Using the local description of $Z_{\pi,k}$, one can show that L is contained in a translate of L_π under the torus action.

Theorem 1 has a number of applications. For example, we recover the following result of Casagrande and Di Rocco [1] (for $k = 1$) and Ito [3] (for k arbitrary):

Corollary 2. *The toric variety $X_{\mathcal{A}}$ is covered by k-planes if and only if there is a Cayley structure $\pi\colon \mathcal{A} \to \Delta_k$.*

We also obtain the following statement concerning smoothness of the components of $\mathbf{F}_k(X_{\mathcal{A}})$:

Corollary 3. *Suppose that the dimension of the singular locus of $X_{\mathcal{A}}$ is less than k. Than every irreducible component of $\mathbf{F}_k(X_{\mathcal{A}})$ is smooth when considered in its reduced structure.*

It is not difficult to see that components $Z_{\pi_1,k}$ and $Z_{\pi_2,k}$ have non-trivial intersection if and only if there is a Cayley structure $\pi'\colon \tau' \to \Delta_k$ such that $\pi' \leq \pi_i$ for $i = 1, 2$. This leads to a combinatorial description of the connected components of $\mathbf{F}_k(X_{\mathcal{A}})$. Finally, we also study the non-reduced structure of $\mathbf{F}_k(X_{\mathcal{A}})$ in some special cases, generalizing the results of [2] for toric surfaces. In particular, we provide combinatorial formulas describing the non-reduced structure of $\mathbf{F}_k(X_{\mathcal{A}})$ whenever $X_{\mathcal{A}}$ is projectively normal and $k = \dim X_{\mathcal{A}} - 1$.

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Frobenius semisimplicity and proper toric morphisms

Mark Andrea de Cataldo

This talk is a discussion of ideas, definition and results contained in three ArXiv papers: [1] Proper toric maps over finite fields (de Cataldo), [2] The combinatorics and topology of proper toric maps (de Cataldo, Mustata, Migliorini), [3] Frobenius semisimplicity for convolution morphisms (de Cataldo, Haines, Li).

I will first discuss the standard semisimplicity conjecture (of Tate type) for cohomology and how it relates to the same conjecture for intersection cohomology via the following theorem:

Theorem 1 ([3]). *Let $f_0 : X_0 \to Y_0$ be a proper map over a finite field F_0. Then the intersection complex $IC_{f_0(X_0)}$ is a direct summand of $Rf_{0,*}IC_{X_0}$.*

I will define the notions of semisimple and of Frobenius semisimple complex over a variety defined over a finite field and discuss the following:

Theorem 2 ([3]). *Let $f_0 : X_0 \to Y_0$ be a proper map over a finite field F_0 and let K_0 be a mixed complex on X_0. If $Rf_{0,*}K_0$ is Frobenius semisimple, then it is semisimple.*

The above is thus a criterion for the semisimplicity of the direct image over F_0, not just over F (as it is usual in the literature). I will then state and sketch the proof of the following:

Theorem 3. *Let $f_0 : X_0 \to Y_0$ be a proper toric of toric varieties over a finite field F_0. Then $Rf_{0,*}IC_{X_0}$ is Frobenius semisimple and thus semisimple.*

The key point is the following result, which is of independent interest.

Theorem 4. *Let $f_0 : X_0 \to Y_0$ be a proper toric fibration (i.e. the corresponding maps of lattices is surjective) of toric varieties over a finite field F_0. Let $y_1 \in Y$ be a closed point (possibly defined over a finite extension F_1 of F_0). Let U_0 be any Zariski dense open subvariety of a toric completion X_0' that contains $f^{-1}(y_0)$. Then the natural map $IH^*(U) \to H^*(f^{-1}(y), IC_X)$ is surjective.*

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Toric MESSI biochemical systems

Alicia Dickenstein
(joint work with Mercedes Pérez Millán)

Many processes within cells involve some kind of post-translational modification of proteins. A subclass of these mechanisms which present modifications of type Enzyme-Substrate or Swap with Intermediates, has attracted considerable theoretical attention due to its abundance in nature and the special characteristics in the topologies. In my talk, I introduced a general framework for these biological systems, developed in joint work with Mercedes Pérez Millán [9], which we call MESSI networks.

MESSI biochemical systems are MESSI chemical reaction networks endowed with mass-action kinetics. The set of species can be partitioned into a subset $S^{(0)}$ of *intermediate* species and different subsets $S^{(1)}, \dots, S^{(m)}$ of *core* species, in such a way that the associated autonomous polynomial dynamical system is linear in the variables of each $S^{(i)}$ union some subset $S_i^{(0)}$ of the intermediate variables. The union of these subsets $S_i^{(0)}$ equals $S^{(0)}$, but they are in general not disjoint, which accounts for several important properties of the systems. We characterize with algebro-geometric and combinatorial tools general properties of MESSI systems (as compactness of invariant subspaces and permanence) and we concentrate on the important question of multistationarity, that is, on the ocurrence of more than one positive steady state with the same conserved quantities.

Many post-translational modification networks are MESSI networks. For example: the motifs in [2], sequential distributive multisite phosphorylation networks [10], sequential processive multisite phosphorylation networks [1], phosphorylation cascades or the bacterial EnvZ/OmpR network in [12]. Our work is inspired by and extends some results in several previous articles [3, 4, 5, 6, 7, 8, 10, 11, 13].

We show that the steady states of most popular MESSI systems (including all those recalled above) present a toric structure, and we give in this case a characterization of the capacity for multistationarity, which leads to an algorithmic approach that we implemented with tools from oriented matroid theory. The statement of our precise results would need a long glossary together with clarifying examples, that we omit in this account.

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Hodge theory in combinatorics

Eric Katz

(joint work with Karim Adiprasito, June Huh)

This abstract describes recent work [1] resolving Rota's conjecture on the log-concavity of the characteristic polynomial of a matroid.

1. The Characteristic Polynomial

We begin by considering the case of realizable matroids. Let $\mathbf{k}$ be a field. Let $V \subset \mathbf{k}^{n+1}$ be an $(r+1)$-dim linear subspace not contained in any coordinate hyperplane. We would like to use inclusion/exclusion to express $[V \cap (\mathbf{k}^*)^{n+1}]$ as a linear combination of $[V \cap L_I]$ where L_I is the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$. You may interpret the brackets as sets of geometric points.

To identify the intersections, we have to discuss flats. A subset $I \subset \{0, \ldots, n\}$ is said to be a *flat* if for any $J \supset I$, we have $V \cap L_J \neq V \cap L_I$. The rank of a flat is $\rho(I) = \operatorname{codim}(V \cap L_I \subset V)$. The flats uniquely label the intersections $V \cap L_I$. We can now write for unique choices $\nu_I \in \mathbb{Z}$,

$$[V \cap (\mathbf{k}^*)^{n+1}] = \sum_{\text{flats } I} \nu_I [V \cap L_I].$$

The characteristic polynomial of V is

$$\chi_V(q) = \sum_{i=0}^{r+1} \left(\sum_{\substack{\text{flats } I \\ \rho(I)=i}} \nu_I \right) q^{r+1-i} \equiv \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1} \mu_{r+1}.$$

2. Matroids

We may abstract the linear space to a matroid which we define as a rank function $\rho : 2^{\{0,\ldots,n\}} \to \mathbb{Z}$ satisfying

(1) $0 \leq \rho(I) \leq |I|$
(2) $I \subset J$ implies $\rho(I) \leq \rho(J)$
(3) $\rho(\{0,\ldots,n\}) = r+1$.
(4) $\rho(I \cup J) + \rho(I \cap J) \leq \rho(I) + \rho(J)$

The first three properties are obvious ones that a notion of codimension must satisfy while item (4) abstracts subadditivity of codimension under intersection. We set $r+1 = \rho(\{0,\ldots,n\})$ to be the rank of the matroid. For matroids, ν_I and hence $\chi(q)$ can be defined combinatorially by Möbius inversion without reference to any linear space. This leads us to a conjecture made by Rota in his 1970 ICM address:

Theorem 1 (Adiprasito-Huh-K '15)**.** *For any matroid, $\chi(q)$ is log-concave.*

Here, a polynomial with coefficients $\mu_0, \ldots, \mu_{r+1}$ is said to be *log-concave* if for all i, we have $|\mu_{i-1}\mu_{i+1}| \leq \mu_i^2$. The logarithms of the coefficients form a concave sequence. This implies unimodality if the sequence is internally zero-free. A polynomial with coefficients $\mu_0, \ldots, \mu_{r+1}$ is said to be *unimodal* if the coefficients are unimodal in absolute value, i.e. there is a j such that

$$|\mu_0| \leq |\mu_1| \leq \cdots \leq |\mu_j| \geq |\mu_{j+1}| \geq \cdots \geq |\mu_{r+1}|.$$

Note that the characteristic polynomial has no symmetry properties. We do not know where the mode is! This makes the unimodality in the Rota conjecture different from that of the g-theorem.

3. The Matroidal Chow Ring

We define a Stanley-Reisnerish ring, the matroidal Chow ring:

Definition 2. Let x_F be indeterminates indexed by proper flats. Let I_M be the ideal in $\mathbb{R}[x_F]$ generated by

(1) For each $i, j \in \{0, 1, \ldots, n\}$,
$$\sum_{F \ni i} x_F - \sum_{F \ni j} x_F,$$
(2) For incomparable flats F, F',
$$x_F x_{F'}.$$

Let $A^*(M) = \mathbb{R}[x_F]/I_M$.

This ring is an abstract stand-in for the cohomology ring of the algebraic variety $\tilde{V}$ that is obtained from $\mathbb{P}(V)$ by blowing up (the proper transforms of) the intersections $\mathbb{P}(V) \cap \mathbb{P}(L_I)$. This ring behaves like the cohomology ring of a smooth compact variety:

Theorem 3. *The ring $A^*(M)$ has a fundamental class: there is an isomorphism:*

$$\deg : A^r(M) \to \mathbb{R}.$$

The ring $A^*(M)$ obeys Poincaré duality: the following pairing is perfect

$$\begin{array}{rcl} A^p(M) \times A^{r-p}(M) & \to & \mathbb{R} \\ (x, y) & \mapsto & \deg(xy). \end{array}$$

4. Outline of the proof

Now, let us outline the proof of log-concavity. First, we use the reduced characteristic polynomial. From the fact $\chi(1) = 0$, we can set

$$\overline{\chi}(q) = \frac{\chi(q)}{q-1} = \mu^0 q^r - \mu^1 q^{r-1} + \cdots + (-1)^r \mu^r q^0.$$

The log-concavity of $\overline{\chi}$ implies the log-concavity of χ.

There are two important elements of $A^*(M)$: for arbitrary $j \in \{0, 1, \ldots, n\}$, set

$$\alpha = \sum_{F \ni j} x_F, \ \beta = \sum_{F \not\ni j} x_F.$$

By a short combinatorial argument, we show that the coefficients of the reduced characteristic polynomial are the mixed degrees of α and β: $\mu^i = \deg(\alpha^i \beta^{r-i})$.

To prove log-concavity, we must establish the Khovanskii-Teissier inequality for α, β: for all $1 \le i \le r-1$,

$$\deg(\alpha^{r-i+1}\beta^{i-1}) \deg(\alpha^{r-i-1}\beta^{i+1}) \le \deg(\alpha^{r-i}\beta^i)^2.$$

We do this by developing the Kahler package for $A^*(M)$. This involves defining an *ample cone* $K \subset A^1(M)$ with α and $\beta \in \overline{K}$. The ring will obey the Hard Lefschetz theorem and the Hodge-Riemann relations with respect to K.

Let R^* be a graded commutative ring in degrees $0, \ldots, r$ satisfying Poincaré duality with respect to $\deg : R^r \to R$. We say that R^* satisfies the *Hard Lefschetz property* with respect to a convex cone $K \subset R^1$ if for all $\ell \in K$ and all $p \le \frac{r}{2}$, the map $L^p : R^p \to R^{r-p}$ given by $c \mapsto \ell^{r-2p} \cdot c$ is an isomorphism. One defines primitive subspace $P^p \subset R^p$ by $P^p = \ker(\ell^{r-2p+1} \cdot : R^p \to R^{r-p+1})$.

For $\ell \in R^1$, we define the quadratic form Q^p_ℓ on R^p for $p \le \frac{r}{2}$ by

$$Q^p_\ell(c_1, c_2) = (-1)^p \deg(\ell^{r-2p} c_1 c_2).$$

We say that the ring R^* obeys *the Hodge-Riemann relations* with respect to K if for all $\ell \in K$ and $p \le \frac{r}{2}$, Q^p_ℓ restricted to the primitive subspace P^p is positive definite. The Hodge-Riemann relations imply a form of the Khovanskii-Teissier inequality that is sufficient for our purposes.

We establish the Kahler package for $A^*(M)$ with respect to a particular cone $K \subset A^1(M)$. Our strategy is to interpolate between the polynomial ring $\mathbb{R}[x]/(x^{r+1})$ and $A^*(M)$. We modify the rings by adding one flat at a time. This corresponds to blowing-up subvarieties in algebraic geometry. We have to leave the world of matroids and work with intermediate Chow rings. The proof makes use of an inductive argument introduced by McMullen [3] in his work on the g-theorem.

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When is a variety the quotient of a smooth variety by a finite group?

Matthew Satriano
(joint work with Anton Geraschenko)

We discuss a local-to-global question concerning quotient singularities posed by William Fulton. A variety X over a field k has *(tame) quotient singularities* if there is an étale cover $\{X_i \to X\}_i$ with each $X_i = U_i/G_i$ where U_i is smooth and G_i is a finite group (whose order is relatively prime to the characteristic of k).[1] We say X has abelian quotient singularities if we can take the G_i to be abelian. It is clear from the definition that every global quotient U/G with U smooth and G finite, has quotient singularities. It is therefore natural to ask if the converse holds:

Question 1 (Fulton). *If X is a variety with quotient singularities over an algebraically closed field, then can we write $X = U/G$ with U smooth and G finite?*

We answer a special case of Question 1 as a consequence of the following result, characterizing when a variety is a global quotient by a finite abelian group.

Theorem 2 ([1, Theorem 1.2]). *Let X be a quasi-projective variety with tame abelian quotient singularities over an algebraically closed field k. Then the following are equivalent:*

(1) *X is a quotient of a smooth quasi-projective variety by a finite abelian group.*
(2) *X is the geometric quotient (in the sense of* [3]*) of a smooth quasi-projective variety by a torus acting with finite stabilizers.*

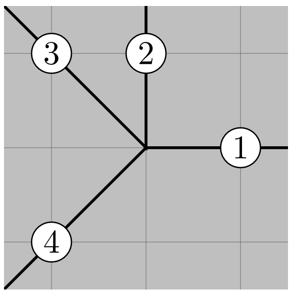
[1] There are several equivalent definitions of (tame) quotient singularities. One can instead take the $\{X_i \to X\}_i$ to be a Zariski cover, or alternatively require that each complete local ring $\widehat{\mathcal{O}}_{X,x}$ be isomorphic to the invariant ring $k(x)[[t_1, \ldots, t_n]]^G$ with G a finite group (with order prime to the characteristic).

(3) *X has Weil divisors $D_1, \ldots, D_r$ whose images generate* $\mathrm{Cl}(\widehat{\mathcal{O}}_{X,x})$ *for all closed points x of X.*
(4) *The canonical smooth tame Deligne-Mumford stack X^{can} as constructed in* [5, 2.9] *is a stack quotient of a quasi-projective variety by a torus.*

Remark 3. Theorem 2 follows from a more technical result [1, Theorem 5.2], which applies to algebraic spaces over infinite fields (in contrast to quasi-projective varieties over algebraically closed fields).

Remark 4. Those readers familiar with stacks should not confuse Question 1 with the question "is every smooth Deligne-Mumford stack of the form $[U/G]$ with U smooth and G a finite group?" The answer to this latter question is "no" as demonstrated by the square root stack of $\mathbb{P}^1$ at a point. However, our proof of Theorem 2 shows that if $\mathcal{X}$ is a global quotient by a split torus and has quasi-projective coarse space, there is a relative coarse space map $[U/G] \to \mathcal{X}$ with U smooth and G finite.

We emphasize that even for toric varieties, the answer to Question 1 is not obvious, although an immediate corollary of our result settles Question 1 for all quasi-projective toric varieties. To better understand why the case of toric varieties is not immediate, consider the blow-up of $\mathbb{P}(1,1,2)$ at a smooth torus-fixed point. Its fan is given by



If one wishes to write this variety as a *toric* quotient U/G with U smooth and G finite, this amounts to refining the above lattice in such a way that all cones become smooth. However, it is an easy exercise to show that every refinement makes the cone generated by rays 1 and 2 singular or keeps the cone generated by rays 3 and 4 singular. Thus, in order to answer Question 1 for toric varieties, one is forced to use non-toric techniques.

Although Question 1 and Theorem 2 (1)–(3) are inherently about varieties, our proof uses stack-theoretic techniques. In the case of toric varieties, however, we are able to unravel our proof and obtain a combinatorial procedure for constructing the U and G of Question 1. We demonstrate this algorithm on the blow-up of $\mathbb{P}(1,1,2)$ at a smooth torus-fixed point at the end of this article.

Before demonstrating the algorithm, we note that there are several natural variants of Question 1 that one can pose. For example, one can ask:

Question 5. *If X is a variety over an algebraically closed field with tame* abelian *quotient singularities, then is it of the form U/G with U a smooth variety and G a finite* abelian *group?*

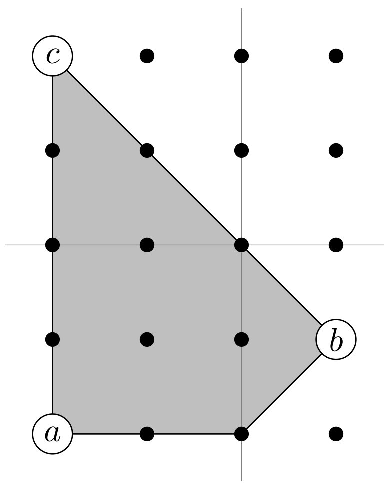
It follows from the theorem below that Question 5 has a negative answer.

Theorem 6. *Let k be an algebraically closed field with* $\mathrm{char}(k) \nmid 60$*, and let V be an irreducible 3-dimensional representation of the alternating group A_5. Then $X = (V \setminus 0)/A_5$ has abelian quotient singularities, but is not a quotient of a smooth variety by a finite abelian group.*

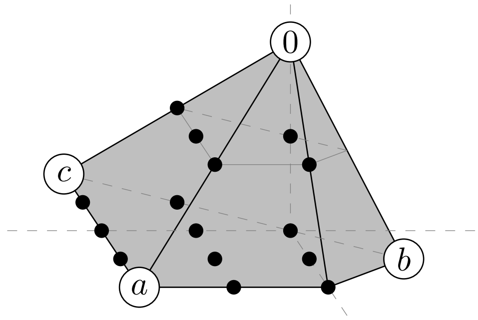
Other natural variants of Question 1 are to drop the requirement that G be finite, or to replace a group quotient by a finite surjection. Both of these are known to have positive answers. The first is given by [2, Corollary 2.20], which states that if X is a quasi-projective variety with quotient singularities over a field of characteristic 0, then $X = U/G$ where U is a smooth scheme and G is a linear algebraic group. The second is given by [4, Theorem 1] and [2, Theorem 2.18]: for an irreducible quasi-projective variety X with quotient singularities over a field k, there is a finite surjection from a smooth variety to X. From this perspective, Question 1 therefore asks if there is a common refinement of these two results.

Lastly, as promised, we give an explicit description of the U and G of Question 1 when X is the blow-up of $\mathbb{P}(1, 1, 2)$ at a torus fixed point. For the explicit procedure in the case of an arbitrary quasi-projective toric variety, see Theorem 7 below.

Step 1: Choosing very ample divisors and sections. Our first step is to choose a collection of very ample divisors satisfying condition (3) of Theorem 2. In this particular case, we need only choose one divisor D. Letting D_{ρ_i} denote the divisor corresponding to the i-th ray of the fan pictured above, we choose $D = D_{\rho_1} + D_{\rho_2} + D_{\rho_3}$. We must next choose a section s of $\mathcal{O}_X(2D)$ with the property that its vanishing locus $V(s)$ misses the singular locus of X. More generally, we seek sections of $\mathcal{O}_X(ND_i)$ with a slightly more complicated intersection property, where N is the lcm of the orders of all complete local class groups $\mathrm{Cl}(\widehat{\mathcal{O}}_{X,x})$. In our case, we choose $s = s_a + s_b + s_c$, where s_p denotes the section corresponding to the lattice point p of the polytope pictured below.



Step 2: Computing U and G. Consider the projective toric variety in $\mathbb{P}^{18}$ defined by the following polytope, which is a height 2 cone over the polytope shown above.



Let U be the hyperplane slice of this toric variety defined by $x_0 - (x_a + x_b + x_c)$ and let $G = \mathbb{Z}/2$. We obtain a G-action on $\mathbb{P}^{18}$ as follows: if χ is the non-trivial character of G and x is a lattice point of the above polytope with height h, then G acts the coordinate corresponding to x through the character χ^h. Then U is smooth, is invariant under the G-action, and $X = U/G$. Note that this is a completely toric procedure except for taking the hyperplane slice.

For an arbitrary quasi-projective toric variety, the presentation $X = U/G$ is obtained explicitly using the following result.

Theorem 7 ([1, Theorem 3.1])**.** *Let X be a quasi-projective toric variety with tame quotient singularities over an infinite field k. Let Σ be the fan of X, let $Z \subseteq X$ be the singular locus, and let $X = V/H$ be the Cox construction.*

1. *There exist Weil divisors $D_1, \ldots, D_r$ which generate the class groups of all torus-invariant open affine subvarieties of X. Letting n_i be integers so that $n_i D_i$ is Cartier for each i, the D_i can be chosen so that $n_i D_i$ is very ample.*
2. *There are sections $\{s_{i,j}\}_{1 \leq j \leq c_i}$ of $\mathcal{O}_X(n_i D_i)$ so that the preimages of the vanishing loci $\{V(s_{i,j})\}_{i,j}$ in V are smooth with simple normal crossings, and $\bigcap_j V(s_{i,j})$ is disjoint from Z for each i.*
3. *Let W be the toric variety with fan $\widehat{\Sigma}$, as described in* [1, §3.2]*, and let $U_{i,j} \subseteq W$ be the $s_{i,j}$-cut together with its μ_{n_i}-action* [1, Definition 3.12]. *Then the scheme-theoretic intersection $U = \bigcap_{i,j} U_{i,j}$ in W is a smooth variety with an action of $G = \prod_i \mu_{n_i}^{c_i}$, such that $X \cong U/G$.*

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Additive actions on toric varieties

Ivan Arzhantsev
(joint work with Elena Romaskevich)

Let X be an irreducible algebraic variety of dimension n over an algebraically closed field $\mathbb{K}$ of characteristic zero and $\mathbb{G}_a = (\mathbb{K}, +)$ be the additive group of the field. Consider the commutative unipotent group $\mathbb{G}_a^n = \mathbb{G}_a \times \ldots \times \mathbb{G}_a$ (n times). By an additive action on X we mean a regular action $\mathbb{G}_a^n \times X \to X$ with an open orbit. Equivalently, one may consider algebraic varieties with an additive action as equivariant embeddings of the vector group $(\mathbb{K}^n, +)$.

A systematic study of additive actions was initiated by Hassett and Tschinkel [12]. They established a remarkable correspondence between additive actions on the projective space $\mathbb{P}^n$ and local $(n+1)$-dimensional commutative associative algebras with unit. This correspondence has allowed to classify additive actions on $\mathbb{P}^n$ for small n. The same technique was used by Sharoiko [13] to prove that an additive action on a non-degenerate projective quadric is unique. Further modification of Hassett-Tschinkel correspondence led to characterization of additive actions on arbitrary projective hypersurfaces, in particular, on degenerate projective quadrics [3], [2].

The study of additive actions was originally motivated by problems of arithmetic geometry. Chambert-Loir and Tschinkel [4] gave asymptotic formulas for the number of rational points of bounded height on smooth projective equivariant compactifications of the vector group. More generally, asymptotic formulas for the number of rational points of bounded height on quasi-projective equivariant embeddings of the vector group are obtained in [5].

In [1] all generalized flag varieties G/P admitting an additive action are found. Here G is a semisimple linear algebraic group and P is a parabolic subgroup. It turns out that if G/P admits an additive action then the parabolic subgroup P is maximal.

Feigin [10] proposed a construction based on the PBW-filtration to degenerate an arbitrary generalized flag variety G/P to a variety with an additive action. Recently Fu-Hwang [11] and Devyatov [9] have proved that if G/P is not isomorphic to the projective space, then up to isomorphism there is at most one additive action on G/P. Classification of additive actions on singular del Pezzo surfaces is obtained by Derenthal and Loughran [8].

The problem of classification of additive actions on toric varieties is raised in [3, Section 6]. Some instructive examples of such actions are given in [12, Proposition 5.5]. It is natural to divide the problem into two parts. The first one deals with additive actions on a toric variety X of dimension n normalized by the acting torus T. In this case an additive action splits into n pairwise commuting $\mathbb{G}_a$-actions on X normalized by T. It is proved in [7] that $\mathbb{G}_a$-actions on a toric variety X normalized by T are in bijection with some vectors defined in terms of the fan Σ_X associated with X. Such vectors are called the Demazure roots of a fan. Cox [6] observed that normalized $\mathbb{G}_a$-actions on a toric variety can be interpreted as certain $\mathbb{G}_a$-subgroups of automorphisms of the Cox ring $R(X)$ of the variety X. In turn, such subgroups correspond to homogeneous locally nilpotent derivations of the Cox ring. In these terms the Demazure root is nothing but the degree of the derivation.

We prove that additive actions on a toric variety X normalized by the acting torus T are in bijection with complete collections of Demazure roots of the fan Σ_X. Also we show that any two normalized additive actions on X are isomorphic.

The second part of the problem concerns non-normalized additive actions. Our result states that if a complete toric variety admits an additive action, then it admits an additive action normalized by the acting torus.

It is well known that a toric variety is projective if and only if its fan is a normal fan of a convex polytope. We characterize polytopes corresponding to projective toric varieties with an additive action.

Also we give explicit examples of additive actions on toric varieties in terms of their Cox rings and formulate several open problems.

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Tropical currents and the Hodge conjecture

June Huh
(joint work with Farhad Babaee)

Let X be a smooth compact toric variety over the complex numbers. We introduce some results from [2, 3]. From toric perspective, the most important assertion is that every cohomology class $\mathcal{C}$ of X has a canonical representative $\mathcal{T}_\mathcal{C}$ in the space of closed currents on X. The representative $\mathcal{T}_\mathcal{C}$ is called the *tropical current* associated to $\mathcal{C}$. We summarize here two main properties of the connection between the cohomology class $\mathcal{C}$ and its tropical current $\mathcal{T}_\mathcal{C}$:

(1) The representatives $\mathcal{T}_\mathcal{C}$ are geometric, and can be intersected in a geometric way. This should be compared with the harmonic representatives of cohomology classes, which are neither geometric nor closed under the wedge product.
(2) The representatives $\mathcal{T}_\mathcal{C}$ reflect the positivity of the class $\mathcal{C}$ in an interesting way. More precisely, $\mathcal{C}$ is nef if and only if $\mathcal{T}_\mathcal{C}$ is positive, and $\mathcal{C}$ is extremal in the nef cone of X if and only if $\mathcal{T}_\mathcal{C}$ is extremal in the cone of positive closed currents on X.

One can show the above connection between $\mathcal{C}$ and $\mathcal{T}_\mathcal{C}$ to produce examples of positive closed currents on smooth projective varieties with unexpected properties. For example, one can construct a 4-dimensional smooth projective complex variety X and a $(2,2)$-dimensional positive closed current $\mathcal{T}$ on X with integral cohomology class that is not a weak limit of effective algebraic cycles, see [3]. The variety X can be chosen to be toric, and the current $\mathcal{T}$ can be chosen to be a tropical current which is extremal in the cone of positive closed currents on X.

Particularly interesting examples of extremal positive closed currents arise when we take X to be the toric variety of the permutohedron and $\mathcal{T}_\mathcal{C}$ to be the tropical current associated to a loopless matroid $\mathcal{C}$, see [4]. Is it true that such a tropical current $\mathcal{T}_\mathcal{C}$ is a weak limit of effective algebraic cycles on X? The answer is "yes" if the matroid $\mathcal{C}$ is representable over the complex numbers, but unknown in general. Several numerical consequences of the affirmative answer to the question is recently proved in [1] by other methods, so there is no obvious numerical reason indicating otherwise.

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Toric structures in nontoric varieties

Mateusz Michałek
(joint work with Luke Oeding, Piotr Zwiernik)

For a projective variety $X \subset \mathbb{P}^N$ we define the k-th secant variety by:

$$\sigma_k(X) = \overline{\bigcup_{x_1,\ldots,x_k \in X} < x_1, \ldots, x_k >},$$

where $< \cdot >$ denotes the smallest projective space containing the given set. Even when X is a well-understood variety, the secant variety can be very complicated from the point of view of algebra and geometry. One of the most important examples is the Segre product of projective spaces that parameterizes tensors of rank one. In this case, the k-th secant variety is the locus of tensors of border rank at most k (the closure of the locus of tensors of rank k). Determining equations of secant varieties is crucial e.g. in computational complexity [6]. Further information about the secant and tangential variety can be found in [16].

When X is the Segre product, from the toric point of view it corresponds to a polytope P that is the product of simplices. The torus action distinguishes coordinates on $\mathbb{P}^N$ corresponding to lattice points of P. Let us fix a vertex $v \in P$. This distinguishes a hyperplane $H_v \subset \mathbb{P}^N$ and an affine open set $A_v = \mathbb{P}^N \setminus H_v$.

Theorem 1. [8] *When X is the Segre product then $\sigma_2(X) \cap A_v$ is a product of an affine space and an affine toric variety T (that is an affine cone over a projectively normal projective toric variety).*

The polytope $\tilde{P}$ representing the toric variety from the theorem, under the assumption that $v = 0$, is defined by $\{(x_i) \in P : \sum_i x_i \geq 2\}$. The 'classical' parametrisation of $\sigma_2(X)$ does not indicate that $\sigma_2(X) \cap A_v$ could be toric. The result is obtained using a special automorphism of the affine space, inspired by calculation of cumulants in statistics - cf. [13, 17]. It relies on the fact that by choosing $v \in P$ we induce a partial order on the lattice points of $P \setminus \{v\}$. The change of coordinates is (a composition of two) triangular (automorphisms), of the form:

$$y_w = x_w + \text{polynomials in } x\text{'s strictly smaller than } x_w.$$

In forthcoming joint work with Hendrik Suess and Alexander Perepechko we extend these results to Segre-Veronese varieties. This allows to conclude that cones of such secant varieties are *flexible.*

Similar results were also obtained in a joint work with Laurent Manivel [7] for Grassmannians (resp. spinor varieties). Here however, instead of a monomial parametrization we get a parametrization by minors (resp. subpfaffians) of a generic (resp. skew symmetric) matrix. The case of Grassmannians is more complicated than the Segre-Veronese case. In particular, we do not know the defining equations of $\sigma_2(G(k,n))$, contrary to the Segre-Verones case [3, 11]. Also the relations among minors are far more complicated than the equations defining the toric variety. For more information about relations among minors of a generic matrix we refer the reader to [1] (note however, that from the point of view of secant varieties we are more interested in relations among *all* minors of size at least 2 of a generic matrix, as opposed to relations among minors of a fixed size).

The polytopes obtained for secants of Segre varieties have nice properties. They are normal, possess a quadratic Groebner basis and are related to other families previously studied - cf. [12, Section 14A]. Christian Haase related the obtained quadratic Groebner basis to known unimodular triangulations.

Our results allow also for a full classification of (locally) Gorenstein secants.

Theorem 2. [8] *The variety $\sigma_2(\mathbb{P}^{a_1}\times\cdots\times\mathbb{P}^{a_n})$ is (locally) Gorenstein if and only if it fills the ambient space or:*

(1) $n = 2$ *and* $a_1 = a_2$,
(2) $n = 3$ *and* (a_i) *equals* $(1, 1, 3)$, $(1, 3, 3)$ *or* $(3, 3, 3)$,
(3) $n = 5$ *and* $(a_i) = (1, 1, 1, 1, 1)$.

The first family $a_1 = a_2$ is classical. Recently, affine cones in the second family were confirmed to be Gorenstein [10].

Conjecture 3. *Is the affine cone over $\sigma_2(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1)$ Gorenstein?*

All of the mentioned results apply also for tangential varieties, i.e. unions of tangent spaces. For the toric case, after the 'cumulant' change of coordinates, the tangential varieties are parameterized by the same monomials as the secant varieties. Recall, that the secant variety is given as an affine *cone*, hence the usual containment of a tangential variety in the secant as a divisor. Moreover, we see that the tangential variety equals the secant if the parameterizing monomials (in the cumulant coordinates) are all of the same degree. This is specially useful, when we consider the following generalization from the forthcoming work with Piotr Zwiernik:

Theorem 4. *Consider a simplicial complex C with variables associated to vertices (possibly noninjectively). Consider the embedding of the n dimensional affine space given by all the monomials corresponding to faces in C (including vertices). The secant variety is isomorphic to a product of an n dimensional affine space with a (possibly nonnormal) toric variety parametrized by all monomials corresponding to faces in C of positive dimension.*

In particular, taking a product with the affine space, all toric varieties associated to graphs in the sense of Hibi and Ohsugi [4] are secant (and tangential) varieties of reembeddings of affine spaces.

During the lecture two conjectures were also presented:

Conjecture 5.

(1) *Is the convex hull of all lattice points in any ball a normal polytope?* [2, Question 7.2 b)]
(2) *Let f be a homogeneous polynomial in $n+1$ variables. Let L be a generic subspace of $\mathbb{P}^n$. Do we always have $L^\perp\cap\overline{\nabla f(L)}=\emptyset$?* [9, Conjecture 5.5]

In Oberwolfach, Bernard Teissier observed that the answer to the second question is positive. Indeed, it is a special case of his results obtained in [14, 15], which strengthen the classical Bertini theorem. For recent application of the same results and a new proof we refer to [5]. The first point of Conjecture 5 still awaits the answer.

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Toric vector bundles and polytopes

Sandra Di Rocco

(joint work with Kelly Jabbusch, Gregory G. Smith)

Exact criteria for positivity of line bundles L on a toric variety X are well understood, mainly due to the combinatorial underline structure and the convex geometry of the associated lattice polytope P_L. Let X_Σ be a smooth toric variety of dimension n. It is well known that:

- L is *nef* if and only if $L \cdot C \geq 0$ for all invariant curves C, which is equivalent to L being *globally generated.*
- L is *ample* if and only if $L \cdot C > 0$ for all invariant curves C, which is equivalent to L being *very ample.*

It is natural to ask to which extent such a characterization holds for higher rank vector bundles. This question has been posed and investigated in [2]. In this paper the authors characterize intersection positivity of *toric vector bundles* generalizing the criteria for rank one. In particular they prove that given a toric vector bundle $\mathcal{E}$ of rank r :

- $\mathcal{E}$ is *nef* if and only if $\mathcal{E}|_C = \oplus_1^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$ for $i = 1 \dots r$ and for all the invariant curves C.
- $\mathcal{E}$ is *ample* if and only if $\mathcal{E}|_C = \oplus_1^r \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i > 0$ for $i = 1 \dots r$ and for all the invariant curves C.

Toric vector bundles carry a nice combinatorial structure, as described in [3]. They are in one-to-one correspondence to decreasing fibrations of vector subspaces of the fiber $\mathbb{E} = \mathcal{E}_{x_0}$ at the generic point x_0 in the torus: $\{\mathbb{E} \supseteq \cdots \supseteq \mathcal{E}^{v_i}(j) \cdots \supseteq \mathcal{E}^{v_i}(j+1) \supseteq \cdots\}$, for each primitive vector v_i on a ray in Σ and satisfying certain compatibility conditions.

In order to further investigate the implications of nefness and ampleness on properties for global sections, i.e. global generation and very ampleness, we introduced associated lattice polytopes, see [1].

We show that a toric vector bundle $\mathcal{E}$ defines a canonical minimal matroid $\mathcal{M}_\mathcal{E}$ whose lattice of flats is given by intersections of the filtration subspaces ordered by inclusion. The ground set of this matroid, $\mathcal{G}(\mathcal{E})$, consists of a finite set of vectors of $\mathbb{E}$, $\mathcal{G}(\mathcal{E}) = \{e_1, \dots, e_s\}$ with $s \geq r$, containing all the basis vectors of the subspaces in the filtration. Moreover the compatibility conditions imply that every fixed fiber is spanned by a distinguished r-tuple of vectors in $\mathcal{G}(\mathcal{E})$. This means that for each n-dimensional cone $\sigma \in \Delta$ there is a basis $(e_\sigma^1, \dots, e_\sigma^r)$, of vectors in $\mathcal{G}(\mathcal{E})$, for the fiber $\mathcal{E}_{x(\sigma)}$ over the fixed point $x(\sigma)$.

We define a polytope for each element $e \in \mathcal{G}(\mathcal{E})$:

$$P_e = \{m \in \mathbb{R}^n \text{s. t. } < m, v_i > \leq \max\{j \text{ s.t. } e \in \mathcal{E}^{v_i}(j)\}\}.$$

and the *parliament of polytopes*: $\mathcal{P}_\mathcal{E} = \{P_e\}_{e \in \mathcal{G}(\mathcal{E})}$.

First we prove that:

Theorem 1 ([1]). *The lattice points in the parliament of polytopes $\mathcal{P}_\mathcal{E}$ form a generating set for the global sections $H^0(X, \mathcal{E})$.*

We then use this characterization to give a criterion of global generation, very ampleness and generation of higher jets in terms of the polytopes, generalizing the corresponding criteria for line bundles. We report here the global generation theorem.

For each σ we denote by $m(\sigma) = (m^1_\sigma, \dots, m^r_\sigma)$ the distinguished characters coming from the given filtration and corresponding to a local trivialization of $\mathcal{E}$ at the open orbit U_σ. After fixing a local basis around the fixed point $x(\sigma)$ one can define a bijection between the r-tuple $m(\sigma) = (m^1_\sigma, \dots, m^r_\sigma)$ and the basis vectors $(e^1_\sigma, \dots, e^r_\sigma)$. In particular every m^i_σ has a corresponding e^i_σ under this correspondence.

Theorem 2 ([1]). *The toric vector bundle $\mathcal{E}$ is globally generated if for each n-dimensional cone $\sigma \in \Sigma$ the character m^i_σ is a vertex of the polytope $P_{e^i_\sigma}$ for all $i = 1, \dots, r$.*

The three polytopes below form the parliament of polytopes of $T_{\mathbb{P}^2}$. The characters of each shape (circle, rectangle and diamond) are associated to one of the three two-dimensional cones. One sees that the tangent bundle is indeed globally generated.

Using this visual criterion we were able to construct examples of ample toric vector bundles which are not very ample and not even globally generated, demonstrating that the nice equivalences (nefness and global generation, ample and very ampleness) in rank one do not extend to higher rank.

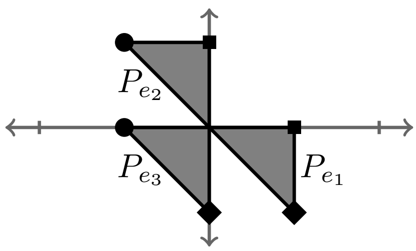

FIGURE 1. The parliament of polytopes for $T_{\mathbb{P}^2}$

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A geometric characterization of toric varieties

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(joint work with James McKernan, Roberto Svaldi, Runpu Zong)

Let X be a projective variety and let Δ be an effective $\mathbb{Q}$-divisor on X. We say that the pair (X, Δ) is a log Calabi-Yau pair if (X, Δ) is log canonical and $K_X + \Delta$ is numerically trivial.

For example, if X is a normal projective toric variety and Δ is the sum of the invariant divisors, then (X, Δ) is a log Calabi-Yau pair.

The goal of the present work is to give a simple characterization of toric pairs among log Calabi-Yau pairs. For simplicity, assume X is $\mathbb{Q}$-factorial. We consider an invariant called the complexity: Let ρ be the rank of the subvector space of $NS_{\mathbb{Q}}(X)$ spanned by components of Δ, and let n be the dimension of X. Finally set $|\Delta|$ to be the sum of the coefficients of the components. Then the complexity is $\gamma(X, \Delta) = \rho + n - |\Delta|$.

When X is a simplicial projective toric variety, the number of torus invariant divisors is the sum of the Picard rank of X and the dimension of X. Thus if Δ is the sum of the torus invariant divisors, the complexity is $\gamma(X, \Delta) = 0$. The following theorem shows that this property characterizes toric pairs. In this way we give a precise sense in which toric varieties are the simplest log Calabi-Yau varieties.

Theorem 1. *Let X be a projective variety over a field of characteristic 0, and suppose (X, Δ) is a log canonical pair such that $-(K_X + \Delta)$ is nef. Then the complexity γ is nonnegative, and if $\gamma < 1$, then X is a split toric variety, and we may choose the torus action such that all but at most one of the torus invariant divisors is a component of Δ.*

As an example, consider a quadric in projective space $Q \subset \mathbb{P}^{n+1}$. The canonical divisor the quadric is $-n$ times the hyperplane class. If we let Δ be the sum of n generic hyperplanes then $K_Q + \Delta = 0$, (Q, Δ) is log canonical, and $\gamma(X, \Delta) = 1$. Thus the theorem is sharp. Likewise, consider a quadric cone Y in $\mathbb{P}^3$. If Γ is the union of 4 lines through the cone point, we have that $K_Y + \Gamma = 0$, and $\gamma(Y, \Gamma) = -1$. This example does not violate the theorem because (Y, Γ) is not log canonical. So in fact the singularity condition is necessary.

Theorem 1 partially answers a conjecture of Shokurov [7, 6], which was originally stated in the relative setting. Yao [8] proved the case where (X, Δ) is a simple normal crossing pair.

Our techniques involve reducing to the local case by studying the Cox ring. Suppose (X, Δ) is a log Calabi-Yau pair, and X is a Mori dream space. Let S be $\mathrm{Spec}(\mathrm{Cox}(X))$, and let Γ be the divisor on S corresponding to Δ. Then (S, Δ) is a log canonical pair [2, 4]. Now a local version of the theorem lets us deduce that if the complexity is small, then S is smooth.

We then apply the results of Cox [3] and Hu-Keel [5], that a projective variety is toric if and only if its Cox ring is a polynomial ring. The main technical hurdle in our work is reducing to the case where X is a Mori dream space. To do this we need a version of Fano varieties for pairs. A pair (Y, Γ) is a log Fano pair if it satisfies the singularity condition Kawamata log terminal (klt) and $-(K_Y + \Gamma)$ is an ample divisor. Log Fano pairs are very important in the minimal model program, and provide a large class of examples of Mori dream spaces [1].

In a reduction step, we are able to show that a log Calabi-Yau pair (X, Δ) of low complexity is very close to being a log Fano pair; when $\gamma(X, \Delta) < 1$ we are able to find X' birational to X and Δ' close to the strict transform of Δ such that (X', Δ') is log Fano.

Our techniques are also able to produce some results when the complexity is larger:

Theorem 2. *Let (X, Δ) be a projective log canonical pair over $\mathbb{C}$. Suppose $K_X + \Delta$ is numerically trivial, the components of Δ generate $NS_{\mathbb{Q}}(X)$, and $\gamma < \frac{3}{2}$. Then if $\mathrm{Cl}(X)$ has no 2-torsion, X is rational.*

To prove Theorem 2, we again reduce to a statement on the Cox ring. In this case we show that the condition on γ implies that the Cox ring has a single quadric relation of the form $x_0^2 + x_1^2 + \ldots = 0$. The condition on 2-torsion is needed to deduce that x_0 and x_1 have the degree in the Cox ring, and from there we construct a birational map from X to a toric variety.

The condition on the class group of X is in fact necessary, as we are able to produce an irrational 3-fold example satisfying all of the other conditions of Theorem 2. A stronger version of Theorem 2 was conjectured by McKernan [6].

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The Decomposition Theorem for toric morphisms

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(joint work with M. de Cataldo, Luca Migliorini)

We describe, following [2], the Decomposition Theorem of Beilinson, Bernstein, Deligne, and Gabber in the context of toric morphisms. The goal is to give a purely combinatorial description of the invariants that come up in this context.

As a consequence of this study, we obtain some invariants of toric morphisms that turn out to be nonnegative and satisfy subtle properties that come from relative Poincaré Duality and Hard Lefschetz theorems. Related results have been obtained independently in a more combinatorial setting by Katz and Stapledon in [3].

Suppose that $f\colon X \to Y$ is a proper, equivariant morphism of complex toric varieties. We denote by N_X and N_Y the lattices of X and Y, respectively, and by Σ_X and Σ_Y the respective fans. The morphism f corresponds to a lattice map $\phi\colon N_X \to N_Y$ such that for every cone $\alpha \in \Sigma_X$ there is a cone $\sigma \in \Sigma_Y$ such that $\phi(\alpha) \subseteq \sigma$. The smallest such σ will be denoted by $\phi_*(\alpha)$. We assume that f is a fibration, that is, f is surjective and has connected fibers. This translates in the surjectivity of ϕ. With our notation, if $O(\sigma)$ and $V(\sigma)$ denote the orbit, respectively, orbit closure corresponding to a cone σ, then we have

$$f(O(\alpha)) = O(\phi_*(\alpha)) \quad \text{and} \quad f(V(\alpha)) = V(\phi_*(\alpha)).$$

The Decomposition Theorem takes the following form in the toric setting.

Theorem 1. *There is an isomorphism*

$$Rf_*IC_X \simeq \bigoplus_{\tau\in\Sigma_Y}\bigoplus_{b\in\mathbf{Z}} IC_{V(\tau)}[-b]^{s_{\tau,b}},$$

such that the nonnegative integers $s_{\tau,b}$ satisfy restrictions coming from relative Poincaré Duality and, when f is projective, Hard Lefschetz.

In the above statement, we write IC_Z for the intersection cohomology complex of a variety Z. Note that by comparison with the general statement of the Decomposition Theorem in [1], in Theorem 1 we have the following two special features:

1) All varieties that appear in the decomposition (the *support varieties*) are torus-invariant subvarieties, and
2) The local systems that usually appear in the Decomposition Theorem are trivial in our case (here is where we use the fact that f is a fibration).

Our goal is to give a description of the invariants $s_{\tau,b}$ in terms of combinatorics. We here focus on the case when both X and Y are simplicial varieties. This implies that X, Y, and all invariant subvarieties of Y have quotient singularities. As a result, the decomposition in Theorem 1 takes the following form:

$$Rf_*\mathbf{Q}_X[\dim(X)] \simeq \bigoplus_{\tau\in\Sigma_Y}\bigoplus_{b\in\mathbf{Z}} \mathbf{Q}_{V(\tau)}[\operatorname{codim}(\tau)-b]^{s_{\tau,b}}. \tag{1}$$

We explain how to obtain the description of the integers $s_{\tau,b}$ in this case, but for simplicity, only write down explicit formulas for

$$\delta_\tau := \sum_{b\in\mathbf{Z}} s_{\tau,b}.$$

Note that the positivity of δ_τ is equivalent to the fact that $V(\tau)$ is a support variety for the Decomposition Theorem.

Suppose now that $y = x_\sigma$, the origin of the torus $O(\sigma)$, for some $\sigma \in \Sigma_Y$. Note that $y \in V(\tau)$ if and only if $\tau \subseteq \sigma$. Consider now the equality (1), take the i^{th} cohomology of the complexes on each side, and compute the stalks at y to get

$$H^{i+\dim(X)}(f^{-1}(x_\sigma), \mathbf{Q}) \simeq \bigoplus_{\tau \subseteq \sigma} \mathbf{Q}^{s_{i+\operatorname{codim}(\tau)}}. \tag{2}$$

In order to use this, we need to compute the cohomology of the fibers of f. The first step consists in computing the Hodge-Deligne polynomial of $f^{-1}(x_\sigma)$. This fiber has a partition by locally closed subsets Z_α, with α varying over the cones in Σ_X such that $\phi_*(\alpha) = \sigma$. Moreover, Z_α is isomorphic to the kernel of $O(\alpha) \to O(\sigma)$, hence it is a torus of dimension equal to $\operatorname{codim}(\alpha) - \operatorname{codim}(\sigma)$. We thus obtain the following:

Theorem 2. *For every toric fibration $f\colon X \to Y$ and for every $\sigma \in \Sigma_Y$, the Hodge-Deligne polynomial of $f^{-1}(x_\sigma)$ is equal to*

$$E(f^{-1}(x_\sigma); u, v) = \sum_{\alpha, \phi_*(\alpha)=\sigma} (uv-1)^{\operatorname{codim}(\alpha)-\operatorname{codim}(\sigma)}.$$

In order to apply this, we also make use of the following result.

Theorem 3. *For every toric fibration $f\colon X \to Y$, with X and Y simplicial, and for every $y \in Y$, the mixed Hodge structure on $H^i(f^{-1}(y), \mathbf{Q})$ is pure of weight i. In particular, we have*

$$E(f^{-1}(y); t, t) = \sum_{i \geq 0} (-1)^i \dim_{\mathbf{Q}} H^i(f^{-1}(y), \mathbf{Q}) t^i.$$

It is clear that by combining Theorems 2 and 3 we obtain explicit formulas for the Betti numbers of the fibers of f. For simplicity, we only state the following Corollary.

Corollary 4. *For every toric fibration $f\colon X \to Y$, with X and Y simplicial, the following hold for every $\sigma \in \Sigma_Y$:*

i) *For every odd i, we have $H^i(f^{-1}(x_\sigma), \mathbf{Q}) = 0$.*
ii) *The Euler-Poincaré characteristic of $f^{-1}(x_\sigma)$ is given by*

$$\chi(f^{-1}(x_\sigma)) = d_0(\sigma),$$

where $d_0(\sigma)$ is the number of cones $\alpha \in \Sigma_X$ such that $\phi_*(\alpha) = \sigma$ and $\operatorname{codim}(\alpha) = \operatorname{codim}(\sigma)$.

By combining the assertions in Corollary 4 with the equality (2) we obtain $d_0(\sigma) = \sum_{\tau \subseteq \sigma} \delta_\tau$. An application of Möbius Inversion then gives

Theorem 5. *For every toric fibration $f\colon X \to Y$, with X and Y simplicial, and for every $\tau \in \Delta_Y$, we have*

$$\delta_\tau = \sum_{\sigma \subseteq \tau} (-1)^{\operatorname{codim}(\tau)-\operatorname{codim}(\sigma)} d_0(\sigma).$$

In particular, it follows that the alternating sum in Theorem 5 is always nonnegative. Similar expressions with the one in Theorem 5 can be obtained for each of the numbers $s_{\tau,b}$. Finally, when X and Y are not simplicial, one can describe these numbers building on the existing combinatorial descriptions for the intersection cohomology, see [2].

Example 6. Consider the Losev-Manin space $f\colon X = \widetilde{\mathbf{P}^n} \to \mathbf{P}^n = Y$, that is, f is the blow-up of all (strict transforms) of the torus-fixed subvarieties of Y, beginning with dimension 0 and ending with dimension $n-2$. Let $e_0, \ldots, e_n$ denote the rays of Σ_Y, hence $\sum_i e_i = 0$. The cones in Σ_X are in bijection with sequences of strict inclusions

$$\mathcal{F}:\ \emptyset \subset F_1 \subset F_2 \subset \ldots \subset F_k \subset [n] = \{0, 1, \ldots, n\}.$$

The cone $\sigma_{\mathcal{F}}$ corresponding to $\mathcal{F}$ as above is generated by $e_{F_1}, \ldots, e_{F_k}$, where for $I \subseteq [n]$ we put $e_I := \sum_{i \in I} e_i$. It is clear that in this case $\phi_*(\sigma_{\mathcal{F}})$ is the convex cone generated by $\{e_i \mid i \in F_k\}$.

It follows from definition that if $\sigma_J \in \Sigma_Y$ is the convex cone generated by $\{e_i \mid i \in J\}$, for some proper subset J of $[n]$, then $d_0(\sigma_J) = (\#J)!$. We thus conclude using Theorem 5 that

$$\delta_{\sigma_I} = \sum_{J \subseteq I} (-1)^{\#I - \#J} (\#J)! = (\#I)! \cdot \sum_{p=0}^{\#I} \frac{(-1)^p}{p!},$$

which is the number of derangements (permutations without fixed points) on the set I.

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On the toric ideal of a matroid

Michał Lasoń

(joint work with Mateusz Michałek)

Let M be a matroid on a ground set E with the set of bases $\mathfrak{B}$ and the rank function $r : \mathcal{P}(E) \to \mathbb{N}$. The rank of M, that is $r(E)$, we denote simply by r.

For a fixed field $\mathbb{K}$ consider a $\mathbb{K}$-homomorphism φ_M between polynomial rings:

$$\varphi_M : \mathbb{K}[y_B : B \in \mathfrak{B}] \ni y_B \to \prod_{e \in B} x_e \in \mathbb{K}[x_e : e \in E].$$

The *toric ideal of a matroid* M, denoted by I_M, is the kernel of the map φ_M. For a realizable matroid M the toric variety associated with the ideal I_M has a very nice embedding as a subvariety of a Grassmannian [5]. It is the closure of the torus orbit of the point of the Grassmannian corresponding to the matroid M.

Neil White in 1980 made three conjectures of growing difficulty that describe generators of the ideal I_M.

Conjecture 1 ([15]). *The toric ideal of a matroid is generated in degree* 2.

The family $\mathfrak{B}$ of bases of M satisfies *symmetric exchange property* (for more exchange properties see [10]). That is, for every bases B_1, B_2 and $e \in B_1 \setminus B_2$ there exists $f \in B_2 \setminus B_1$, such that both sets $B_1' = (B_1 \setminus e) \cup f$ and $B_2' = (B_2 \setminus f) \cup e$ are bases. In this case we say that the quadratic binomial $y_{B_1}y_{B_2} - y_{B_1'}y_{B_2'}$ *corresponds to symmetric exchange.* It is clear that such binomials belong to the ideal I_M.

Conjecture 2 ([15]). *The toric ideal of a matroid is generated by quadratic binomials corresponding to symmetric exchanges.*

The strongest among White's conjectures describing generators of the ideal I_M, turned out to be equivalent to Conjecture 2 (see the discussion in Section 4 of [9]).

Since every toric ideal is generated by binomials, it is not hard to rephrase the above conjectures in the combinatorial language. Conjecture 1 asserts that if two multisets of bases of a matroid have equal union (as a multiset), then one can pass between them by a sequence of steps, in each step exchanging two bases for another two bases of the same union (as a multiset). In Conjecture 2 additionally each step corresponds to a symmetric exchange. Actually, this is the original formulation due to White. We immediately see that the conjectures do not depend on the field $\mathbb{K}$.

Conjectures 1 and 2 are known to be true for many special classes of matroids: graphic matroids [1], strongly base orderable matroids [9] (so also for transversal matroids), sparse paving matroids [3], and for matroids of rank at most 3 [8] (see also other related papers [2, 4, 6, 12]). We give first results valid for arbitrary matroids.

We prove White's conjectures 'up to saturation'. Let $\mathfrak{m}$ be the ideal generated by all variables in the polynomial ring $\mathbb{K}[y_B : B \in \mathfrak{B}]$ (so-called *irrelevant ideal*). Recall that $I : \mathfrak{m}^\infty = \{a \in S_M : a\mathfrak{m}^n \subset I \text{ for some } n \in \mathbb{N}\}$ is called the *saturation* of an ideal I with respect to the ideal $\mathfrak{m}$. Notice that the ideal I_M, as a prime ideal, is saturated. Let J_M be the ideal generated by quadratic binomials corresponding to symmetric exchanges. Clearly, $J_M \subset I_M$ and Conjecture 2 asserts that the ideals J_M and I_M are equal. We prove that their saturations are equal.

Theorem 3 ([9]). *For every matroid M, its toric ideal I_M and the ideal J_M generated by quadratic binomials corresponding to symmetric exchanges, have equal saturations with respect to irrelevant ideal $\mathfrak{m}$. This means exactly that the homogeneous parts of I_M and J_M are equal starting from some degree.*

As a corollary we get that both ideals have equal radicals and the same affine set of zeros (since both I_M and J_M are contained in $\mathfrak{m}$). Moreover, it follows that in order to prove Conjecture 2 it is enough to show that the ideal J_M is saturated, radical or prime.

Ideals are one of the central objects of commutative algebra. From the point of view of algebraic geometry one is interested in schemes defined by them. A homogeneous ideal (both I_M and J_M are homogeneous) defines two schemes – affine and projective. Ideals define the same affine scheme if and only if they are equal. Thus Conjecture 2 asserts equality of affine schemes defined by I_M and J_M. Homogeneous ideals define the same projective scheme if and only if their saturations with respect to the irrelevant ideal are equal. Thus we prove equality of projective schemes defined by them, $\mathrm{Proj}(S_M/I_M) = \mathrm{Proj}(S_M/J_M)$. The projective toric variety $\mathrm{Proj}(S_M/I_M)$ has been already studied (see [5, 7]). White proved that it is projectively normal [14].

We bound the degree in which the toric ideal of a matroid is generated. By Hilbert's basis theorem the ideal I_M is finitely generated. However, it is not easy to give any explicit bound. A bound follows from a more general theorem about toric ideals. If a graded set $\mathcal{A} \subset \mathbb{Z}^d$ generates a normal semigroup, then the corresponding toric ideal $I_{\mathcal{A}}$ is generated in degree at most d (see Theorem 13.14 in [13]). For the matroid M we consider the set $\mathcal{A} = \{\chi_B : B \in \mathfrak{B}\} \subset \mathbb{Z}^{|E|}$, where χ_B is a characteristic function of B in E. By [14] it generates a normal semigroup (it is also an easy consequence of matroid union theorem). The toric ideal corresponding to $\mathcal{A}$ is the ideal I_M. Hence, the toric ideal of a matroid is generated in degree at most the size of its ground set.

If we fix the size of the ground set, then there are only finitely many matroids on it. So a common bound is not surprising. But, when we fix only the rank, then the number of matroids of that rank is infinite. We prove that in this case there is also a common bound on degree.

Theorem 4 ([11])**.** *The toric ideal of a matroid of rank r is generated in degree at most $(r+3)!$.*

As a corollary we get that checking if Conjecture 2 is true for matroids of a fixed rank is a decidable problem (it is enough to check connectivity of a finite number of graphs).

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Toric degenerations of complete local domains equipped with a rational valuation

Bernard Teissier

In this note we work with algebraic varieties over an algebraically closed field k. An approach to embedded resolution of singularities of an affine variety $X \subset \mathbf{A}^N(k)$ by a single toric map after a suitable re-embedding $\mathbf{A}^N(k) \subset \mathbf{A}^M(k)$ was proposed in [5]. It received serious encouragement after a 2009 Oberwolfach workshop when Jenia Tevelev proved in [7] a theorem for projective embeddings. It states that *any* embedded resolution of an irreducible $X \subset \mathbf{P}^N(k)$ is obtained by base change from an embedded resolution of $X \subset \mathbf{P}^M(k)$ by a single toric birational map $Z(\Sigma) \to \mathbf{P}^M(k)$ of non singular toric varieties through a suitable embedding $\mathbf{P}^N(k) \subset \mathbf{P}^M(k)$. Here suitable means in particular that the toric structure on $\mathbf{P}^M(k)$ is such that X meets the torus, and of course the embedding $\mathbf{P}^N(k) \subset \mathbf{P}^M(k)$ depends on the given embedded resolution.

Tevelev's result means in particular that embedded resolution by a suitable toric birational map is not only possible whenever embedded resolution is, and in particular in characteristic zero, but that it is also in some sense "universal".

How can one find suitable re-embeddings when no embedded resolution is known to exist? One possibility, going back to the local case $X \subset \mathbf{A}^N(k)$, is to try to find local re-embeddings $\mathbf{A}^N(k) \subset \mathbf{A}^M(k)$ and toric maps which will uniformize a given valuation centered at a point x of X. This means that we can find, after re-embedding, a birational toric map such that the strict transform $X' \subset Z(\Sigma)$ of X will be non singular and transversal to the toric boundary but only at the point $x' \in X'$ picked by the valuation. According to [4] the valuations which are *rational* concentrate the difficulty from this viewpoint. Rational valuations are those which are such that the inclusion $\mathcal{O}_{X,x} = R \subset R_\nu$ which determines the valuation, where R_ν is the valuation ring, satisfies $m_\nu \cap R = m$ and $R/m \simeq R_\nu/m_\nu$. Here we assume that $\mathcal{O}_{X,x} = R$ is an integral domain, which for the problems at hand is permissible.

What suggests to look at valuations is that if X is a germ of analytically irreducible curve X in $\mathbf{A}^N(k)$ and x its singular point, its local ring R has a unique valuation with value group $\mathbf{Z}$ and the semigroup of values $\Gamma = \nu(R \setminus \{0\})$ is therefore finitely generated, say that it is minimally generated by $\gamma_1, \ldots, \gamma_s$. If we choose a system of generators $\xi_1, \dots, \xi_s$ of the maximal ideal of R having those valuations, they determine a re-embedding $X \subset \mathbf{A}^s(k)$ which has an embedded resolution by a single toric map. The reason is that in $\mathbf{A}^s(k)$ the curve X can degenerate (or specialize) in an *overweight* manner (see [6]) to the monomial curve $\mathrm{Spec} k[t^\Gamma] \subset \mathbf{A}^s(k)$ which is an affine toric variety and as such has toric embedded resolutions *in any characteristic*, some of which also resolve X (see [2]). The ring $k[t^\Gamma]$ is the associated graded ring of R with respect to the valuation.

When one tries to generalize this to higher dimension things begin well: given a rational valuation ν on R one defines a filtration of R by the valuation ideals

$$\mathcal{P}_\phi(R) = \{x \in R \setminus \{0\} | \nu(x) \geq \phi\} \cup \{0\}, \quad \mathcal{P}^+_\phi(R) = \{x \in R \setminus \{0\} | \nu(x) > \phi\} \cup \{0\},$$

and the associated graded ring $\mathrm{gr}_\nu R = \sum_{\phi \in \Gamma} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$.

Because of the properties of rational valuations, all components of this graded algebra are 1-dimensional vector spaces over k. This implies that if we take any system of homogeneous generators $(\overline{\xi}_j)_{j\in J}$ of the k-algebra $\mathrm{gr}_\nu R$, the surjective map of k-algebras

$$k[(U_j)_{j\in J}] \to \mathrm{gr}_\nu R, \quad U_j \mapsto \overline{\xi}_j$$

has a kernel generated by binomials; in fact it is isomorphic to the semigroup algebra $k[t^\Gamma]$. So we have our toric variety to which we want to degenerate, although it may be of infinite embedding dimension as we are going to see.

Here, one meets the difficulty that the semigroup $\Gamma = \nu(R \setminus \{0\})$ is not finitely generated in general. Since R is noetherian, Γ has nevertheless some important properties:

(1) It is well ordered, which implies that it has a minimal system of generators $\Gamma = \langle \gamma_1, \gamma_2, \dots, \gamma_i, \dots \rangle$ where the generators $(\gamma_i)_{i\in I}$ are indexed by an ordinal I which is $\leq \omega^{\mathrm{dim} R}$ by results of Krull and Zariski.
(2) By a result of Campillo-Galindo (see [1]), it is combinatorially finite, which means that the number of distinct ways of writing an element of Γ as a sum of other elements is finite.
(3) By a result of Piltant (see [4]), even though $\mathrm{gr}_\nu R$ may not be noetherian, its Krull dimension is the rational rank of the value group Φ of the valuation.
(4) In view of this, Abhyankar's inequality reads, for rational valuations: $\mathrm{dim}\,\mathrm{gr}_\nu R \leq \mathrm{dim} R$.

The lack of noetherianity makes it difficult to use the valuation algebra of [4]:

$$\mathcal{A}_\nu(R) = \sum_{\phi \in \Phi} \mathcal{P}_\phi(R) v^{-\phi} \subset R[v^\Phi]$$

which describes the "natural" specialization of R to its associated graded ring; in fact in the absence of noetherianity we cannot get equations to describe this specialization.

We are going to describe this specialization in another way, assuming that the ring R is complete. The reduction to the complete case is a separate issue which is treated separately and involves an assumption of excellence on the ring R, which is satisfied in our case; see [3] and [6].

Assume now that the equicharacteristic noetherian local domain R is complete and endowed with a rational valuation with semigroup of values $\Gamma = \langle(\gamma_i)_{i\in I}\rangle$. Choose a field of representatives $k \subset R$.

Let $(u_i)_{i\in I}$ be variables indexed by the elements of the minimal system of generators $(\gamma_i)_{i\in I}$ of the semigroup Γ. Give each u_i the weight $w(u_i) = \gamma_i$ and let us consider the k-vector space of power series $\sum_{e\in E} d_e u^e$ where $(u^e)_{e\in E}$ is any set of monomials in the variables u_i and $d_e \in k$. Since Γ is combinatorially finite, for any given series the map $w\colon E \to \Gamma,\ e \mapsto w(u^e)$ has finite fibers. Each of these fibers is a finite set of monomials in variables indexed by a totally ordered set, and so can be given the lexicographical order and order-embedded into an interval $1 \leq i \leq n$ of $\mathbf{N}$. This defines an injection of the set E into $\Gamma \times \mathbf{N}$ equipped with the lexicographical order and thus induces a total order on E, for which it is well ordered. When E is the set of all monomials, this gives a total monomial order.

The combinatorial finiteness also implies that this vector space of series is a k-algebra, which we denote by $k[\widehat{(u_i)_{i\in I}}]$.

The weight of a series is defined to be the lowest weight of its terms. The filtration by weight determines a topology on our ring. It has many nice properties, in particular of completeness with respect to this topology. We can think of it as a generalized power series ring with weights on the variables.

Theorem 1 (The valuative Cohen theorem)**.** 1) *There exist choices of representatives $\xi_i \in R$ of the $\overline{\xi}_i$ minimally generating the k-algebra $\mathrm{gr}_\nu R$ such that the application $u_i \mapsto \xi_i$ determines a surjective map of k-algebras*

$$\pi\colon k[\widehat{(u_i)_{i\in I}}] \longrightarrow R$$

which is continuous with respect to the topologies associated to the filtrations by weight and by valuation respectively. The associated graded map with respect to these filtrations is the surjective map

$$\mathrm{gr}_w\pi\colon k[(U_i)_{i\in I}] \longrightarrow \mathrm{gr}_\nu R \simeq k[t^\Gamma], \quad U_i \mapsto \overline{\xi}_i = \mathrm{in}_\nu \xi_i$$

whose kernel is a prime ideal generated by binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L},\ \lambda_\ell \in k^*$.

2) *There exist elements $F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(p)>w(m^\ell)} c_p^{(\ell)} u^p$ which, as the binomials run through a set of generators of the kernel of $\mathrm{gr}_w\pi$, topologically generate the kernel F of π.*

Now we have equations for our degeneration! If the valuation is of rank one or if Γ is finitely generated, any choice of the representatives ξ_i is allowed.

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On Cox rings of elliptic del Pezzo varieties

Antonio Laface

(joint work with Jürgen Hausen, Simon Keicher, Andrea L. Tironi, Luca Ugaglia)

1. Cox rings

In what follows a *variety* X will be always a normal projective variety, defined over an algebraically closed field $\mathbb{K}$ of characteristic zero, such that the divisor class group $\mathrm{Cl}(X)$ is finitely generated. The *Cox sheaf* and the *Cox ring* of a variety X are respectively [5, 1]:

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D) \qquad \mathcal{R}(X) := \Gamma(X, \mathcal{R}).$$

A variety is a *Mori dream space* if its Cox ring is a finitely generated $\mathbb{K}$-algebra. Let $f\colon X \to Y$ be a birational morphism of varieties. If f is a small modification, that is an isomorphism in codimension one, then f induces an isomorphism of Cox rings. Moreover all the small modifications of X can be read off from its Cox ring $\mathcal{R}(X)$. So one can assume f to be a divisorial contraction, whose exceptional divisor E is prime. The Riemann-Roch space $\Gamma(X, \mathcal{O}_X(E))$ is one dimensional since E is contractible. Let σ be a generator of this space.

Proposition 1 ([3])**.** *With the above notation the pushforward map induces an isomorphism of* $\mathrm{Cl}(Y)$*-graded algebras* $\mathcal{R}(X)/\langle \sigma_E - 1\rangle \to \mathcal{R}(Y)$.

Assume now that f is a blowing-up of an irreducible and reduced subvariety $C \subseteq Y$ which is contained in the smooth locus of Y. The second condition guarantees the existence of the pullback of any Weil divisor of Y. Assume $R = \mathcal{R}(Y)$ to be finitely generated, let $\widehat{Y} = \mathrm{Spec}_Y(\mathcal{R})$ be the relative spectrum of the Cox sheaf of Y and let $\overline{Y} = \mathrm{Spec}(\mathcal{R}(Y))$. Denote by $J \subseteq R$ the *irrelevant ideal*, that is the ideal of $\overline{Y} \setminus \widehat{Y}$. Both $\widehat{Y}$ and $\overline{Y}$ are acted by the quasitorus $H = \operatorname{Spec} \mathbb{K}[\operatorname{Cl}(Y)]$ and there is a good quotient map $p\colon \widehat{Y} \to Y$. Denote by $I \subseteq R$ the ideal of $p^{-1}(C)$. The *extended saturated Rees algebra* of I is:

$$R[I]^{\text{sat}} := \bigoplus_{d\in\mathbb{Z}} (I^d : J^\infty)t^{-d},$$

where $I^d = R$ if $d < 0$. The second result is the following.

Proposition 2 ([3]). *With the above notation there is an isomorphism of* $\operatorname{Cl}(X)$*-graded algebras:* $R[I]^{\text{sat}} \to \mathcal{R}(X)$ *defined by* $g \cdot t^d \mapsto f^* g \cdot \sigma_E^d$.

The above isomorphism can be used to provide an algorithm for computing the Cox ring of X which terminates if and only if the latter is finitely generated. First of all one constructs a subalgebra S of $R[I]^{\text{sat}}$ in the following way. We want S to contain R and the variable t, so that the containment $R[I] \subseteq S$ holds, where the left hand side is the extended Rees algebra, without taking saturation. Let $g_i \in I^{m_i} : J^\infty$ with $1 \leq i \leq k$ be a finite set of homogeneous elements in the saturated powers of I. The exponent m_i is the *Rees degree* of g_i and is the multiplicity of $V(g_i)$ at the generic point of $p^{-1}(C)$. The subalgebra S is

$$S := R[g_1 t^{-m_1}, \ldots, g_k t^{-m_k}, t] \simeq \frac{R[z_1, \ldots, z_k, t]}{\langle z_1 t^{m_1} - g_1, \ldots, z_k t^{m_k} - g_k \rangle : \langle t \rangle^\infty}.$$

Now, observe that the inclusion $S \subseteq R[I]^{\text{sat}}$ is strict if and only if there exists a homogeneous element $g \in R$ such that $gt^{-d} \in R[I]^{\text{sat}} \setminus S$ and $gt^{-d+1} \in S$. Equivalently the following holds:

$$gt^{-d+1} \in S \cap \langle t \rangle_{R[I]^{\text{sat}}} \setminus \langle t \rangle_S.$$

The ideals $S \cap \langle t \rangle_{R[I]^{\text{sat}}}$ and $\langle t \rangle_S$ have the same dimension, the first is prime and the second is principal. The key observation now is that if $h \in J \setminus I$ is any homogeneous element then the localizations of $R[I]$ and $R[I]^{\text{sat}}$ at h are equal and thus both are equal to S_h. In particular the prime components of $\langle t \rangle_S$, which are all hypersurfaces being the ideal principal, are contained in $V(h)$. We deduce that the equality $S = R[I]^{\text{sat}}$ holds if and only if

$$\dim \langle t \rangle_S > \dim \langle t, h \rangle_S.$$

We use the above characterisation to provide an algorithm for computing Cox rings of blowing-ups. Such algorithm thus consists of a saturation step, for computing S, and a dimension test. The geometric meaning of the saturation step is the following. A homogeneous presentation for the Cox ring $R = \mathcal{R}(Y)$ defines a closed embedding $\overline{Y} \to \mathbb{A}^r$, where the codomain is the spectrum of the Cox ring of a, not unique, toric variety Z where Y embeds. The map

$$\mathbb{A}^r \to \mathbb{A}^{r+k} \qquad (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, g_1, \ldots, g_k)$$

embeds Y into another toric variety such that C is cut out by the torus orbit where the last k coordinates vanish. The blowing-up map in Cox coordinates is the following:

$$\mathbb{A}^{r+k+1} \to \mathbb{A}^{r+k} \qquad (x_1, \ldots, x_{r+k}, t) \mapsto (x_1, \ldots, x_r, x_{r+1} t^{m_1}, \ldots, x_{r+k} t^{m_k}).$$

Hence saturating with respect to $\langle t\rangle^\infty$ means to compute the strict transform of $\overline{Y}$. This operation is called a *toric ambient modification.*

Example 3. Let Y be the weighted projective plane $\mathbb{P}(7,13,18)$ and let $\pi\colon X\to Y$ be the blowing-up at $p=(1,1,1)$. The divisor class group $\mathrm{Cl}(X)$ is generated by the class of $H=\pi^*\mathcal{O}_Y(1)$ and the exceptional divisor E. It is not difficult to show that $H^2=(7\cdot 13\cdot 18)^{-1}$. By applying the above algorithm one finds the following elements in Rees degrees $1,1,1,2,3$ respectively:

$$f_1=-x^3z+y^3,\quad f_2=x^7-yz^2,\quad f_3=x^4y^2-z^3\quad f_4=x^{11}y-3x^4y^2z^2+xy^5z+z^5$$
$$f_5=x^{18}-3x^{11}yz^2-x^8y^4z+x^5y^7+5x^4y^2z^4-2xy^5z^3-z^7.$$

The polynomial f_1 defines a negative curve C of X which is linearly equivalent to $39H-E$. Since $-K_X\sim 38H-E$ the anticanonical class is not pseudoeffective. The Cox ring $\mathcal{R}(X)$ is isomorphic to $\mathbb{K}[T_1,\dots,T_9]/\mathfrak{I}$, where $\mathfrak{I}$ is the ideal generated by the following elements:

$$\begin{array}{lll}
T_1T_4^2T_6-T_2T_8+T_5T_7, & T_1T_3T_4^2-T_2T_7+T_5T_6, & T_1^2T_4^3T_5-T_6T_8+T_7^2\\
T_1^4T_4-T_2T_6+T_3T_5, & T_1T_4T_5^2+T_3T_8-T_6T_7, & T_1T_2T_4T_5+T_3T_7-T_6^2\\
-T_1^3T_8+T_4T_6^2+T_5^3, & -T_1^3T_7+T_2T_5^2+T_3T_4T_6, & -T_1^3T_6+T_2^2T_5+T_3^2T_4\\
-T_1^3T_3+T_2^3-T_4T_9, & T_1^7-T_2T_3^2-T_5T_9, & T_1^4T_2^2-T_3^3-T_6T_9\\
T_1^4T_5^2+T_1T_2^2T_4T_6-T_3^2T_7-T_8T_9, & & \\
T_1^4T_2T_5+T_1T_2^2T_3T_4-T_3^2T_6-T_7T_9 & &
\end{array}$$

and the degree matrix is the following:

$$\begin{bmatrix} 7 & 13 & 18 & 39 & 49 & 54 & 90 & 126 & 0\\ 0 & 0 & 0 & -1 & -1 & -1 & -2 & -3 & 1\end{bmatrix}$$

2. Del Pezzo elliptic varieties

A del Pezzo variety Y is a smooth n-dimensional projective variety such that $-K_Y\simeq(n-1)H$, with H ample. These varieties are classified [2, §12.1]. The degree of a del Pezzo variety Y is $d=\deg(Y)=H^n$ and lies in between 1 and 7. These varieties exist in any dimension only for degree at most 4. Basic examples are cubic hypersurfaces and complete intersection of two quadrics. Given such a $Y\subseteq\mathbb{P}^{n+d-2}$ one defines an elliptic fibration by resolving the indeterminacy of the linear projection from a linear space $L\subseteq\mathbb{P}^{n+d-2}$ of dimension $d-2$. The situation is summarised in the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\sigma"'] & \mathbb{P}^{n-1} \\ Y \arrow[ur, dotted, "\pi_L"'] & \end{tikzcd}$$

where $\sigma\colon X\to Y$ is the resolution of indeterminacy of π_L and it consists of blowing up d points, possibly infinitely near. The elliptic fibration is given by the complete linear system of the divisor $F:=H-E_1-\cdots-E_d$, where H is the pull-back $\sigma^*\mathcal{O}_Y(1)$ of a hyperplane section of Y. It is easy to show that

$$-K_X=(n-1)F.$$

The variety X is a *del Pezzo elliptic variety* and its degree is the degree of Y. The Mordell-Weil group $\mathrm{Mw}(\pi)$ of π is the group of $\mathbb{C}(\mathbb{P}^{n-1})$-rational points of the generic fiber of π. It can be determined by means of the Shioda-Tate-Wazir exact sequence [4]:

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Mw}(\pi) \longrightarrow 0,$$

where $\mathcal{T}$ is the subgroup generated by the classes of prime vertical divisors, i.e. prime divisors D such that $\pi(D)$ is a divisor, and the image of a rational section of π. By exploiting the geometry of Y one can describe the prime vertical divisors of Y for any such variety when $d \leq 4$. The first result is given in the following table (for the notation see [6]).

Deg	Type	Mw(π)	Deg	Type	Mw(π)
1	X_1	$\langle 0\rangle$	4	X_{40}	$\mathbb{Z}^3$
2	X_{11}	$\mathbb{Z}$		X_{41}, X_{30}	$\mathbb{Z}^2$
	X_{SS}	$\mathbb{Z}/2\mathbb{Z}$		X_{42}	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
	X_2	$\langle 0\rangle$		X_{31}, X_{20}, X_{21}	$\mathbb{Z}$
3	X_{111}	$\mathbb{Z}^2$		X_{43}	$(\mathbb{Z}/2\mathbb{Z})^2$
	X_{S11}, X_{12}	$(\mathbb{Z}/2\mathbb{Z})^2$		X_{21}, X_{22}	$\mathbb{Z}/2\mathbb{Z}$
	X_{SSS}	$\mathbb{Z}/3\mathbb{Z}$		X_{10}, X_{11}	$\langle 0\rangle$
	X_{S2}	$\mathbb{Z}/2\mathbb{Z}$			
	X_3, X_S	$\langle 0\rangle$			

Theorem 4 ([4, 6])**.** *Let X be a del Pezzo elliptic variety of degree at most four with elliptic fibration $\pi\colon X \to \mathbb{P}^{n-1}$. The Cox ring $\mathcal{R}(X)$ is finitely generated if and only if the Mordell-Weil group of π is a finite group.*

To prove the theorem when $d \leq 3$ we explicitly compute the Cox rings by means of the above algorithm. When $d = 4$ the above computations can be performed only for some examples and thus we use Hu and Keel theorem [5]: we show that the moving cone $\mathrm{Mov}(X)$ is union of polyhedral semiample cones of finitely many flop images of X. The idea for determining $\mathrm{Mov}(X)$ in each case is the following. The bilinear form on $\mathrm{Pic}(X)$

$$\langle A, B\rangle = \Big(-\frac{1}{n-1}K_X\Big)^{n-2} \cdot A \cdot B$$

has the following property: if A is effective, B is irreducible and $\langle A, B\rangle < 0$ then B is contained in the stable base locus of A. Thus it immediately follows that the following containment holds

$$\mathrm{Mov}(X) \subseteq \mathrm{Eff}(X) \cap \mathrm{Eff}(X)^\vee,$$

where the dual is with respect to the above bilinear form. We compute the cone $\mathcal{C} \subseteq \mathrm{Eff}$ generated by the classes of prime vertical divisors and the classes of the elements of the Mordell-Weil group of π. The inclusion $\mathcal{C} \subseteq \mathrm{Eff}(X)$ implies the inclusion $\mathrm{Eff}(X)^\vee \subseteq \mathcal{C}^\vee$. We then show, with a case by case analysis, that any class in $\mathcal{C}^\vee$ is movable proving the equalities $\mathcal{C} = \mathrm{Eff}(X)$ and $\mathcal{C}^\vee = \mathrm{Mov}(X)$.

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Minimal rational curves on group compactifications

Michel Brion
(joint work with Baohua Fu)

Rational curves play an important role in the birational geometry of algebraic varieties. In this talk, we present some questions and results on minimal rational curves; these are intrinsically defined, and include lines in a projective embedding. We begin with some basic definitions, referring to [10] for a detailed exposition of rational curves on algebraic varieties, and to [6] for a survey of recent developments about minimal rational curves.

Let X be a projective algebraic variety over the field $\mathbb{C}$ of complex numbers. A *rational curve* on X is the image of a non-constant morphism $f : \mathbb{P}^1 \to X$; we may assume that f is birational over its image (an irreducible curve, possibly singular). A parameter space for all rational curves on X may be constructed as follows. Start with the space of morphisms $\mathrm{Hom}(\mathbb{P}^1, X)$, an infinite disjoint union of quasi-projective schemes. The morphisms that are birational over their image are parameterized by an open subscheme $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$. The algebraic group $\mathrm{PGL}(2) = \mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}(\mathbb{P}^1, X)$ and stabilizes $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X)$; moreover, there exists a quotient morphism $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X) \to \mathrm{RatCurves}(X)$, which is a principal $\mathrm{PGL}(2)$-bundle. The normalization $\mathrm{RatCurves}^{\mathrm{n}}(X)$ is an infinite disjoint union of normal quasiprojective varieties, called *families of rational curves*. A family V is called *covering* if the subvariety V_x consisting of curves through x is non-empty for any general point $x \in X$. If in addition V_x is proper (or equivalently, projective), then V is called *minimal*. Minimal families exist whenever X is covered by rational curves, as seen by considering rational curves through a general point, which are of minimal degree relative to a fixed projective embedding of X, or equivalently, to a fixed ample line bundle.

For example, when X is the projective space $\mathbb{P}^n$, the rational curves of any prescribed degree form a family in the above sense. Each such family is covering; also, there is a unique minimal family V, consisting of all lines on $\mathbb{P}^n$. Thus, V is the grassmannian of 2-planes in $\mathbb{C}^{n+1}$, and $V_x \cong \mathbb{P}^{n-1}$ for any $x \in \mathbb{P}^n$.

More generally, the families of minimal rational curves on an arbitrary (smooth, projective) toric variety X have been described by Chen, Fu and Hwang in [4]. Part of their results can be summarized as follows.

Theorem 1. *Let X be a smooth projective toric variety with torus $T \cong (\mathbb{C}^*)^n$, fan Σ and base point x.*

(i) *For any family V of minimal rational curves on X, there exists a T-stable open subset $U \subset X$ such that $U \cong (\mathbb{C}^*)^{n-m} \times \mathbb{P}^m$ equivariantly, and V_x consists of all lines in $\mathbb{P}^m$ through x (in particular, $V_x \cong \mathbb{P}^{m-1}$).*

(ii) *There is a bijection between the families of minimal rational curves on X and the primitive collections of Σ with zero sum.*

We recall from [2] that a *primitive collection* of Σ is a finite set $\mathfrak{P} = \{v_0, \ldots, v_m\}$ of primitive elements in the lattice $N \cong \mathbb{Z}^n$ of one-parameter subgroups of T, such that $\mathfrak{P} \setminus \{v_i\}$ generates a cone of Σ for $i = 0, \ldots, m$, and $\mathfrak{P}$ does not generate a cone of Σ. The latter condition is fulfilled whenever $v_0 + \cdots + v_m = 0$; we then say that $\mathfrak{P}$ *has zero sum*.

The proof of assertion (i) in the above Theorem uses an algebro-geometric result of Araujo (see [1]); assertion (ii) is a direct consequence. It would be interesting to obtain a direct proof of the Theorem by methods of toric geometry, and to extend it to possibly singular toric varieties.

In another direction, an open question is to describe minimal families of rational curves on any (smooth, projective) *almost homogeneous* variety X, that is, X is equipped with an action of a connected linear algebraic group G with an open orbit, say $G\cdot x$. Then G is generated by copies of additive or multiplicative groups; thus, the orbit closures of the corresponding one-parameter subgroups and their translates yield rational curves which cover X. Also, G acts on every family V of rational curves on X; the isotropy group $H := \mathrm{Stab}_G(x)$ acts on V_x, and V is covering if and only if V_x is non-empty.

Baohua Fu and I treated the case where X is the *wonderful compactification* of a semisimple algebraic group G with trivial center, introduced by De Concini and Procesi in [5]. This is a smooth projective variety equipped with an action of $G \times G$ and containing G as an open orbit, where $G \times G$ acts on G by left and right multiplication. Moreover, the boundary $X \setminus G$ is a union of ℓ irreducible divisors $D_1, \ldots, D_\ell$ with smooth normal crossings, where ℓ is the rank of G, and the $G \times G$-orbit closures in X are exactly the partial intersections $D_{i_1} \cap \cdots \cap D_{i_s}$. Note that X comes with a base point x (the identity element of G), with isotropy group G embedded diagonally in $G \times G$.

Every algebraic group as above satisfies $G \cong G_1 \times \cdots \times G_m$, where $G_1, \ldots, G_m$ are simple; we have accordingly $X \cong X_1 \times \cdots \times X_m$, and each family of minimal rational curves on X lives on some factor X_i. Thus, we may assume that G is simple; then our main result may be stated as follows (see [3]):

Theorem 2. *Let X be the wonderful compactification of a simple algebraic group G with trivial center.*

(i) *There exists a unique family of minimal rational curves V on X.*

(ii) *If* $G = \mathrm{PGL}_{\ell+1}$*, where* $\ell \geq 3$*, then* V_x *is equivariantly isomorphic to* $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^*$*, where* G *acts diagonally.*
(iii) *If* $G \neq \mathrm{PGL}_{\ell+1}$ *for* $\ell \geq 3$*, then* V_x *is equivariantly isomorphic to the closed* G*-orbit in the projectivization of the Lie algebra of* G*.*
(iv) $L \cdot C \geq \ell$ *for any ample line bundle* L *on* X *and for any curve* C *on* X*.*

By (ii) and (iii), V_x is smooth and consists of 1 or 2 orbits of G. Also, in view of (iv), X contains no line in any projective embedding, if $\ell \geq 2$. On the other hand, if $\ell = 1$ then $G \cong \mathrm{PGL}_2$ and hence X is equivariantly isomorphic to the projectivization of the space M_2 of 2×2 matrices, on which $G \times G$ acts via the action of $\mathrm{GL}_2 \times \mathrm{GL}_2$ on M_2 by left and right multiplication. Thus, $X \cong \mathbb{P}^3$ and the family of minimal rational curves consists of all lines.

More generally, the projectivization of the space $M_{\ell+1}$ of square matrices of size $\ell + 1$ yields a $G \times G$-equivariant compactification Y of $G := \mathrm{PGL}_{\ell+1}$. The $G \times G$-orbit closures in Y are exactly the projectivizations $Y_1, \ldots, Y_{\ell+1} = Y$ of the determinantal varieties (consisting of matrices of rank at most $1, \ldots, \ell+1$). Since the singular locus of Y_i is Y_{i-1} for $i = 1, \ldots, \ell$, we see that Y is not wonderful for $\ell \geq 2$. The wonderful compactification X is obtained from Y by blowing up first Y_1, then the strict transform of Y_2, and so on until $Y_{\ell-1}$ (see [12]). The secant lines to Y_1 through x form a subvariety of the grassmannian, equivariantly isomorphic to Y_1 (the image of the Segre embedding $\mathbb{P}^\ell \times (\mathbb{P}^\ell)^* \to \mathbb{P}(\mathbb{C}^\ell \otimes (\mathbb{C}^\ell)^*) \cong \mathbb{P}(M_{\ell+1})$). The strict transforms of these lines in X yield the minimal family.

The structure of minimal families of rational curves on any projective rational *homogeneous* variety X is also known, by work of Hwang and Mok (see [7, 8, 9]). We then have $X = G \cdot x \cong G/P$, where G is semisimple and $P = \mathrm{Stab}_G(x)$ is a parabolic subgroup of G. When P is maximal, there is a unique family of minimal rational curves V on X; moreover, V_x is smooth and consists of 1 or 2 orbits of P. Also, V consists of the lines in the minimal projective embedding of X; the linear subspaces of X in that embedding have been studied by Landsberg and Manivel (see [11]).

The general case of almost homogeneous varieties is much less understood, already for the complete symmetric varieties of [5].

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