TORIC STACKS II: INTRINSIC CHARACTERIZATION OF TORIC STACKS – ERRATUM

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1. Introduction

The purpose of this note is to correct an error in Theorem 6.1 of [GS2]. We thank W.D. Gillam and Sam Molcho for pointing out the mistake, see Example 4.3.3 and Corollary 4.3.9 of [GM]. Before discussing the error and how to modify the affected results, we wish to emphasize:

- The results in [GS1] are not affected.
- We have checked that every paper which cited [GS2] is not affected.
- Theorem 6.1 of [GS2] is correct for smooth $\mathcal{X}$; this is stated as [GS2, Theorem 5.2].

As we have just mentioned, Theorem 6.1 is true for smooth stacks; the goal of this note is to explain how Theorem 6.1 needs to be modified for singular $\mathcal{X}$. In order to state this result, we introduce the following definition which is a conglomerate of Tyomkin's toric stacky data [Tyo, Definition 4.1] (cf. [GM, Definition 2.2.3]) and our stacky fans [GS1, Definition 2.4].

Definition 1.1. A *generalized stacky fan* is a triple $(\Sigma, \beta, \mathcal{L}')$ such that:

(1) Σ is a fan on a lattice L,
(2) $\beta\colon L \to N$ is a homomorphism to a lattice N so that $\operatorname{cok}\beta$ is finite,
(3) $\mathcal{L}' = \{L'_\sigma \mid \sigma \in \Sigma\}$ such that $L'_\sigma \subset L_\sigma := \mathrm{Span}(\sigma) \cap L$ is a finite index lattice,
(4) $L'_\sigma \cap \mathrm{Span}(\sigma \cap \tau) = L'_\tau \cap \mathrm{Span}(\sigma \cap \tau)$ for all $\sigma, \tau \in \Sigma$.

To each such generalized stacky fan, we associate a stack as follows. For each $\sigma \in \Sigma$, since $\iota_\sigma\colon L'_\sigma \subset L_\sigma$ is finite index and since β has finite cokernel, the composite map $\beta\iota_\sigma$ also has finite cokernel. Since $(L'_\sigma)_{\mathbb{R}} = (L_\sigma)_{\mathbb{R}}$, we may view σ as a cone on L'_σ and hence $(\sigma, \beta\iota_\sigma)$ is a stacky fan. We thus obtain a (cohomologically affine) toric stack $\mathcal{X}_{\sigma,\beta\iota_\sigma}$. By the compatibility condition, these toric stacks glue to give a stack $\mathcal{X}_{\Sigma,\beta,\mathcal{L}'}$.

Definition 1.2. We refer to stacks of the form $\mathcal{X}_{\Sigma,\beta,\mathcal{L}'}$ as *generalized toric stacks*.

We these definitions in place, we can now state the modified version of [GS2, Theorem 6.1].

Theorem 1.3 (Corrected version of [GS2, Theorem 6.1])**.** *Let $\mathcal{X}$ be an Artin stack of finite type over an algebraically closed field k of characteristic 0. Suppose $\mathcal{X}$ has an action of a torus T and a dense open substack which is T-equivariantly isomorphic to T. If $\mathcal{X}$ has finite quotient singularities (e.g. a smooth Artin stack), then $\mathcal{X}$ is a generalized toric stack if and only if the following conditions hold:*

(1) *$\mathcal{X}$ is normal,*
(2) *$\mathcal{X}$ has affine diagonal, and*
(3) *geometric points of $\mathcal{X}$ have linearly reductive stabilizers*

2. Summary of the affected statements

The mistake occurs in [GS2, Lemma 2.10], specifically D^1 is not necessarily join-closed. This error affects Corollary 2.12, Theorem 2.13, Remark 2.14, Theorem 6.1, Remark 6.2, Remark 6.3, Proposition 7.2 of [GS2] as well as [GS1, Corollary 6.5]. We explain in Section 3 how all of these results need to be modified; we additionally show that [GS1, Corollary 6.5] is correct as stated.

3. Modified statements and their proofs

In this section, we explain how to appropriately modify the aforementioned affected results of Section 2. We present a direct proof of [GS2, Corollary 2.12], so are no longer in need of [GS2, Lemma 2.10].

Corollary 3.1 (Corrected version of [GS2, Corollary 2.12])**.** Let D be a tight diagram of simplicial toric monoids with colimit M. Then for each i, there exists a face F_i of M such that $D_i \to M$ factors as $D_i \subset F_i \subset M$ with $D_i^{gp} \subset F_i^{gp}$ a finite index inclusion of lattices.

Proof. To show that $D_i \to M$ is an inclusion, it suffices to show that $D_i^{\mathrm{gp}} \to M^{\mathrm{gp}}$ is an inclusion, for which it suffices to show that the dual map has finite cokernel. So, we must show that for every linear functional χ on D_i^{gp}, there is a positive integer m for which $m\chi$ extends to a linear functional on M^{gp}. Now, a linear functional on D_j is nothing more than a Cartier divisor on the affine toric variety X_j corresponding to D_j. Similarly, a linear functional on M is a compatible choice of Cartier divisors on each X_j. Since each D_j is simplicial, $\mathbb{Q}$-Cartier divisors are the same as $\mathbb{Q}$-Weil divisors. Since we are allowing ourslves to scale, it is enough to extend some $m\chi$ to a Weil divisor on each D_j. But this is clearly possible, as we may arbitrary assign a value to each 1-dimensional face of $D \smallsetminus D_i$.

Next, let $F_i \subset M$ be the saturation of $D_i \subset M$. Since $D_i \subset F_i$ is finite index, it suffices to show $F_i \subset M$ is a face. By the previous paragraph, the zero functional on D_i extends arbitrarily to a linear functional on M^{gp}. By assigning a positive value to each 1-dimensional face of $D \smallsetminus D_i$, we may choose the extension to be non-negative on M and vanish exactly on F_i. So, F_i is a face. □

Theorem 3.2 (Corrected version of [GS2, Theorem 2.13])**.** *Let $\mathcal{X}$ be a stack with finite quotient singularities and an action of a torus T and a dense open T-orbit which is T-equivariantly isomorphic to T. Let $\mathcal{Y} \to \mathcal{X}$ be a morphism from a toric stack. Suppose $\mathcal{X}$ has a cover by T-invariant open substacks $\mathcal{X}_i$ which are toric stacks with torus T, and that the maps $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_i \to \mathcal{X}_i$ are canonical stack morphisms (see* [GS1, Definition 5.1]*). Then $\mathcal{X}$ is a generalized toric stack.*

Proof. The proof is exactly the same as in [GS2, Theorem 2.13], but the last paragraph needs to be slightly modified. We again know that the colimit of the diagram of toric monoids $\sigma_i \cap L_i$ is of the form $\sigma \cap L$, where L is the colimit of the L_i. Since L is the colimit of the L_i, we see the $\beta_i \colon L_i \to N$ extend to $\beta \colon L \to N$. By Corollary 3.1 (our corrected version of [GS2, Corollary 2.12]), the maps $\sigma_i \cap L_i \to \sigma \cap L$ factor as $\sigma_i \cap L_i \to \sigma_i \cap \widetilde{L}_i \to \sigma \cap L$ where $\sigma_i \cap \widetilde{L}_i \to \sigma \cap L$ is the inclusion of a face and $L_i \to \widetilde{L}_i$ is a finite index sublattice. Since $\mathcal{X}$ is covered by the $\mathcal{X}_{\sigma,\beta}$, we have that by definition $\mathcal{X} = \mathcal{X}_{\Sigma,\beta,\mathcal{L}'}$, where Σ is the fan consisting of the σ_i and $\mathcal{L}'$ is the collection of finite index sublattices $L_i \to \widetilde{L}_i$. □

Remark 3.3 (Corrected version of [GS2, Remark 2.14]). This remark is correct, but only applies to the situation where $\mathcal{X}$ happens to be a toric stack with finite quotient singularities. Furthermore, the corrected proof of [GS2, Theorem 2.13] shows that for generalized toric stacks $\mathcal{X}$ with finite quotient singularities, there is a canonical coarse space map to a toric stack.

Proof of Theorem 1.3, i.e. corrected version of [GS2, Theorem 6.1]. The proof is exactly the same, however we now appeal to Theorem 3.2, i.e. the corrected version of [GS2, Theorem 2.13]. Furthermore, by [AHR], condition (4) in the original statement of Theorem 6.1 is no longer needed. ☐

Remark 3.4 (Corrected version of [GS2, Remark 6.2]). This remark should state that the toric stacks defined by Tyomkin in [Tyo, §4] are generalized toric stacks. They need not be toric stacks since they need not be global quotients, as demonstrated by Gillam and Molcho in [GM, Example 4.3.3].

We can replace [GS2, Remark 6.3] by the following stronger statement.

Remark 3.5 (Corrected version of [GS2, Remark 6.3]). Suppose $\mathcal{X}$ is a normal algebraic space with affine diagonal, that there is an action of a torus T, and a dense open subspace which is T-equivariantly isomorphic to T. Then the proof of [GS2, Theorem 6.1] shows that $\mathcal{X}$ has an open cover by cohomology affine toric stacks which must therefore by toric varieties. Thus, $\mathcal{X}$ is a possibly non-separated toric scheme.

Proposition 3.6 (Corrected version of [GS2, Proposition 7.2])**.** *Suppose $\mathcal{X}$ is a toric stack, $f\colon \mathcal{X} \to Y$ is a good moduli space morphism, and Y is a separated algebraic space. Then Y is a toric variety and f is a toric morphism.*

Proof. The hypothesis of [GS2, Proposition 7.2] is that $\mathcal{X}$ is a toric stack (as opposed to a generalized toric stack) and so the proof is largely the same. The only modification to the proof is in the second paragraph: appealing to Remark 3.5 (the corrected version of [GS2, Remark 6.3]) instead of [GS2, Theorem 6.1] shows that Y is a toric scheme. Since we have additionally assumed Y is separated, we know it is a toric variety. ☐

Finally, we note that the error only affects [GS1, Corollary 6.5] through its appeal to [GS2, Proposition 7.2] in the case where Y is a variety, so [GS1, Corollary 6.5] remains correct as stated.

References

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