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A rational map with infinitely many points of distinct arithmetic degrees

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Abstract. Let $f : X \dashrightarrow X$ be a dominant rational self-map of a smooth projective variety defined over $\overline{\mathbb{Q}}$. For each point $P \in X(\overline{\mathbb{Q}})$ whose forward f-orbit is well defined, Silverman introduced the arithmetic degree $\alpha_f(P)$, which measures the growth rate of the heights of the points $f^n(P)$. Kawaguchi and Silverman conjectured that $\alpha_f(P)$ is well defined and that, as P varies, the set of values obtained by $\alpha_f(P)$ is finite. Based on constructions by Bedford and Kim and by McMullen, we give a counterexample to this conjecture when $X = \mathbb{P}^4$.

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1. *Introduction*

Let $f : X \dashrightarrow X$ be a dominant rational map of a smooth projective variety defined over $\overline{\mathbb{Q}}$. We let I_f denote the indeterminacy locus of f, and $X_f(\overline{\mathbb{Q}})$ denote the set of $\overline{\mathbb{Q}}$-points of X whose forward f-orbit is well defined, that is, those $P \in X(\overline{\mathbb{Q}})$ such that $f^n(P) \notin I_f$ for all $n \geq 0$. To each point $P \in X_f(\overline{\mathbb{Q}})$, Silverman [**Sil14**] introduced the following quantity which measures the arithmetic growth rate of $f^n(P)$. Fix an ample divisor H on X and a logarithmic Weil height function $h_H : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ for H. Letting $h_H^+(P) = \max(h_H(P), 1)$, consider the quantities

$$\underline{\alpha}_f(P) = \liminf_{n\to\infty} h_H^+(f^n(P))^{1/n}, \quad \overline{\alpha}_f(P) = \limsup_{n\to\infty} h_H^+(f^n(P))^{1/n}.$$

Kawaguchi and Silverman proved in [**KS16b**, Proposition 12] that these quantities are independent of the choice of ample divisor H. When $\underline{\alpha}_f(P) = \overline{\alpha}_f(P)$, the *arithmetic* *degree* $\alpha_f(P)$ is defined to be the common limit. Kawaguchi and Silverman made the following conjecture and proved it in the case when f is a morphism [**KS16a**, Theorem 3].

CONJECTURE 1. [**KS16b**, Conjecture 6abc] *If* $P \in X_f(\overline{\mathbb{Q}})$, *then the limit* $\alpha_f(P)$ *exists. Moreover,*

$$\{\alpha_f(Q) \mid Q \in X_f(\overline{\mathbb{Q}})\}$$

is a finite set of algebraic integers.

We prove the following result which gives a counterexample to Conjecture 1.

THEOREM 2. *Let* $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ *be the birational map defined by*

$$[X;\, Y;\, Z;\, A;\, B] \mapsto [XY + AX;\, YZ + BX;\, XZ;\, AX;\, BX].$$

Then there exists a sequence of points $P_n \in X_f(\overline{\mathbb{Q}})$ *for which* $\alpha_f(P_n)$ *exists, and* $\{\alpha_f(P_n)\}_n$ *is an infinite set.*

The strategy we use to prove Theorem 2 is actually inspired by another conjecture of Kawaguchi and Silverman [**KS16b**, Conjecture 6d], namely that if $P \in X_f(\overline{\mathbb{Q}})$ and P has Zariski dense orbit under f, then $\alpha_f(P)$ is equal to the first dynamical degree $\lambda_1(f)$. Consider a family $\pi : X \to T$ and a dominant rational map $f : X \dashrightarrow X$ which preserves fibers and induces a dominant rational map $f_t : X_t \dashrightarrow X_t$ on every fiber. For generic values of t, the first dynamical degrees $\lambda_1(f)$ and $\lambda_1(f_t)$ agree, but it is possible to have a countable union of subvarieties $\mathcal{T} \subset T$ such that $\lambda_1(f_t) < \lambda_1(f)$ for all $t \in \mathcal{T}$, and for which infinitely many distinct values arise as $\lambda_1(f_t)$. Suppose that for all $t \in \mathcal{T}$ we can find $P_t \in X_t(\overline{\mathbb{Q}})$ whose forward orbit under f_t is well defined and Zariski dense in X_t. Then we would expect that $\alpha_f(P_t) = \alpha_{f_t}(P_t) = \lambda_1(f_t)$. Since the set of $\lambda_1(f_t)$ is infinite, this would achieve infinitely many different values for α_f.

There are a few issues we must handle in order to turn the above strategy into a counterexample to Conjecture 1. First, we must produce a suitable map f, and ensure that there are points P_t with dense orbit under f_t and whose orbits avoid the indeterminacy of f. Second, one would expect that $\alpha_{f_t}(P_t) = \lambda_1(f_t)$, but this requires a proof. The easiest way to show this is to work in a case where [**KS16b**, Conjecture 6d] is already known to hold. For this reason, we consider a family of surface maps with f birational and where f_t extends to an automorphism of a birational model of X_t, so that we can appeal to [**Kaw08**, **KS14**], which proves that $\alpha_f(P) = \lambda_1(f)$ in this case. We implement this strategy based on constructions by Bedford and Kim [**BK06**] and by McMullen [**McM07**].

2. *Proof of Theorem 2*

We begin by taking the strategy described in the introduction and codifying it as the following result.

PROPOSITION 3. *Let* X *be a smooth projective variety over* $\overline{\mathbb{Q}}$, *and* $\pi : X \to T$ *be a projective morphism of* $\overline{\mathbb{Q}}$*-varieties with two-dimensional fibers. Let* $f : X \dashrightarrow X$ *be a birational map defined over* T *and suppose there is an infinite sequence of parameters* $t_n \in T(\overline{\mathbb{Q}})$ *satisfying the following:*

(1) *for each* n, *there exists a birational model* $\pi_{t_n} : \widetilde{X}_{t_n} \to X_{t_n}$ *so that* f_{t_n} *extends to an automorphism* $\widetilde{f}_{t_n} : \widetilde{X}_{t_n} \to \widetilde{X}_{t_n}$;

(2) *for each n, there exists a $\overline{\mathbb{Q}}$-point P_n of X_{t_n}, contained in the open set where π_{t_n} is an isomorphism, and with well-defined f-orbit that is Zariski dense in X_{t_n};*
(3) *the set of values $\lambda_1(f_{t_n})$ is infinite.*

Then the set of values of $\alpha_f(P_n)$ is infinite.

Proof. Fix an ample divisor H on X. Since H restricts to an ample on X_t, we see $\alpha_f(P) = \alpha_{f_t}(P)$ for all $P \in X_t(\overline{\mathbb{Q}})$ such that the arithmetic degree is well defined. So to complete the proof, it is enough to show $\alpha_{f_{t_n}}(P_n) = \lambda_1(f_{t_n})$.

Let $\widetilde{P}_n$ be the unique point of $\widetilde{X}_{t_n}$ with $\pi_{t_n}(\widetilde{P}_n) = P_n$. We have $\alpha_{f_{t_n}}(P_n) = \alpha_{\widetilde{f}_{t_n}}(\widetilde{P}_n)$ by [**MSS17**, Theorem 3.4], and $\lambda_1(f_{t_n}) = \lambda_1(\widetilde{f}_{t_n})$ by [**Dan17**, Theorem 1.(2)] and the discussion that follows it. Since $\widetilde{f}_{t_n}$ is a surface automorphism and $\widetilde{P}_n$ has dense orbit, [**KS14**, Theorem 2c] tells us that $\alpha_{\widetilde{f}_{t_n}}(\widetilde{P}_n) = \lambda_1(\widetilde{f}_{t_n})$, completing the proof. □

We next use a construction due in various guises to Bedford and Kim [**BK06**] and to McMullen [**McM07**]. The relation between these two constructions is explained in the introduction of [**BK09**] as well as their remark on page 578. We collect the relevant facts from these papers in the following proposition.

PROPOSITION 4. *Let $X = \mathbb{P}^2 \times \mathbb{A}^2$ and consider the map $f: X \dashrightarrow X$ whose fiber over $(a, b) \in \mathbb{A}^2$ is given in affine coordinates by $f_{a,b}(x, y) = (y + a, y/x + b)$. There is a sequence $t_n = (a_n, b_n) \in \mathbb{A}^2(\overline{\mathbb{Q}})$ indexed by the integers $n \geq 10$ with the following properties:*

(1) *the first dynamical degree $\lambda_1(f_{t_n})$ is given by the largest real root δ_n of the polynomial $x^{n-2}(x^3 - x - 1) + x^3 + x^2 - 1$;*
(2) *the numbers δ_n increase monotonically in n to $\delta_* \approx 1.32472\ldots$, the real root of $x^3 - x - 1$;*
(3) *there is an f_{t_n}-invariant cuspidal cubic curve $C_{t_n} \subset \mathbb{P}^2$ with cusp q_{t_n} which is invariant under f_{t_n};*
(4) *there is a birational model $\pi_{t_n}: \widetilde{X}_{t_n} \to \mathbb{P}^2$ such that f_{t_n} extends to an automorphism of $\widetilde{X}_{t_n}$; specifically, π_{t_n} is a blow-up at n points in the smooth locus of C_{t_n};*
(5) *the point q_{t_n} is not contained in the indeterminacy locus of f;*
(6) *the derivative of f_{t_n} at q_{t_n} is given in suitable coordinates by $\begin{pmatrix} \delta_n^{-2} & 0 \\ 0 & \delta_n^{-3} \end{pmatrix}$.*

Proof. First note that the indeterminacy locus of f is $\{(1:0:0), (0:1:0), (0:0:1)\} \times \mathbb{A}^2$.

Let $t_n = (a_n, b_n)$ be as on page 39 of [**McM07**]. Let $p_1 = (0:0:1)$, $p_2 = (1:0:0)$, $p_3 = (0:1:0)$, and $p_{4+i} = f_{t_n}^i(a_n : b_n : 1)$ for $0 \leq i \leq n-4$. By construction (see §7), the p_j lie in the smooth locus of a cuspidal cubic curve C_{t_n}, and letting $\pi_{t_n}: \widetilde{X}_{t_n} \to \mathbb{P}^2$ be the blow-up at the p_j, the map f_{t_n} extends to an automorphism $\widetilde{f}_{t_n}$ of $\widetilde{X}_{t_n}$.† Moreover, $\widetilde{f}_{t_n}$ preserves an irreducible curve $Y_n \subset \widetilde{X}_{t_n}$ in the complete linear system of the anti-canonical bundle, and $C_{t_n} = \pi_{t_n}(Y_n)$. Since the cusp q_{t_n} of C_{t_n} is not a smooth point of the curve, q_{t_n} is necessarily distinct from the p_j. In particular, π_{t_n} is an isomorphism in a neighborhood of q_{t_n}. Since Y_n is preserved by $\widetilde{f}_{t_n}$, we see q_{t_n} is fixed by $\widetilde{f}_{t_n}$ and hence f_{t_n}. Finally, q_{t_n} is not in the indeterminacy locus of f as $q_{t_n} \notin \{p_1, p_2, p_3\}$. This handles statements (3)–(5).

† For reference, McMullen denotes C_{t_n}, $\widetilde{X}_{t_n}$, and $\widetilde{f}_{t_n}$ by X_n, S_n, and F_n, respectively.

By equation (9.1) of [**McM07**], the derivative of f_{t_n} at q_{t_n} has eigenvalues $\lambda_1(f_{t_n})^{-2}$ and $\lambda_1(f_{t_n})^{-3}$, so (6) will follow upon showing $\lambda_1(f_{t_n}) = \delta_n$ in (1).

Finally, taking into account differences in notation explained in the remark on page 578 of [**BK09**], we see (a_n, b_n) belongs to the locus V_{n-3} as defined in their equation (0.2). Statements (1) and (2) then follow from [**BK06**, Theorem 2] by taking $\alpha = (a, 0, 1)$ and $\beta = (b, 1, 0)$. □

We now prove the main result.

Proof of Theorem 2. We retain the notation of Proposition 4. By construction, f gives a rational self-map of $\mathbb{A}^4$ sending (x, y, a, b) to $(y + a, y/x + b, a, b)$. Taking projective coordinates $[X; Y; Z; A; B]$ on $\mathbb{P}^4$, our map extends to the birational map $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ given by

$$[X; Y; Z; A; B] \mapsto [XY + AX; YZ + BX; XZ; AX; BX].$$

To prove the theorem, we apply Proposition 3. Condition (1) of the proposition is met by virtue of Proposition 4(4), and condition (3) follows from Proposition 4(1) and (2). So we need only find $P_n \in X_{t_n}(\overline{\mathbb{Q}})$ whose forward orbit under f is well defined and Zariski dense in $X_{t_n} = \mathbb{P}^2$, and for which P_n lies in the locus where π_{t_n} is an isomorphism.

Notice that by Proposition 4(1) and (2), for each $n \geq 10$ we have $\lambda_1(f_{t_n}) = \delta_n \geq \delta_{10} > 1$. From [**Dan17**, Theorem 1.(2)], we see $\lambda_1(\widetilde{f_{t_n}}) = \lambda_1(f_{t_n}) > 1$. Theorem 1.1(1) and Lemma 2.4(1) of [**Zha10**] then show there are only finitely many $\widetilde{f_{t_n}}$-periodic curves.

By Proposition 4(3) and (6), q_{t_n} is an attracting fixed point of f_{t_n}. Fixing a metric d on $\mathbb{P}^2(\mathbb{C})$, we find that there exists an analytic open set $U_n \subset \mathbb{P}^2(\mathbb{C})$ containing q_{t_n} for which $f_{t_n}(U_n) \subseteq U_n$ and for which there exists a constant $C < 1$ so that for any u in U_n, we have $d(f_{t_n}(u), q_{t_n}) < C\, d(u, q_{t_n})$. In particular, the set U_n does not contain any f_{t_n}-periodic point other than q_{t_n}. By (4) and (5), we can choose U_n so that it avoids the indeterminacy locus of f and such that $\pi_{t_n} : \widetilde{U}_n = \pi_{t_n}^{-1}(U_n) \to U_n$ is an isomorphism.

Let $\widetilde{P}_n$ be any $\overline{\mathbb{Q}}$-point of $\widetilde{U}_n \setminus \bigcup_{C \text{ is } \widetilde{f_{t_n}}\text{-periodic}} C$, and $P_n = \pi_{t_n}(\widetilde{P}_n)$. Notice that the f-orbit of P_n is contained in U_n, so the orbit is well defined and contained in the locus over which π_{t_n} is an isomorphism. By construction, $\widetilde{P}_n$ is not contained in any $\widetilde{f_{t_n}}$-periodic curve. At last, since $\widetilde{P}_n$ lies in $\widetilde{U}_n$, it is not $\widetilde{f_{t_n}}$-periodic. Since $\widetilde{P}_n$ is not periodic and does not lie on any $\widetilde{f_{t_n}}$-periodic curve, it must have Zariski dense orbit under $\widetilde{f_{t_n}}$, so that P_n has dense orbit under f_{t_n}. □

Remark 5. One can imagine various corrections to Conjecture 1 to circumvent the counterexample of Theorem 2. For example, one might ask that the map $f : X \dashrightarrow X$ does not preserve any fibration. This does not seem sufficient, however. Indeed, the map $g : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ defined by

$$[X; Y; Z; A; B; C] \mapsto [XY + AX; YZ + BX; XZ; AX + CY; BX + CZ; C^2]$$

does not appear to preserve a fibration, but the hyperplane $C = 0$ is g-invariant, and the restriction of g to this hyperplane is the map $f : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ of Theorem 2. One might instead attempt to correct Conjecture 1 by requiring one of the following properties.

(1) There is no subvariety $Z \subset X$ such that $f|_Z$ preserves a fibration.
(2) The points P_n are of bounded degree over $\mathbb{Q}$.

We know of no counterexamples in these settings.

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