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Dynamical Uniform Bounds for Fibers and a Gap Conjecture

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We prove a uniform version of the Dynamical Mordell–Lang Conjecture for étale maps; also, we obtain a gap result for the growth rate of heights of points in an orbit along an arbitrary endomorphism of a quasiprojective variety defined over a number field. More precisely, for our 1st result, we assume X is a quasi-projective variety defined over a field K of characteristic 0, endowed with the action of an étale endomorphism Φ, and $f\colon X \longrightarrow Y$ is a morphism with Y a quasi-projective variety defined over K. Then for any $x \in X(K)$, if for each $y \in Y(K)$, the set $S_{x,y} := \{n \in \mathbb{N}\colon f(\Phi^n(x)) = y\}$ is finite, then there exists a positive integer N_x such that $\sharp S_{x,y} \le N_x$ for each $y \in Y(K)$. For our 2nd result, we let K be a number field, $f : X \dashrightarrow \mathbb{P}^1$ is a rational map, and Φ is an arbitrary endomorphism of X. If $\mathcal{O}_\Phi(x)$ denotes the forward orbit of x under the action of Φ, then either $f(\mathcal{O}_\Phi(x))$ is finite, or $\limsup_{n\to\infty} h(f(\Phi^n(x)))/\log(n) > 0$, where $h(\cdot)$ represents the usual logarithmic Weil height for algebraic points.

1 Introduction

As usual in algebraic dynamics, given a self-map $\Phi\colon X \longrightarrow X$ of a quasi-projective variety X, we denote by Φ^n the n-th iterate of Φ. Given a point $x \in X$, we let $\mathcal{O}_\Phi(x) = \{\Phi^n(x)\colon n \in \mathbb{N}\}$ be the orbit of x. Recall that a point x is periodic if there exists some $n \in \mathbb{N}$ such that $\Phi^n(x) = x$; a point y is preperiodic if there exists $m \in \mathbb{N}$ such that $\Phi^m(y)$ is periodic. Our 1st result is the following.

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Theorem 1.1. Let X and Y be quasi-projective varieties defined over a field K of characteristic 0, let $f\colon X \longrightarrow Y$ be a morphism defined over K, let $\Phi\colon X \longrightarrow X$ be an étale endomorphism, and let $x \in X(K)$. If $|\mathcal{O}_\Phi(x) \cap f^{-1}(y)| < \infty$ for each $y \in Y(K)$, then there is a constant N (depending only on X, Y, Φ and x, but independent of y) such that

$$|\mathcal{O}_\Phi(x) \cap f^{-1}(y)| < N$$

for each $y \in Y(K)$.

Theorem 1.1 offers a uniform statement for the Dynamical Mordell–Lang Conjecture. Indeed, the Dynamical Mordell–Lang Conjecture (see [4, 10]) predicts the following: given a quasi-projective variety X defined over a field K of characteristic 0, endowed with an endomorphism Φ, for any point $x \in X(K)$ and any subvariety $V \subset X$, the set

$$S(X, \Phi, V, x) := \{n \in \mathbb{N}\colon \Phi^n(x) \in V(K)\}$$

is a finite union of arithmetic progressions $\{ak + b\colon k \in \mathbb{N}\}$ for some suitable integers a and b, where the case $a = 0$ yields a singleton instead of an infinite arithmetic progression. We also note that $a = 0$ is the typical case since $a > 0$ would mean that V contains a positive dimensional periodic subvariety.

In particular, assuming x is not preperiodic, if $V \subset X$ contains no periodic positive-dimensional subvariety intersecting the orbit of x, then the Dynamical Mordell–Lang Conjecture predicts that V intersects the orbit of x in finitely many points, see [2,§3.1.3]. The Dynamical Mordell–Lang Conjecture is still open in its full generality, though several partial results are known; for a full account of the known results prior to 2016, see [4]. One important case for which the dynamical Mordell–Lang conjecture is known is the case of étale endomorphisms, see [2]. Our Theorem 1.1 yields a uniform statement for the dynamical Mordell–Lang conjecture in the case of étale endomorphisms, as follows.

Let X be a quasi-projective variety defined over a field K of characteristic 0, endowed with an étale endomorphism Φ. Let $\{X_y\}_{y\in Y}$ be an algebraic family of subvarieties of X parametrized by some quasi-projective variety Y (i.e., the family of fibers of a morphism $X \longrightarrow Y$). Let $x \in X(K)$ be a non-preperiodic point with the property that its orbit under Φ meets each subvariety X_y in finitely many points, that is, no subvariety X_y contains a periodic positive-dimensional subvariety intersecting $\mathcal{O}_\Phi(x)$. Then Theorem 1.1 proves that there exists a uniform upper bound N for the number of points from the orbit $\mathcal{O}_\Phi(x)$ on the subvarieties X_y as y varies in $Y(K)$. We believe the same statement would hold more generally (for an arbitrary endomorphism), as stated in the following uniform version of the dynamical Mordell–Lang conjecture.

Conjecture 1.2. (Uniform dynamical Mordell–Lang conjecture) Let $f\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties defined over a field K of characteristic 0, let $\Phi\colon X \longrightarrow X$ be an endomorphism defined over K and let $x \in X(K)$. If $|\mathcal{O}_\Phi(x) \cap f^{-1}(y)| < \infty$ for all $y \in Y(K)$, then there is a constant N (depending only on X, Y, Φ, and x, but independent of y) such that

$$|\mathcal{O}_\Phi(x) \cap f^{-1}(y)| < N$$

for all $y \in Y(K)$.

Theorem 1.1 answers Conjecture 1.2 in the case of étale endomorphisms. Furthermore, at the expense of replacing Φ by an iterate and also, replacing X by $\Phi^\ell(X)$ for a suitable ℓ, we see that Theorem 1.1 yields a positive answer for Conjecture 1.2 for unramified endomorphisms Φ of a smooth quasi-projective variety X; thus, the conclusion of Theorem 1.1 applies to any endomorphism of a semiabelian variety X defined over a field of characteristic 0. We note that a uniform version of the Dynamical Mordell–Lang Conjecture was suggested by the *automatic uniformity* feature from the classical Mordell–Lang conjecture (see [12]); also, there were various special cases which suggested that a uniform dynamical Mordell–Lang conjecture might hold (see [9]).

One of the key lemmas from the proof of Theorem 1.1 (see Lemma 1) provides the motivation for our next result (which is also motivated in its own right by the dynamical Mordell–Lang conjecture, as we will explain after its statement).

Theorem 1.3. Let X be a quasi-projective variety defined over $\overline{\mathbb{Q}}$, let $\Phi\colon X \longrightarrow X$ be an endomorphism, and let $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational function. Then for each $x \in X(\overline{\mathbb{Q}})$ with the property that the set $f(\mathcal{O}_\Phi(x))$ is infinite, we have

$$\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} > 0,$$

where $h(\cdot)$ is the logarithmic Weil height for algebraic numbers.

Note that if $X = \mathbb{A}^1$, the map $\Phi\colon X \longrightarrow X$ is given by $\Phi(x) = x + 1$, and $f\colon X \hookrightarrow \mathbb{P}^1$ is the usual embedding, then $h(f(\Phi^n(0))) = \log(n)$ for $n \in \mathbb{N}$. This example shows that Theorem 1.3 is, in some sense, the best possible. However, we believe that this gap result should hold more generally for rational self-maps. Specifically, we make the following conjecture.

Conjecture 1.4. (Height gap conjecture) Let X be a quasi-projective variety defined over $\overline{\mathbb{Q}}$, let $\Phi\colon X \dashrightarrow X$ be a rational self-map, and let $f\colon X \dashrightarrow \mathbb{P}^1$ be a rational function. Then for $x \in X(\overline{\mathbb{Q}})$ with the property that $\Phi^n(x)$ avoids the indeterminacy locus of Φ for every $n \geq 0$, if $f(\mathcal{O}_\Phi(x))$ is infinite then

$$\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} > 0.$$

Remark 1.5. We note that since in our proof of Theorem 1.3 we use the fact that there exist finitely many points of bounded height in a given finite extension of our ground field, then our argument does not extend to the function field case (i.e., replacing $\overline{\mathbb{Q}}$ with the algebraic closure of $L(t)$ for a field L of characteristic 0). Also, in order to state a counterpart of our Conjecture 1.4 in the function field setting, one would also need to take into account the isotriviality issues since the orbit of x might be infinite but contain only points defined over the constant field.

Theorem 1.3 proves this conjecture in the case of endomorphisms. Many interesting number theoretic questions fall under the umbrella of the gap conjecture stated above. As an example, we recall that a power series $F(x) \in \overline{\mathbb{Q}}[[x]]$ is called *D-finite* if it is the solution to a nontrivial homogeneous linear differential equation with rational function coefficients. It is known that if $\sum_{n\geq 0} a(n)x^n$ is a D-finite power series over a field of characteristic zero, then there is some $d \geq 2$, a rational endomorphism $\Phi\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^d$, a point $c \in \mathbb{P}^d$, and a rational map $f\colon \mathbb{P}^d \dashrightarrow \mathbb{P}^1$ such that $a(n) = f \circ \Phi^n(c)$ for $n \geq 0$, see [4, Section 3.2.1]. Heights of coefficients of D-finite power series have been studied independently, notably by van der Poorten and Shparlinski [11], who showed a gap result holds in this context that is somewhat weaker than what is predicted by our height gap conjecture above; specifically, they showed that if $\sum_{n\geq 0} a(n)x^n \in \overline{\mathbb{Q}}[[x]]$ is D-finite and

$$\limsup_{n\to\infty} \frac{a(n)}{\log\log(n)} = 0,$$

then the sequence $\{a(n)\}$ is eventually periodic. This was improved recently [5], where it is shown that if $\limsup_{n\to\infty} \frac{a(n)}{\log(n)} = 0$, then the sequence $\{a(n)\}$ is eventually periodic. We see this then gives additional underpinning to Conjecture 1.4. Furthermore, with the notation as in Theorem 1.3, assume now that

$$\limsup_{n\to\infty} \frac{h\big(f(\Phi^n(x))\big)}{\log(n)} = 0. \tag{1.1}$$

Then Theorem 1.3 asserts that Equation (1.1) yields that $f(\mathcal{O}_\Phi(x))$ is finite. We claim that actually this means that the set $\{f(\Phi^n(x))\}_{n\in\mathbb{N}}$ is eventually periodic. Indeed, for each $m \in \mathbb{N}$, we let Z_m be the Zariski closure of $\{\Phi^n(x)\}_{n\geq m}$. Then $Z_{m+1} \subseteq Z_m$ for each m and thus, by the Noetherian property, we get that there exists some $M \in \mathbb{N}$ such that $Z_m = Z_M$ for each $m \geq M$. So, there exists a suitable positive integer ℓ such that Φ^ℓ induces an endomorphism of each irreducible component of Z_M; moreover, each irreducible component of Z_M contains a Zariski dense set of points from the orbit of x. Furthermore, because $f(\mathcal{O}_\Phi(x))$ is a finite set, we get that f must be constant on each irreducible component of Z_M and thus, in particular, f is constant on each orbit $\mathcal{O}_{\Phi^\ell}(\Phi^r(x))$ for r sufficiently large. Hence, Theorem 1.3 actually yields that once Equation (1.1) holds, then $\{f(\Phi^n(x))\}_{n\in\mathbb{N}}$ is eventually periodic.

It is important to note that one cannot replace lim sup with lim inf in Conjecture 1.4, even in the case of endomorphisms. To see this, consider the map $\Phi\colon \mathbb{A}^3 \to \mathbb{A}^3$ given by $(x, y, z) \mapsto (yz, xz, z + 1)$. Then, letting $c = (0, 1, 1)$, it is easily shown by induction that for $n \geq 0$, we have

$$\Phi^{2n}(c) = (0, (2n)!\,, 2n + 1) \qquad \text{and} \qquad \Phi^{2n+1}(c) = ((2n + 1)!\,, 0, 2n + 2).$$

Consequently, if $f\colon \mathbb{A}^3 \to \mathbb{A}^1$ is given by $f(x, y, z) = x$, then we see that $f(\Phi^{2n}(c)) = 0$ and $f(\Phi^{2n+1}(c)) = (2n + 1)!$ for every $n \geq 0$, and so

$$\liminf_{n\to\infty} \frac{h(f(\Phi^n(c)))}{\log(n)} = 0, \qquad \text{while} \qquad \limsup_{n\to\infty} \frac{h(f(\Phi^n(c)))}{\log(n)} = \infty.$$

Despite the fact that the conjecture does not hold when one replaces lim sup with lim inf, we believe the following variant of Conjecture 1.4 holds if we were to add the hypothesis that the orbit of x under Φ is Zariski dense in X (note that in the above example, the orbit $\mathcal{O}_\Phi(x)$ lies inside the union of the two lines $x = 0$ and $y = 0$ of $\mathbb{A}^3$).

Conjecture 1.6. Let X be an irreducible quasi-projective variety defined over $\overline{\mathbb{Q}}$, let $\Phi\colon X \dashrightarrow X$ be a rational self-map, and let $f\colon X \dashrightarrow \mathbb{P}^1$ be a non-constant rational function. Let $x \in X(\overline{\mathbb{Q}})$ with the property that $\Phi^n(x)$ avoids the indeterminacy locus of Φ for every $n \geq 0$, and further suppose that $\mathcal{O}_\Phi(x)$ is Zariski dense in X. Then

$$\liminf_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} > 0.$$

We point out that, if true, this would be a powerful result and would imply the dynamical Mordell–Lang conjecture for rational self-maps when we work over a number field. To see this, let Z be a quasi-projective variety defined over $\overline{\mathbb{Q}}$, let $\Phi\colon Z \dashrightarrow Z$ be a rational self-map, Y be a subvariety of Z, and suppose that the orbit of $x \in Z(\overline{\mathbb{Q}})$ avoids the indeterminacy locus of Φ. As before, denote by Z_n the Zariski closure of $\{\Phi^j(x)\colon j \geq n\}$. Since Z is a Noetherian topological space, there is some m such that $Z_n = Z_m$ for every $n \geq m$. Letting $X = Z_m$, and replacing Y with $Y \cap X$, it suffices to show that the conclusion to the Dynamical Mordell–Lang conjecture holds for the data (X, Φ, x, Y). We let $X_1, \ldots, X_d$ denote the irreducible components of X and let $Y_i = Y \cap X_i$. Since $\Phi|_X$ is a dominant self-map, it permutes the components X_i, so there is some b such that $\Phi^b(X_i) \subset X_i$ for each i. Then if we let $x_1, \ldots, x_d$ be elements in the orbit of x with the property that $x_i \in X_i$, then it suffices to show that the conclusion to the statement of the Dynamical Mordell–Lang conjecture holds for the data (X_i, Φ^b, x_i, Y_i) for $i = 1, \ldots, d$. Then by construction, the orbit of x_i under Φ^b is Zariski dense. We prove that either $\mathcal{O}_{\Phi^b}(x_i) \subset Y_i$ or that $\mathcal{O}_{\Phi^b}(x_i)$ intersects Y_i finitely many times. If $Y_i = X_i$ or $Y_i = \varnothing$ then the result is immediate; thus, we may assume without loss of generality that Y_i is a non-empty proper subvariety of X_i. We pick a non-constant morphism $f_i\colon X_i \longrightarrow \mathbb{P}^1$ such that $f_i(Y_i) = 1$ (we find such f_i by choosing first a non-constant rational function F_i vanishing on Y_i and then letting $f_i := F_i + 1$). If $\Phi^{bn}(x_i) \in Y_i$, then $h(f_i(\Phi^{bn}(x_i))) = 0$. Conjecture 1.6 implies that this can only happen finitely many times, and so $\{n\colon \Phi^{bn}(x_i) \in Y_i\}$ is finite.

2 Proof of Our Main Results

We recall the following definitions. The ring of strictly convergent power series $\mathbb{Q}_p\langle z\rangle$ is the collection of elements $P(z) := a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{Q}_p[[z]]$ such that $|a_n|_p \to 0$ as $n \to \infty$ and which thus consequently converge uniformly on $\mathbb{Z}_p$. The *Gauss norm* is given by $|P(z)|_{\mathrm{Gauss}} := \max_{n\geq 0} |a_n|_p$. The ring $\mathbb{Z}_p\langle z\rangle \subset \mathbb{Q}_p\langle z\rangle$ is the set of $P(z)$ with $|P(z)|_{\mathrm{Gauss}} \leq 1$, that is, the set of P with $a_i \in \mathbb{Z}_p$.

Proof. of Theorem 1.1 Clearly, we may reduce immediately (at the expense of replacing Φ by an iterate of it) to the case X and Y are irreducible.

A standard spreading out argument (similar to the one employed in the proof of [2, Theorem 4.1]) allows us to choose a model of X, Y, f, Φ, and x over an open subset $U \subseteq \operatorname{Spec} R$, where R is an integral domain, which is a finitely generated $\mathbb{Z}$-algebra. In other words, K is a field extension of the fraction field of R, we can find a map $\mathcal{X} \longrightarrow \mathcal{Y}$ over U, a section $U \longrightarrow \mathcal{X}$, and an étale endomorphism $\mathcal{X} \longrightarrow \mathcal{X}$ over U which base change over K to be $f\colon X \longrightarrow Y$, $x\colon \mathrm{Spec}K \longrightarrow X$, and $\Phi\colon X \longrightarrow X$, respectively. After replacing U by a possibly smaller open subset, we can assume $U = \mathrm{Spec}R[g^{-1}]$ for some $g \in R$. Since $R[g^{-1}]$ is a finitely generated $\mathbb{Z}$-algebra, it is of the form $\mathbb{Z}[u_1, \ldots, u_r]$. Applying [1, Lemma 3.1], we can find a prime $p \geq 5$ and an embedding $R[g^{-1}]$ into $\mathbb{Q}_p$ that maps the u_i into $\mathbb{Z}_p$. Base changing by the resulting map $\mathrm{Spec}\mathbb{Z}_p \longrightarrow U$, we can assume $U = \mathrm{Spec}\mathbb{Z}_p$. We will abusively continue to denote the map $\mathcal{X} \longrightarrow \mathcal{Y}$ by f, the étale endomorphism $\mathcal{X} \longrightarrow \mathcal{X}$ by Φ, and the section $\mathrm{Spec}\mathbb{Z}_p = U \longrightarrow \mathcal{X}$ by x. We let $\overline{\mathcal{X}} = \mathcal{X} \times_{\mathbb{Z}_p} \mathbb{F}_p$, let $\overline{\Phi}\colon \overline{\mathcal{X}} \longrightarrow \overline{\mathcal{X}}$ be the reduction of Φ, and let $\overline{x} \in \mathcal{X}(\mathbb{F}_p)$ be the reduction of $x \in \mathcal{X}(\mathbb{Z}_p)$.

Notice that if $f(\Phi^n(x)) = y$, then since x extends to a $\mathbb{Z}_p$-point of $\mathcal{X}$, necessarily $y \in Y(K)$ extends to a $\mathbb{Z}_p$-point of $\mathcal{Y}$ as well. In particular, it suffices to give a uniform bound on the sets $\{n : f(\Phi^n(x)) = y\}$ as y varies through the elements $\mathcal{Y}(\mathbb{Z}_p)$.

To prove Theorem 1.1, we may replace x by $\Phi^\ell(x)$ for some $\ell \in \mathbb{N}$; similarly, we can replace Φ by Φ^D for some $D \in \mathbb{N}$. Since $|\mathcal{X}(\mathbb{F}_p)| < \infty$, there exist integers $i \geq 0$ and $j \geq 1$ such that $\overline{\Phi}^{i+j}(\overline{x}) = \overline{\Phi}^i(\overline{x})$; therefore, at the expense of replacing x by $\Phi^i(x)$ and also replacing Φ by Φ^j, we may assume that $\overline{x}$ is fixed by $\overline{\Phi}$. Applying the p-adic Arc Lemma (see, e.g., Remark 2.3 and Theorem 3.3 of [2]) we can assume there are p-adic analytic functions $\phi_1, \ldots, \phi_d \in \mathbb{Z}_p\langle z\rangle$ such that letting $B \subset \mathcal{X}(\mathbb{Z}_p)$ be the set of points whose reduction mod p is $\overline{x}$, then there is a bijection $\iota\colon B \longrightarrow \mathbb{Z}_p^d$, such that

$$\iota\big(\Phi^n(x)\big) = (\phi_1(n), \ldots, \phi_d(n)) := \phi(n)$$

for each positive integer n.

Next, fix an embedding $\mathcal{Y} \subset \mathbb{P}^r_{\mathbb{Z}_p}$, let $\{V_i\}_i$ be an open affine cover of $\mathcal{Y}$, and for each i, let $\{U_{ij}\}_j$ be an open affine cover of $f^{-1}(V_i)$. We can further assume that each V_i is contained in one of the coordinate spaces $\mathbb{A}^r_{\mathbb{Z}_p} \subset \mathbb{P}^r_{\mathbb{Z}_p}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are quasi-compact, we can assume the $\{U_{ij}\}_{i,j}$ and $\{V_i\}_i$ are finite covers. Then we can view $f|_{U_{ij}}\colon U_{ij} \longrightarrow V_i \subseteq \mathbb{A}^r_{\mathbb{Z}_p}$ as a tuple of polynomials $(p_{ij0}, \ldots, p_{ijr})$. Letting $P_{ijk}(z) = p_{ijk}\iota^{-1}\phi(z)$, we see $f|_{\mathcal{O}_\Phi(x)}$ is given by the following piecewise analytic function:

$$f(\Phi^n(x)) = (P_{ij0}(n), \ldots, P_{ijr}(n))$$

whenever $\Phi^n(x) \in U_{ij}$.

It therefore suffices to prove that for each i, j, there exists N_{ij} such that for all $(y_1, \ldots, y_r) \in V_i(\mathbb{Z}_p) \subseteq \mathbb{A}^r(\mathbb{Z}_p)$, the number of simultaneous roots of $P_{ijk}(z) - y_k$ (for $k = 1, \ldots, r$) is bounded by N_{ij}. In other words, we have reduced to proving the lemma below, where $S = \{n : \Phi^n(x) \in U_{ij}\}$ and $V = V_i(\mathbb{Z}_p)$.

Lemma 2.1 Let r be a positive integer, let $V \subset \mathbb{Z}_p^r$, and let $S \subset \mathbb{N}$ be an infinite subset. For each $1 \le k \le r$, let $P_k \in \mathbb{Z}_p\langle z\rangle$ and consider the function $P\colon S \longrightarrow \mathbb{Z}_p^r$ given by

$$P(n) := (P_1(n), \dots, P_r(n)).$$

Suppose the set $\{n \in S : P(n) = y\}$ is empty if $y \in \mathbb{Z}_p^r \setminus V$ and is finite if $y \in V$. Then there exists a positive integer N depending on V, $P_1 \dots, P_r$, but independent of y, such that

$$|\{n \in S : P(n) = y\}| \le N$$

for all $y \in V$. ■

Proof. We may assume S is infinite since otherwise we can take $N = |S|$. We claim that $P_k(z)$ is not a constant power series for some k. Suppose to the contrary that $P_k(z) = c_k \in \mathbb{Z}_p$ for each k. If $y := (c_1, \dots, c_r) \in \mathbb{Z}_p^r \setminus V$, then we can take $N = 0$. If $y \in V$, then $\{n \in S : P(n) = y\} = S$ which is infinite, contradicting the hypotheses of the lemma.

We have therefore shown that some $P_k(z)$ is non-constant. Let $\mathcal{K}$ be the set of k for which $P_k(z) := \sum_{m\ge 0} c_{k,m} z^m$ is non-constant. Given any non-constant element $Q(z) := \sum_{m\ge 0} c_m z^m$ of $\mathbb{Z}_p\langle z\rangle$, let

$$D(Q) := \max\{m : |c_m| = |Q|_{\mathrm{Gauss}}\}. \tag{2.1}$$

Recall from Strassman's theorem (see [14] or [8, Theorem 4.1, p. 62]) that the number of zeros of $Q(z)$ is bounded by $D(Q)$. We can obtain a slight strengthening of Strassman's theorem as follows.

Proposition 2.3. Let $Q(z) := \sum_{m\ge 0} c_m z^m \in \mathbb{Z}_p\langle z\rangle$ be a non-constant power series. Then there exists a positive integer $D_{\max}(Q)$ such that for any $\alpha \in \mathbb{Z}_p$, there are at most $D_{\max}(Q)$ zeros for the power series $Q(z) - \alpha$. ■

Proof of Proposition 2.3 We claim that the desired conclusion holds with $D_{\max}(Q) := D(Q(z) - Q(0))$. Indeed, $D(Q(z) - Q(0))$ is a positive integer since $Q(z)$ is a non-constant power series and therefore, $Q(z) - Q(0)$ is a non-constant power series with its constant term equal to 0. Now, this means that for any $\alpha \in \mathbb{Z}_p$, we have that the power series $Q(z) - \alpha$ has at most $D(Q(z) - \alpha)$ zeros (according to Strassman's Theorem). However, for each $\alpha \in \mathbb{Z}_p$, we have that $D(Q(z) - \alpha) \le D(Q(z) - Q(0))$ because the constant term of $Q(z) - Q(0)$ is zero and therefore, it has absolute value less than the absolute value of the constant term of any other power series $Q(z) - \alpha$ (while the other corresponding coefficients of the two power series $Q(z) - \alpha$ and $Q(z) - Q(0)$ are equal to each other). So, the conclusion in Proposition 2.3 holds with $D_{\max}(Q) := D(Q(z) - Q(0))$, as desired. ■

We let $N := \min_{k\in\mathcal{K}} D_{\max}(P_k)$ and then we see then that for all $(y_1, \ldots, y_r) \in \mathbb{Z}_p^r$, the number of simultaneous zeros of $P_1(z) - y_1, \ldots, P_r(z) - y_r$ is bounded by N. In particular, $|\{n \in S : P(n) = y\}| \le N$ for all $y \in V$, as desired in the conclusion of Lemma 2.1.

This concludes our proof of Theorem 1.1.

Proof of Theorem 1.3 As before, at the expense of replacing Φ by an iterate, we may assume X is irreducible. Furthermore, arguing as in the last paragraph of the introduction, we may assume $\mathcal{O}_\Phi(x)$ is Zariski dense.

Let K be a number field such that X, Φ, and f are defined over K and moreover, $x \in X(K)$. As proven in [13], there exists a constant $c_0 > 0$ such that for each real number $N \ge 1$, there are fewer than c_0N^2 algebraic points in K of logarithmic height bounded above by $\log(N)$. So, there exists a constant $c_1 > 1$ such that for each real number $N \ge 1$, there are fewer than c_1^N points in K of logarithmic height bounded above by N.

Arguing as in the proof of Theorem 1.1, we can find a suitable prime number p, a model $\mathcal{X}$ of X over some finitely generated $\mathbb{Z}$-algebra R which embeds into $\mathbb{Z}_p$ such that the endomorphism Φ extends to an endomorphism of $\mathcal{X}$, and a section $\mathrm{Spec}(\mathbb{Z}_p) \longrightarrow \mathcal{X}$ extending x; we continue to denote by Φ and x the endomorphism of $\mathcal{X}$ and the section $\mathrm{Spec}(\mathbb{Z}_p) \longrightarrow \mathcal{X}$, respectively. At the expense of replacing both Φ and x by suitable iterates, we may assume the reduction of x modulo p (called $\overline{x}$) is fixed under the induced action of $\overline{\Phi}$ on the special fiber of $\mathcal{X}$. Consider the p-adic neighborhood $B \subset \mathcal{X}(\mathbb{Z}_p)$ consisting of all points whose reduction modulo p is $\overline{x}$. Then there is an analytic isomorphism $\iota\colon B \to \mathbb{Z}_p^m$ so that in these coordinates

$$\overline{x} = (0, \ldots, 0) \in \mathbb{F}_p^m$$

and Φ is given by $(x_1, \cdots, x_m) \mapsto (\phi_1(x_1, \ldots, x_m), \cdots, \phi_m(x_1, \ldots, x_m))$, where

$$\phi_i(x_1, \ldots, x_m) \equiv \sum_{j=1}^{m} a_{i,j}x_j \pmod{p}$$

for each $i = 1, \ldots, m$, for some suitable constants $a_{i,j} \in \mathbb{Z}_p$ (for more details, see [4, Section 11.11]). Applying [4, Theorem 11.11.1.1] (see also the proof of [4, Theorem 11.11.3.1]), there exists a p-adic analytic function $G\colon \mathbb{Z}_p \longrightarrow \mathbb{Z}_p^m$ such that for each $n \ge 1$, we have

$$\|\Phi^n(x) - G(n)\| \le p^{-n}, \tag{2.2}$$

where for any point $(x_1, \ldots, x_m) \in \mathbb{Z}_p^m$, we let

$$\|(x_1, \ldots, x_m)\| := \max_{1 \le i \le m} |x_i|_p.$$

As in the proof of Theorem 1.1, let $V_1 \simeq \mathbb{A}^1$ and $V_2 \simeq \mathbb{A}^1$ be the standard affine cover of $\mathbb{P}^1$, and let $\{U_{ij}\}$ be a finite open affine cover of $\mathcal{X}$ minus the indeterminacy locus of f such that $f(U_{ij}) \subset V_i \simeq \mathbb{A}^1$. Let

$$S_{ij} := \{n : \Phi^n(x) \in U_{ij}\}.$$

Since $f|_{U_{ij}}$ is given by a polynomial with p-adic integral coefficients, there exist $H_{ij}(z) \in \mathbb{Z}_p\langle z\rangle$ such that

$$f(G(n)) = H_{ij}(n)$$

whenever $n \in S_{ij}$. Notice that if $f(\Phi^n(x)) = y$, then since x extends to a $\mathbb{Z}_p$-point of $\mathcal{X}$, necessarily $y \in \mathbb{P}^1(K)$ extends to a $\mathbb{Z}_p$-point of $\mathbb{P}^1$ as well. Thus, we need only concern ourselves with roots of $H_{ij}(z) - t$ for $t \in \mathbb{Z}_p$.

Lemma 2.5. The following holds:

1. for all i and j, we have that $\{f(\Phi^n(x)) : n \in S_{ij}\}$ is an infinite set,
2. for all i and j, we have that $\mathbb{N} \setminus S_{ij}$ has upper Banach density zero,
3. there exist i and j, there exists a constant κ and a sequence $M_1 < M_2 < \ldots$ such that

$$\sharp\{n \in S_{ij} : n \le \kappa M_\ell\} \ge M_\ell.$$

∎

Proof of Lemma 2.5. We first prove property 1 for all i, j. If $\{f(\Phi^n(x)) : n \in S_{ij}\} = \{t_1, \ldots, t_k\}$ is a finite set, then

$$\mathcal{O}_\Phi(x) \subset (X \setminus U_{ij}) \cup \bigcup_{1 \le \ell \le k} f^{-1}(t_\ell),$$

which contradicts the fact that $\mathcal{O}_\Phi(x)$ is Zariski dense.

We next prove property 2 for all i, j. Since $Z := X \setminus U_{ij}$ is a closed subvariety, [3, Corollary 1.5] tells us $\mathbb{N} \setminus S_{ij}$ is a union of at most finitely many arithmetic progressions and a set of upper Banach density zero. Since $\mathcal{O}_\Phi(x)$ is Zariski dense, $\mathbb{N} \setminus S_{ij}$ cannot contain any nontrivial arithmetic progressions. Indeed, if there exists $0 \le b < a$ such that $\{an+b : n \in \mathbb{N}\} \subset \mathbb{N} \setminus S_{ij}$, then writing $\mathcal{O}_\Phi(x) = \bigcup_{0 \le \ell < a} \Phi^\ell \mathcal{O}_{\Phi^a}(x)$, we see $\Phi^\ell \mathcal{O}_{\Phi^a}(x)$ is

Zariski dense for some ℓ; applying a suitable iterate of Φ, we see $\Phi^b\mathcal{O}_{\Phi^a}(x)$ is also Zariski dense, contradicting the fact that $\Phi^b\mathcal{O}_{\Phi^a}(x)$ is contained in the proper closed subvariety Z. Thus, $\mathbb{N} \setminus S_{ij}$ has upper Banach density zero.

Next, we turn to property 3. Let $\mathcal{U}$ denote our set of affine patches U_{ij}, and let $\kappa = |\mathcal{U}|$. For each $M \geq 2$, there must exist some element $g(M) \in \mathcal{U}$ and $0 \leq n_1 < n_2 < \cdots < n_M \leq (M-1)\kappa$ such that $\Phi^{n_\ell}(x) \in g(M)$ for $1 \leq \ell \leq M$. Let i and j be such that $g(M) = U_{ij}$ for infinitely many $M \geq 2$. With this choice of i and j, by construction, there is a sequence $M_1 < M_2 < \ldots$ such that S_{ij} contains at least M_ℓ of the integers $0 \leq n \leq (M_\ell - 1)\kappa$. As a result,

$$\sharp\{n \in S_{ij} : n \leq \kappa M_\ell\} \geq M_\ell$$

finishing the proof of lemma. ■

Let i and j be as in Lemma 2.5. For ease of notation, let $H(z) = H_{ij}(z)$ and $S = S_{ij}$. Since

$$\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} \geq \limsup_{n\in S, n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)}$$

it suffices to show the latter is positive.

We split our proof now into two cases which are analyzed separately in Lemmas 2.6 and 2.10. Before giving the proof of Lemma 2.6, we recall that by the Weierstrass Preparation Theorem [7, 5.2.2], if $P(z) := a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{Q}_p\langle z\rangle$ is nonzero and $D = D(P)$ as in (1), then $P(z) = Q(z)u(z)$, where $u(z)$ is a unit in $\mathbb{Q}_p\langle z\rangle$ with $|u(z)|_{\mathrm{Gauss}} = 1$, and $Q(z)$ is a polynomial of degree D whose leading coefficient has p-adic norm equal to $|P(z)|_{\mathrm{Gauss}}$. Combined with [7, 5.1.3 Proposition 1], we see $u(z) = c + pu_0(z)$ with $|c|_p = 1$ and $|u_0|_{\mathrm{Gauss}} \leq 1$. In particular, $|u(n)|_p = 1$ for all $n \in \mathbb{N}$.

Lemma 2.6. If $H(z)$ is non-constant, then the conclusion of Theorem 1.3 holds.

Proof of Lemma 2.6 Writing $H(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in \mathbb{Z}_p\langle z\rangle$, there exists some $L \geq 1$ such that $|a_L|_p > |a_j|_p$ for all $j > L$. As proven in Lemma 2.1, since $H(z)$ is not constant, there exists a uniform bound C such that for each $t \in \mathbb{Z}_p$, the number of solutions to $H(z) = t$ is at most C. Furthermore, if n is an element of S such that $f(\Phi^n(x)) = t$, then equation (2.2) yields

$$|H(n) - t|_p \leq p^{-n}.$$

As mentioned above, by the Weierstrass preparation theorem, we can write

$$H(z) - t = q_t(z)u_t(z)$$

with $q_t(z)$ a polynomial of degree $D(H-t) \le L$ and $u_t(z)$ a unit of Gauss norm 1; moreover, the leading coefficient of $q_t(z)$ has p-adic norm equal to the Gauss norm of $H-t$. Hence, we can write

$$q_t(z) = b_t(z - \beta_{1,t}) \cdots (z - \beta_{D(H-t),t})$$

with $b_t \in \mathbb{Q}_p$, the $\beta_{j,t} \in \overline{\mathbb{Q}}_p$, and

$$|b_t|_p = |H - t|_{\mathrm{Gauss}} \ge |a_L|_p.$$

We have therefore bounded $|b_t|_p$ below independent of $t \in \mathbb{Z}_p$. As noted before the proof of the lemma, we know $|u_t(n)|_p = 1$ for all $t \in \mathbb{Z}_p$ and $n \in \mathbb{N}$. Hence, there is a constant $c_2 > 0$ (independent of t) such that for all $t \in \mathbb{Z}_p$, if $|H(n) - t|_p \le p^{-n}$ then there exists $1 \le j \le D(H-t)$ such that

$$|n - \beta_{j,t}|_p < c_2 p^{-\frac{n}{D(H-t)}} \le c_2 p^{-\frac{n}{L}}.$$

So, if $n_1, \ldots, n_{L+1}$ are distinct elements of S with $|H(n_i)-t|_p \le p^{-n_i}$ for $i = 1, \ldots, L+1$ then there exist k_1, k_2 with $k_1 \neq k_2$ and j such that $|n_{k_1} - \beta_{j,t}|_p < c_2 p^{-n_{k_1}/L}$ and $|n_{k_2} - \beta_{j,t}|_p < c_2 p^{-n_{k_2}/L}$. Consequently,

$$|n_{k_1} - n_{k_2}|_p < c_2 p^{-\min(n_{k_1}, n_{k_2})/L}.$$

Hence, letting $|\cdot|$ be the usual Archimedean absolute value, we have that

$$|n_{k_1} - n_{k_2}| > c_2 p^{\min(n_{k_1}, n_{k_2})/L};$$

therefore, there exists a positive constant c_3 (independent of t, since both L and c_2 are independent of t) such that for all $M \ge 1$ and all $t \in \mathbb{P}^1(K)$,

$$\sharp\{n \le M : n \in S \text{ and } f(\Phi^n(x)) = t\} \le c_3 \log(M). \tag{2.3}$$

In fact, we have a substantially better bound. Let $\exp^k$ denote the k-th iterate of the exponential function and let $L_p(M)$ be the smallest integer k such that $\exp^k(p) > M$. Then $\sharp\{n \le M : n \in S \text{ and } f(\Phi^n(x)) = t\} \le c_3 L_p(M)$; however, we will not need this stronger bound. As an aside, we note that this stronger bound is similar to the one obtained for the Dynamical Mordell–Lang problem in Theorem 1.4 of [6].

Now, let κ be as in Lemma 2.5, and choose a constant $c_4 > 1$ such that

$$c_3 \cdot \log(\kappa c_4^r) \cdot c_1^r < c_4^{r-1} \tag{2.4}$$

for all sufficiently large $r \in \mathbb{R}$, for example, we may take $c_4 := 2c_1$. Let $M_1 < M_2 < \ldots$ be as in Lemma 2.5, and let

$$N_\ell = \lceil \log_{c_4}(M_\ell) \rceil.$$

Property 3 of Lemma 2.5 implies

$$\sharp\{n \le \kappa c_4^{N_\ell} : n \in S\} \ge M_\ell > c_4^{N_\ell - 1}. \tag{2.5}$$

To conclude the proof, we show that for all ℓ sufficiently large, there exists some $n_\ell \le \kappa c_4^{N_\ell}$ with the property that $n_\ell \in S$ and $h(f(\Phi^{n_\ell}(x))) \ge N_\ell$. If this were not the case, then since there are fewer than $c_1^{N_\ell}$ algebraic numbers $t \in \mathbb{P}^1(K)$ of logarithmic Weil height bounded above by N_ℓ, by (2.5) there would be such an algebraic number t with

$$\sharp\{n \le \kappa c_4^{N_\ell} : n \in S \text{ and } f(\Phi^n(x)) = t\} > \frac{c_4^{N_\ell - 1}}{c_1^{N_\ell}} > c_3 \log(\kappa c_4^{N_\ell})$$

and this violates inequality (2.3). We have therefore proven our claim that for all ℓ sufficiently large, there exists a positive integer $n_\ell \le \kappa c_4^{N_\ell}$ with $h(f(\Phi^{n_\ell}(x))) \ge N_\ell$. So,

$$\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} \ge \lim_{\ell\to\infty} \frac{N_\ell}{\log(\kappa) + N_\ell \log(c_4)} = \frac{1}{\log(c_4)} > 0$$

as desired in the conclusion of Theorem 1.3. ∎

Lemma 2.10. If $H(z)$ is a constant, then $\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} = \infty$.

Proof of Lemma 2.10 By property 1 of Lemma 2.5, we can find a sequence $n_1 < n_2 < \ldots$ with the $n_i \in S$ such that

$$f\left(\Phi^{n_{2k-1}}(x)\right) \ne f\left(\Phi^{n_{2k}}(x)\right) \quad \text{and} \quad \{n \in S : n_{2k-1} < n < n_{2k}\} = \varnothing$$

for all $k \ge 1$.

Let $t_0 := H(n)$ (for all $n \in \mathbb{N}$) and for each $i \ge 1$, let $t_i := f\left(\Phi^{n_i}(x)\right)$. Then (2.2) yields that

$$|t_i - t_0|_p \le p^{-n_i}.$$

So, $|t_{2k} - t_{2k-1}|_p \le p^{-n_{2k-1}}$ and since $t_{2k} \ne t_{2k-1}$, we have that for all $k \ge 1$,

$$h\left(t_{2k} - t_{2k-1}\right) = h\left((t_{2k} - t_{2k-1})^{-1}\right) \ge c_5 n_{2k-1}, \tag{2.6}$$

for a constant c_5 depending only on the number field K and on the particular embedding of K into $\mathbb{Q}_p$ (e.g., for the usual embedding of K into $\mathbb{Q}_p$, we may take $c_5 := \frac{1}{2[K:\mathbb{Q}]}$). Inequality (2.6) yields that

$$\max\{h(t_{2k}), h(t_{2k-1})\} \geq \frac{1}{2}(c_5 n_{2k-1} - \log(2)) \tag{2.7}$$

since $h(a+b) \leq h(a) + h(b) + \log(2)$ for any $a, b \in \overline{\mathbb{Q}}$.

First suppose that $\max\{h(t_{2k}), h(t_{2k-1})\} = h(t_{2k-1})$ for infinitely many k. Then consider a subsequence $k_1 < k_2 < \ldots$ where $\max\{h(t_{2k_j}), h(t_{2k_j-1})\} = h(t_{2k_j-1})$. Letting $m_j = n_{2k_j-1}$, we see

$$h\big(f\big(\Phi^{m_j}(x)\big)\big) \geq \frac{1}{2}(c_5 m_j - \log(2)),$$

which shows $\limsup_{n\to\infty} \frac{h(f(\Phi^n(x)))}{\log(n)} = \infty$.

Thus, we may assume that $\max\{h(t_{2k}), h(t_{2k-1})\} = h(t_{2k})$ for all k sufficiently large. We claim that

$$\limsup_{k\to\infty} \frac{h(f(\Phi^{n_{2k}}(x)))}{\log(n_{2k})} = \infty. \tag{2.8}$$

If this is not the case, then there is some $C' > 0$ such that for all sufficiently large k, we have

$$C' > \frac{h(f(\Phi^{n_{2k}}(x)))}{\log(n_{2k})} \geq \frac{1}{2\log(n_{2k})}(c_5 n_{2k-1} - \log(2)),$$

where we have made use here of inequality (2.7). In particular, there is a constant $C > 1$ such that for all k sufficiently large,

$$n_{2k} > C^{n_{2k-1}}. \tag{2.9}$$

Recalling that S does not contain any positive integers between n_{2k-1} and n_{2k}, inequality (2.9) implies that $\mathbb{N} \setminus S$ has positive upper Banach density. This contradicts property 2 of Lemma 2.5, and so our initial assumption that $C' > \frac{h(f(\Phi^{n_{2k}}(x)))}{\log(n_{2k})}$ is incorrect. This proves equation (2.8), and hence Lemma 2.10. ■

Clearly, Lemmas 2.6 and 2.10 finish the proof of Theorem 1.3.

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