New classes of examples satisfying the three matrix analog of Gerstenhaber's theorem

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A B S T R A C T

In 1961, Gerstenhaber proved the following theorem: if k is a field and X and Y are commuting $d \times d$ matrices with entries in k, then the unital k-algebra generated by these matrices has dimension at most d. The analog of this statement for four or more commuting matrices is false. The three matrix version remains open. We use commutative–algebraic techniques to prove that the three matrix analog of Gerstenhaber's theorem is true for some new classes of examples.

In particular, we translate this three commuting matrix statement into an equivalent statement about certain maps between modules, and prove that this commutative–algebraic reformulation is true in special cases. We end with ideas for an inductive approach intended to handle the three matrix analog of Gerstenhaber's theorem more generally.

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Contents

1. Introduction and statement of results

1.1. An overview of Gerstenhaber's theorem and its three matrix analog

Let k be a field and let $\mathrm{M}_d(k)$ denote the space of $d \times d$ matrices with entries in k.

Question 1.1. *Let $X_1, \ldots, X_n \in \mathrm{M}_d(k)$ be pairwise commuting matrices. Must the unital k-algebra that they generate be a finite-dimensional vector space of dimension at most d?*

When $n = 1$, the answer to Question 1.1 is "yes" by the Cayley–Hamilton theorem: $X \in \mathrm{M}_d(k)$ satisfies its characteristic polynomial, thus $I, X, X^2, \ldots, X^{d-1}$ is a vector space spanning set for the algebra generated by X.

When $n \geq 4$, the answer is "no" in general. The standard $n = 4$ counter-example is given by the matrices $E_{13}, E_{14}, E_{23}, E_{24} \in \mathrm{M}_4(k)$, where E_{ij} denotes the matrix with a 1 in position (i, j) and 0s elsewhere. These 4 matrices generate a 5-dimensional algebra.

The first interesting case is $n = 2$. Here the answer to Question 1.1 is "yes." This result is often called **Gerstenhaber's theorem** and was proved in [7]. Gerstenhaber's proof was algebro-geometric, and relied on the irreducibility of the commuting scheme $\mathcal{C}(2, d)$ of pairs of $d \times d$ commuting matrices (a fact also proved in the earlier paper [16]). Some years later, linear algebraic proofs (see [2,15]) and commutative algebraic proofs (see [22,1]) of Gerstenhaber's theorem were found. More detailed summaries on the history and approaches to Gerstenhaber's theorem can be found in [19,12,18].

The case $n = 3$ is still open, and is the subject of this paper. We refer to the following statement (Statement 1.2) as the **three matrix analog of Gerstenhaber's theorem**. Determining whether or not Statement 1.2 is true is sometimes called the **Gerstenhaber problem**.

Statement 1.2. *If $X, Y, Z \in \mathrm{M}_d(k)$ are matrices which pairwise commute, then the unital k-algebra generated by X, Y, and Z is a finite-dimensional vector space of dimension at most d.*

To prove Statement 1.2, one might try to mimic the algebro-geometric proof of Gerstenhaber's theorem. This approach succeeds whenever the affine scheme of triples of commuting $d \times d$ matrices, denoted $\mathcal{C}(3, d)$, is irreducible. Consequently, Statement 1.2 is true when $d \leq 10$ and k is of characteristic 0 (see [21] and references therein).[3] However, since $\mathcal{C}(3, d)$ has multiple irreducible components when $d \geq 29$ (see [9,11,18]), a different approach is necessary to handle the general case.

1.2. Summary of the main results

In this paper, we use commutative–algebraic methods to study the three matrix analog of Gerstenhaber's theorem. We do so by reformulating Statement 1.2 in terms of morphisms of modules, our key technical tools being Propositions 1.8 and 1.10. Although this is nothing more than a simple reformulation, an approach along these lines appears to be new.

We work over an arbitrary field k and let $S = k[x_1, \ldots, x_n]$ denote a polynomial ring in n variables. One can easily rephrase Question 1.1 on n commuting matrices in terms of S-modules. We provide a proof of this in Section 2 (see also [1, §5]).

Proposition 1.3. *Question 1.1 has answer "yes" if and only if for all S-modules N which are finite-dimensional over k, we have*

$$\dim S / \operatorname{Ann}(N) \leq \dim N. \tag{1.4}$$

Thus, Statement 1.2 is true if and only if inequality (1.4) holds for all finite-dimensional $k[x, y, z]$-modules N.

From the perspective of Proposition 1.3, one can approach Statement 1.2 by successively considering modules of increasing complexity. The simplest modules to consider are cyclic ones: when $N = S/I$ it is obvious that equation (1.4) holds. The next simplest case to consider is extensions

$$0 \to S/I \to N \to S/\mathfrak{m} \to 0,$$

where $\mathfrak{m} \subseteq S$ is a maximal ideal. This case is much less obvious, and is the central focus of this paper. Our main result is that Statement 1.2 holds for such modules:

Theorem 1.5. *Let $S = k[x, y, z]$, $I \subseteq S$ an ideal of finite colength, and $\mathfrak{m} \subseteq S$ a maximal ideal. If N is an extension of S-modules*

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[3] In [12], the authors remark that $\mathcal{C}(3, d)$ is also known to be irreducible when $d = 11$ although a proof of this result has not yet been publicly circulated.

$$0 \to S/I \to N \to S/\mathfrak{m} \to 0,$$

then the inequality (1.4) holds for N.

Using Theorem 1.5, we also obtain the following more general result.

Theorem 1.6. Let $S = k[x, y, z]$ and N an S-module which is finite dimension over k. Suppose $N = N_1 \oplus \cdots \oplus N_r$ where for each i, one of the following holds:

(1) N_i is a cyclic module, or
(2) N_i is local with socle of dimension 1 over k, or
(3) there is an extension

$$0 \to M_i \to N_i \to S/\mathfrak{m}_i \to 0$$

where $\mathfrak{m}_i \subseteq S$ is a maximal ideal and $M_i = \bigoplus_\ell M_{i,\ell}$ with each $M_{i,\ell}$ a cyclic module or having socle of dimension 1 over k.

Then the inequality (1.4) holds for N.

Before discussing the technique of proof, it is worth remarking why Theorems 1.5 and 1.6 are more difficult than the cyclic case $N = S/I$. One reason is that the cyclic case holds over polynomial rings $S = k[x_1, \ldots, x_n]$ in arbitrarily many variables, whereas Theorem 1.5 is specific to 3 variables. Indeed, the technique of proof *must* be specific to 3 variables since there are counter-examples of this form in 4 variables:

Example 1.7 (Theorems 1.5 and 1.6 are false for 4 variables). Let $S = k[x, y, z, w]$ and $\mathfrak{m} = (x, y, z, w)$ and consider the S-module N given as follows: we have an extension

$$0 \to S/((x, y) + \mathfrak{m}^2) \to N \xrightarrow{\pi} S/\mathfrak{m} \to 0$$

so that N is generated by the elements of $M := S/((x, y) + \mathfrak{m}^2)$ and an additional element $f \in N$ such that $\pi(f) = 1 \in S/\mathfrak{m}$. The S-module structure is given by

$$zf = wf = 0, \quad xf = z \in M, \quad yf = w \in M.$$

One checks $\mathrm{Ann}(N) = \mathfrak{m}^2$ so that

$$\dim S/\mathrm{Ann}(N) = 5 > 4 = \dim N$$

which violates the inequality of Proposition 1.3.

Observe that $z, w, 1 \in M$, and f form an (ordered) basis of N. In this basis, multiplication by x, y, z, and w are given by the 4×4 matrices E_{14}, E_{24}, E_{13}, and E_{23} respectively.

These are the matrices from the standard $n = 4$ counterexample discussed above. See the proof of Proposition 1.3 for further detail on the connection between commuting matrices and S-modules. ◇

The proof of Theorem 1.5 consists of two steps. The first (see Theorem 3.2) is showing that if a counter-example of this form exists, then we can reduce to the case where $I = (F_{12}, F_{13}, F_{23}, g)$, the F_{ij} are the maximal minors of a specific 2×3 matrix, g is a non-zero divisor, and N satisfies certain additional properties. The second step (see §4) consists of showing that no such counter-example exists.

The case of modules with 1-dimensional socle, and extensions of $S/\mathfrak{m}$ by such modules, are handled in Section 5. This, together with some preliminary results in Section 2, quickly reduces Theorem 1.6 to the case of Theorem 1.5.

As mentioned above, the following is a key tool we use in the proofs of Theorems 1.5 and 1.6.

Proposition 1.8. *Let $S = k[x, y, z]$ be a polynomial ring and $\mathfrak{m} = (x, y, z)$. Then Statement 1.2 is true if and only if for all finite-dimensional S-modules M with $\sqrt{\operatorname{Ann} M} = \mathfrak{m}$, and all S-module maps $\beta\colon \mathfrak{m} \to M$, we have*

$$\dim S/\operatorname{Ann}(M) + \dim \beta(\operatorname{Ann} M) \leq \dim M + 1. \tag{1.9}$$

From the perspective of Proposition 1.8, Statement 1.2 is an assertion about bounding the dimension of the image of $\operatorname{Ann}(M)$ under a module map. Along these lines, we obtain an inductive approach to Statement 1.2:

Proposition 1.10. *Let $S = k[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. Then Statement 1.2 is equivalent to the following assertion: for all finite-dimensional S-modules M with $\sqrt{\operatorname{Ann} M} = \mathfrak{m}$, all submodules $M' \subseteq M$ with $\dim(M/M') = 1$, all ideals $J \subseteq \mathfrak{m}$, and all S-module maps $\beta\colon J \to M'$, if*

$$\dim J/(J \cap \operatorname{Ann} M') + \dim \beta(J \cap \operatorname{Ann} M') \leq \dim M'$$

then

$$\dim J/(J \cap \operatorname{Ann} M) + \dim \beta(J \cap \operatorname{Ann} M) \leq \dim M.$$

We prove Proposition 1.8 in Section 2, and Proposition 1.10 in Section 7.

In Section 8, we end the paper by contrasting the present commutative–algebraic approach to the Gerstenhaber problem with other approaches in the literature. We do this via a series of examples.

Notation. All rings in this paper are commutative with unit. We will let $\operatorname{Supp}(M)$ and $\operatorname{Soc}(M)$ denote the support, respectively the socle, of a module M. Unless otherwise

specified, $\dim(M)$ and the word dimension will refer to the dimension of a module M as a vector space over a given base field.

2. Reformulating Statement 1.2 in terms of module morphisms

Throughout this section, we fix a field k, let $S = k[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. Although our primary focus is the three matrix analog of Gerstenhaber's theorem, our proofs work for arbitrary fields and arbitrarily many matrices. For convenience, consider the following:

Statement 2.1. *The algebra generated by commuting matrices $X_1, X_2, \ldots, X_n \in \mathrm{M}_d(k)$ has dimension at most d.*

For $n > 3$ this statement is false, $n < 3$ it is true, and the case when $n = 3$ is Statement 1.2. Our goal in this section is to prove the following version of Proposition 1.8 which is valid for n commuting matrices.

Proposition 2.2. *Statement 2.1 is true if and only if for all finite-dimensional S-modules M with $\sqrt{\mathrm{Ann}\, M} = \mathfrak{m}$, and S-module maps $\beta \colon \mathfrak{m} \to M$, we have*

$$\dim S/\operatorname{Ann}(M) + \dim \beta(\operatorname{Ann} M) \leq \dim M + 1. \tag{2.3}$$

We begin with a different module-theoretic reformulation of Statement 2.1. This appeared as Proposition 1.3 in the introduction.

Proposition 2.4. *Statement 2.1 holds if and only if for all S-modules N which are finite-dimensional over k, we have*

$$\dim S/\operatorname{Ann}(N) \leq \dim N.$$

Proof. Given commuting matrices $X_1, \ldots, X_n \in \mathrm{M}_d(k)$, we obtain an S-module structure on $k^{\oplus d}$ where we define multiplication by x_i to be the action of X_i. Conversely, given any S-module N of dimension d over k, after fixing a basis we have $N \simeq k^{\oplus d}$ as k-vector spaces; multiplication by $x_1, \ldots, x_n$ on N can then be viewed as matrices $X_1, \ldots, X_n \in \mathrm{M}_d(k)$ which commute since $x_i x_j = x_j x_i$.

Let A denote the unital k-algebra generated by our commuting matrices. We then have a surjection $\pi\colon S \to A$ given by $\pi(x_i) = X_i$, and so $A \simeq S/\ker\pi$ as S-modules. Under the correspondence in the previous paragraph, we have $\ker\pi = \operatorname{Ann}(N)$ and so $A = S/\operatorname{Ann}(N)$. Therefore, the inequality $\dim A \le d$ is equivalent to the inequality $\dim S/\operatorname{Ann}(N) \le \dim N$. $\quad\square$

Example 2.5. Following the above proof, we describe some triples of commuting matrices associated to the modules from Theorem 1.5. Let $S = k[x, y, z]$ and $\mathfrak{m} = (x, y, z)$. Fix an ideal $I \subseteq S$ of colength $d - 1$, and an ordered basis $m_1, \ldots, m_{d-1}$ of S/I. Each module N that fits into a short exact sequence

$$0 \to S/I \xrightarrow{i} N \xrightarrow{\pi} S/\mathfrak{m} \to 0$$

has an ordered basis of the form $i(m_1), \ldots, i(m_{d-1}), f$ where $\pi(f) = 1$ in $S/\mathfrak{m}$. Let $n_j = i(m_j)$. Let $X, Y, Z \in \operatorname{M}_d(k)$ be the matrices of multiplication by x, y, z in N in the given basis. Then X, Y, and Z take the form

$$X = \left(\begin{array}{c|c} X' & \begin{matrix} a_1 \\ \vdots \\ a_{d-1} \end{matrix} \\ \hline 0\cdots0 & 0 \end{array}\right), \quad Y = \left(\begin{array}{c|c} Y' & \begin{matrix} b_1 \\ \vdots \\ b_{d-1} \end{matrix} \\ \hline 0\cdots0 & 0 \end{array}\right), \quad Z = \left(\begin{array}{c|c} Z' & \begin{matrix} c_1 \\ \vdots \\ c_{d-1} \end{matrix} \\ \hline 0\cdots0 & 0 \end{array}\right), \quad (2.6)$$

where $X', Y', Z' \in \operatorname{M}_{d-1}(k)$ are determined by multiplication in S/I by x, y, and z respectively, and the final column of each of X, Y, and Z is determined by the particular choice of extension N: the extension which yields the matrices in (2.6) is defined via

$$\begin{aligned} xf &= a_1 n_1 + \cdots + a_{d-1} n_{d-1}, \\ yf &= b_1 n_1 + \cdots + b_{d-1} n_{d-1}, \\ zf &= c_1 n_1 + \cdots + c_{d-1} n_{d-1}. \end{aligned} \qquad (2.7)$$

Conversely, one may ask which triples of the form appearing in (2.6) correspond to S-module extensions of S/I by $S/\mathfrak{m}$. Since S/I along with basis $m_1, \ldots, m_{d-1}$ has been fixed, the matrices X', Y', and Z' are pre-determined. However, the last column of X, Y, and Z can be chosen almost arbitrarily; the only restriction is that X, Y, and Z must pairwise commute. The corresponding S-module extension N is then determined by the equations (2.7).

See Section 8 for some explicit examples. Our examples will show that X', Y', and Z' can have interesting matrix-theoretic properties. $\quad\diamond$

The next observation allows us to handle the case of direct sums of modules and in particular, reduce to the local case.

Lemma 2.8. *Let $N = N_1 \oplus \cdots \oplus N_r$ where each N_i is an S-module which is finite-dimensional over k. If $\dim S/\operatorname{Ann}(N_i) \le \dim N_i$ for all i, then $\dim S/\operatorname{Ann}(N) \le \dim N$.*

Proof. We have $\operatorname{Ann}(N) = \bigcap_i \operatorname{Ann}(N_i)$, so the diagonal map

$$S/\operatorname{Ann}(N) \to \bigoplus_i S/\operatorname{Ann}(N_i)$$

is injective. As a result,

$$\dim S/\operatorname{Ann}(N) \le \sum_i \dim S/\operatorname{Ann}(N_i) \le \sum_i \dim N_i = \dim N. \qquad \square$$

Corollary 2.9. *If N is an S-module that is finite-dimensional over k, and $\dim S/\operatorname{Ann}(N') \le \dim N'$ for all localizations N' of N, then $\dim S/\operatorname{Ann}(N) \le \dim N$. In particular, Statement 2.1 is true if and only if $\dim S/\operatorname{Ann}(N) \le \dim N$ for all S-modules N which are finite-dimensional over k and such that $\operatorname{Supp} N$ is a point.*

Proof. Since N is finite-dimensional as a k-vector space, its support consists of finitely many points, and N can be written as a direct sum $N_1 \oplus \cdots \oplus N_r$ where N_i are the localizations of N. By Lemma 2.8, the desired inequality for N is implied by that for each of the N_i. In particular, $\dim S/\operatorname{Ann}(N) \le \dim N$ for all N if and only if it holds for those N which are supported at a point. The corollary then follows from Proposition 2.4. $\square$

Remark 2.10. Corollary 2.9 is equivalent to the following statement about commuting matrices: "Statement 2.1 holds for all choices of pairwise commuting matrices $X_1, X_2, \ldots, X_n \in \mathrm{M}_d(k)$ if and only if it holds for all choices of n pairwise commuting *nilpotent* matrices." To see this, we follow the proof of Proposition 2.4, using the same notation: if the matrices X_i are nilpotent then one easily checks that $S/\operatorname{Ann} N$ is a local ring with maximal ideal $\mathfrak{m}$. Conversely, if $S/\operatorname{Ann} N$ is local with maximal ideal $\mathfrak{m}$ then $\operatorname{Ann} N \supseteq \mathfrak{m}^c$ for some large enough c. Identifying the variable $x_i \in S$ with the matrix X_i as in the proof of Proposition 2.4, and recalling that $S/\operatorname{Ann} N \cong A$, we see that each X_i satisfies $X_i^c = 0$, hence is nilpotent.

We next prove Proposition 2.2, and hence Proposition 1.8.

Proof of Proposition 2.2. By Corollary 2.9, we need only show that the inequality in the statement of the theorem is equivalent to $\dim S/\operatorname{Ann}(N) \le \dim N$ for all S-modules N which are finite-dimensional over k and such that $\operatorname{Supp} N$ is a point. Let N be such a module. By translation, we can assume without loss of generality that its support is the origin, i.e. $\sqrt{\operatorname{Ann} N} = \mathfrak{m}$. Then the Jordan–Hölder filtration yields a short exact sequence

$$0 \to M \to N \to S/\mathfrak{m} \to 0$$

and this corresponds to a class $\alpha \in \mathrm{Ext}^1(S/\mathfrak{m}, M)$.

From the short exact sequence

$$0 \to \mathfrak{m} \to S \to S/\mathfrak{m} \to 0$$

we have a long exact sequence

$$0 \to \mathrm{Hom}(S/\mathfrak{m}, M) \to M \to \mathrm{Hom}(\mathfrak{m}, M) \to \mathrm{Ext}^1(S/\mathfrak{m}, M) \to 0.$$

So, $\mathrm{Ext}^1(S/\mathfrak{m}, M) \simeq \mathrm{Hom}(\mathfrak{m}, M)/M$. Thus, our class α lifts to an S-module map $\beta \colon \mathfrak{m} \to M$. It is easy to check that $\mathrm{Ann}(N) = \mathrm{Ann}(M) \cap \ker \beta$, so

$$\dim S/\mathrm{Ann}(N) = \dim S/\mathrm{Ann}(M) + \dim \mathrm{Ann}(M)/(\mathrm{Ann}(M) \cap \ker \beta)$$
$$= \dim S/\mathrm{Ann}(M) + \dim \beta(\mathrm{Ann}\, M).$$

Since $\dim N = \dim M + 1$, the inequality

$$\dim S/\mathrm{Ann}(M) + \dim \beta(\mathrm{Ann}\, M) \le \dim M + 1$$

is equivalent to $\dim S/\mathrm{Ann}(N) \le \dim N$. $\quad\square$

Remark 2.11. The proof of Proposition 2.2 shows that if

$$0 \to M \to N \to S/\mathfrak{m} \to 0$$

is the extension corresponding to the map $\beta \colon \mathfrak{m} \to M$, then the inequality (2.3) holds if and only if the inequality (1.4) holds for N. Note that N only depends on the class $[\beta] \in \mathrm{Ext}^1(S/\mathfrak{m}, M)$. Consequently, whether or not the inequality (2.3) holds depends only on the class $[\beta]$ rather than the map β itself.

In light of Proposition 2.2, we make the following definition:

Definition 2.12. Let M be an S-module which is finite-dimensional as a k-vector space with $\sqrt{\mathrm{Ann}\, M} = \mathfrak{m}$. If $\beta \colon \mathfrak{m} \to M$ is an S-module map, we say that (β, M) is a *counter-example* if $\dim S/\mathrm{Ann}(M) + \dim \beta(\mathrm{Ann}\, M) > \dim M + 1$.

Corollary 2.13. *If $I \subseteq S$ is an ideal with S/I finite-dimensional over k, then $(\beta, S/I)$ is a counter-example if and only if $\dim \beta(I) \ge 2$.*

Proof. Let $M = S/I$ and notice that $\mathrm{Ann}(S/I) = I$. By definition, $(\beta, S/I)$ is a counter-example if and only if

$$\dim S/I + \dim \beta(I) = \dim S/\operatorname{Ann}(M) + \dim \beta(\operatorname{Ann} M) > \dim M + 1 = \dim S/I + 1.$$

In other words, it is a counter-example if and only if $\dim \beta(I) > 1$. $\quad\square$

We end with the following result which reduces our search for counter-examples (β, M) to the case where M is indecomposable. Note the distinction from Lemma 2.8: the lemma concerns the inequality $\dim S/\operatorname{Ann}(M) \le \dim M$ whereas the proposition below concerns the inequality $\dim S/\operatorname{Ann}(\widetilde{M}) \le \dim \widetilde{M}$, where $\widetilde{M}$ is the extension of $S/\mathfrak{m}$ by M defined by β.

Proposition 2.14. *Let $\beta\colon \mathfrak{m} \to M$ be an S-module map with M finite-dimensional over k and satisfying $\sqrt{\operatorname{Ann} M} = \mathfrak{m}$. Suppose $M = N_1 \oplus N_2$ and let $\pi_j\colon M \to N_j$ be the two projections. If neither $(\pi_1\beta, N_1)$ nor $(\pi_2\beta, N_2)$ is a counter-example, then (β, M) is also not a counter-example.*

Proof. Let $\widetilde{M}$ be the extension defined by β, and let $\widetilde{M_j}$ be the extension defined by $\pi_j\beta$. We must show $\dim S/\operatorname{Ann}(\widetilde{M}) \le \dim \widetilde{M}$. We know that

$$\operatorname{Ann}(\widetilde{M}) = \ker \beta \cap \operatorname{Ann}(M) = \bigcap_j \ker(\pi_j\beta) \cap \bigcap_j \operatorname{Ann}(N_j) = \bigcap_j \operatorname{Ann}(\widetilde{M_j}),$$

so we have

$$\dim S/\operatorname{Ann}(\widetilde{M}) = \sum_j \dim S/\operatorname{Ann}(\widetilde{M_j}) - \dim S/(\operatorname{Ann}(\widetilde{M_1}) + \operatorname{Ann}(\widetilde{M_2})).$$

Since $(\pi_j\beta, N_j)$ is not a counter-example, we have $\dim S/\operatorname{Ann}(\widetilde{M_j}) \le \dim \widetilde{M_j}$. Also notice that $\sqrt{\operatorname{Ann} M} = \mathfrak{m}$ implies $\operatorname{Ann}(\widetilde{M_1}) + \operatorname{Ann}(\widetilde{M_2}) \subseteq \mathfrak{m}$. Hence, $\dim S/(\operatorname{Ann}(\widetilde{M_1}) + \operatorname{Ann}(\widetilde{M_2})) \ge 1$. Since

$$\dim \widetilde{M} = 1 + \sum_j \dim N_j = -1 + \sum_j \dim \widetilde{M_j},$$

we have our desired inequality $\dim S/\operatorname{Ann}(\widetilde{M}) \le \dim \widetilde{M}$. $\quad\square$

3. Reducing Theorem 1.5 to a special case

Throughout this section, we fix a field k, let $S = k[x_1, x_2, x_3]$ and $\mathfrak{m} = (x_1, x_2, x_3)$. The goal of the next two sections is to prove Theorem 1.5. It suffices to replace k by an extension field, and so we may assume k is infinite. By Lemma 2.8, we reduce immediately to the case where $\operatorname{Supp}(S/I) = \mathfrak{m}$, so we must prove:

Theorem 3.1. *If M is cyclic and $\operatorname{Supp} M = \mathfrak{m}$, then (β, M) is not a counter-example in the sense of Definition 2.12.*

The focus of this section is to prove the following theorem, which reduces Theorem 3.1 to a special case:

Theorem 3.2. *Suppose (β, M) is a counter-example over k with M cyclic and $\sqrt{\operatorname{Ann} M} = \mathfrak{m}$. Then there exist $f_1, f_2, f_3 \in \mathfrak{m}$ and $g, h \in \operatorname{Ann}(M)$ with the following properties:*

(1) *letting $F_{ij} = x_i f_j - x_j f_i$ and $J = (F_{12}, F_{13}, F_{23}, g)$, we have $h \notin J$, the module S/J is finite-dimensional over k, and $\mathfrak{m} \in \operatorname{Supp}(S/J)$,*

(2) *letting $\beta' : \mathfrak{m} \to S/J$ be the S-module map defined by $\beta'(x_i) = f_i$, the elements $\beta'(g)$ and $\beta'(h)$ are linearly independent in the localization of $S/(J + \langle h \rangle)$ at $\mathfrak{m}$, and*

(3) $\dim \operatorname{Soc}(S_\mathfrak{m}/J_\mathfrak{m}) = 2$.

This theorem is proved over the course of §§3.1–3.2. We begin with some preliminary results. Since M is cyclic, it is of the form S/I where I is an ideal of S with $\sqrt{I} = \mathfrak{m}$. By Corollary 2.13, we know $\dim \beta(I) \geq 2$. Furthermore, we can make a minimality assumption: we may assume that S/I is the cyclic module of smallest dimension for which there exists a counter-example $(\beta, S/I)$, i.e. for all cyclic modules S/K with $\dim S/K < \dim S/I$ and all pairs $(\gamma, S/K)$, we can assume $\dim \gamma(K) \leq 1$.

Let $f_i \in S$ such that $\beta(x_i) = f_i \bmod I$. Letting $F_{ij} = x_i f_j - x_j f_i$, we see $F_{ij} \in I$. So, we can write

$$I = (F_{12}, F_{13}, F_{23}, g_1, \ldots, g_s)$$

for some polynomials $g_1, \ldots, g_s \in \mathfrak{m}$.

Lemma 3.3. $\beta(I) \subseteq \operatorname{Soc}(S/I)$ *is the vector space spanned by the $\beta(g_i)$. Moreover, there exist $i \neq j$ such that $\beta(g_i)$ and $\beta(g_j)$ are linearly independent.*

Proof. Notice that if $p \in I$, then $x_i \beta(p) = p\beta(x_i) = 0$ in S/I. This shows that $\beta(I) \subseteq \operatorname{Soc}(S/I)$. Next observe that

$$\beta(F_{ij}) = \beta(x_i)f_j - \beta(x_j)f_i = f_i f_j - f_j f_i = 0.$$

As a result, $\beta(I)$ is the ideal generated by the $\beta(g_j)$ and since the $\beta(g_j)$ are contained in $\operatorname{Soc}(S/I)$, this ideal is nothing more than the vector space they span. Since $\dim \beta(I) \geq 2$, there must be $i \neq j$ such that $\beta(g_i)$ and $\beta(g_j)$ are linearly independent. □

Proposition 3.4. *For each $1 \leq i \leq 3$, we have $f_i \in \mathfrak{m}$.*

Proof. Without loss of generality, assume that $f_1 \notin \mathfrak{m}$. Since $\sqrt{I} = \mathfrak{m}$, we see S/I is an Artin local ring, and so f_1 is a unit in S/I. Since $x_2 f_1 - x_1 f_2 = F_{12}$ is 0 in S/I, we see $x_2 = x_1 f_2 f_1^{-1} \in (x_1)$. Similarly for x_3.

Thus, the maximal ideal $\mathfrak{m}/I$ of S/I is principally generated by x_1. Any Artin local ring with principal maximal ideal has the property that all ideals are of the form $(\mathfrak{m}/I)^n$, hence principal. Since $\mathrm{Soc}(S/I)$ is an ideal, it is principal and so 1-dimensional. This contradicts Lemma 3.3 which shows that $\dim \mathrm{Soc}(S/I) \geq 2$. □

3.1. Showing the existence of a non-zero divisor

In this subsection, we show the existence of g and h in the statement of Theorem 3.2. Let us give an outline of how we proceed. We begin by showing that $S/(F_{12}, F_{13}, F_{23})$ is Cohen–Macaulay of Krull dimension 1. Now there are of course many choices of g such that $S/(F_{12}, F_{13}, F_{23}, g)$ has Krull dimension 0 (i.e. finite dimensional over k), however to prove Theorem 3.2, we need to guarantee that $\dim \mathrm{Soc}(S_\mathfrak{m}/J_\mathfrak{m}) = 2$ and that $\beta'(g)$ and $\beta'(h)$ are linearly independent. This is accomplished by choosing g to be a suitable non-zero divisor in $S/(F_{12}, F_{13}, F_{23})$.

Proposition 3.5. *Every minimal prime of S over (F_{12}, F_{13}, F_{23}) has height 2.*

Proof. For convenience, let $L = (F_{12}, F_{13}, F_{23})$. As F_{12}, F_{13}, F_{23} are the minors of the 2×3 matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ f_1 & f_2 & f_3 \end{pmatrix},$$

the height of each minimal prime over L is at most 2. So assume that there is a minimal prime $\mathfrak{p}$ over L of height ≤ 1.

If $\mathfrak{p}$ has height 0, then $\mathfrak{p} = \{0\}$ and so $L = \{0\}$. Then $F_{ij} = 0$, so $x_i f_j = x_j f_i$ for each i, j. From this it follows that there exists $q \in S$ such that $f_i = x_i q$ for all i. Thus, given any $h \in \mathfrak{m}$, we have $\beta(h) = hq$. In particular, for $h \in I$ we see $\beta(h) = hq \in I$, so $\dim \beta(I) = 0$, which contradicts $\dim \beta(I) \geq 2$.

If $\mathfrak{p}$ has height 1, then there exists $p \in S$ irreducible with $\mathfrak{p} = (p)$. By Bertini's theorem, we know that for a generic linear combination $y = \sum_i \lambda_i x_i$, the ideal (p, y) is prime. Choose y so that (p, y) is prime and so that (the open conditions) $\lambda_3 \neq 0$ and $y \notin (p)$ are satisfied. Let $f_y = \sum_i \lambda_i f_i$, and let $F_{1y} = x_1 f_y - y f_1, F_{2y} = x_2 f_y - y f_2$. Observe that $L = (F_{12}, F_{1y}, F_{2y})$.

Next, since $F_{1y}, F_{2y} \in L \subseteq (p) \subsetneq (p, y)$, we have $x_1 f_y, x_2 f_y \in (p, y)$. Recalling that (p, y) is prime, we see $f_y \in (p, y)$, or both $x_1, x_2 \in (p, y)$. In the latter case, we have that $(x_1, x_2, x_3) = (x_1, x_2, y) \subseteq (p, y)$, which is impossible as the vanishing locus $V(p, y) \subseteq \mathbb{A}^3$ is irreducible of dimension at least 1. We must therefore have $f_y \in (p, y)$.

Since $f_y \in (p, y)$, we have $f_y = qy + r_y p$ for some $q, r_y \in S$. Using that p divides $F_{1y} = x_1 f_y - y f_1 = (x_1 q - f_1) y + x_1 r_y p$, we see that p divides $(x_1 q - f_1) y$. Since (p) is prime and $y \notin p$, we have $x_1 q - f_1 \in (p)$ and so $f_1 = q x_1 + r_1 p$ for some r_1. Similarly, $f_2 = q x_2 + r_2 p$ for some r_2. As a result, $\beta = \beta' + \beta''$ where $\beta'(h) = qh$ for all $h \in \mathfrak{m}$, and $\beta''(x_i) = p r_i$. By Remark 2.11, whether or not $(\beta, S/I)$ is a counter-example depends

only on the value of $[\beta] \in \mathrm{Ext}^1(S/\mathfrak{m}, S/I)$ and since $[\beta'] = 0$, we can assume $\beta = \beta''$. As a result, we can assume the image of β factors through $(p) \subsetneq S/I$. Since (p) is generated by a single element, we have $(p) \simeq S/J$ where $J = \mathrm{Ann}(p)$. Since $\dim S/J < \dim S/I$, by our minimality assumption at the start of §3, we know that $\beta \colon \mathfrak{m} \to (p) = S/J$ is not a counter-example, and so $\dim \beta(J) \leq 1$. But, $I \subseteq J$ because I kills p. So, $\dim \beta(I) \leq \dim \beta(J) \leq 1$. $\square$

Corollary 3.6. $S/(F_{12}, F_{13}, F_{23})$ *is Cohen–Macaulay of Krull dimension* 1.

Proof. By Proposition 3.5, the variety cut out by (F_{12}, F_{13}, F_{23}) has codimension 2 in $\mathbb{A}^3$, so $S/(F_{12}, F_{13}, F_{23})$ has Krull dimension 1. Since F_{12}, F_{13}, F_{23} are the 2×2 minors of a 2×3 matrix, we know that $S/(F_{12}, F_{13}, F_{23})$ is Cohen–Macaulay, see [5, Theorem 18.18] or [10]. $\square$

The following proposition establishes the existence of our desired $g, h \in \mathfrak{m}$.

Proposition 3.7. *There exist* $g, h \in I$ *such that* g *is a non-zero divisor in* $S/(F_{12}, F_{13}, F_{23})$ *and* $\beta(g)$ *and* $\beta(h)$ *are linearly independent. Furthermore, we necessarily have* $h \notin (F_{12}, F_{13}, F_{23}, g)$.

Proof. Recall our notation $I = (F_{12}, F_{13}, F_{23}, g_1, \ldots, g_s)$. Our first goal is to show that I contains a non-zero divisor in $S/(F_{12}, F_{13}, F_{23})$.

We know from Corollary 3.6 that $S/(F_{12}, F_{13}, F_{23})$ is Cohen–Macaulay, so its associated primes are its minimal primes. Consequently, the set of zero divisors of $S/(F_{12}, F_{13}, F_{23})$ is the union of minimal primes over (F_{12}, F_{13}, F_{23}). If I is contained in this union of minimal primes then, by the prime avoidance lemma, I is contained in one of these minimal primes. But this is impossible as $S/(F_{12}, F_{13}, F_{23})$ is Cohen–Macaulay of Krull dimension 1 by Corollary 3.6, and S/I has Krull dimension 0 by assumption. Thus, I is not contained in the union of minimal primes over (F_{12}, F_{13}, F_{23}) and so I contains a non-zero divisor of $S/(F_{12}, F_{13}, F_{23})$.

Next, by Lemma 3.3, there exist $i \neq j$ such that $\beta(g_i)$ and $\beta(g_j)$ are linearly independent in S/I. Let $q \in I$ be a non-zero divisor of $S/(F_{12}, F_{13}, F_{23})$. If $c, d \in k \smallsetminus \{0\}$ with $c \neq d$, and if $q + cg_i$ and $q + dg_i$ are in the same minimal prime over (F_{12}, F_{13}, F_{23}), then q and g_i are also in that minimal prime, a contradiction to q being a non-zero divisor. Thus, since there are only finitely many minimal primes, we see that for all but finitely many $c \in k$, the polynomial $q + cg_i$ is a non-zero divisor.

Since $q \in I$, we know from Lemma 3.3 that $\beta(q) \in \mathrm{Soc}(S/I)$. Then $\beta(g_i)$ and $\beta(q)$ are elements of the vector space $\mathrm{Soc}(S/I)$ which has dimension at least 2 and $\beta(g_i) \neq 0$, so for infinitely many $c \in k$, we see $\beta(q + cg_i) = \beta(q) + c\beta(g_i) \neq 0$. Combining this with our conclusion from the previous paragraph that $q + cg_i$ is a non-zero divisor for all but finitely many $c \in k$, we see we can find a non-zero divisor $g := q + cg_i \in I$ such that $\beta(g) \neq 0$. Now choose $h \in \mathfrak{m}$ to be any linear combination of g_i and g_j such that $\beta(h)$

is not a scalar multiple of $\beta(g)$; this is possible by Lemma 3.3 as $\beta(g_i)$ and $\beta(g_j)$ span a 2-dimensional subspace of $\mathrm{Soc}(S/I)$.

Lastly, we show $h \notin J := (F_{12}, F_{13}, F_{23}, g)$. From Lemma 3.3, we know $0 \neq \beta(g) \in \mathrm{Soc}(S/I)$ and we see that $\beta(F_{ij}) = 0$. Thus, $\beta(J)$ is 1-dimensional, generated by $\beta(g)$. Since $\beta(g)$ and $\beta(h)$ are linearly independent, it follows that $h \notin J$.   □

Remark 3.8 *(Hypothesis that k is infinite).* Proposition 3.7 is the only step in the proof of Theorem 3.1 that assumes that k is infinite.

With the choice of g and h from Proposition 3.7, we have:

Corollary 3.9. *Conditions (1) and (2) of Theorem 3.2 hold.*

Proof. From Proposition 3.7, g is a non-zero divisor of $S/(F_{12}, F_{13}, F_{23})$, which is Cohen–Macaulay of Krull dimension 1 by Corollary 3.6. Thus, $S/J = S/(F_{12}, F_{13}, F_{23}, g)$ has Krull dimension 0, so it is finite-dimensional over k. Since S/J surjects onto S/I and $\sqrt{I} = \mathfrak{m}$, we know that $\mathfrak{m} \in \mathrm{Supp}(S/J)$, which proves condition (1).

Next since $h \in I$, we have surjections $S/J \to S/(J + \langle h \rangle) \to S/I$. After localizing at $\mathfrak{m}$, these remain surjections. Since $\sqrt{I} = \mathfrak{m}$, we know $S_\mathfrak{m}/I_\mathfrak{m} = S/I$ and so

$$\mathfrak{m} \xrightarrow{\beta'} S/J \to S_\mathfrak{m}/J_\mathfrak{m} \to S_\mathfrak{m}/(J + \langle h \rangle)_\mathfrak{m}$$

is a lift of β, meaning that after post-composing the above map by $S_\mathfrak{m}/(J + \langle h \rangle)_\mathfrak{m} \to S_\mathfrak{m}/I_\mathfrak{m} = S/I$, we obtain β. Since $\beta(g)$ and $\beta(h)$ are linearly independent in S/I, it must also be the case that $\beta'(g)$ and $\beta'(h)$ are linearly independent in $S_\mathfrak{m}/(J + \langle h \rangle)_\mathfrak{m}$, proving condition (2).   □

3.2. Computing the socle

In this subsection we compute the socle of $S_\mathfrak{m}/J_\mathfrak{m}$, thereby showing condition (3) of Theorem 3.2 and finishing the proof.

Proposition 3.10. $\dim \mathrm{Soc}(S_\mathfrak{m}/J_\mathfrak{m}) = 2$.

Proof. From Corollary 3.6 we know that $S/(F_{12}, F_{13}, F_{23})$ is Cohen–Macaulay of dimension 1. Thus, the Eagon–Northcott complex yields a minimal free resolution [6, Theorem A2.60]:

$$0 \to S^2 \xrightarrow{A} S^3 \xrightarrow{B} S \to S/\langle F_{12}, F_{13}, F_{23} \rangle \to 0,$$

$$A = \begin{pmatrix} x_3 & f_3 \\ x_2 & f_2 \\ x_1 & f_1 \end{pmatrix}, \quad B = \begin{pmatrix} F_{12} & -F_{13} & F_{23} \end{pmatrix}$$

is an exact sequence.

Since $g \in \mathfrak{m}$ is a non-zero divisor in $S/(F_{12}, F_{13}, F_{23})$, we can use the above resolution to obtain the minimal free resolution of S/J:

$$0 \to S^2 \xrightarrow{A'} S^5 \xrightarrow{B'} S^4 \xrightarrow{C'} S \to S/J \to 0,$$

$$A' = \begin{pmatrix} g & 0 \\ 0 & g \\ x_3 & f_3 \\ x_2 & f_2 \\ x_1 & f_1 \end{pmatrix}, \quad B' = \begin{pmatrix} x_3 & f_3 & -g & 0 & 0 \\ x_2 & f_2 & 0 & -g & 0 \\ x_1 & f_1 & 0 & 0 & -g \\ 0 & 0 & F_{12} & -F_{13} & F_{23} \end{pmatrix}, \quad C' = \begin{pmatrix} F_{12} & -F_{13} & F_{23} & g \end{pmatrix}.$$

Localizing at $\mathfrak{m}$, we obtain the minimal free resolution

$$0 \to S_{\mathfrak{m}}^2 \to S_{\mathfrak{m}}^5 \to S_{\mathfrak{m}}^4 \to S_{\mathfrak{m}} \to (S/J)_{\mathfrak{m}} \to 0.$$

Consequently, $\dim \operatorname{Soc}(S_{\mathfrak{m}}/J_{\mathfrak{m}}) = 2$, as the left-most term in the above resolution is of rank 2. □

4. Completing the proof of Theorem 1.5

Having now proved Theorem 3.2, we have reduced Theorem 3.1 to showing the following. Let $f_1, f_2, f_3, g \in \mathfrak{m}$, $F_{ij} = x_i f_j - x_j f_i$, and $J = (F_{12}, F_{13}, F_{23}, g)$ such that S/J is finite-dimensional with $\dim \operatorname{Soc}(S_{\mathfrak{m}}/J_{\mathfrak{m}}) = 2$. Let $\beta \colon \mathfrak{m} \to S/J$ be defined by $\beta(x_i) = f_i$. Then it is impossible to find $h \in \mathfrak{m} \smallsetminus J$ such that $\beta(g)$ and $\beta(h)$ are linearly independent in $S_{\mathfrak{m}}/(J_{\mathfrak{m}} + \langle h \rangle)$.

We begin with two well-known lemmas.

Lemma 4.1. *If R is an Artinian Gorenstein local ring, and I_1 and I_2 are ideals of R, then $\operatorname{Ann}(I_1) + \operatorname{Ann}(I_2) = \operatorname{Ann}(I_1 \cap I_2)$.*

Proof. In any commutative ring, we have the equality $\operatorname{Ann}(K_1 + K_2) = \operatorname{Ann}(K_1) \cap \operatorname{Ann}(K_2)$ for all ideals K_1 and K_2. For Artinian Gorenstein local rings, we have $\operatorname{Ann}(\operatorname{Ann}(K)) = K$ for all ideals K, see *e.g.* [3, Exercise 3.2.15]. So letting $K_j = \operatorname{Ann}(I_j)$, we see in our case that $\operatorname{Ann}(\operatorname{Ann}(I_1) + \operatorname{Ann}(I_2)) = I_1 \cap I_2$. Taking annihilators of both sides then proves the result. □

Lemma 4.2. *Let R be any Artinian local ring and $0 \neq r \in R$. If $s_1, \ldots, s_m$ form a basis for $\operatorname{Soc}(R)$, then there is a linear dependence relation among the s_i in R/r.*

Proof. Since every non-zero ideal of R intersects the socle non-trivially, $(r) \cap \operatorname{Soc}(R)$ contains a non-trivial element s. We can write $s = \sum a_i s_i$ with $(a_1, \ldots, a_m) \in k^m \smallsetminus 0$. Then in R/r we have the linear dependence relation $\sum a_i s_i = 0$. □

Proposition 4.3. *Let $K \subseteq S = k[x_1, x_2, x_3]$ be an ideal with S/K an Artinian Gorenstein local ring, and $\gamma \colon \mathfrak{m} \to S/K$ an S-module map. If $\gamma(q)$ is divisible by q for all $q \in \mathfrak{m}$, then there exists $r \in S/K$ such that for all $q \in \mathfrak{m}$, we have $\gamma(q) = qr$.*

Proof. For every $r \in S/K$, let $\delta_r \colon \mathfrak{m} \to S/K$ be the map $\delta_r(q) = qr$. To prove the result, it suffices to replace γ by $\gamma - \delta_r$ for any r. Let $\gamma(x_i) = x_i p_i$. Replacing γ by $\gamma - \delta_{p_3}$, we can assume $p_3 = 0$, i.e. $\gamma(x_3) = 0$. Then $x_1 x_3 p_1 = x_3 \gamma(x_1) = x_1 \gamma(x_3) = 0$. In other words,

$$p_1 \in \mathrm{Ann}(x_1 x_3).$$

Similarly,

$$p_2 \in \mathrm{Ann}(x_2 x_3) \quad \text{and} \quad p_1 - p_2 \in \mathrm{Ann}(x_1 x_2).$$

Our first goal is to show that $p_2 \in \mathrm{Ann}(x_2) + \mathrm{Ann}(x_3)$. By Lemma 4.1, we have $\mathrm{Ann}(x_2) + \mathrm{Ann}(x_3) = \mathrm{Ann}((x_2) \cap (x_3))$. So, let $q \in (x_2) \cap (x_3)$ and we must show $p_2 q = 0$. Since q is divisible by both x_2 and x_3, we can write $q = x_3 q'$ and $q = x_2 q''$. We can further write $q'' = a(x_2) + x_3 b(x_2, x_3) + x_1 c(x_1, x_2, x_3)$ where $a \in k[x_2]$, $b \in k[x_2, x_3]$, and $c \in k[x_1, x_2, x_3]$. Then using $p_2 \in \mathrm{Ann}(x_2 x_3)$ and $p_1 - p_2 \in \mathrm{Ann}(x_1 x_2)$, we have

$$p_2 q = p_2 x_2 a(x_2) + p_2 x_1 x_2 c(x_1, x_2, x_3) = p_2 x_2 a(x_2) + p_1 x_1 x_2 c(x_1, x_2, x_3)$$
$$= \gamma(x_2 a(x_2) + x_1 x_2 c(x_1, x_2, x_3) + x_3 d)$$

for any choice of d. Choosing $d = x_2 b(x_2, x_3) - q'$, we see

$$s := x_2 a(x_2) + x_1 x_2 c(x_1, x_2, x_3) + x_3 d = q - x_3 q' = 0.$$

But, by assumption $p_2 q = \gamma(s)$ is divisible by $s = 0$, and hence $p_2 q = 0$. We have therefore shown

$$p_2 \in \mathrm{Ann}(x_2) + \mathrm{Ann}(x_3).$$

Thus, we can write $p_2 = (p_2 - r) + r$ with $r \in \mathrm{Ann}(x_3)$ and $p_2 - r \in \mathrm{Ann}(x_2)$. Then $(\gamma - \delta_r)(x_2) = x_2(p_2 - r) = 0$ and $(\gamma - \delta_r)(x_3) = -x_3 r = 0$, and so we can assume

$$p_2 = p_3 = 0.$$

Then

$$p_1 \in \mathrm{Ann}(x_1 x_2) \cap \mathrm{Ann}(x_1 x_3).$$

To finish the proof we need only show that $p_1 \in \mathrm{Ann}(x_1) + \mathrm{Ann}(x_2, x_3)$. Indeed, upon doing so, we can write $p_1 = (p_1 - r) + r$ with $r \in \mathrm{Ann}(x_2, x_3)$ and $p_1 - r \in \mathrm{Ann}(x_1)$. Then

$(\gamma - \delta_r)(x_1) = x_1(p_1 - r) = 0$, $(\gamma - \delta_r)(x_2) = -x_2 r = 0$, and $(\gamma - \delta_r)(x_3) = -x_3 r = 0$. In other words, we will have found $r \in S$ such that $\gamma - \delta_r = 0$, i.e. $\gamma(q) = qr$ for all $q \in \mathfrak{m}$.

To show that $p_1 \in \mathrm{Ann}(x_1) + \mathrm{Ann}(x_2, x_3)$, and thereby finish the proof, we again note by Lemma 4.1 that $\mathrm{Ann}(x_1) + \mathrm{Ann}(x_2, x_3) = \mathrm{Ann}((x_1) \cap (x_2, x_3))$. We let $q \in (x_1) \cap (x_2, x_3)$ and must show that $p_1 q = 0$. We can then write $q = x_1(a(x_1) + x_2 b(x_1, x_2) + x_3 c(x_1, x_2, x_3))$ and $q = x_2 q' + x_3 q''$. Then using that $p_1 \in \mathrm{Ann}(x_1 x_2) \cap \mathrm{Ann}(x_1 x_3)$, we have

$$p_1 q = p_1 x_1 a(x_1) = \gamma(x_1 a(x_1) + x_2 d + x_3 e)$$

for any choice of d and e. As before, choosing $d = x_1 b(x_1, x_2) - q'$ and $e = x_1 c(x_1, x_2, x_3) - q''$ yields $s := x_1 a(x_1) + x_2 d + x_3 e = 0$, and since $p_1 q = \gamma(s)$ is divisible by $s = 0$, we have $p_1 q = 0$. $\square$

Finally, we turn to the proof of the main theorem.

Proof of Theorem 3.1. Since S/J is finite-dimensional over k, we know that $S_\mathfrak{m}/J_\mathfrak{m} = S/K$ for some ideal K. Recall that $\dim \mathrm{Soc}(S/K) = 2$ and our goal is to show that for all $h \in \mathfrak{m} \smallsetminus J$ there is a linear dependence relation between $\beta(g)$ and $\beta(h)$ in $S_\mathfrak{m}/(J_\mathfrak{m} + \langle h \rangle) = S/(K + \langle h \rangle)$. In particular, we may assume $\beta(g) \neq 0$ in $S/(K + \langle h \rangle)$.

To begin, we show $\beta(h) \notin \mathrm{Soc}(S/K)$. If $\beta(h)$ were in the socle, then since $\dim \mathrm{Soc}(S/K) = 2$ and $\beta(g) \in \mathrm{Soc}(S/K)$, either we have our desired linear dependence relation between $\beta(g)$ and $\beta(h)$ in S/K (and hence in $S/(K + \langle h \rangle)$), or $\beta(g)$ and $\beta(h)$ form a basis for $\mathrm{Soc}(S/K)$. In the latter case, Lemma 4.2 shows there is a linear dependence relation between $\beta(g)$ and $\beta(h)$ in $S/(K + \langle h \rangle)$. So, we have shown our claim that $\beta(h) \notin \mathrm{Soc}(S/K)$. Further note that $h \in \mathrm{Soc}(S/K)$ implies $\beta(h) \in \mathrm{Soc}(S/K)$, since $x_i \beta(h) = h\beta(x_i) \in h\mathfrak{m} = 0$. So, we conclude

$$h, \beta(h) \notin \mathrm{Soc}(S/K).$$

Next, notice that $\beta(h)$ does not divide $\beta(g)$ in S/K. Indeed, suppose to the contrary that $\beta(g) = q\beta(h)$ with $q \in S$. If $q \in k$, then we have a linear dependence relation in S/K and hence in $S/(K + \langle h \rangle)$. If $q \in \mathfrak{m}$, then $\beta(g) = q\beta(h) = h\beta(q) \in \langle h \rangle$ so $\beta(g) = 0$ in $S/(K + \langle h \rangle)$, which is again a linear dependence relation. This shows our claim that

$$\beta(g) \notin S\beta(h).$$

Now, $S\beta(h)$ is an ideal of S/K, so it intersects $\mathrm{Soc}(S/K)$ non-trivially. Since $\dim \mathrm{Soc}(S/K) = 2$, we know $S\beta(h) \cap \mathrm{Soc}(S/K)$ has dimension 1 or 2, but $\beta(g) \notin S\beta(h) \cap \mathrm{Soc}(S/K)$, and so $S\beta(h) \cap \mathrm{Soc}(S/K)$ is 1-dimensional. Let $q_0 \in S$ such that $q_0 \beta(h)$ is a basis vector for $S\beta(h) \cap \mathrm{Soc}(S/K)$. Then

$$\beta(g) \text{ and } q_0 \beta(h) \text{ form a basis for } \mathrm{Soc}(S/K).$$

Since $h \notin \text{Soc}(S/K)$, we can induct on the smallest ℓ for which $\mathfrak{m}^\ell h \in \text{Soc}(S/K)$. That is, we can assume the result for qh for all $q \in \mathfrak{m}$, i.e. we can assume that $\beta(g)$ and $\beta(qh)$ are linearly dependent in $S/(K + \langle qh \rangle)$ for all $q \in \mathfrak{m}$. So for all $q \in \mathfrak{m}$, there exists $p \in S/K$ and $a, b \in k$ such that $(a, b) \neq (0, 0)$ and

$$a\beta(g) + b\beta(qh) = pqh.$$

Note that $\beta(qh) = h\beta(q) \in \langle h \rangle$, so the above equality shows $a\beta(g) \in \langle h \rangle$. This yields our desired linear dependence relation among $\beta(g)$ and $\beta(h)$ in $S/(K + \langle h \rangle)$ unless $a = 0$, in which case after rescaling we can assume $b = 1$. We can therefore assume that

$$\beta(qh) \in Sqh \quad \forall q \in \mathfrak{m}.$$

Next, let

$$\gamma \colon \mathfrak{m} \to \langle h \rangle \subseteq S/K, \quad \gamma(q) = \beta(qh).$$

Since $\langle h \rangle \cap \text{Soc}(S/K)$ is non-trivial, it has dimension 1 or 2. If $\beta(g)$ is in this intersection, then we have our desired linear dependence relation among $\beta(g)$ and $\beta(h)$ in $S/(K + \langle h \rangle)$, so we can assume this is not the case. Thus, $\langle h \rangle \cap \text{Soc}(S/K)$ does not contain $\beta(g)$, so it is 1-dimensional, and hence $\langle h \rangle$ is Gorenstein. Notice that $\langle h \rangle \simeq S/\text{Ann}_{S/K}(h)$ and via this identification, the condition $\beta(qh) \in \langle qh \rangle$ is equivalent to the condition that q divides $\gamma(q)$. Applying Proposition 4.3, there is $r \in S/\text{Ann}_{S/K}(h)$ such that $\gamma(q) = qr$. Translating this back into a statement about $\langle h \rangle$ via our identification with $S/\text{Ann}(h)$, this says

$$\exists\, r \in S \text{ such that } \beta(qh) = qhr \quad \forall q \in \mathfrak{m}.$$

As a result, $\beta(h) - hr \in \text{Ann}(\mathfrak{m}) = \text{Soc}(S/K)$, and so there exist $a, b \in k$ such that

$$\beta(h) - hr = a\beta(g) + bq_0\beta(h).$$

If $q_0 \in \mathfrak{m}$, then we see $\beta(h) - a\beta(g) = h(b\beta(q_0) + r) = 0$ in $S/(K + \langle h \rangle)$ and gives our linear dependence relation. So, $q_0 \notin \mathfrak{m}$, i.e. q_0 is a unit. By construction $\beta(g)$ and $q_0\beta(h)$ form a basis for $\text{Soc}(S/K)$, and q_0 is a unit, so $\beta(g)$ and $\beta(h)$ form a basis for $\text{Soc}(S/K)$. Then by Lemma 4.2, they have a linear dependence relation in $S/(K + \langle h \rangle)$. $\square$

5. Proof of Theorem 1.6

We begin with the following result which holds for arbitrarily many variables:

Proposition 5.1. *Let* $S = k[x_1, \ldots, x_n]$ *and let* $\mathfrak{m} = (x_1, \ldots, x_n)$. *Suppose that* M *is a finite-dimensional* S-*module with* $\text{Supp}\, M = \mathfrak{m}$ *and* $\dim \text{Soc}(M) = 1$. *Then the following hold:*

(1) $\dim S/\operatorname{Ann} M \leq \dim M$,
(2) *for every short exact sequence*

$$0 \to M \to N \to S/\mathfrak{m} \to 0,$$

we have $\dim S/\operatorname{Ann} N \leq \dim N$.

Proof. We first prove (1) by induction on the dimension of M. If $\dim M = 1$ then $M \simeq S/\mathfrak{m}$ and the result holds trivially. For higher dimensional M, recall that M has a composition series

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} \subseteq M$$

where $M_i/M_{i-1} \simeq S/\mathfrak{m}$. Since the socle of M_{r-1} is contained in the socle of M, we also have that $\dim \operatorname{Soc}(M_{r-1}) = 1$, and we may apply induction to see that $\dim S/\operatorname{Ann} M_{r-1} \leq \dim M_{r-1}$. Then we have a short exact sequence

$$0 \to M_{r-1} \to M_r \to S/\mathfrak{m} \to 0$$

which corresponds to a map $\beta \colon \mathfrak{m} \to M_{r-1}$. By Remark 2.11, we need only show

$$\dim S/\operatorname{Ann} M_{r-1} + \dim \beta(\operatorname{Ann} M_{r-1}) \leq \dim M_{r-1} + 1.$$

By induction, we know $\dim S/\operatorname{Ann} M_{r-1} \leq \dim M_{r-1}$. Furthermore, $\beta(\operatorname{Ann} M_{r-1}) \subseteq \operatorname{Soc}(M_{r-1})$, so has dimension at most 1. This proves the desired inequality.

The proof of (2) is entirely analogous. We know that the short exact sequence defining N corresponds to a map $\beta \colon \mathfrak{m} \to M$. By (1), we know $\dim S/\operatorname{Ann}(M) \leq \dim M$. Since $\beta(\operatorname{Ann} M) \subseteq \operatorname{Soc}(M)$, we have $\dim \beta(\operatorname{Ann} M) \leq 1$. Combining these two statements, inequality (2.3) holds for β, and so $\dim S/\operatorname{Ann} N \leq \dim N$ by Remark 2.11. $\square$

We now turn to the proof of the second main result of this paper.

Proof of Theorem 1.6. By Lemma 2.8, we reduce immediately to the case where: (i) N is either cyclic, or (ii) N has 1-dimensional socle, or (iii) there is an extension

$$0 \to M \to N \to S/\mathfrak{m} \to 0 \tag{5.2}$$

where $\mathfrak{m} \subseteq S$ is a maximal ideal and $M = \bigoplus_\ell M_\ell$ with each M_ℓ a cyclic module or having 1-dimensional socle. For case (i), inequality (1.4) holds trivially. Case (ii) is handled by Proposition 5.1 (1).

It remains to handle case (iii). Further decomposing if necessary, we can assume each M_ℓ is local. Next, letting L be the set of ℓ for which $\operatorname{Supp}(M_\ell) = \mathfrak{m}$, we can write $N = N' \oplus \bigoplus_{\ell \notin L} M_\ell$ where we have a short exact sequence

$$0 \to \bigoplus_{\ell \in L} M_\ell \to N' \to S/\mathfrak{m} \to 0.$$

Since M_ℓ is cyclic or has 1-dimensional socle for every $\ell \notin L$, another application of Lemma 2.8 combined with cases (i) and (ii) above allows us to assume $N = N'$, i.e. we can assume that $\mathrm{Supp}(M_\ell) = \mathfrak{m}$ for all ℓ. Proposition 2.14 then reduces us to the case where there is only one ℓ; that is, we need only consider extensions (5.2) where $\sqrt{\mathrm{Ann}\, M} = \mathfrak{m}$ and M itself is cyclic or has 1-dimensional socle. If M has 1-dimensional socle, then inequality (1.4) holds by Proposition 5.1 (2). If M is cyclic, then the inequality holds by Theorem 1.5. $\quad\square$

6. Some other instances where Statement 1.2 holds

In this short section we record a couple of additional situations where a finite-dimensional module M over $S = k[x_1, \ldots, x_n]$ satisfies $\dim S/\mathrm{Ann}\, M \leq \dim M$.

Proposition 6.1. *If $\beta : \mathfrak{m} \to M$ is surjective, then the pair (β, M) is not a counterexample. Moreover, if N is the extension of M defined by β then*

$$\dim S/\mathrm{Ann}(N) = \dim N.$$

Proof. Since $\mathrm{Ann}(N) = \mathrm{Ann}(M) \cap \ker(\beta)$, we see

$$\dim S/\mathrm{Ann}(N) = \dim S/\mathfrak{m} + \dim \mathfrak{m}/\ker(\beta) + \dim \ker(\beta)/(\mathrm{Ann}(N) \cap \ker(\beta))$$
$$= 1 + \dim M + \dim \ker(\beta)/(\mathrm{Ann}(N) \cap \ker(\beta)).$$

Since $\dim N = 1 + \dim M$, we must show $\ker(\beta) \subseteq \mathrm{Ann}(M)$. Given $m \in M$ and $f \in \ker(\beta)$, we know β is surjective, so $m = \beta(g)$ for some $g \in \mathfrak{m}$. Then $fm = f\beta(g) = g\beta(f) = 0$, and so $f \in \mathrm{Ann}(M)$. $\quad\square$

Proposition 6.2. *If there exists $m \in N$ such that $\mathrm{Ann}(m) = \mathrm{Ann}(N)$, then*

$$\dim S/\mathrm{Ann}\, N \leq \dim N.$$

Proof. In this case, $\dim S/\mathrm{Ann}(N) = \dim S/\mathrm{Ann}(m)$ and since $S/\mathrm{Ann}(m)$ is isomorphic to the cyclic submodule $Sm \subseteq N$, we necessarily have $\dim S/\mathrm{Ann}(m) = \dim Sm \leq \dim N$. $\quad\square$

We end this section with an example where Theorem 1.5 applies but Proposition 6.2 does not.

Example 6.3. Let $I = (x, y^2, z) \subseteq k[x, y, z]$. Let $N = (xy, z)/(yz, x^2, z^2, xy^2 - xz)$ and observe that N fits into a short exact sequence

$$0 \to S/I \to N \to S/\mathfrak{m} \to 0$$

where the injective map sends 1 to xy. We know by Theorem 1.5 that inequality (1.4) holds for N. Proposition 6.2, however does not apply here: every element of N can be represented as $m = az + bxy + cxz$, for some $a, b, c \in k$, and one checks that there is no choice of $a, b, c \in k$ such that $\mathrm{Ann}(m)$ agrees with $\mathrm{Ann}\, N = (z, y^2, xy, x^2)$. Indeed, if $a \neq 0$, then $y - (b/a)x \in \mathrm{Ann}(m)$ and if $a = 0$, then $x \in \mathrm{Ann}(m)$. $\diamond$

7. An inductive approach to Statement 1.2

Our goal is to prove Proposition 1.10 stated in the introduction. We do so after a preliminary lemma.

Lemma 7.1. *Let $S = k[x_1, \ldots, x_n]$ and $\mathfrak{m} = (x_1, \ldots, x_n)$. Then Statement 2.1 is true if and only if for all ideals $J \subseteq \mathfrak{m}$, all finite-dimensional S-modules M with $\sqrt{\mathrm{Ann}\, M} = \mathfrak{m}$, and all S-module morphisms $\beta \colon J \to M$, we have*

$$\dim J/(J \cap \mathrm{Ann}\, M) + \dim \beta(J \cap \mathrm{Ann}\, M) \leq \dim M. \tag{7.2}$$

Proof. Notice that if $J = \mathfrak{m}$, then $J \cap \mathrm{Ann}\, M = \mathrm{Ann}\, M$, and so the inequalities (2.3) and (7.2) are equivalent. So by Proposition 2.2, the inequality (7.2) implies Statement 2.1.

Thus, it remains to show that Statement 2.1 implies inequality (7.2). To see this, fix $J \subseteq \mathfrak{m}$ and an S-module map $\beta \colon J \to M$. As in the proof of Proposition 2.2, the map β defines an extension

$$0 \to M \to N \to S/J \to 0$$

with $\mathrm{Ann}(N) = \mathrm{Ann}(M) \cap \ker \beta$. By Proposition 2.4, we know Statement 2.1 implies inequality (1.4) for N. Now note that

$$\dim S/\mathrm{Ann}(N) = \dim S/(J \cap \mathrm{Ann}\, M) + \dim(J \cap \mathrm{Ann}\, M)/(\ker \beta \cap \mathrm{Ann}\, M)$$
$$= \dim S/(J \cap \mathrm{Ann}\, M) + \dim \beta(J \cap \mathrm{Ann}\, M).$$

Since $\dim N = \dim M + \dim S/J$, we obtain inequality (7.2) by subtracting $\dim S/J$ from both sides of the inequality (1.4). $\square$
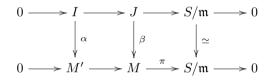
Remark 7.3. The proof of Lemma 7.1 shows that if

$$0 \to M \to N \to S/J \to 0$$

is the extension corresponding to the map $\beta \colon J \to M$, then the inequality (7.2) holds if and only if the inequality (1.4) holds for N.

Proof of Proposition 1.10. Let $S = k[x, y, z]$ be a polynomial ring and $\mathfrak{m} = (x, y, z)$. We know by Lemma 7.1 that Statement 1.2 is true if and only if inequality (7.2) holds for all J, M, and maps $\beta \colon J \to M$. So, if Statement 1.2 is true, then both of the inequalities in the statement of Proposition 1.10 are true, hence the first inequality implies the second.

We now show that if the implication of inequalities in the statement of Proposition 1.10 holds, then Statement 1.2 is true. By virtue of Lemma 7.1, we need only show that inequality (7.2) holds for all J, M, and $\beta \colon J \to M$. We prove this latter statement by induction on $\dim M$, the base case being trivial. So, we need only handle the induction step. For this, we can choose a submodule $M' \subseteq M$ such that $\dim(M/M') = 1$. Then $M/M' \simeq S/\mathfrak{m}$ and we let $\pi \colon M \to S/\mathfrak{m}$ be the quotient map.

Suppose first that $\pi\beta \colon J \to S/\mathfrak{m}$ is surjective. Then letting $I = \ker(\pi\beta)$, we have a map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & J & \longrightarrow & S/\mathfrak{m} & \longrightarrow & 0 \\
 & & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \overset{\pi}{\longrightarrow} & S/\mathfrak{m} & \longrightarrow & 0
\end{array}
$$

The maps α and β define extensions

$$0 \to M' \to N' \to S/I \to 0 \quad \text{and} \quad 0 \to M \to N \to S/J \to 0,$$

respectively, and one checks that $N' \simeq N$. Since $\dim M' < \dim M$, we can assume by induction that

$$\dim I/(I \cap \operatorname{Ann} M') + \dim \alpha(I \cap \operatorname{Ann} M') \leq \dim M'.$$

By Remark 7.3, this is equivalent to the inequality $\dim S/\operatorname{Ann}(N') \leq \dim N'$. Since $N \simeq N'$ we obtain the inequality $\dim S/\operatorname{Ann}(N) \leq \dim N$, and applying Remark 7.3 again, we have

$$\dim J/(J \cap \operatorname{Ann} M) + \dim \beta(J \cap \operatorname{Ann} M) \leq \dim M.$$

It remains to handle the case when $\pi\beta \colon J \to S/\mathfrak{m}$ is not surjective. In this case $\pi\beta = 0$ and so β factors through M'. We are thus in the situation $\beta \colon J \to M' \subseteq M$. By induction, we can assume

$$\dim J/(J \cap \operatorname{Ann} M') + \dim \beta(J \cap \operatorname{Ann} M') \leq \dim M'$$

and we want to show

$$\dim J/(J \cap \operatorname{Ann} M) + \dim \beta(J \cap \operatorname{Ann} M) \leq \dim M.$$

This is precisely the implication of inequalities in the hypothesis of Proposition 1.10. □

8. Contrasting the present approach with established approaches: a series of examples

As outlined in the introduction, the two primary methods used to attack the Gerstenhaber problem have been matrix-theoretic and algebro-geometric. In this section, we show how our approach produces examples which are new from each of these perspectives. Many of the results we reference in this section are only stated over $\mathbb{C}$ and so we assume here that the field $k = \mathbb{C}$.

There are a number of interesting results on the Gerstenhaber problem stated in terms of matrix-theoretic properties. For example, the three matrix analog of Gerstenhaber's theorem holds for a commuting triple of nilpotent matrices $X, Y, Z \in \mathrm{M}_d(k)$ whenever one of the three matrices has nullity strictly less than 4 (see [20], also [17]), or index strictly less than 3 [11]. Recall that the index of a nilpotent matrix A is the smallest $k \in \mathbb{N}$ for which $A^k = 0$. Via our method, it is straightforward to produce examples of triples of nilpotent matrices all of which have arbitrarily large nullity and index, as illustrated in the following example.

Example 8.1 *(Arbitrarily large nullity and index).* Let $I = (x^r, y^s, z^t)$ and let $N = S/I$. Let $X, Y, Z \in \mathrm{M}_{rst}(k)$ be the matrices of multiplication by x, y, z in N in some fixed basis of N. The kernel of the multiplication by x map is $(I : (x))/I$. Thus, the nullity of X is $\dim(I : (x))/I$, which is st. Similarly, the nullities of Y and Z are rt and rs respectively. The indexes of X, Y, and Z are r, s, and t respectively.

Notice that N is cyclic, but it already illustrates how to produce triples of matrices with arbitrarily large nullity and index where the Gerstenhaber problem is satisfied. Furthermore, since index and nullity can only increase when taking extensions, we know by Theorem 1.5 that any module M sitting within an extension

$$0 \to N \to M \to S/\mathfrak{m} \to 0$$

also yields matrices satisfying the Gerstenhaber problem, and can be made to have arbitrarily large rank and nullity, i.e. we can produce triples of matrices of the form (2.6) satisfying the Gerstenhaber problem and having arbitrarily large rank and nullity. ◇

We next show that the Weyr form of X can be chosen arbitrarily. If we furthermore wish for nullities and indexes of X, Y, Z to be large, then the Weyr form of X can be chosen nearly arbitrarily.

Example 8.2 *(Arbitrary Weyr form and arbitrarily large nullity and index).* We begin this example by showing how to produce triples of commuting matrices (X, Y, Z) where the Gerstenhaber problem holds and where X has arbitrary Weyr form. Let $n_0 \geq n_1 \geq \cdots \geq n_\ell$ be any sequence of positive integers. Let $I \subset S$ be any monomial ideal with the following property: for each $i \geq 0$, there are precisely n_i linearly independent monomials $x^i y^j z^k \in N := S/I$; such a monomial ideal I always exists. Let X, Y, Z be the matrices

of multiplication by x, y, z in N in some fixed basis of N. It is not difficult to see that $\dim(I : (x^i))/I = n_0 + n_1 + \cdots + n_{i-1}$, and so the Weyr form for X is $(n_0, n_1, \ldots, n_\ell)$.

We next show that if we wish for the nullities and indexes of X, Y, and Z to be arbitrarily large, then we can choose the Weyr form of X to be almost arbitrary. To this end, fix positive integers r, s, t, and fix any sequence of positive integers $n_0 \geq n_1 \geq \cdots \geq n_\ell$ with $n_{r-1} \geq st$. We can then find a monomial ideal I as in the previous paragraph with the additional property that $I \subseteq (x^r, y^s, z^t)$. It follows that:

(1) X, Y, and Z have nullities at least st, rt, and rs respectively, and indexes at least r, s, and t respectively,
(2) X has Weyr form $(n_0, n_1, \ldots, n_\ell)$.

Finally, applying Theorem 1.5 to an extension of $N = S/I$ of the form

$$0 \to N \to M \to S/\mathfrak{m} \to 0$$

yields matrices X, Y, Z satisfying the Gerstenhaber problem. If we write these matrices in the form (2.6), then X', Y', Z' correspond to multiplication by x, y, z on N. In particular, we see that X', Y', Z' satisfy properties (1) and (2) above, i.e. the nullities and indexes of X', Y', Z' can be chosen to be arbitrarily large and the Weyr form of X' is arbitrary subject to the one constraint $n_{r-1} \geq st$. $\diamond$

A concrete example of an extension module appears below. We provide the nullity and index of each of the resulting commuting matrices, as well as the Weyr form of X.

Example 8.3. Let $I = (x^4, y^4, z^4, x^2 y^2, x^2 z^2, y^2 z^2, xyz)$. This is a colength 25 ideal. Let $a_1, \ldots, a_8$, $b_1, \ldots, b_8$, and $c_1, \ldots, c_8$ be arbitrary elements of k. Consider the extension

$$0 \to S/I \to N \xrightarrow{\pi} S/\mathfrak{m} \to 0$$

where N is generated by the elements of S/I and an element f satisfying $\pi(f) = 1$ and

$$xf = a_1 x^2 y + a_2 x^2 z + a_3 xy^3 + a_4 x^3 y + a_5 y^3 z + a_6 x^3 z + a_7 xz^3 + a_8 yz^3,$$
$$yf = b_1 xy^3 + b_2 xy^2 + b_3 x^3 y + b_4 x^3 z + b_5 y^2 z + b_6 y^3 z + b_7 xz^3 + b_8 yz^3,$$
$$zf = c_1 xy^3 + c_2 x^3 y + c_3 x^3 z + c_4 y^3 z + c_5 xz^2 + c_6 yz^2 + c_7 xz^3 + c_8 yz^3.$$

Let X, Y, Z be the matrices given by multiplication by x, y, z on N. Then X, Y, and Z all have nullity 12 and index 4. Moreover, after choosing an appropriate basis, we can assume X is a Weyr matrix. If $a_1 = a_2 = 0$ then X has Weyr type $(12, 8, 3, 3)$; otherwise, it has Weyr type $(12, 7, 4, 3)$. $\diamond$

Next we consider results on the Gerstenhaber problem stated in algebro-geometric language. Recall that $\mathcal{C}(3, d)$ denotes the commuting scheme of triples of $d \times d$ commuting

matrices; let $\mathcal{P}(3, d)$ denote the irreducible component of $\mathcal{C}(3, d)$ consisting of those triples which can be perturbed to simultaneously diagonalizable matrices. (See [18] or [12] for further explanation.) We refer to $\mathcal{P}(3, d)$ as the *main component* of $\mathcal{C}(3, d)$. The three matrix analog of Gerstenhaber's theorem is known to hold for any commuting triple of matrices on the main component $\mathcal{P}(3, d)$ (see [18, Section 7]), but it is not known whether the three matrix analog of Gerstenhaber's theorem holds on other irreducible components of $\mathcal{C}(3, d)$.

By considering cyclic modules of length $d \geq 78$, one naturally obtains infinitely many examples of triples of commuting matrices on irreducible components other than $\mathcal{P}(3, d)$. Indeed, using the close relationship between $\mathcal{C}(3, d)$ and the Hilbert scheme of d-points in affine 3-space established in [4], one notes that if N_1 and N_2 are length d cyclic modules which are on two different components of the Hilbert scheme of d points in 3-space, then a triple of commuting matrices corresponding to N_1 is necessarily on a different component of $\mathcal{C}(3, d)$ than a triple of commuting matrices corresponding to N_2. Since the Hilbert scheme of d points in 3-space is known to be reducible for $d \geq 78$ [13,14], this allows us to produce infinitely many examples which are not on $\mathcal{P}(3, d)$ and where the three matrix analog of Gerstenhaber holds. Let us give a specific such example for $d = 96$.

Example 8.4. Consider the Hilbert scheme $\text{Hilb}^{96}(\mathbb{A}^3)$ of 96 points in affine 3-space. Let $I = \mathfrak{m}^8 + (r_1, \ldots, r_{24})$ with the following properties: each $r_i \in \mathfrak{m}$ is a homogeneous degree 7 polynomial, $r_1, \ldots, r_{24}$ generate a 24-dimensional subspace of $\mathfrak{m}^7/\mathfrak{m}^8$, and $N = S/I$ is a smooth point of $\text{Hilb}^{96}(\mathbb{A}^3)$. Let $X, Y, Z \in \text{M}_{96}(k)$ be the triple of matrices corresponding to multiplication by x, y, z on N after fixing a choice of basis for N. As shown in [13], N cannot be deformed to the coordinate ring of 96 reduced points, i.e. N does not live on the main component of $\text{Hilb}^{96}(\mathbb{A}^3)$. By our discussion preceding Example 8.4, it follows that the triple of matrices X, Y, Z is a point of $\mathcal{C}(3, 96)$ which does not live on the component $\mathcal{P}(3, 96)$.

Observe that this is an example where the three matrix analog of Gerstenhaber's theorem trivially holds from the commutative–algebraic perspective, but the triple of matrices has nullity at least 4, index at least 3, and cannot be perturbed to simultaneously diagonalizable matrices. ◇

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