EXERCISES

Tuesday

1. Given $i \in [n]$ we define $\phi_i : \mathcal{B}_{n,k} \to \{0,1\}$ by
$$\phi_i(b_1 b_2 \ldots b_n) = b_i.$$
   (a) Show that, for all $i \in [n]$, the following diagram commutes
$$\begin{array}{ccc} \mathcal{B}_{n,k} & \xrightarrow{\text{Rot}} & \mathcal{B}_{n,k} \\ & \searrow^{\phi_{i+1}} & \downarrow \phi_i \\ & & \{0,1\} \end{array}$$
   where $i+1$ is taken modulo n.

   (b) Use part (a) and the fact that ϕ_1 is k/n-mesic to prove that ϕ_i is k/n-mesic for all $i \in [n]$.

2. Consider the multiset
$$M = \{\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}\}$$
where i^{n_i} means that the element i is repeated n_i times, that is, with multiplicity n_i. Define $n = n_1 + n_2 + \cdots + n_m$. Let
$$\mathcal{P}_M = \{\pi = \pi_1 \pi_2 \ldots \pi_n \mid \pi \text{ a permutation of } M\}.$$
Then Rot acts on $\mathcal{P}_M$ by rotation just as it does on $\mathcal{B}_{n,k}$. Define a statsistic $\psi : \mathcal{P}_M \to \{0,1\}$ by
$$\psi(\pi_1 \pi_2 \ldots \pi_n) = \begin{cases} 1 & \text{if } \pi_1 = 1, \\ 0 & \text{else.} \end{cases}$$
Show that ψ is n_1/n-mesic in two ways.

   (a) By using a super orbit argument.

   (b) By using the fact that ϕ_1 from Exercise 1 is homomesic on $\mathcal{B}_{n,k}$.

3. As a special case of the action in Exercise 2 when all the multiplicities are 1, we have that Rot acts on the symmetric group $\mathfrak{S}_n$. Is the inv statistic homomesic under this action?

4. Suppose that G acts on X and so on $\binom{X}{k}$. Show that is $S \in \binom{K}{k}$ is a union of cycles of $g \in G$ then $gS = S$.

5. Prove that
$$\sum_{b \in \mathcal{B}_{n,k}} q^{\operatorname{inv} b} = \left[\begin{array}{c} n \\ k \end{array} \right]_q$$
in two ways.

   (a) By induction. See Theorem 3.2.3 in The Art of Counting for recurrences satisfied by the q-binomial coefficients.

   (b) By using the fact that $\left[\begin{array}{c} n \\ k \end{array} \right]_q$ is the generating function for integer partitions inside a $k \times (n-k)$ rectangle. See Theorem 3.2.5 in The Art of Counting for details.

Wednesday

1. Suppose that $o(\omega) = d$. Prove that when $0 < r < d$ we have
$$[r]_\omega \neq 0.$$

2. (a) Prove that the down map $d : \mathcal{A}(P) \to \mathcal{I}(P)$ is well defined, that is, if $A \in \mathcal{A}(P)$ then $d(A) \in \mathcal{I}(P)$.
   (b) Prove that the down map d is a bijection.
   (c) Repeat the first two parts for the up map u and the complement map c.

3. Prove that Row acting on C_n has a single orbit and its length is $n + 1$.

4. Let $\mathcal{O}$ be an antichain orbit of Row on a fence F, and let T be the corresponding tiling. Suppose that there is a red tile covering rows i and $i + 1$ in a column of T.
   (a) Show that if i is odd then rows i and $i + 1$ of the next column contain yellow tiles.
   (b) Show that if i is even then rows i and $i + 1$ of the previous column contain yellow tiles.

5. Consider Row acting on a fence $F = \breve{F}(\alpha_1, \alpha_2, \ldots, \alpha_t)$.
   (a) Consider an orbit $\mathcal{O}$ and corresponding tiling T. Show that for any $i \in [t]$ we have
$$\#\mathcal{O} = b_i\alpha_i + r_i + r_{i-1}.$$
   (b) Use part (a) to show that if $\alpha_i = 2$ for all i then χ is $t/2$-mesic.

6. Consider Row acting on a fence $F = \breve{F}(a, a)$.
   (a) Describe the orbits $\mathcal{O}$ of length a and show that the total number of orbits (including $\mathcal{O}'$ which has length $a + 1$) is a.
   (b) Show that we have the homometry
$$\chi(\mathcal{O}) = \begin{cases} 2a - 2 & \text{if } \mathcal{O} \neq \mathcal{O}', \\ 2a - 1 & \text{if } \mathcal{O} = \mathcal{O}'. \end{cases}$$
   (c) Show that we have the homometry
$$\hat{\chi}(\mathcal{O}) = \begin{cases} a^2 - a & \text{if } \mathcal{O} \neq \mathcal{O}', \\ a^2 + a - 1 & \text{if } \mathcal{O} = \mathcal{O}'. \end{cases}$$

Thursday

1. For all sets X, Y and elements y prove the following properties of the symmetric difference.

   (a) $(X \bigtriangleup Y) \bigtriangleup Y = X$

   (b) $X \bigtriangleup \{y\} = \begin{cases} X \cup \{y\} & \text{if } y \notin X, \\ X - \{y\} & \text{if } y \in X. \end{cases}$

2. For all posets P and $x, y \in P$ prove the following properties of toggles.

   (a) $t_x^2 = \text{id}$, the identity map.

   (b) If we have neither x covers y, nor y covers x, then $t_x t_y = t_y t_x$.

   (c) Give an example where $t_x t_y \neq t_y t_x$.

3. Complete the proof of the theorem that if $x_1, x_2, \ldots, x_n$ is a reverse linear extension of P and $I \in \mathcal{I}(P)$ then
$$\operatorname{Row} I = t_{x_n} \cdots t_{x_2} t_{x_1}(I).$$
   Letting $T_i = t_{x_i} \cdots t_{x_2} t_{x_1}(I)$, the remaining cases are:

   (a) x_i in both T_{i-1} and T_i,

   (b) x_i in neither T_{i-1} nor T_i,

   (c) x_i in T_{i-1} but not T_i.

4. Let P be a poset with $\#P = n$ and let $L : P \to [n]$ be a bijection. Show that L is a linear extension of P if and only if for every cover $x \lessdot y$ we have $L(x) < L(y)$.

5. Let P be a poset consider the Bender-Knuth toggles $s_i : \mathcal{L}(P) \to \mathcal{L}(P)$.

   (a) Show that $s_i^2 = 1$ (the identity element) for all i.

   (b) Show that if $|i - j| > 1$ then $s_i s_j = s_j s_i$.

   (c) Use (a) and (b) to show that if $|i - j| > 1$ then $s_i s_j s_i^{-1} = s_j$.

6. Show that equivariance is an equivalence relation.

7. Suppose that $(g, X) \equiv (h, Y)$ via a bijection $f : X \to Y$. Complete the proof that f induces a size-preserving bijection between the orbits of (g, X) and those of (h, Y).

Friday

1. Let ϕ^{-1} be the map defined in the proof that $(\text{Pro}, \mathcal{L}([m] \uplus [n])) \equiv (\text{Rot}, \mathcal{B}_{m+n,n})$.
   (a) Show that ϕ^{-1} is well defined.
   (b) Show that $\phi^{-1} \circ \phi$ and $\phi \circ \phi^{-1}$ are identity maps.
2. In the proof that $(\text{Pro}, \mathcal{L}([m] \times [n])) \equiv (\text{Rot}, \mathcal{B}_{m+n,n})$, provide the details of the last two cases in proving that ϕ is equivariant.
3. Show that the equivariance $(\text{Pro}, \mathcal{L}([m] \uplus [n])) \equiv (\text{Row}, \mathcal{I}([m] \times [n]))$ can be derived as a corollary to the fact that $(\text{Pro}, \mathcal{I}(P)) \equiv (\text{Row}, \mathcal{I}(P))$ for any rc-poset P.
4. Suppose we have partitions $\mu \subseteq \lambda$. Show that the map $\mathcal{L}(\lambda/\mu) \to \mathcal{C}(\mu, \lambda)$ defined in class is a bijection.
5. Suppse that P is an rc-poset. Show that there is a drawing of P in the plane $\mathbb{R}^2$ such that the following hold.
   (a) $(i, j) \in \mathbb{Z}^2$ for all $(i, j) \in P$.
   (b) $i \equiv j (\text{mod}\, 2)$ for all $(i, j) \in P$.
   (c) The rows and columns of P are $R_1, R_2, \ldots, R_s$ and $C_1, C_2, \ldots, C_t$ for some s, t, where we allow empty rows and columns.
6. Show how in the proof of Lemma 15 (Hoffman-Humphries) one can move g_{n-1} in g' so that, possibly by using conjugation, the product is of the form $g'' = \cdots g_{n-1} g_n$.
7. Prove that $(\text{Pro}, \mathcal{I}([m] \times [n])) \equiv (\text{Pro}, \mathcal{L}([m] \uplus [n]))$.