POLYNOMIALS

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Contents

Abstract.

We give eight[1] *linear bases of the ring of polynomials in n indeterminates :* Schubert polynomials, Grothendieck polynomials, flag elementary/complete functions, Demazure characters (key polynomials) for types A, B, C, D, Macdonald polynomials.

All these bases are triangular in the basis of monomials, with respect to appropriate orders. We introduce different scalar products and compute the adjoint bases of the previous polynomials.

*We provide recursions (*transition formulas*) which allow to cut these polynomials into smaller ones of the same family.*

We recover the multiplicative structure of the ring of polynomials by describing the multiplication by a single variable.

In type A we lift the Schubert polynomials and Demazure characters to the free algebra.

We recover by symmetrisation Schur functions and symmetric Macdonald polynomials in type A, and symplectic and orthogonal Schur functions in types B, C, D.

[1] In fact, counting adjoint bases and deformations, many more, but the next lucky number, 88, seems out of reach for the moment.

Introduction

Polynomials appeared since the beginnings of algebra, and it may seem that there is not much to say, nowadays, about the space of polynomials as a vector space. In the case of a single variable x, many linear bases of $\mathfrak{Pol}(x)$ other than the powers of x have been described, starting with the Newton's interpolation polynomials. The theory of orthogonal polynomials flourished during the whole XIX^e century, providing many more bases.

In the case of symmetric polynomials, Newton, again, gave a basis of products of elementary functions. The transition matrices between these functions and the monomial functions were already considered in the $XVIII^e$ century by Vandermonde in particular. Later, the chevalier Faa de Bruno, Cayley, Kostka spent much energy computing different other transition matrices. It happens in fact that there is a fundamental basis, the basis of Schur functions. A great majority of the classical problems in the theory of symmetric functions involve this basis, and leads to a combinatorics of diagrams of partitions and Young tableaux.

The picture is not so bright when one relaxes the condition of symmetry and consider $\mathfrak{Pol}(x_1, \ldots, x_n)$ in full generality. In fact, computer algebra systems like Maple or Mathematica do not know the ring of polynomials in several variables with coefficients in $\mathbb{Z}$, but only the ring $\mathbb{Z}[x_1] \otimes \mathbb{Z}[x_2] \otimes \cdots \otimes \mathbb{Z}[x_n]$. Since 40 years, geometry and representation theory provided a new incentive for describing linear bases of polynomials. The cohomology theory and the K-theory flag manifolds lead to different bases related to Schubert varieties: *Demazure characters, Schubert polynomials, Grothendieck polynomials*. Independently, the theory of orthogonal polynomials, in conjunction with root systems, developed in the direction of several variables, with the work of Koornwinder, Macdonald and many others.

In these notes, we shall mostly restrict to *Schubert polynomials, Grothendieck polynomials, Demazure characters (key polynomials), Macdonald polynomials.* These objects will be obtained using simple operators such as *Newton's divided differences* and their deformations. Such operators act on two consecutive variables at a time, say x_i, x_{+1}, and commute with multiplication with symmetric functions in x_i, x_{i+1}. Therefore, they are characterized by their action on $1, x_{i+1}$ (which is a basis of $\mathfrak{Pol}(x_i, x_{i+1})$ as a free $\mathfrak{Sym}(x_i, x_{i+1})$-module). In type A, computations will not require more than the rules figuring in the following tableau, which expresses the images of $1, x_{i+1}$ under different operators, and indicates the related polynomials.

operator	$s_i+\partial_i$	∂_i	π_i	$\widehat{\pi}_i$	$(1-x_{i+1})\partial_i$	T_i
1	1	0	1	0	1	t
x_{i+1}	x_i-1	-1	0	$-x_{i+1}$	$x_i+x_{i+1}-1$	x_i
polynoms	*Jack*	*Schubert*	*Demazure Grothendieck*	*Demazure*	$\widetilde{G}$ *Grothendieck*	*Macdonald*

To be complete, we have to add to this list the operators π_n^B, π_n^C and π_i^D in the case of key polynomials for types B, C, D, and the translation $f(x_1,\ldots,x_n) \to f(x_n/q, x_1,\ldots,x_{n-1})(x_n-1)$ in the case of Macdonald polynomials, but this does not change the picture: it is remarkable that such simple rules suffice to generate interesting families of polynomials. As a matter of fact, one also needs initial polynomials. In the case of Demazure characters, one starts with dominant monomials $x^\lambda = x_1^{\lambda_1}\ldots x_n^{\lambda_n}$, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. For Schubert polynomials, one introduces another set of variables, and one takes $Y_\lambda := \prod_{i=1..n, j=1..\lambda_i}(x_i - y_j)$. For Grothendieck polynomials, one takes $G_\lambda := \prod_{i=1..n, j=1..\lambda_i}(1 - y_j x_i^{-1})$, still with the requirement that $\lambda_1 \ge \cdots \ge \lambda_n$. In the case of Macdonald polynomials, one needs only one starting point, which is 1, because the translation operator increases degree and allows to generate polynomials of any degree.

Schubert and Macdonald polynomials can also be defined by interpolation properties. Indeed, to each $v \in \mathbb{N}^n$, one associates a *spectral vector* $\langle v\rangle^y$ (which is a permutation of $y_1, y_2, \ldots$), and another spectral vector $\langle v\rangle^{tq}$ (with components which are monomials in t, q). Now the Schubert polynomial Y_v and the Macdonald polynomial M_v are the only polynomials, up to normalization, of degree $d = |v| = v_1+\ldots+v_n$, such that

$$Y_v\Big(\langle u\rangle^y\Big) = 0 \qquad \& \qquad M_v\Big(\langle u\rangle^{tq}\Big) = 0 \ \forall u:\ |u| \le d,\ u \ne v\,.$$

it is easy to check that the vanishing conditions imply a recursion on polynomials, the image of a Schubert polynomial under ∂_i being another Schubert polynomial (when it is not 0), and the image of a Macdonald polynomial under T_i+c being another Macdonald polynomial (when choosing appropriately the constant c).

Divided differences are discrete analogues of derivatives. One can thus expect a discrete analogue of the *multivariate Taylor formula*. In the case of functions of a single variable, this discrete analogue is the *Newton interpolation formula*. In the multivariate case, the universal coefficients appearing as coefficients of products of divided differences are precisely the Schubert polynomials, and this is a direct consequence of their vanishing properties.

In these notes, we have put the emphasis on Grothendieck polynomials, because the literature on this subject is rather scanty , apart from the *Graßmannian* case, which the case where the polynomials are symmetric and can be treated as deformations of Schur functions. We do not touch the subject of Schubert polynomials for types B, C, D (see [9, 40, 42, 34, 111, 112]). They require introducing the operation $x_n \to -x_n$, while, for Demazure characters and K-theory, one must use $x_n \to x_n^{-1}$. In type A on the contrary, cohomology and K-theory can be mixed, operators like $\pi_i + \partial_i$ make sense.

Linear algebra is not enough, the ring $\mathfrak{Pol}(x_1, \ldots, x_n)$ has also a multiplicative structure that one needs to describe. We mostly restrict to multiplication by a single variable, which is enough to determine the multiplicative structure in each of the bases that we consider. Already this simple case involves fine properties of the Ehresmann-Bruhat order on the symmetric group (or on the affine symmetric group in the case of Macdonald polynomials). It is clear, however, that more work should be invested in that direction, the product of two general Schubert polynomials or two Grothendieck polynomials having, for example, many geometrical consequences . Fomin and Kirillov [33] have introduced an quadratic algebra to explain the connections between the Ehresmann-Bruhat order and Schubert calculus.

Having different bases, one may look for the relations between them. We consider the relations between Schubert and Grothendieck, Schubert and Demazure, Macdonald and key polynomials, but this subject is far from being exhausted.

Polynomials can be written uniquely as linear combination of *flag elementary functions*) (products of the type $\ldots e_i(x_1, x_2, x_3)e_j(x_1, x_2)e_k(x_1)$). Since the natural way to lift an elementary function of degree k in the free algebra is to take the sum of all strictly decreasing words of degree k, one has therefore a natural embedding, as a $\mathbb{Z}$-module, of $\mathfrak{Pol}(x_1, \ldots, x_n)$ in the free algebra on n letters. We shall rather use a distinguished quotient of the free algebra, the *plactic algebra* $\mathfrak{Plac}(n)$, quotient by the relations

$$cab \equiv acb,\ bac \equiv bca,\ baa \equiv aba,\ bab \equiv bba,\ a < b < c\,.$$

The lift of $\mathfrak{Sym}(x_1 \ldots, x_n)$ in $\mathfrak{Plac}(n)$ has now recovered its multiplicative structure, compared to the lift in the free algebra where one must have recourse to operations like shuffle instead of concatanation of words. In others words, one has an embedding of $\mathfrak{Sym}(x_1 \ldots, x_n)$ into a non-commutative algebra, and therefore any identity on symmetric polynomials translates automatically into a statement in the non-commutative world. Combinatorists will have no difficulty in going one step further in the translation and use Young tableaux, Dyck paths or non-intersecting paths instead of mere words.

Simple transpositions can be lifted to the free algebra, inducing an action of the symmetric group on the free algebra. The isobaric divided differences π_i can also be lifted to the free algebra, but they do not satisfy the braid relations any more. This does not prevent using them on the lifts of Schubert polynomials and of Demazure characters. In particular, this is the most sensible way of understanding the decomposition of Schubert polynomials as a positive sum of key polynomials.

We use two structures on the ring of polynomials in $x_1, \ldots, x_n$, with coefficients in $\mathbf{y}$: as a module over $\mathbb{Z}[\mathbf{y}]$ with basis the infinite family of Schubert polynomials

$\{Y_v(\mathbf{x}_n, \mathbf{y}) : v \in \mathbb{N}^n\}$, or as a free module of dimension $n!$ over $\mathbb{Z}[\mathbf{y}] \otimes \mathfrak{Sym}(\mathbf{x}_n)$, with basis $\{Y_v(\mathbf{x}_n, \mathbf{y}) : v \le \rho = [n{-}1, \ldots, 0]\}$. We show how to extend this finite Schubert basis in type C so as to obtain a pair of adjoint bases for $\mathfrak{Pol}(x_1^{\pm}, \ldots, x_n^{\pm})$ as a free-module under the invariants of the Weyl group of type C, but do not treat the case of type D for lack of energy.

The Hecke algebra is used in the generation of Macdonald polynomials. We say a word about the Kazhdan-Lusztig basis and its relation with key polynomials.

Chapter 1

Operators on polynomials

1.1 A, B, C, D

What are the simplest operations on vectors ?

- add
- concatanate
- transpose two consecutive components
- multiply a component by -1

Thus, acting on vectors $v \in \mathbb{Z}^n$ one has the following operators (denoted on the right) corresponding to the root systems of type A, B, C, D :

$$\begin{aligned} v\, s_i &= [\ldots, v_{i+1}, v_i, \ldots]\ ,\ 1 \le i < n, \\ v\, s_i^B = v\, s_i^C &= [\ldots, -v_i, \ldots],\ 1 \le i \le n, \\ v\, s_i^D &= [\ldots, -v_i, -v_{i-1}, \ldots],\ 2 \le i \le n\,. \end{aligned}$$

The groups generated by $s_1, \ldots, s_{n-1}$ (resp. $s_1, \ldots, s_{n-1}, s_n^B$, resp. $s_1, \ldots, s_{n-1}, s_n^D$) are the Weyl groups of type A, BC, D. We shall distinguish between B and C later, when acting on polynomials.

The orbit of the vector $[1, 2, \ldots, n]$ consists of all *permutations* of $1, \ldots, n$ for type A, all *signed permutations* for type B, C, and all signed permutations with an even number of "-" in type D. The elements of the different groups can be denoted by these objects.

The generators satisfy the *braid relations* (or *Coxeter relations*)

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \& \quad s_i s_j = s_j s_i\,,\ |i-j| \neq 1\,, \tag{1.1.1}$$

$$s_{n-1}s_n^B s_{n-1}s_n^B = s_n^B s_{n-1}s_n^B s_{n-1} \quad \& \quad s_i s_n^B = s_n^B s_i,\ i \le n-2\,, \tag{1.1.2}$$

$$s_{n-2}s_n^D s_{n-2} = s_n^D s_{n-2}s_n^D \quad \& \quad s_i s_n^B = s_n^B s_i,\ i \ne n-2\,. \tag{1.1.3}$$

An expression of an element w of the group as a product of generators is called a *decomposition*, and when this product is of minimal length, it is called a *reduced decomposition*, the length being called the *length* of w and denoted $\ell(w)$.

By recursion on n, it is easy to write reduced decompositions of the maximal element w_0 of the group for type A_{n-1}, B_n, C_n, D_n. Write $1,\dots,n$ for $s_1,\dots,s_{n-1}$ and s_n^B or s_n^D. Then w_0 admits the following reduced decompositions (that we have cut into self-explanitory blocks; read blocks from left to right)

- type A $\quad \emptyset \quad \begin{array}{|c|}\hline n{-}1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline n{-}2 & n{-}1 \\ \hline\end{array} \quad \cdots \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n{-}1 \\ \hline\end{array}$

- type BC $\quad \begin{array}{|c|}\hline n \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline n{-}1 & n \\ \hline \multicolumn{1}{c|}{} & n{-}1 \\ \cline{2-2}\end{array} \quad \cdots \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n \\ \hline \multicolumn{3}{c|}{} & \vdots \\ \cline{4-4} \multicolumn{3}{c|}{} & 2 \\ \cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \cline{4-4}\end{array}$

- type D $\quad \boxed{\binom{n-1}{n}} \quad \boxed{n{-}2\binom{n-1}{n}n{-}2} \quad \cdots \quad \boxed{1\,2\cdots n{-}2\binom{n-1}{n}n{-}2\cdots 2\,1}$

In the case of type D we have written $\binom{n-1}{n}$ for the commutative product $s_{n-1}s_n^D$.

Erase in each block a right factor[1]. The resulting decomposition is still reduced, and the group elements are in bijection with these decompositions. Therefore, the sequence of lengths of the remaining left factors *codes* the elements for type A and B. In type D, one has to use an extra symbol to distinguish between a factor $s_k\cdots s_{n-2}s_{n-1}$ and a factor $s_k\cdots s_{n-2}s_n$.

Many combinatorial properties of permutations are more easily seen by taking, in type A, another decomposition. Instead of reading the successive rows of

[1] In type D_3, for example, the right factors of the block $1\binom{2}{3}1$ are $\emptyset$, 1, 21, 31, $\binom{2}{3}1$, $1\binom{2}{3}1$.

$$\begin{array}{|c|cccc}
\cline{1-1}
n{-}1 & & & \\
\cline{1-2}
n{-}2 & \multicolumn{1}{|c|}{n{-}1} & & \\
\cline{1-3}
\cdots & \multicolumn{1}{|c|}{\cdots} & \multicolumn{1}{c|}{\cdots} & \\
\cline{1-4}
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{n{-}1} \\
\cline{1-4}
\end{array}$$

one takes the successive columns, and thus chooses the decomposition

$$(n{-}1, \ldots, 1)(n{-}1, \ldots, 2) \ldots (n{-}1) \;\leftrightarrow\; \begin{array}{|c|}\hline n{-}1 \\ \hline \vdots \\ \hline 2 \\ \hline 1 \\ \hline \end{array}\;\; \begin{array}{|c|}\hline n{-}1 \\ \hline \vdots \\ \hline 2 \\ \hline \end{array}\;\; \cdots \;\; \begin{array}{|c|}\hline n{-}1 \\ \hline \end{array}\,.$$

It is easy to check that the decompositions obtained by taking arbitrary right factors of the successive blocks (= bottom parts of the columns) are reduced and in bijection with permutations.

For example, for $n = 5$,

$$\begin{array}{cccccc} & (\bullet\, 3\, 2\, 1) & (\bullet \bullet \bullet) & (\bullet\, 3) & (4) & () \\ \textit{code} & 3 & 0 & 1 & 1 & 0 \end{array} \quad \Longleftrightarrow \quad \begin{array}{|c|}\hline \bullet \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}\; \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \end{array}\; \begin{array}{|c|}\hline \bullet \\ \hline 3 \\ \hline \end{array}\; \begin{array}{|c|}\hline 4 \\ \hline \end{array}$$

is a reduced decomposition, that we shall call *canonical reduced decomposition*, of the permutation $s_3 s_2 s_1 s_3 s_4 = [4, 1, 3, 5, 2]$, and the sequence $[3, 0, 1, 1, 0]$ of lengths of the right factors is called the *code* of the permutation. Given σ in the symmetric group $\mathfrak{S}_n$, its code $\mathfrak{c}(\sigma)$ can also be described as the vector v of components $v_i := \#\{j : j > i \;\&\; \sigma_i > \sigma_j\}$, which describes the inversions of σ. The sum $|v| = v_1 + \cdots + v_N$ is therefore the length $\ell(\sigma)$ of σ.

Having groups, one has also *group algebras*. Instead of enumerating the elements of the group W, together with their lengths one can now write a generating series which is called the *Poincaré polynomial*

$$\sum_{w \in W} q^{\ell(w)} \,.$$

From the preceding canonical decompositions, denoting by $[i]$ the q-integer $(q^i - 1)/(q - 1)$, one obtains the following Poincaré polynomials :

- type A $\quad [1]\,[2] \cdots [n] \quad ,$
- type BC $\quad [2]\,[4] \cdots [2n] \quad ,$
- type D $\quad [2]\,[4] \cdots [2n-2]\,[n] \quad .$

One can embed a Weyl group of type B_n, C_n, D_n into $\mathfrak{S}_{2n}$, as a subgroup, by sending s_i to $s_i s_{2n-i}$, $1 \le i \le n{-}1$, s_n^B and s_n^C to s_n, and s_n^D to $s_n s_{n+1} s_{n-1} s_n$. This amounts transforming a signed permutation v by $v_i \to \sigma_i = v_i$ if $v_i > 0$, and $v_i \to \sigma_i = 2n{+}1{+}v_i$ if $v_i < 0$, $i = 1, \ldots, n$, and completing by symmetry: $\sigma_{2n-i} = 2n{+}1 - \sigma_i$, thus obtaining a permutation in $\mathfrak{S}_{2n}$.

An *inversion* of a permutation $\sigma \in \mathfrak{S}_n$ is a pair (i,j) such that $i < j$ and $\sigma_i > \sigma_j$. One inherits from the embedding into $\mathfrak{S}_{2n}$, taking into account symmetries, inversions for type B, C, D. If w is sent to σ, then an inversion is a pair $i, j : 1 \le i < j \le n$ such that $\sigma_i > \sigma_j$ or such that $\sigma_i > \sigma_{2n+1-j}$. In type B, C, the indices $i : 1 \le i \le n$ such that $w_i < 0$ (equivalently, $\sigma_i > \sigma_{2n+1-i}$) are also inversions. It is easy to see by recursion that the length coincides with the number of inversions.

1.2 Reduced decompositions in type A

In type A, we shall use graphical displays to handle more easily the braid relations. A *column* is defined to be a strictly decreasing sequence of integers. Any two-dimensional display of integers must be read columnwise, from left to right, each integer i being interpreted as s_i (or some other operators indexed by integers, depending on the context). A display is *reduced* if the corresponding product of s_i's is reduced. For example, $\begin{smallmatrix} 1 & 3 & \\ & 2 & 3 \\ & 1 & 2 \end{smallmatrix}$ must be read (1)(321)(32) and interpreted as $s_1 s_3 s_2 s_1 s_3 s_2$ (which happens to be a reduced decomposition of the permutation $[4,3,2,1]$). With these conventions, the braid relation $s_1 s_2 s_1 = s_2 s_1 s_2$ becomes $\begin{smallmatrix} 1 & 2 \\ & 1 \end{smallmatrix} = \begin{smallmatrix} 2 & \\ 1 & 2 \end{smallmatrix}$. More generally, one has the following commutation lemma.

Lemma 1.2.1. *Let u, v be two columns such that uv is reduced and each letter of u also occurs in v. Then $uv = vu^+$, where u^+ is obtained from u by increasing each letter of u by* 1.

Proof. By induction on the size of u, the statement reduces to the case where $u = i$ is a single letter. Because iv is reduced, v must be of the type $v = v'\, i{+}1\, i\, v"$, with all the letters of v' bigger or equal to $i{+}2$, and all the letters of $v"$ less or equal to $i{-}1$. In that case,

$$iv = v'\, i\, i{+}1\, i\, v" = v'\, i{+}1\, i\, i{+}1\, v" = v'\, i{+}1\, i\, v"\, i{+}1\,,$$

as wanted. QED

For example, starting from the canonical reduced decomposition of $\omega = [5,4,3,2,1]$, one obtains the decompositions

$$\begin{smallmatrix} 4 & & & \\ 3 & 4 & & \\ 2 & 3 & 4 & \\ 1 & 2 & 3 & 4 \end{smallmatrix} = \begin{smallmatrix} 3 & 4 & & \\ 2 & 3 & & \\ 1 & 2 & 4 & \\ & 1 & 3 & 4 \end{smallmatrix} = \begin{smallmatrix} 3 & & & \\ 2 & 3 & 4 & \\ 1 & 2 & 3 & \\ & & 2 & \\ & & 1 & 4 \end{smallmatrix} = \begin{smallmatrix} 2 & 3 & 4 & \\ 1 & 2 & 3 & \\ & 1 & 2 & \\ & & 1 & 4 \end{smallmatrix} = \begin{smallmatrix} 2 & 3 & & \\ 1 & 2 & & \\ & 1 & 3 & 4 \\ & & & 3 \\ & & & 2 \\ & & & 1 \end{smallmatrix} = \begin{smallmatrix} 2 & & & \\ 1 & 2 & 3 & 4 \\ & & 2 & 3 \\ & & 1 & 2 \\ & & & 1 \end{smallmatrix} = \begin{smallmatrix} 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 \\ & & 1 & 2 \\ & & & 1 \end{smallmatrix}.$$

(these are 7 among the $2^8 \times 3$ reduced decompositions of ω).

1.3 Acting on polynomials with the symmetric group

Of course, considering vectors as exponents of monomials: $x^v = x_1^{v_1} x_2^{v_2} \cdots$, we get operators on polynomials: $v \to vs_i$ induces the simple transposition of x_i, x_{i+1} : $x^v \to x^{vs_i}$, and similarly for types B, D. No need to point out that addition of exponents corresponds to product of monomials, and that concatenation corresponds to a shifted product that we shall use when considering non-commutative symmetric functions:

$$u \in \mathbb{Z}^n, v \in \mathbb{Z}^m \ \to\ x^{u,v} = x_1^{u_1} \cdots x_n^{u_n} x_{n+1}^{v_1} \cdots x_{n+m}^{v_m} .$$

If v is such that $v_1 \geq \cdots \geq v_n$, then v is called *dominant* (we also say that v is a *partition*, terminal zeros being allowed). When $v_1 \leq \cdots \leq v_n$, then v is *antidominant*. The reversed vector $[v_n, \ldots, v_1]$ is denoted $v\omega$. Reordering v increasingly (resp. decreasingly) is denoted $v\uparrow$ (resp. $v\downarrow$).

Instead of vectors in $\mathbb{N}^n$, one may use permutations. We have just to reverse the correspondence seen above between permutations and *codes*[2]. One identifies $\sigma \in \mathfrak{S}_N$ and $[\sigma, N{+}1, N{+}2, \ldots]$; this corresponds to concatenating 0's to the right of the code of σ. For example, one identifies the two permutations $[2,4,1,5,3]$ and $[2,4,1,5,3,6,7,\ldots]$, as well as their codes $[1,2,0,1,0]$ and $[1,2,0,1,0,0,0,\ldots]$.

Let us consider in more details the space $\mathfrak{Pol}(x_1, x_2)$ of polynomials in $x_1^{\pm}, x_2^{\pm}$, with the simple transposition s of x_1, x_2. One remarks that s commutes with multiplication with symmetric functions in x_1, x_2 (whose space is denoted $\mathfrak{Sym}(x_1, x_2)$).

Every $f \in \mathfrak{Pol}(x_1, x_2)$ can be written

$$f = \frac{f + f^s}{2} + \frac{f - f^s}{2} = \frac{f + f^s}{2} + (x_1 - x_2)\left(\frac{f - f^s}{2(x_1{-}x_2)}\right) .$$

This means that every polynomial in $\mathfrak{Pol}(x_1, x_2)$ can be written uniquely as a linear combination of the polynomials 1 and $(x_1{-}x_2)$, with coefficients in $\mathfrak{Sym}(x_1, x_2)$. In other words $\mathfrak{Pol}(x_1, x_2)$ is a free $\mathfrak{Sym}(x_1, x_2)$-module of rank 2, and one can choose as natural bases $\{1, x_1{-}x_2\}$, $\{1, x_2\}$ or $\{1, x_1\}$.

The last choice corresponds to writing f as

$$f = \left(\frac{f - f^s}{x_1{-}x_2}\right) + x_1 \left(\frac{x_1 f^s - x_2 f}{x_1{-}x_2}\right) ,$$

the action of s being determined by

$$\{1, x_1\} \longrightarrow \{1, x_2 = -x_1 + (x_1{+}x_2)\}$$

[2]This correspondence is in fact due to Rothe (1800), who defined a planar diagram representing the inversions of a permutation.

and represented by the matrix

$$\begin{bmatrix} 1 & x_1+x_2 \\ 0 & -1 \end{bmatrix} .$$

Since a 2×2 matrix has 4 entries, this is not a big step to consider more general actions, such as

$$\{1,x_1\} \longrightarrow \{0,1\} ,$$

which, for a general polynomial f, translate into

$$f \longrightarrow (f-f^s)\frac{1}{x_1-x_2} := f\,\partial_1 ,$$

and is called *Newton divided difference.*

Similarly

$$\{1,x_2\} \to \{1,0\} \quad \text{induces} \quad f \to (x_1 f - x_2 f^s)\frac{1}{x_1-x_2} := f\,\pi_1 ,$$

$$\{1,x_1\} \to \{0,x_2\} \quad \text{induces} \quad f \to (f-f^s)\frac{x_2}{x_1-x_2} := f\,\widehat{\pi}_1 ,$$

$$\{1,x_2\} \to \{t,x_1\} \quad \text{induces} \quad f \to f\pi_1(t-1)+f^s := f\,T_1 ,$$

$$\{1,x_1\} \to \{1,tx_2\} \quad \text{induces} \quad f \to f\widehat{\pi}_1(t-1)+f^s := f\,\widehat{T}_1 ,$$

which are, respectively, two kinds of *isobaric divided differences*, and two choices of a generator of the *Hecke algebra* $\mathcal{H}_2$ of the symmetric group $\mathfrak{S}_2$.

Of course, for every pair of consecutive variables x_i, x_{i+1}, one defines similar operators $\partial_i, \pi_i, \widehat{\pi}_i, T_i, \widehat{T}_i$. The following table summarizes their action on the basis $\{1, x_{i+1}\}$ of $\mathfrak{Pol}(x_i,x_{i+1})$ as a free $\mathfrak{Sym}(x_i,x_{i+1})$-module :

operator	s_i	∂_i	π_i	$\widehat{\pi}_i$	T_i	$\widehat{T}_i$
equivalent form		$(1{-}s_i)\frac{1}{x_i-x_{i+1}}$	$x_i\partial_i$	$\partial_i x_{i+1}$	$\pi_i(t{-}1)+s_i$	$\widehat{\pi}_i(t{-}1)+s_i$
1	1	0	1	0	t	1
x_{i+1}	x_i	-1	0	$-x_{i+1}$	x_i	$x_i{+}x_{i+1}{-}tx_{i+1}$

Equivalently, these different operators are represented, in the basis $\{1, x_{i+1}\}$ of the free module $\mathfrak{Pol}(x_i,x_{i+1})$, by the matrices

$$s_i = \begin{bmatrix} 1 & x_i{+}x_{i+1} \\ 0 & -1 \end{bmatrix} ,\ \partial_i = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} ,\ \pi_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ,$$

$$\widehat{\pi}_i = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} ,\ T_i = \begin{bmatrix} t & x_i{+}x_{i+1} \\ 0 & -1 \end{bmatrix} ,\ \widehat{T}_i = \begin{bmatrix} 1 & x_i{+}x_{i+1} \\ 0 & -t \end{bmatrix} .$$

All these operators are of the type

$$D_i = \mathbf{1}\, P(x_i, x_{i+1}) + s_i\, Q(x_i, x_{i+1})\,, \tag{1.3.1}$$

with P, Q rational functions, that is to say, they are linear combination of the identity operator and a simple transposition with rational coefficients. The operators $\partial_i, \pi_i, \widehat{\pi}_i, T_i, \widehat{T}_i$ all satisfy the type *A-braid relations*

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \quad \& \quad D_i D_j = D_j D_i\,,\ |i-j| \neq 1\,.$$

One discovers that these operators also satisfy a *Hecke relation*

$$s_i s_i = 1,\ \partial_i\partial_i = 0,\ \pi_i\pi_i = \pi_i,\ \widehat{\pi}_i\widehat{\pi}_i = -\widehat{\pi}_i,\ (T_i - t)(T_i + 1) = 0,\ (\widehat{T}_i + t)(\widehat{T}_i - 1) = 0.$$

Let us check for example the relation $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$. These two operators commute with symmetric functions in x_1, x_2, x_3, and decrease degree by 3. We can take as a basis of $\mathfrak{Pol}(\mathbf{x}_n)$ (as a free module over $\mathfrak{Sym}(\mathbf{x}_3)$) the 6 monomials $\{x^v : [0,0,0] \leq v \leq [2,1,0]\}$. The first five are sent to 0 by $\partial_1\partial_2\partial_1$ and $\partial_2\partial_1\partial_2$ for degree reason, there remains only to check that $x^{210}\partial_1\partial_2\partial_1 = x^{210}\partial_2\partial_1\partial_2 = 1$ to conclude that, indeed, $\partial_1\partial_2\partial_1 = \partial_2\partial_1\partial_2$.

As a consequence of the braid relations, there exists operators $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma, T_\sigma$, indexed by permutations σ, which are obtained by taking any reduced decomposition of σ and the corresponding product of operators D_i.

1.4 Commutation relations

Divided differences satisfy Leibnitz[3] formulas[4], as easily seen from the definition:

$$fg\partial_i = f\,(g\partial_i) + f\partial_i\, g^{s_i} = g\,(f\partial_i) + g\partial_i\, f^{s_i}\ . \tag{1.4.1}$$

Iterating, one obtains the image of fg under any product of divided differences :

$$fg\,\partial_i\partial_j \ldots \partial_h = \sum_{\epsilon_i,\ldots\epsilon_h \in\{0,1\}} \Big(f\partial_i^{\epsilon_i}\partial_j^{\epsilon_j}\cdots\partial_h^{\epsilon_h}\Big)\Big(g s_i^{\epsilon_i}\partial_i^{1-\epsilon_i} s_j^{\epsilon_j}\partial_j^{1-\epsilon_j}\cdots s_h^{\epsilon_h}\partial_h^{1-\epsilon_h}\Big)\,. \tag{1.4.2}$$

It may be appropriate to use a tensor notation, the above formula being the expansion of

$$f\otimes g\,(\partial_i\otimes s_i + 1\otimes\partial_i)(\partial_j\otimes s_j + 1\otimes\partial_j)\ldots(\partial_h\otimes s_h + 1\otimes\partial_h)\,.$$

[3]For fear of being called Leinisse, Leibnitz chosed the spelling “Leibnitz” in his letters to the Académie des Sciences. We shall respect his choice.

[4]Notice that formulas are disymmetrical in f, g, one has two expressions for the image of a product.

In particular, when $g = x_i$, relations (1.4.1) may be seen as commutation relations :

$$x_i\partial_i = \partial_i x_{i+1} + 1 \quad \& \quad x_i\pi_i = \pi_i x_{i+1} + x_i \quad \& \quad x_i\widehat{\pi}_i = \widehat{\pi}_i x_{i+1} + x_{i+1}\,, \tag{1.4.3}$$

the relations $x_iT_i = T_ix_{i+1} + (t-1)x_i$ together with the trivial commutations $x_jT_i = T_ix_j$, when $|j-i| \neq 1$, being taken as axioms of the *affine Hecke algebra*[5].

Since $\widehat{\pi}_i = \partial_i x_{i+1}$, one has also $\widehat{\pi}_i x_i = \partial_i x_{i+1} x_i = x_{i+1} x_i \partial_i = x_{i+1}\pi_i$, and by iteration, reading the objects by successive columns,

$$\begin{array}{|c|}\hline \widehat{\pi}_n \\ \widehat{\pi}_{n-1} \\ \vdots \\ \widehat{\pi}_i \\ \hline \end{array}\; x_i = x_{n+1} \begin{array}{|c|}\hline \pi_n \\ \pi_{n-1} \\ \vdots \\ \pi_i \\ \hline \end{array}\;, \qquad \begin{array}{|cc|}\hline \widehat{\pi}_j & \cdots \widehat{\pi}_n \\ \vdots & \vdots \\ \widehat{\pi}_{i+j-n} & \cdots \widehat{\pi}_i \\ \hline \end{array}\; x_i = x_{j+1} \begin{array}{|cc|}\hline \pi_j & \cdots \pi_n \\ \vdots & \vdots \\ \pi_{i+j-n} & \cdots \pi_i \\ \hline \end{array}$$

We shall need some more commutation rules. For example,

$$\pi_1\pi_2\pi_3x_1x_2x_3 = x_2x_3x_4\pi_1\pi_2\pi_3 + x_1x_2x_3x_4\pi_1\pi_2 + x_1x_2x_4\pi_1\pi_3 + x_1x_3x_4\pi_2\pi_3$$

and to iterate such relations, we prefer to represent them graphically as

$$\begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \\ x_1 & x_2 & x_3 \end{array} = \begin{array}{ccc} x_2 & x_3 & x_4 \\ \boxed{1} & \boxed{2} & \boxed{3} \end{array} + x_1 \begin{array}{ccc} x_2 & x_3 & \\ \boxed{1} & \boxed{2} & \boxed{\bullet} \end{array}$$
$$+ x_1 \begin{array}{ccc} x_2 & & x_4 \\ \boxed{1} & \boxed{\bullet} & \boxed{3} \end{array} + x_1 \begin{array}{ccc} & x_3 & x_4 \\ \boxed{\bullet} & \boxed{2} & \boxed{3} \end{array}$$

In general, given an antidominant $v \in \mathbb{N}^k$, the *v-diagram* $\mathcal{V}$ is the array with columns of length $v_1, \ldots, v_n$ filled by decreasing integers as follows :

$$\mathcal{V} = \begin{array}{cccc} \cline{4-4} & & & \multicolumn{1}{|c|}{u_k} \\ \cline{2-4} & \multicolumn{1}{|c|}{u_2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\vdots} \\ \cline{2-4} & \multicolumn{1}{|c|}{\vdots} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\vdots} \\ \hline \multicolumn{1}{|c|}{u_1} & \multicolumn{1}{c|}{\vdots} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\vdots} \\ \hline \multicolumn{1}{|c|}{\vdots} & \multicolumn{1}{c|}{\vdots} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\vdots} \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{k} \\ \hline \end{array}\;,$$

where $u = v + [0, 1, \ldots, k{-}1]$, and $\pi^{\mathcal{V}}$, $\widehat{\pi}^{\mathcal{V}}$, are the columnwise-reading of $\mathcal{V}$, interpreting i as π_i or $\widehat{\pi}_i$ respectively.

Iterating the preceding commutation rules, one obtains the following lemma.

Lemma 1.4.1. *Let $v \in \mathbb{N}^k$ be antidominant, $\mathcal{V}$ its associated diagram, n be an integer such $n > v_k{+}k$. Then*

$$\pi^{\mathcal{V}} \frac{1}{x_1 \cdots x_k} = \frac{1}{x_{v_1+1} \cdots x_{v_k+k}}\, \widehat{\pi}^{\mathcal{V}}\,.$$

[5] For the *double affine Hecke algebra* for the type A, omnipresent in the work of Cherednik, one needs also to define T_0 or an affine operation.

Equivalently, multiplying by the factor $x_1 \dots x_n$ which commutes with $\pi_i, \widehat{\pi}_i$ for $i < n$, one has

$$\pi^{\mathcal{V}}\, x_{k+1} \dots x_n = \left(\frac{x_1 \dots, x_n}{x_{v_1+1} \cdots x_{v_k+k}} \right)\, \widehat{\pi}^{\mathcal{V}} . \tag{1.4.4}$$

A *punched v-diagram $\mathcal{U}$* is what results after punching holes in a v-diagram, in such a way that there are no two holes in the same row or same column, and such that no two holes occupy the South-West and North-East corner of a rectangle contained in the diagram. We forbid $\begin{array}{|c|c|c|}\hline \bullet & & \bullet \\ \hline \end{array}$, $\begin{array}{|c|}\hline \bullet \\ \hline \\ \hline \bullet \\ \hline \end{array}$, $\begin{array}{|c|c|c|c|}\hline & & & \bullet \\ \hline & & & \\ \hline \bullet & & & \\ \hline \end{array}$.

Label the rows of a v-diagram by the first entry of each row, and the columns by $v_1+1, \dots, v_n+n$. The *weight* of a punched v-diagram $\mathcal{U}$, that we denote $x^{\mathcal{U}}$, is the product $\prod_{rows} x_i \prod_{columns} x_j$, keeping the indices of punched rows, and of full columns. By $\pi^{\mathcal{U}}$ we mean the reading of $\mathcal{U}$ columnwise, from left to right, interpreting each i as π_i and ignoring the holes.

Let us give an example of a punched diagram for $v = [2, 2, 4, 4, 4]$.

$$\begin{array}{ccccc|c} x_3 & x_4 & x_7 & x_8 & x_9 & \\ & & 6 & 7 & 8 & x_6 \\ & & 5 & 6 & 7 & x_5 \\ 2 & 3 & 4 & 5 & 6 & x_2 \\ 1 & 2 & 3 & 4 & 5 & x_1 \end{array} \qquad \begin{array}{ccccc|c} x_3 & & & x_8 & & \\ & & \bullet & 7 & 8 & x_6 \\ & & 5 & 6 & 7 & \\ 2 & \bullet & 4 & 5 & 6 & x_2 \\ 1 & 2 & 3 & 4 & \bullet & x_1 \end{array}$$

coordinates and filling weight of a punched diagram

The punched 133-diagrams with two holes, together with their weights, are

$$\underset{x_1\,x_4\,x_6}{\begin{array}{ccc} & \bullet & 5 \\ & 3 & 4 \\ \bullet & 2 & 3 \end{array}} \quad \underset{x_1\,x_3\,x_6}{\begin{array}{ccc} & 4 & 5 \\ & \bullet & 4 \\ \bullet & 2 & 3 \end{array}} \quad \underset{x_1\,x_5\,x_4}{\begin{array}{ccc} & 4 & \bullet \\ & 3 & 4 \\ \bullet & 2 & 3 \end{array}} \quad \underset{x_1\,x_5\,x_3}{\begin{array}{ccc} & 4 & 5 \\ & 3 & \bullet \\ \bullet & 2 & 3 \end{array}}$$

$$\underset{x_2\,x_4\,x_3}{\begin{array}{ccc} & \bullet & 5 \\ & 3 & \bullet \\ 1 & 2 & 3 \end{array}} \quad \underset{x_2\,x_4\,x_1}{\begin{array}{ccc} & \bullet & 5 \\ & 3 & 4 \\ 1 & 2 & \bullet \end{array}} \quad \underset{x_2\,x_3\,x_1}{\begin{array}{ccc} & 4 & 5 \\ & \bullet & 4 \\ 1 & 2 & \bullet \end{array}} .$$

We shall need more commutation relations.

Lemma 1.4.2. *For any positive integer n, one has*

$$\frac{1}{x_1 \cdots x_{n+1}} \pi_1 \cdots \pi_n\, x_1 \cdots x_n = \frac{1}{x_1} \pi_1 \cdots \pi_n + \sum_{i=1}^{n} \frac{1}{x_{i+1}} \pi_1 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_n\, , \tag{1.4.5}$$

$$\pi_1 \cdots \pi_n\, x_2 \cdots x_n\, \pi_1 \cdots \pi_{n-1} = x_3 \cdots x_{n+1}\, \pi_1 \cdots \pi_n \pi_1 \cdots \pi_{n-1} . \tag{1.4.6}$$

Given $v \in \mathbb{N}^n$ antidominant, $\mathcal{V}$ its associated diagram, then

$$\pi^{\mathcal{V}}\, x_1 \cdots x_n = \sum_{\mathcal{U}} x^{\mathcal{U}}\, \pi^{\mathcal{U}}\,, \tag{1.4.7}$$

sum over all the punched v-diagrams.

Proof. The first two assertions are obtained by iterating the relation $\pi_i x_i = x_{i+1}\pi_i + x_i$. Let us check the last one by recursion, adding a top row to the diagram $\mathcal{V}$.

One therefore has to evaluate a product of the type $\pi_r \cdots \pi_m x^{\mathcal{U}}\, \pi^{\mathcal{U}}$, where the restriction of $x^{\mathcal{U}}$ to $\{x_r, \ldots, x_{m+1}\}$ is a subword of $x_r \cdots x_m$ which points out full columns in $\mathcal{U}$.

Let us first examine the case where $x_r \notin \mathcal{U}$. Taking specific values to simplify the exposition, ignoring the left part figured by hearts, one has to evaluate

$$\begin{array}{cc|c|c|c|c|c|}
 & & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & \pi_{19} \\
 & & \cdot & x_{16} & x_{17} & \cdot & x_{19} \\ \cline{3-7}
 & \heartsuit & 14 & 15 & 16 & 17 & 18 \\ \cline{3-7}
\heartsuit & \heartsuit & \bullet & 14 & 15 & 16 & 17 \\ \cline{3-7}
\heartsuit & \heartsuit & 12 & 13 & 14 & 15 & 16 \\ \cline{3-7}
\heartsuit & \heartsuit & 11 & 12 & 13 & \bullet & 15 \\ \cline{3-7}
\end{array}$$

By commutation of the incomplete columns with the complete ones, one obtains

$$\begin{array}{cc|c|c|c|c|c|c|}
 & & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & \pi_{19} & \\
 & & \cdot & x_{16} & x_{17} & \cdot & x_{19} & \\ \cline{3-7}
 & \heartsuit & \cdot & 15 & 16 & \cdot & 18 & \\ \cline{3-8}
\heartsuit & \heartsuit & \bullet & 14 & 15 & 16 & 17 & 18 \\ \cline{3-8}
\heartsuit & \heartsuit & 12 & 13 & 14 & \cdot & 16 & 17 \\ \cline{3-8}
\heartsuit & \heartsuit & 11 & 12 & 13 & \bullet & 15 & 16 \\ \cline{3-8}
\end{array}\ ,$$

from which one extracts the left factor $(\pi_{15}\pi_{16}\pi_{17}\, x_{16}x_{17}\, \pi_{15}\pi_{16})(\pi_{18}\pi_{19}\, x_{19}\, \pi_{18})$, which, thanks to (1.4.6), is equal to $x_{17}x_{18}x_{20}\, (\pi_{15}\pi_{16}\pi_{15}\pi_{17}\pi_{16})(\pi_{18}\pi_{19}\pi_{18})$. We therefore have transformed $x^{\mathcal{U}}\, \pi^{\mathcal{U}}$ into $x^{\mathcal{U}^+}\, \pi^{\mathcal{U}^+}$, where $\mathcal{U}^+$ is obtained from $\mathcal{U}$ by adding a top row.

Let us consider now the case where $x_r \in\ x^{\mathcal{U}}$. Still with the same example, one has to evaluate

$$
\begin{array}{cc|c|c|c|c|c|}
 & & \pi_{15} & \pi_{16} & \pi_{17} & \pi_{18} & \pi_{19} \\
 & & x_{15} & x_{16} & x_{17} & \cdot & x_{19} \\ \cline{3-7}
 & \heartsuit & 14 & 15 & 16 & 17 & 18 \\ \cline{3-7}
\heartsuit & \heartsuit & 13 & 14 & 15 & 16 & 17 \\ \cline{3-7}
\heartsuit & \heartsuit & 12 & 13 & 14 & 15 & 16 \\ \cline{3-7}
\heartsuit & \heartsuit & 11 & 12 & 13 & \bullet & 15 \\ \cline{3-7}
\end{array}
$$

Thanks to (1.4.5), the factor $\pi_{15}\pi_{16}\pi_{17}\,(x_{15}x_{16}x_{17})$ is equal to the sum

$$
\begin{matrix} x_{16} & x_{17} & x_{18} \\ \pi_{15} & \pi_{16} & \pi_{17} \end{matrix}
+
\begin{matrix} & & x_{17} & x_{18} \\ x_{15} & \bullet & \pi_{16} & \pi_{17} \end{matrix}
+
\begin{matrix} & x_{16} & & x_{18} \\ x_{15} & \pi_{15} & \bullet & \pi_{17} \end{matrix}
+
\begin{matrix} & x_{16} & x_{17} & \\ x_{15} & \pi_{15} & \pi_{16} & \bullet \end{matrix}\,.
$$

Adding a top row to the diagram $\mathcal{V}$ has resulted in adding a top row to $\mathcal{U}$, or adding a row with only one hole, in all possible manners such that the new hole is left of the already existing holes in the last block of columns. This finishes the proof of the lemma. QED

For example, for $v = [1,2,2]$, one has

$$
\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}\, x_1x_2x_3 = x_2x_4x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}
$$
$$
+ x_1x_4x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline \bullet & 2 & 3 \\ \hline \end{array}
+ x_2x_3x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline 1 & 2 & 3 \\ \hline \end{array}
+ x_1x_2x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & \bullet & 3 \\ \hline \end{array}
$$
$$
+ x_2x_3x_4 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & \bullet \\ \hline 1 & 2 & 3 \\ \hline \end{array}
+ x_1x_2x_4 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & 4 \\ \hline 1 & 2 & \bullet \\ \hline \end{array}
+ x_1x_3x_5 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline \bullet & 2 & 3 \\ \hline \end{array}
$$
$$
+ x_1x_3x_4 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & 3 & \bullet \\ \hline \bullet & 2 & 3 \\ \hline \end{array}
+ x_1x_2x_3 \begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bullet & 4 \\ \hline 1 & 2 & \bullet \\ \hline \end{array}\,.
$$

Comparing the relations $\pi_1x_2 = x_1\pi_1 - x_2$ and $x_1(-\widehat{\pi}_1) = (-\widehat{\pi}_1)x_2 - x_2$, one obtains a symmetry between commuting any π_σ with a polynomial f, and commuting f^ω and $\widehat{\pi}_{\omega\sigma^{-1}\omega}$:

Lemma 1.4.3. *Given n, $\sigma \in \mathfrak{S}_n$, and a polynomial $f(\mathbf{x}_n)$, suppose known the commutation*

$$\pi_\sigma f(\mathbf{x}_n) = \sum\nolimits_\zeta g_\zeta(\mathbf{x}_n)\,\pi_\zeta\,.$$

Then one has

$$f(\mathbf{x}_n^\omega)\,\widehat{\pi}_{\omega\sigma^{-1}\omega} = \sum\nolimits_\zeta (-1)^{\ell(\sigma)-\ell(\zeta)}\widehat{\pi}_{\omega\zeta^{-1}\omega}\,g_\zeta(\mathbf{x}_n^\omega)\,. \tag{1.4.8}$$

Similarly,

$$\widehat{\pi}_\sigma f(\mathbf{x}_n) = \sum\nolimits_\zeta g_\zeta(\mathbf{x}_n)\,\widehat{\pi}_\zeta$$

implies

$$f(\mathbf{x}_n^\omega)\,\pi_{\omega\sigma^{-1}\omega} = \sum\nolimits_\zeta (-1)^{\ell(\sigma)-\ell(\zeta)} \pi_{\omega\zeta^{-1}\omega}\, g_\zeta(\mathbf{x}_n^\omega)\,. \tag{1.4.9}$$

For example, for $n = 3$, one has

$$\begin{aligned} \pi_1\pi_2 x_2 &= x_3\pi_1\pi_2 + x_1\pi_1 - x_1\,,\\ x_2\widehat{\pi}_1\widehat{\pi}_2 &= \widehat{\pi}_1\widehat{\pi}_2 x_1 - \widehat{\pi}_2 x_3 - x_3 \end{aligned}$$

and

$$\begin{aligned} \widehat{\pi}_1\widehat{\pi}_2 x_2^2 &= x_3^2\widehat{\pi}_1\widehat{\pi}_2 + x_3(x_1{+}x_3)\widehat{\pi}_1 + x_3x_2\,,\\ x_2^2\pi_1\pi_2 &= \pi_1\pi_2 x_1^2 - \pi_2(x_1(x_1+x_3) - x_1x_2\,. \end{aligned}$$

Punched diagrams can also be used to describe the commutation of a product $\widehat{\pi}^{\mathcal{V}}$ with a monomial. For an antidominant $v \in \mathbb{N}^k$, $n = v_k{+}k$, $\mathcal{V}$ associated to v, let us take the monomial $x_{k+1}\dots x_n$. Transposing diagrams along the main diagonal, and introducing signs exchange the two cases. For example, for $v = [2,2]$, one has

$$\underset{x_1\ x_2}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}} = \overset{x_3\ x_4}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}} + \overset{x_2\ x_4}{\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&2\\\hline\end{array}} + \overset{x_3\ x_2}{\begin{array}{|c|c|}\hline 2&\bullet\\\hline 1&2\\\hline\end{array}} + \overset{x_1\ x_4}{\begin{array}{|c|c|}\hline 2&3\\\hline \bullet&2\\\hline\end{array}} + \overset{x_3\ x_1}{\begin{array}{|c|c|}\hline 2&3\\\hline 1&\bullet\\\hline\end{array}} + \overset{x_2\ x_1}{\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&\bullet\\\hline\end{array}}\,,$$

that is,

$$\begin{aligned} \pi_2\pi_3\pi_1\pi_2 x_1x_2 = x_3x_4\pi_2\pi_3\pi_1\pi_2 + x_2x_4\pi_3\pi_1\pi_2 + x_3x_2\pi_2\pi_1\pi_2 &\\ + x_1x_4\pi_2\pi_3\pi_2 + x_3x_1\pi_2\pi_3\pi_1 + x_1x_2\pi_3\pi_1\,,& \end{aligned}$$

while

$$\begin{aligned} \widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_1\widehat{\pi}_2 x_3x_4 = x_2x_1\widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_1\widehat{\pi}_2 - x_3x_1\widehat{\pi}_3\widehat{\pi}_1\widehat{\pi}_2 - x_4x_1\widehat{\pi}_2\widehat{\pi}_1\widehat{\pi}_2 &\\ - x_2x_3\widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_2 - x_2x_4\widehat{\pi}_2\widehat{\pi}_3\widehat{\pi}_1 + x_3x_4\widehat{\pi}_3\widehat{\pi}_1& \end{aligned}$$

can be displayed as

$$\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}\genfrac{}{}{0pt}{}{x_4}{x_3} = \genfrac{}{}{0pt}{}{x_2}{x_1}\begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array} - \genfrac{}{}{0pt}{}{x_3}{x_1}\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&2\\\hline\end{array} - \genfrac{}{}{0pt}{}{x_4}{x_1}\begin{array}{|c|c|}\hline 2&\bullet\\\hline 1&2\\\hline\end{array} - \genfrac{}{}{0pt}{}{x_2}{x_3}\begin{array}{|c|c|}\hline 2&3\\\hline \bullet&2\\\hline\end{array} - \genfrac{}{}{0pt}{}{x_2}{x_4}\begin{array}{|c|c|}\hline 2&3\\\hline 1&\bullet\\\hline\end{array} + \genfrac{}{}{0pt}{}{x_3}{x_4}\begin{array}{|c|c|}\hline \bullet&3\\\hline 1&\bullet\\\hline\end{array}\,.$$

The operators of the type (1.3.1) and preserving polynomials are characterized in [106]. They are essentially deformations of divided differences, though their explicit expression can look more frightening. For example, the operators (depending on the parameters $u_1,\dots,u_4,p,q,r$)

$$f \to f\,\frac{((qu_1+pu_3)x_i + (qu_2+pu_4))(u_3x_{i+1}+u_4)}{u_1u_4 - u_2u_3}\,\partial_i + r f^{s_i} := f\,D_i$$

do satisfy the braid relations.

1.5 Maximal operators for type A

The operators associated to the maximal permutation $\omega = [n, \dots, 1]$ play a proeminent role. In fact, they all come from the projector onto the alternating 1-dimensional representation of $\mathfrak{S}_n$, already used by Cauchy and Jacobi :

$$f \to \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} f^\sigma \ .$$

Indeed, writing Δ for the Vandermonde $\prod_{1 \le i < j \le n} (x_i - x_j)$, and x^ρ for $x_1^{n-1} x_2^{n-2} \cdots x_n^0$, one has the following proposition.

Proposition 1.5.1. *Given* $\mathbf{x}$ *of cardinality* n*, the divided differences* ∂_ω*,* π_ω *and* $\widehat{\pi}_\omega$ *verify :*

$$\partial_\omega = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma \frac{1}{\Delta} \ , \tag{1.5.1}$$

$$\pi_\omega = x^\rho \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma \frac{1}{\Delta} \ , \tag{1.5.2}$$

$$\widehat{\pi}_\omega = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} \sigma \frac{(x^\rho)^\omega}{\Delta} \ . \tag{1.5.3}$$

Proof. The monomials $x^u : u \le \rho$ are a basis of $\mathfrak{Pol}(n)$ as a free $\mathfrak{Sym}(n)$-module. They all are sent to 0 by ∂_ω or $\sum \pm \sigma \Delta^{-1}$ for degree reasons, except x^ρ which is sent to 1 (this is the only computation to make) by both operators. This proof can be adapted for π_ω and $\widehat{\pi}_\omega$. QED

We have not mentionned T_ω in the proposition, because this is not a symmetrizer, since, for $n = 2$ for example, $x_2 T_1 = x_1$. However, $x_2(T_1 + 1) = x_1 + x_2$ and $1(T_1 + 1) = t + 1$. This indicates that one has to take the Yang-Baxter deformation of T_ω for $v = [1, t, \dots, t^{n-1}]$ if one wants a symmetrizer. Indeed one has, as we shall see in more details in (1.9.8), the following symmetrizer in the Hecke algebra (as shows the last expression):

$$(T_1 + 1) \left(T_2 + \frac{t-1}{t^2-1} \right) \left(T_3 + \frac{t-1}{t^3-1} \right) \cdots (T_1 + 1) \left(T_2 + \frac{t-1}{t^2-1} \right) (T_1 + 1)$$
$$= \sum_{\sigma \in \mathfrak{S}_n} T_\sigma = \prod_{1 \le i < j \le n} (t x_i - x_j) \, \partial_\omega \ .$$

We shall frequently use the action of ∂_ω on a product $f_1(x_1) \cdots f_n(x_n)$ of functions of a single variable. In that case, the sum $\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} \left(f_1(x_1) \cdots f_n(x_n) \right)^\sigma$ is equal to the determinant $\left| f_i(x_j) \right|$, and one may view

$$f_1(x_1) \cdots f_n(x_n) \, \partial_\omega = \left| f_i(x_j) \right|_{i,j=1 \dots n} \Delta^{-1} \tag{1.5.4}$$

as the *discrete Wronskian* of the functions $f_1, \ldots, f_n$. .

Schur functions correspond to the case where $f_1, \ldots, f_n$ are powers of a variable, factorial Schur functions arise when taking instead modified powers $x(x-1)\ldots(x-k)$, while q-factorial Schur functions stem from q-powers $(x-1)(x-q)\ldots(x-q^k)$. More precisely, for any $v \in \mathbb{N}^n$, the *Schur function* $s_v(\mathbf{x}_n)$ is equal to $x^{v+\rho}\,\partial_\omega$, the *factorial Schur function* of index v is equal to

$$\Big(x_1(x_1-1)\ldots(x_1-v_1-n+2)\Big)\ldots\Big(x_n(x_n-1)\ldots(x_n-v_n+1)\Big)\,\partial_\omega$$

and the *q-factorial Schur function* of index v is equal to

$$\Big((x_1-1)(x_1-q)\ldots(x_1-q^{v_1+n-1})\Big)\ldots\Big((x_n-1)(x_n-q)\ldots(x_n-q^{v_n})\Big)\,\partial_\omega\,.$$

For example, when $n=3$ and $v=[5,2,1]$, then the corresponding factorial Schur function is equal to

$$\begin{aligned}(x_1-1)\ldots(x_1-q^6)(x_2-1)(x_2-q)(x_2-q^2)(x_3-1)\partial_{321}&\\ = \frac{1}{\Delta}\begin{vmatrix}(x_1-1)\ldots(x_1-q^6) & (x_2-1)\ldots(x_2-q^6) & (x_3-1)\ldots(x_3-q^6)\\ (x_1-1)\ldots(x_1-q^2) & (x_2-1)\ldots(x_2-q^2) & (x_3-1)\ldots(x_3-q^2)\\ x_1-1 & x_2-1 & x_3-1\end{vmatrix}&\,.\end{aligned}$$

We shall interpret it later as the specialization $y_1=1, y_2=q, y_3=q^2, \ldots$ of the Graßmannian Schubert polynomial $Y_{125}(\mathbf{x},\mathbf{y})$.

Divided differences can be defined for any pair x_i, x_j, and not only consecutive variables :

$$\partial_{i,j} : \ f \to (f - f^{\tau_{ij}})(x_i - x_j)^{-1} ,$$

τ_{ij} being the transposition of x_i, x_j. We shall need these differences to factorize ∂_ω.

Lemma 1.5.2. *Let* $n = 2m$, $\omega' = [m, \ldots, 1, 2m, \ldots, m{+}1]$, $\omega = [2m, \ldots, 1]$. *Then*

$$\partial_{\omega'} \, \partial_{1,m+1} \partial_{2,m+2} \ldots \partial_{m,2m} \, \partial_{\omega'} = (\text{-}1)^{\binom{m}{2}} m! \, \partial_\omega \, . \qquad (1.5.5)$$

Proof. The left-hand side commutes with multiplication by elements of $\mathfrak{Sym}(\mathbf{x}_n)$, and decreases degree by $\binom{m}{2}$. It is therefore sufficient to test its action on x^ρ to characterize it. One has $x^\rho \partial_{\omega'} = x^{m^m, 0^m}$, $x^\rho \partial_{\omega'} \, \partial_{1,m+1} \ldots \partial_{m,2m} = \sum x^v$, sum over all $v \in \mathbb{N}^n$ such that $v_i + v_{m+i} = m\text{-}1$, $i = 1, \ldots, m$. Each such monomial has a non-zero image under $\partial_{\omega'}$ if and only if $v_1, \ldots, v_m$ is a permutation of $[m\text{-}1, \ldots, 0]$. There are $m!$ such monomials, which each contribute to $x^{m-1, \ldots, 0, 0, \ldots, m-1} \partial_{\omega'} = (\text{-}1)^{\binom{m}{2}}$ to the right-hand side. QED

For example, for $n = 4$, one has $\partial_{2143} \partial_{13} \partial_{24} \partial_{2143} = -2 \partial_{4321}$. Many other decompositions are possible, e.g.

$$\partial_{12} \partial_{14} \partial_{34} \partial_{23} \partial_{13} \partial_{24} = \partial_{4321} = \partial_{14} \partial_{13} \partial_{24} \partial_{23} \partial_{24} \partial_{13} = \partial_{23} \partial_{13} \partial_{24} \partial_{14} \partial_{34} \partial_{12} \, .$$

1.6 Littlewood's formulas

One can combine the above operators with change of variables $x_i \to \varphi(x_i)$. The maximal divided difference ∂_ω becomes $\sum (\pm\sigma)\, \Delta(\varphi(\mathbf{x}))^{-1} = \partial_\omega \Delta(\mathbf{x})\Delta(\varphi(\mathbf{x}))^{-1}$, and it remains to find functions φ furnishing an interesting Vandermonde $\Delta(\varphi(\mathbf{x}))$.

Notice that if $\varphi(x_i) = g(x_i)/f(x_i)$, then

$$\left| f(x_i)^{n-1} \quad f(x_i)^{n-2} g(x_i) \quad \cdots \quad g(x_i)^{n-1} \right|_{i=1..n} = \prod_i (f(x_i)^{n-1}\, \Delta(\varphi(\mathbf{x}))\,.$$

Taking $f(x_i) = x_i$, $g(x_i) = 1 + x_i^k$, $k \geq 0$, and remarking that $(1 + x_i^k)/x_i - (1 + x_j^k)/x_j = (x_i^{-1} - x_j^{-1})\big(1 - x_i x_j s_{k-2}(x_i{+}x_j)\big)$, one obtains that

$$\begin{aligned} &\left| x_i^{n-1} \quad x_i^{n-2}(1{+}x_i^k) \quad \cdots \quad (1{+}x_i^k)^{n-1} \right|_{i=1...n} \Delta(\mathbf{x})^{-1} \\ &\qquad = (1 + x_2^k)(1 + x_3^k)^2 \ldots (1 + x_n)^{n-1}\, \pi_\omega \\ &\qquad = \prod_{1 \leq i < j \leq n} \Big(1 - x_i x_j s_{k-2}(x_i{+}x_j)\Big)\,, \end{aligned} \tag{1.6.1}$$

the first equality resulting from the definition of π_ω.

In the case $k = 2$, the preceding determinant can be transformed into

$$\left| x_i^{n-1} \quad x_i^{n-2}(1{+}x_i^2) \quad x_i^{n-2}(1{+}x_i^4) \quad \cdots \quad (1{+}x_i^{2n-2}) \right|_{i=1...n}\,.$$

Since the operator π_ω sends x^v, $v \in \mathbb{N}^n$ onto the Schur function $s_{v(\mathbf{x})}$, the preceding identity, still in the case $k = 2$, can be written as

$$\begin{aligned} \prod_{1 \leq i < j \leq n} \Big(1 - x_i x_j\Big) &= (1 + x_2^2)(1 + x_3^2)^2 \ldots (1 + x_n^2)^{n-1})\, \pi_\omega \\ &= (1 + x_2^2)(1 + x_3^4) \ldots (1 + x_n^{2n-2})\, \pi_\omega \\ = \sum_{\epsilon = [\epsilon_1, \ldots, \epsilon_n] \in \{0,1\}^n} (\text{-}1)^{|\epsilon|} s_{[0\epsilon_1, 2\epsilon_2, \ldots, (2n-2)\epsilon_n]}(\mathbf{x}) &= 1 + s_{02}(\mathbf{x}) + s_{004}(\mathbf{x}) + s_{024}(\mathbf{x}) + \ldots \\ &= 1 - s_{11}(\mathbf{x}) + s_{211}(\mathbf{x}) - s_{222}(\mathbf{x}) + \ldots \\ &= 1 + \sum_{r,\alpha} (-1)^{|\alpha|} s_{(\alpha|\alpha+1^r)}(\mathbf{x})\,, \end{aligned} \tag{1.6.2}$$

sum over all r, all $\alpha = [\alpha_1, \ldots, \alpha_r]$, $\alpha_1 > \alpha_2 > \ldots \alpha_r \geq 0$, using the Frobenius notation[6] for partitions.

Similar identities, known to Littlewood [122], [123, p. 78], can be obtained as easily, the reordering of the indices of the Schur functions being translated into

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properties of diagonal hooks.

$$\begin{aligned}
\prod_i (1-x_i) \prod_{1\le i<j\le n} \Bigl(1-x_ix_j\Bigr) &= (1-x_1)(1-x_2^3)\dots(1-x_n^{2n-1})\,\pi_\omega \\
&= \sum_{\epsilon=[\epsilon_1,\dots,\epsilon_n]\in\{0,1\}^n} (-1)^{|\epsilon|} s_{[\epsilon_1,3\epsilon_2,\dots,(2n-1)\epsilon_n]}(\mathbf{x}) \\
&= 1 - s_1(\mathbf{x}) - s_{03}(\mathbf{x}) + s_{13}(\mathbf{x}) - s_{005}(\mathbf{x}) + s_{105}(\mathbf{x}) + s_{035}(\mathbf{x}) - s_{135}(\mathbf{x}) + \dots \\
&= 1 - s_1(\mathbf{x}) + s_{21}(\mathbf{x}) - s_{22}(\mathbf{x}) - s_{311}(\mathbf{x}) + s_{321}(\mathbf{x}) - s_{332}(\mathbf{x}) + s_{333}(\mathbf{x}) + \dots \\
&= 1 + \sum_\alpha (-1)^{|\alpha|} s_{(\alpha|\alpha)}(\mathbf{x})\,. \qquad (1.6.3)
\end{aligned}$$

$$\begin{aligned}
\prod_{1\le i\le j\le n} \Bigl(1-x_ix_j\Bigr) &= (1-x_1^2)(1-x_2^4)\dots(1-x_n^{2n})\,\pi_\omega \\
&= \sum_{\epsilon=[\epsilon_1,\dots,\epsilon_n]\in\{0,1\}^n} (-1)^{|\epsilon|} s_{[2\epsilon_1,4\epsilon_2,\dots,2n\epsilon_n]}(\mathbf{x}) \\
&= 1 - s_2(\mathbf{x}) - s_{04}(\mathbf{x}) + s_{24}(\mathbf{x}) - s_{006}(\mathbf{x}) + s_{206}(\mathbf{x}) + s_{046}(\mathbf{x}) - s_{246}(\mathbf{x}) + \dots \\
&= 1 - s_2(\mathbf{x}) + s_{31}(\mathbf{x}) - s_{33}(\mathbf{x}) - s_{411}(\mathbf{x}) + s_{431}(\mathbf{x}) - s_{442}(\mathbf{x}) + s_{444}(\mathbf{x}) + \dots \\
&= 1 + \sum_{r,\beta} (-1)^{|\beta|} s_{(\beta+1^r|\beta)}(\mathbf{x})\,. \qquad (1.6.4)
\end{aligned}$$

$$\begin{aligned}
&\prod_{i=1}^n (1-x_i) \prod_{1\le i\le j\le n} \Bigl(1-x_ix_j\Bigr) \\
&\quad = (1-x_1)(1-x_1^2)(1-x_2^2)(1-x_2^3)\dots(1-x_n^n)(1-x_n^{n+1})\,\pi_\omega \\
&\quad = \Bigl(1 - s_1(\mathbf{x}) + s_{11}(\mathbf{x}) - s_{111}(\mathbf{x}) + \dots\Bigr) \sum_{\epsilon_i\in\{0,1\}} (-1)^{|\epsilon|} s_{[2\epsilon_1,4\epsilon_2,\dots,2n\epsilon_n]}(\mathbf{x})\,. \qquad (1.6.5)
\end{aligned}$$

One can generalize these formulas by adding letters to the alphabet $\mathbf{x}$. For example, using $\mathbf{x}\cup\{1\}$ in (1.6.2), one obtains

$$\begin{vmatrix} x_1^n & x_1^{n-1}+x_1^{n+1} & \dots & 1+x_1^{2n} \\ \vdots & \vdots & & \vdots \\ x_n^n & x_n^{n-1}+x_n^{n+1} & \dots & 1+x_n^{2n} \\ 1 & 1 & \dots & 1 \end{vmatrix} \frac{1}{\Delta(\mathbf{x})} = \prod_{i=1}^n (1-x_i)^2 \prod_{1\le i<j\le n} \Bigl(x_ix_j - 1\Bigr)\,, \qquad (1.6.6)$$

the factor $\prod(1-x_i)^2$ being due to $s_{11}(\mathbf{x}+1) = s_{11}(\mathbf{x}) + s_1(\mathbf{x})$ and $\Delta(\mathbf{x}+1) = \Delta(\mathbf{x})\prod(1-x_i)$. More variations of this type can be found in [91].

All the preceding formulas can be interpreted, in terms of λ-rings, as describing the plethysms $\Lambda^i(S^2)$ or $\Lambda^i(\Lambda^2)$, and have counterparts describing $S^i(S^2)$ or $S^i(\Lambda^2)$. Let us show that the symmetrizer π_ω still allow to describe the generating function of this last plethysms.

Proposition 1.6.1. *For a given n, one has*

$$\prod_{i\le j}(1-x_ix_j)^{-1} = \frac{1}{(1-x_1^2)(1-x_1^2x_2^2)\dots(1-x_1^2\dots x_n^2)}\pi_\omega \tag{1.6.7}$$
$$= \sum_{even\,rows} s_\lambda(\mathbf{x})$$
$$\prod_{i<j}(1-x_ix_j)^{-1} = \frac{1}{(1-x_1x_2)(1-x_1\dots x_4)(1-x_1\dots x_6)\dots}\pi_\omega \tag{1.6.8}$$
$$= \sum_{even\,columns} s_\lambda(\mathbf{x})$$
$$\prod(1-x_i)^{-1}\prod_{i<j}(1-x_ix_j)^{-1} = \frac{1}{(1-x_1)(1-x_1x_2)\dots(1-x_1\dots x_n)}\pi_\omega \tag{1.6.9}$$
$$= \sum s_\lambda(\mathbf{x})$$
$$\prod(1-x_i)^{-2}\prod_{i<j}(1-x_ix_j)^{-1} = \frac{1}{(1-x_1)^2(1-x_1x_2)^2\dots(1-x_1\dots x_n)^2}\pi_\omega \tag{1.6.10}$$
$$= \sum(\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)\dots(\lambda_n+1)s_\lambda(\mathbf{x}). \tag{1.6.11}$$

Proof. One can use induction on n, factorizing $\pi_\omega = \pi_{\omega'}\pi_\omega$, with $\omega' = [n{-}1,\dots,1]$. Thus one is left with computing the image under π_ω of the quotient of the two successive denominators appearing in the left-hand sides. For the first formula, it means computing

$$(1-x_1x_n)\dots(1-x_1x_{n-1})(1-x_n^2)(1-x_1\dots x_n)^{-1}\pi_\omega$$
$$= (1-x_ne_1+\dots+(-x_n)^ne_n))(1-x_1\dots x_n)^{-1}\pi_\omega\,,$$

$e_1,\dots,e_n$ being the elementary symmetric functions in $\mathbf{x}_n$, and therefore commuting with π_ω. Since $x_n,\dots,x_n^{n-1}$ are sent to 0, and $(-x_n)^n\pi_\omega = -x_1\dots x_n$, the above expression is equal to 1, thus proving (1.6.7). The other formulas require no more pain. Moreover, the rational functions in the right-hand sides expanding as sums of dominant monomials, the expressions in terms of Schur functions follow immediately. QED

One should try expressions more general than products of factors $(1\pm u)^{\pm1}$, with u monomial. I shall give a single example.

Lemma 1.6.2. *Given n, then*

$$\frac{1}{(1-x^1-x^2)(1-x^{22})(1-x^{111}-x^{222})(1-x^{2222})\dots}\pi_\omega$$
$$= \prod_i \frac{1}{1-x_i-x_i^2}\prod_{i<j}\frac{1}{1-x_ix_j}\,. \tag{1.6.12}$$

Proof. Let G_n be the right-hand side. Using induction on n, one has to compute $G_{n-1}/G_n\pi_\omega$. This depends on parity, and taking $n=4,5$ will be generic enough to follow the proof.

$$
\begin{aligned}
G_3/G_4\pi_{4321} &= (1-x_1x_4)(1-x_2x_4)(1-x_3x_4)(1-x_4-x_4^2)\,\pi_{4321} \\
&= \prod_{i=1}^{4}(1-x_ix_4)\pi_{4321} - x_4(1-x_1x_4)(1-x_2x_4)(1-x_3x_4)\pi_{4321}\,.
\end{aligned}
$$

One has already seen that $\prod(1-x_ix_4)\pi_{4321} = 1-x^{2222}$, and one checks that all the monomials appearing in $x_4(\dots)$ are sent to 0 under π_{4321}. In the case of $G_4/G_5\pi_{54321}$ on the contrary, the monomial $-x^{11003}$ is such that $-x^{11003}\pi_{54321} = -x^{11111}$, and thus, $G_4/G_5\pi_{54321} = 1-x^{22222}-x^{11111}$. In both cases, the resulting factor is what is required by the left-hand side of (1.6.12) to ensure equality. QED

The left-hand side of (1.6.12) expands as a positive sum of Schur functions, which multiplicities that are easily written in terms of the multiplicities of parts in the conjugate partitions.

1.7 Yang-Baxter relations

With a little more work, one can construct operators offering still more parameters.

The uniform shift $D_i \to D_i + 1$, $i = 1, \ldots, n{-}1$, destroys in general the braid relations[7]. For example,

$$(1+s_1)(1+s_2)(1+s_1) = 2 + 2s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1 \\ \neq (1+s_2)(1+s_1)(1+s_2)\ .$$

However

$$(1+s_1)(\frac{1}{2}+s_2)(1+s_1) = (1+s_2)(\frac{1}{2}+s_1)(1+s_2)\ ,$$

because both sides expand (in the group algebra of $\mathfrak{S}_3$) into the sum of all permutations.

Therefore, one abandons uniform shifts, but how to find compatible shifts like $1, 1/2, 1$?

The solution is due to Young [162], and called Yang-Baxter equation [161, 4] because Young-Yang-Baxter would be confusing.

One chooses an arbitrary vector of parameters $v = [v_1, \ldots, v_n]$ (called *spectral vector*), and each time one operates with D_i, $i = 1, \ldots, n{-}1$, one modifies accordingly the spectral vector by $v \to vs_i$.

Now, the shift to use depends only on the difference of the spectral values exchanged, with similar rules for the different varieties of operators D_i.

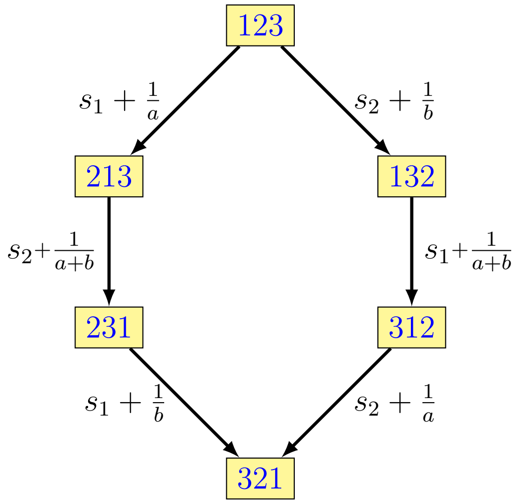
More precisely, given i, let $a = v_i$, $b = v_{i+1}$ the corresponding components of the spectral vector. Then, instead of $s_i, \partial_i, \pi_i, \widehat{\pi}_i, T_i$ respectively, one takes

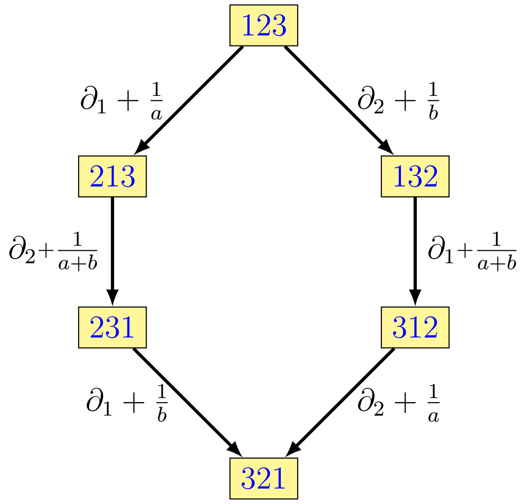
$$s_i + \frac{1}{b-a},\ \partial_i + \frac{1}{b-a},\ \pi_i + \frac{1}{b/a-1},\ \widehat{\pi}_i + \frac{1}{b/a-1},\ T_i + \frac{t-1}{b/a-1}$$

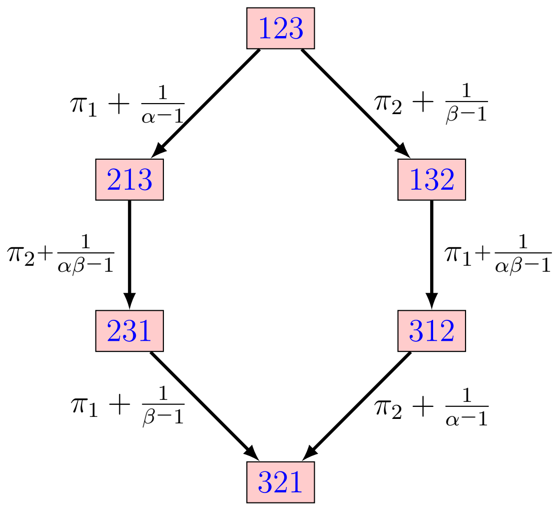
(the careful reader adds "provided $b \neq a$").

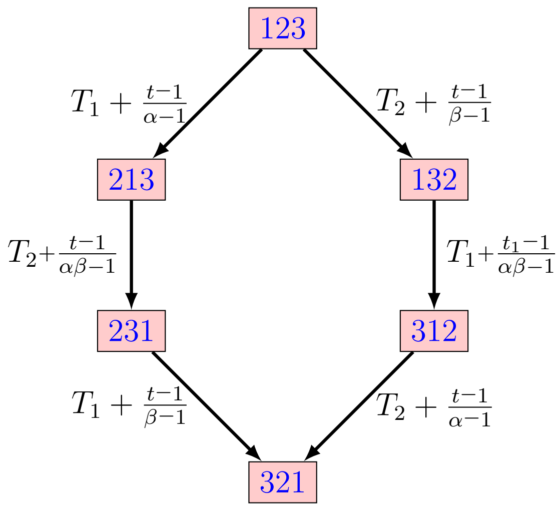
For $n = 3$, the Yang-Baxter relations for s_i, ∂_i, π_i and T_i, and a spectral vector v are, writing $v_2{-}v_1 = a$, $v_3{-}v_2 = b$, $v_2/v_1 = \alpha$, $v_3/v_2 = \beta$,

[7] it only works for $\widehat{\pi}_i \to \widehat{\pi}_i + 1 = \pi_i$.









The fact that each hexagon closes means that the two paths from top to bottom give equal elements when evaluated as products of the labels on the edges.

Thanks to the Yang-Baxter relations, to each spectral vector v, is associated a family of operators $D^v_\sigma : \sigma \in \mathfrak{S}_n$, obtained by taking products corresponding to reduced decompositions.

For example, for $\mathfrak{S}_3$, and $v = [y_1, y_2, y_3]$, one has the operators

$$\partial^v_{123} = 1\,, \quad \partial^v_{213} = \partial_1 + \frac{1}{y_2 - y_1}\,, \quad \partial^v_{132} = \partial_2 + \frac{1}{y_3 - y_2}\,,$$
$$\partial^v_{231} = \partial_1\partial_2 + \partial_2\frac{1}{y_2 - y_1} + \partial_1\frac{1}{y_3 - y_1} + \frac{1}{(y_2 - y_1)(y_3 - y_1)}\,,$$
$$\partial^v_{312} = \partial_2\partial_1 + \partial_1\frac{1}{y_3 - y_2} + \partial_2\frac{1}{y_3 - y_1} + \frac{1}{(y_3 - y_2)(y_3 - y_1)}\,,$$
$$\partial^v_{321} = \partial_1\partial_2\partial_1 + \partial_1\partial_2\frac{1}{y_3 - y_2} + \partial_2\partial_1\frac{1}{y_2 - y_1} + \partial_1\frac{1}{(y_2 - y_1)(y_3 - y_1)}$$
$$+ \partial_2\frac{1}{(y_3 - y_2)(y_3 - y_1)} + \frac{1}{(y_2 - y_1)(y_3 - y_1)(y_3 - y_2)}\,.$$

One recognizes that the product $(1 + s_1)(2^{-1} + s_2)(1 + s_1)$ corresponds to the choice $D_i = s_i$, $\sigma = [3, 2, 1]$, $v = [1, 2, 3]$. The reader will guess, and prove, that for any n, the choice $D_i = s_i$, $\sigma = \omega :== [n, \dots, 1]$, $v = [1, 2, \dots, n]$ gives

$$(1 + s_1)\Big((\frac{1}{2} + s_2)(1 + s_1)\Big)\cdots\Big((\frac{1}{n-1} + s_{n-1})\cdots(\frac{1}{2} + s_2)(1 + s_1)\Big) = \sum_{\sigma\in\mathfrak{S}_n}\sigma\,.$$

One can also twist the action of the symmetric group, and use $D_i = \partial_i + s_i$. The operators D_i still satisfy the braid relations, together with the relations $D_i^2 = 1$. Therefore, the operators $D_1, \dots, D_{n-1}$ provide a twisted action of the symmetric group on $\mathfrak{Pol}(\mathbf{x}_n)$. Since the Yang-Baxter shifts are the same for ∂_i and s_i, they can also be used for $\partial_i + s_i$. In particular, one can take the spectral vector $[1, 2, \dots, n]$.

$$\begin{tikzcd}
 & \boxed{123} \arrow[dl, "\partial_1 + s_1{+}1"'] \arrow[dr, "\partial_2 + s_2{+}1"] & \\
\boxed{213} \arrow[d, "\partial_2 + s_2{+}\frac{1}{2}"'] & & \boxed{132} \arrow[d, "\partial_1 + s_1{+}\frac{1}{2}"] \\
\boxed{231} \arrow[dr, "\partial_1 + s_1{+}1"'] & & \boxed{312} \arrow[dl, "\partial_2 + s_2{+}1"] \\
 & \boxed{321} &
\end{tikzcd}$$

Let us show that the maximal Yang-Baxter element for this choice of spectral vector is still a symmetrizer. In the case $n = 2$, one has indeed

$$\partial_1 + s_1 + 1 = (1 + x_1 - x_2)\,\partial_1\,.$$

Lemma 1.7.1. *Given n, let*

$$\square_\omega = \Big((D_1{+}1)\dots(D_{n-1}+\frac{1}{n-1})\Big)\Big((D_1{+}1)\dots(D_{n-2}+\frac{1}{n-2})\Big)\dots\Big(D_1\Big).$$

Then

$$\square_\omega = \prod_{1\le i<j\le n}(1+x_i-x_j)\,\partial_\omega\,. \tag{1.7.1}$$

Proof. Both sides of (1.7.1) commute with multiplication with symmetric functions, it is therefore sufficient to test their action on a basis of $\mathfrak{Pol}(\mathbf{x}_n)$ as a free $\mathfrak{Sym}(\mathbf{x}_n)$ module. But instead of the basis of monomials $\{x^v : v\le\rho\}$ used above, we shall use a basis of homogeneous polynomials $\{Y_v : v\le\rho\}$ in their linear span, such that each Y_v has a least one symmetry[8] in some x_i,x_{i+1}, except for $Y_{n-1,\dots,0}=x^\rho$. But using symmetric rational functions in $\mathbf{x}_n$ instead of elements of $\mathfrak{Sym}(\mathbf{x}_n)$, we can take the polynomials $Y_v\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)$ as a test cohort. All these elements, except in the case $v=\rho$, are sent to 0 by $\prod_{1\le i<j\le n}(1+x_i-x_j)\,\partial_\omega$ because the factor $\prod_{i\neq j}(1+x_i-x_j)$, being symmetrical, commutes with ∂_ω, and because $Y_v\partial_\omega=0$ for degree reasons.

On the other hand, if Y_v has the symmetry x_i,x_{i+1}, then, by commutation,

$$\begin{aligned} Y_v\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)(\partial_i+s_i+1) &= Y_v\Bigg(\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)\Bigg)(1{+}x_i{-}x_{i+1})\,\partial_i \\ &= Y_v\partial_i\Bigg(\prod_{1\le i<j\le n}(1{+}x_j{-}x_i)\Bigg)(1{+}x_i{-}x_{i+1}) = 0\,. \end{aligned}$$

Since, thanks to Yang-Baxter equation, one can factorize on the left of $\square_\omega$ any D_i+1, the image of $Y_v\prod(1+x_j-x_i)$ under $\square_\omega$ is 0 when $v\neq\rho$. Thus, both sides of (1.7.1) coincide up to multiplication by a rational symmetric function. To determine this constant, it is sufficient to see that

$$1(\partial_1{+}s_1{+}1)(\partial_1{+}s_1{+}2^{-1})\cdots = n! = \prod_{1\le i<j\le n}(1{+}x_i{-}x_j)\,\partial_\omega\,,$$

and this ensures the required equality. QED

The Yang-Baxter rules do not exhaust the realm of interesting factorized expressions. Let us take[9]

$$\begin{aligned}&((1-y_1\partial_1)(1-y_1\partial_2)\cdots(1-y_1\partial_{n-1}))\\ &\qquad\qquad((1-y_2\partial_1)(1-y_2\partial_2)\cdots(1-y_2\partial_{n-2}))\cdots((1-y_{n-1}\partial_1))\end{aligned}$$

[8]tTo show that such a basis exists is easy by induction on n, we shall see later that the Schubert polynomials $Y_v(\mathbf{x},\mathbf{0})$ satisfy such properties.

[9]This product of divided differences is the generating function of Schubert polynomials in the pair of alphabets $\mathbf{y},\mathbf{0}$, in the algebra of divided differences, also called the *Nil-Coxeter algebra* [32] see (8.2.2).

and show that this element can be used to transform the staircase monomial x^ρ, with $\rho = [n{-}1, \ldots, 0]$, into a product of factors of the type $x_i - y_j$.

Let us make the step-by-step computation for $n = 4$, displaying the factors of the polynomials planarly.

$$
\begin{array}{|c|c|c|} \hline x_3 & & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & x_1 \\ \hline \end{array}
\xrightarrow{1-y_1\partial_1}
\begin{array}{|c|c|c|} \hline x_3 & & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & x_1-y_1 \\ \hline \end{array}
\xrightarrow{1-y_1\partial_2}
\begin{array}{|c|c|c|} \hline x_3 & & \\ \hline x_2 & x_2-y_1 & \\ \hline x_1 & x_1 & x_1-y_1 \\ \hline \end{array}
$$

$$
\xrightarrow{1-y_1\partial_3}
\begin{array}{|c|c|c|} \hline x_3-y_1 & & \\ \hline x_2 & x_2-y_1 & \\ \hline x_1 & x_1 & x_1-y_1 \\ \hline \end{array}
\quad \cdots \quad
\xrightarrow{1-y_3\partial_1}
\begin{array}{|c|c|c|} \hline x_3-y_1 & & \\ \hline x_2-y_2 & x_2-y_1 & \\ \hline x_1-y_3 & x_1-y_2 & x_1-y_1 \\ \hline \end{array} .
$$

Each step is of the type $f\, x_i(1 - y\partial_i) = f\,(x_i - y)$, with f symmetrical in x_i, x_{i+1}. In final, we have obtained the function $\prod_{i+j\le 4}(x_i - y_j)$ by using only that $1\partial_i = 0$, $x_i\partial_i = 1$. This function, together with the "staircase monomial" x^{3210}, will play a key role in all the sequel. This identity can be written more compactly, still reading the planar arrays by columns (by rows still works in the present case), as

$$
\begin{array}{|c|c|c|} \hline x_3 & & \\ \hline x_2 & x_2 & \\ \hline x_1 & x_1 & x_1 \\ \hline \end{array}
\;
\begin{array}{ccc} 1-y_1\partial_1 & 1-y_1\partial_2 & 1-y_1\partial_3 \\ & 1-y_2\partial_1 & 1-y_2\partial_2 \\ & & 1-y_3\partial_1 \end{array}
=
\begin{array}{|c|c|c|} \hline x_3-y_1 & & \\ \hline x_2-y_2 & x_2-y_1 & \\ \hline x_1-y_3 & x_1-y_2 & x_1-y_1 \\ \hline \end{array} . \tag{1.7.2}
$$

1.8 Yang-Baxter bases and the Hecke algebra

The Yang-Baxter relations constitute a powerful tool to define linear bases with an explicit action of the Hecke algebra (or of the different algebras obtained by specialization, the first interesting one being the group algebra of the Weyl group).

In this section we shall change the conventions for the Hecke algebra, compared to the preceding section, to bring into prominence some symmetries.

The *Hecke algebra* $\mathcal{H}_n$ of the symmetric group $\mathfrak{S}_n$ is the algebra generated by $T_1, \ldots, T_{n-1}$ satisfying the braid relations together with the Hecke relations

$$
(T_i - t_1)(T_i - t_2) = 0\ ,\ \ i = 1, \ldots, n{-}1\,,
$$

for some fixed generic parameters t_1, t_2. For Macdonald polynomials, one takes $t_1 = t$, $t_2 = -1$. The 0-*Hecke algebra* is the specialisation $t_1 = 0$, $t_2 = -1$ of the Hecke algebra (that one can realize as the algebra generated by $\widehat{\pi}_1, \ldots, \widehat{\pi}_{n-1}$). The 00-*Hecke algebra*, also called *NilCoxeter algebra*, is the specialisation $t_1 = 0$, $t_2 = 0$. It can be realized as the algebra generated by $\partial_1, \ldots, \partial_{n-1}$. .

From the point of view of operators, the Hecke algebra is the algebra generated by operators T_i such that each T_i acts on x_i, x_{i+1} only, commutes with $\mathfrak{Sym}(x_i, x_{i+1})$, and acts on $\{1, x_{i+1}\}$ by

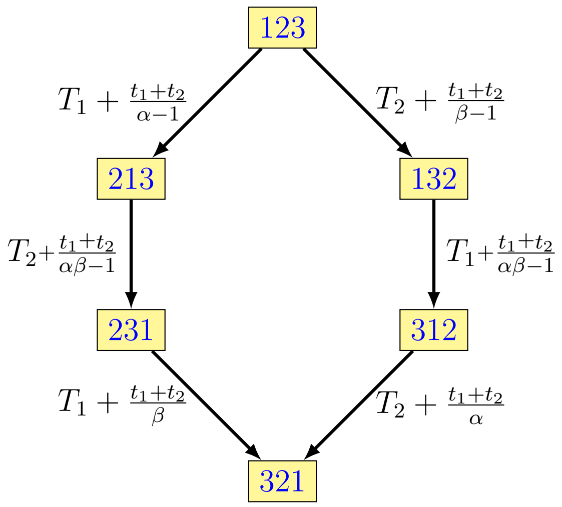
$$1\, T_i = t_1 \qquad \& \qquad x_{i+1}\, T_i = -t_2 x_i \,.$$

One has therefore $T_i = \pi_i(t_1+t_2) - s_i t_2$.

The general *Yang-Baxter equation*[10] depends on two generic parameters α, β :

$$\left(T_1 + \frac{t_1{+}t_2}{\alpha{-}1}\right)\left(T_2 + \frac{t_1{+}t_2}{\alpha\beta{-}1}\right)\left(T_1 + \frac{t_1{+}t_2}{\beta{-}1}\right) = \left(T_2 + \frac{t_1{+}t_2}{\beta{-}1}\right)\left(T_1 + \frac{t_1{+}t_2}{\alpha\beta{-}1}\right)\left(T_2 + \frac{t_1{+}t_2}{\alpha{-}1}\right) . \qquad (1.8.1)$$

Graphically, it reads



Given n, one takes an arbitrary *spectral vector* $[\gamma_1, \ldots, \gamma_n]$ of indeterminates. The *Yang-Baxter basis* $\{\mho_\sigma^\gamma : \sigma \in \mathfrak{S}_n\}$ *corresponding to* $[\gamma_1, \ldots, \gamma_n]$ is defined recursively, as follows, starting from $\mho_\sigma^\gamma = 1$ for the identity permutation:

$$\mho_{\sigma s_i}^\gamma = \mho_\sigma^\gamma \left(T_i + \frac{t_1+t_2}{\gamma_{\sigma_{i+1}}/\gamma_{\sigma_i} - 1}\right) \text{ for } \sigma_i < \sigma_{i+1} \,. \qquad (1.8.2)$$

The consistency of the definition is insured by the Yang-Baxter equation (1.8.1). Notice that arrows are reversible in the generic case. Indeed, for any i, any

[10] The Yang-Baxter relations for the group algebra of $\mathfrak{S}_n$, for the algebra of divided differences, and for the algebra of isobaric divided differences are the limits $t_1 = 1, t_2 = -1$, $t_1 = 0, t_2 = 0$, $t_1 = 1, t_2 = 0$ of (1.8.1 respectively.

$\gamma \neq 0, 1$, one has

$$\left(T_i{+}\frac{t_1{+}t_2}{\gamma{-}1}\right)\left(T_i{+}\frac{t_1{+}t_2}{\gamma^{-1}{-}1}\right) = \left(t_1{+}\frac{t_1{+}t_2}{\gamma{-}1}\right)\left(t_1{+}\frac{t_1{+}t_2}{\gamma^{-1}{-}1}\right) = -\frac{(t_1\gamma{+}t_2)(t_1{+}t_2\gamma)}{(\gamma-1)^2}\,.$$

It is clear that the set $\{\mho_\sigma^\gamma : \sigma \in \mathfrak{S}_n\}$ constitute a linear basis of $\mathcal{H}_n$, because $\mho_\sigma = T_\sigma + \sum_{v:\ell(v)<\ell(\sigma)} c_\sigma^v T_v$. Since this basis is generated using the T_i's, it is immediate to write the matrices representing the Hecke algebra in this basis. The matrices representing each T_i are made of 2×2 blocks corresponding to the spaces $\langle \mho_\sigma^\gamma, \mho_{\sigma s_i}^\gamma \rangle$. They generalize the *semi-normal representation of the symmetric group* due to Young[11].

Indeed, for $\mathfrak{S}_2$, and the spectral vector $[1, \gamma]$, the Yang-Baxter basis is $\{1, T_1 + (t_1{+}t_2)(\gamma{-}1)^{-1}\}$, and the matrix representing T_1 is given on the left, while Young's matrix (which is the limit for $\gamma = (-t_1/t_2)^g, (-t_1/t_2) \to 1$) is given on the right [138] :

$$\begin{bmatrix} -(t_1{+}t_2)(\gamma{-}1)^{-1} & -(t_1\gamma{+}t_2)(t_1{+}\gamma t_2)(\gamma{-}1)^{-2} \\ 1 & (t_1{+}t_2)(\gamma^{-1}{-}1)^{-1} \end{bmatrix}, \begin{bmatrix} -g^{-1} & 1-g^{-2} \\ 1 & g^{-1} \end{bmatrix} \tag{1.8.3}$$

One could write the similar matrices for the other types B, C, D, once the Yang-Baxter relations have been written for these types.

Irreducible representations can be obtained by either degeneration of the spectral vector, or by making the Hecke algebra act on polynomials. For example, in the case of the symmetric group, a *Specht representation* is obtained by acting on a product of Vandermondes on consecutive variables. Similarly, acting on a product of t-t Vandermondes $\prod_{a\le i<j\le b}(x_i - tx_j)$ on blocks of consecutive variables produces an irreducible representation of the Hecke algebra.

Yang-Baxter bases possess many symmetries. Let $f \to \omega \star f \star \omega$ be the automorphism of $\mathcal{H}_n$ induced by $T_\sigma \to \omega \star T_\sigma \star \omega = T_{\omega\sigma\omega}$. Then one has

Lemma 1.8.1. *The Yang-Baxter bases associated to the spectral vectors* $[y_1, \ldots, y_n]$ *and* $[y_n^{-1}, \ldots, y_1^{-1}]$ *satisfy the relations*

$$\mho_\sigma^{y_n^{-1},\ldots,y_1^{-1}} = \omega \star \mho_{\omega\sigma\omega}^{y_1,\ldots,y_n} \star \omega\ , \quad \sigma \in \mathfrak{S}_n\,. \tag{1.8.4}$$

Proof. In the case $n = 2$, this is the identity

$$T_1 + \frac{t_1+t_2}{y_1^{-1}/y_2^{-1}-1} = \omega \star \left(T_1 + \frac{t_1+t_2}{y_2/y_1-1}\right) \star \omega = T_1 + \frac{t_1+t_2}{y_2/y_1-1}\,.$$

For a general σ and i such that $\ell(\sigma s_i) \ge \ell(\sigma)$, putting $\gamma = y_{n-i}^{-1}/y_{n+1-i}^{-1}$, one has

$$\begin{aligned} \mho_\sigma^{y_n^{-1},\ldots,y_1^{-1}}\left(T_i + \frac{t_1+t_2}{\gamma-1}\right) &= (\omega \star \mho_{\omega\sigma\omega}^{y_1,\ldots,y_n} \star \omega)\left(\omega \star \left(T_i + \frac{t_1+t_2}{\gamma-1}\right) \star \omega\right) \\ &= \omega \star \left(\mho_{\omega\sigma\omega}^{y_1,\ldots,y_n}\left(T_i + \frac{t_1+t_2}{y_{n+1-i}/y_{n-i}-1}\right)\right) \star \omega\,, \end{aligned}$$

[11] We have taken generic parameters. To build general representations, one also needs blocks of size 1!.

and this proves the statement by induction on length. QED

We also need another involution $f \to \widehat{f}$ induced by

$$T_i \to \widehat{T_i} = T_i - (t_1{+}t_2)\,,\ t_1 \to \text{-}t_2\,,\ t_2 \to \text{-}t_1\,.$$

Notice that $\widehat{T}_1, \ldots, \widehat{T}_{n-1}$ satisfy the braid relations, together with the Hecke relations

$$\left(\widehat{T_i} + t_2\right)\left(\widehat{T_i} + t_1\right) = 0\,,$$

and that $T_i\widehat{T_i} = -t_1t_2$.

Let now $f \to f^\vee$ be the anti-automorphism induced by $(T_\sigma)^\vee = T_{\sigma^{-1}}$. Define a quadratic form $(\,,\,)^{\mathcal{H}}$ on $\mathcal{H}_n$ by

$$(f\,,\,g)^{\mathcal{H}} = f\,g^\vee \cap T_\omega\,, \tag{1.8.5}$$

i.e. by taking the coefficient of T_ω in the product $f\,g^\vee$.

The basis $\{\widehat{T_\sigma}\}$ is clearly the adjoint of $\{T_{\omega\sigma}\}$, i.e. one has

$$\left(T_{\omega\sigma}\,,\,\widehat{T_\zeta}\right)^{\mathcal{H}} = \delta_{\sigma,\zeta}\,,\ \sigma,\zeta \in \mathfrak{S}_n\,.$$

Testing the statements on the pairs $T_\sigma, \widehat{T}_\zeta$, one checks :

$$(T_i\,f\,,\,g)^{\mathcal{H}} = (f\,,\,T_{n-i}g)^{\mathcal{H}} \quad \& \quad (fT_i\,,\,g)^{\mathcal{H}} = (f\,,\,gT_i)^{\mathcal{H}}\,. \tag{1.8.6}$$

The quadratic form can be restricted to two-dimensional spaces, for which one has the following property of a Yang-Baxter basis.

Lemma 1.8.2. *Let $f, g \in \mathcal{H}_n$, i, γ be such that*

$$\Big(f\,,\,g\Big)^{\mathcal{H}} = 0 \quad \& \quad \Big(f(T_i + \frac{t_1+t_2}{\gamma-1})\,,\,g\Big)^{\mathcal{H}} = 1\,.$$

Then

$$\Big(f\,,\,g(T_i + \frac{t_1+t_2}{\gamma^{-1}-1})\Big)^{\mathcal{H}} = 1 \quad \& \quad \Big(f(T_i + \frac{t_1+t_2}{\gamma-1})\,,\,g(T_i + \frac{t_1+t_2}{\gamma^{-1}-1})\Big)^{\mathcal{H}} = 0\,.$$

Proof. One transfers the factor $(T_i{+}\bullet)$ to the left, and uses that $(T_i{+}(t_1{+}t_2)(\gamma\text{-}1)^{-1})(T_i{+}(t_1{+}t_2)(\gamma^{-1}\text{-}1)^{-1})$ be a scalar. QED

In other words, the two Yang-Baxter bases associated with the spectral vectors $[1,\gamma]$ and $[1,\gamma^{-1}]$ are adjoint of each other with respect to $(\,,\,)^{\mathcal{H}}$.

Combining the Yang-Baxter relations and the preceding lemma, one can evaluate scalar products of factorized elements. For example

$$\left((T_1 + \frac{t_1{+}t_2}{\alpha\text{-}1})(T_2 + \frac{t_1{+}t_2}{\alpha\beta\text{-}1})(T_1 + \frac{t_1{+}t_2}{\beta\text{-}1})\,,\,(T_1 + \frac{t_1{+}t_2}{\frac{1}{\alpha}\text{-}1})(T_2 + \frac{t_1{+}t_2}{\frac{1}{\alpha\beta}\text{-}1})\right)^{\mathcal{H}}$$

$$= \left((T_1 + \frac{t_1{+}t_2}{\alpha\text{-}1})(T_2 + \frac{t_1{+}t_2}{\alpha\beta\text{-}1})(T_1 + \frac{t_1{+}t_2}{\beta\text{-}1})(T_2 + \frac{t_1{+}t_2}{\frac{1}{\alpha\beta}\text{-}1})(T_1 + \frac{t_1{+}t_2}{\frac{1}{\alpha}\text{-}1})\right) \cap T_{321}$$

can be computed by reducing the length of the expression, replacing some factors $T_i + (t_1{+}t_2)(\gamma^{-1}{-}1)^{-1})$ by a sum of two terms $(T_i{+}c_1) + c_2$ to fit the parameters in the Yang-Baxter relations. But it is simpler to move the RHS of the scalar product to the left, obtaining

$$\left((T_2 + \frac{t_1{+}t_2}{\frac{1}{\alpha\beta}{-}1})(T_1 + \frac{t_1{+}t_2}{\frac{1}{\alpha}{-}1})(T_1 + \frac{t_1{+}t_2}{\alpha{-}1})(T_2 + \frac{t_1{+}t_2}{\alpha\beta{-}1})(T_1 + \frac{t_1{+}t_2}{\beta{-}1})\,,\,1 \right)^{\mathcal{H}}$$

which reduces to a scalar multiple of $(T_1 + (t_1{+}t_2)(\beta{-}1)^{-1},\,,\,1)^{\mathcal{H}} = 0$.

This example is some instance of a general orthogonality of Yang-Baxter bases. Let us write $T_i(a,b) = T_i + (t_1{+}t_2)(y_b y_a^{-1} - 1)^{-1}$, $\mathbf{y}\omega = [y_n \ldots, y_1]$, and first settle the case of the maximal Yang-Baxter element.

Lemma 1.8.3. *The element $\mho^{\mathbf{y}}_{\omega}$ satisfies the $n!$ equations*

$$\left(\mho^{\mathbf{y}}_{\omega}\,,\,\mho^{\mathbf{y}\omega}_{\sigma}\right)^{\mathcal{H}} = \delta_{1,\sigma}\,. \tag{1.8.7}$$

Proof. One takes a reduced decomposition $s_{i_1}s_{i_2}\ldots s_{i_r}$ of σ. Then there exists integers such that

$$\mho^{\mathbf{y}\omega}_{\sigma} = T_{i_1}(a_1,b_1)T_{i_2}(a_2,b_2)\ldots T_{i_r}(a_r,b_r)\,.$$

One can factor $\omega = \sigma^{-1}(\sigma\omega)$, and correspondingly write the maximal element as

$$\mho^{\mathbf{y}}_{\omega} = T_{n-i_1}(b_1,a_1)T_{n-i_2}(b_2,a_2)\ldots T_{n-i_r}(b_r,a_r)\ \bullet\bullet\bullet\ .$$

Tanks to (1.8.6) ,

$$\left(\mho^{\mathbf{y}}_{\omega}\,,\,T_{i_1}(a_1,b_1)\ldots T_{i_r}(a_r,b_r)\right)^{\mathcal{H}}$$
$$= \left(T_{n-i_r}(a_r,b_r)\ldots T_{n-i_1}(a_1,b_1)\,T_{n-i_1}(b_1,a_1)\ldots T_{n-i_r}(b_r,a_r)\bullet\bullet\bullet\,,\,1\right)^{\mathcal{H}}$$

is a scalar multiple of $\left(\bullet\bullet\bullet\,,\,1\right)^{\mathcal{H}}$, and therefore null if σ is not the identity permutation. QED

The following duality property of Yang-Baxter bases is given in [98, Th.5.1].

Theorem 1.8.4. *The Yang-Baxter bases associated to the spectral vectors $[y_1,\ldots,y_n]$ and $[y_n,\ldots,y_1]$ satisfy the relations*

$$\left(\mho^{\mathbf{y}}_{\sigma}\,,\,\mho^{\mathbf{y}\omega}_{\zeta}\right)^{\mathcal{H}} = \delta_{\sigma,\omega\zeta}\,, \tag{1.8.8}$$

that is, they are adjoint of each other.

Proof. When $\sigma = \omega$, this is property (1.8.7). One proves the general statement by decreasing induction on $\ell(\sigma)$, using Lemma 1.8.2. QED

Given any product

$$\Big(T_i + \alpha(t_1{+}t_2)\Big) \dots \Big(T_k + \gamma(t_1{+}t_2)\Big) = \sum_\zeta c_\zeta T_\zeta \,,$$

then the product

$$\Big(\widehat{T}_i - \alpha(t_1{+}t_2)\Big) \dots \Big(\widehat{T}_k - \gamma(t_1{+}t_2)\Big)$$

is equal to $\sum_\zeta \widehat{c}_\zeta \widehat{T}_\zeta$. This remark allows to rewrite the orthogonality relation (1.9.4). Define the coefficients c_σ^η by $\mho_\sigma^{\mathbf{y}} = \sum c_\sigma^\eta(\mathbf{y}) T_\eta$, and recall that the involution $c \to \widehat{c}$ acts by $t_1 \to -t_2$, $t_2 \to -t_1$.

Corollary 1.8.5. *Let $\sigma, \zeta \in \mathfrak{S}_n$. Then*

$$\sum_{\eta \in \mathfrak{S}_n} \widehat{c_\sigma^\eta}(y_n^{-1}, \dots, y_1^{-1})\, c_\zeta^{\omega\eta}(\mathbf{y}) \;=\; \delta_{\omega\sigma,\zeta} \tag{1.8.9}$$

$$\sum_{\eta \in \mathfrak{S}_n} \widehat{c_\sigma^\eta}(\mathbf{y})\, c_\zeta^{\eta\omega}(\mathbf{y}) \;=\; \delta_{\sigma,\zeta\omega} \tag{1.8.10}$$

Proof. One uses that

$$\mho_\zeta^{\mathbf{y}\omega} = \sum_\eta \widehat{c_\zeta^\eta}(y_n^{-1}, \dots, y_1^{-1})\, \widehat{T}_\eta \,,$$

and that the symmetry (1.8.4) translates into $c_\sigma^\eta(y_n^{-1}, \dots, y_1^{-1}) = c_{\omega\sigma\omega}^{\omega\eta\omega}(y_1, \dots, y_n)$. QED

Each of the relations (1.8.9) or (1.8.10) can be used to describe the inverse of the matrix of Yang-Baxter coefficients $\Big[c_\sigma^\eta\Big]$.

1.9 t_1t_2-Yang-Baxter bases

For $k > 1$, write

$$[k] = t_1^{k-1} - t_2t_1^{k-2} + \cdots + (-t_2)^{k-1} \quad , \quad [-k] = t_2^{k-1} - t_1t_2^{k-2} + \cdots + (-t_1)^{k-1} ,$$

with the convention that $[0] = 0$, $[1] = 1 = [-1]$. Define, for all $k \in \mathbb{Z}$, $k \neq 0$,

$$T_i(k) = T_i + \frac{t_1 + t_2}{(-t_1/t_2)^k - 1} = \begin{cases} T_i - t_2^k\,[-k]^{-1} , & k > 0 \\ T_i - t_1^{-k}[k]^{-1} , & k < 0 \end{cases} ,$$

adding $T_i(0) = T_i$.

Thus

$$T_i(1) = T_i - t_2,\ T_i(2) = T_i - \frac{t_2^2}{t_2 - t_1},\ T_i(3) = T_i - \frac{t_2^3}{t_2^2 - t_1t_2 + t_1^2} , \ldots ,$$

$$T_i(-1) = T_i - t_1,\ T_i(-2) = T_i - \frac{t_1^2}{t_1 - t_2},\ T_i(-3) = T_i - \frac{t_1^3}{t_1^2 - t_1t_2 + t_2^2} , \ldots$$

We denote $\mathbb{U}_i = T_i(1)$ and $\nabla_i = T_i(-1)$ the two factors of the Hecke relation for T_i. Acting on $\{1, x_i\}$, one checks that

$$\nabla_i = \partial_i(t_2x_i + t_1x_{i+1}) \quad \& \quad \mathbb{U}_i = (t_1x_i + t_2x_{i+1})\partial_i . \tag{1.9.1}$$

Notice that for $k > 0$ one has

$$T_i(k)T_i(-k) = -t_1t_2 \frac{[k{-}1]\,[k{+}1]}{[k]^2} , \tag{1.9.2}$$

so that, for $k \neq \pm 1$, $T_i(k)$ and $T_i(-k)$ are inverse of each other up to a scalar. More generally, the Yang-Baxter equation (1.8.1) implies that, for any $i > 0$, any $k, r \in \mathbb{Z}$ such that $k, r, k{+}r \neq 0$, one has

$$T_i(k)T_{i+1}(k{+}r)T_i(r) = T_{i+1}(r)T_i(k{+}r)T_{i+1}(k) \tag{1.9.3}$$

Taking the spectral vectors $[t_1^{n-1}, -t_1^{n-2}t_2, \ldots, (-t_2)^{n-1}]$, and $[t_2^{n-1}, -t_1t_2^{n-2}, \ldots, (-t_1)^{n-1}]$, one obtains a pair of adjoint Yang-Baxter bases which are exchanged by the involution exchanging t_1, t_2. We shall denote these two bases $\{\nabla_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\mathbb{U}_\sigma : \sigma \in \mathfrak{S}_n\}$ respectively. Here is the basis associated to the spectral vector

$[t_2^2, \text{-}t_1t_2, t_1^2]$

$$\begin{tikzcd}
 & \mathbb{U}_{123}=1 \arrow[dl, dash] \arrow[dr, equal] & \\
\mathbb{U}_{213}=T_1-t_2 \arrow[d, equal] & & \mathbb{U}_{132}=T_2-t_2 \arrow[d, dash] \\
\begin{array}{c}\mathbb{U}_{231}=T_1T_2\text{-}t_2T_2\\ -\frac{t_2^2}{t_2-t_1}T_1+\frac{t_2^3}{t_2-t_1}\end{array} \arrow[dr, dash] & & \begin{array}{c}\mathbb{U}_{312}=T_2T_1\text{-}t_2T_1\\ -\frac{t_2^2}{t_2-t_1}T_2+\frac{t_2^3}{t_2-t_1}\end{array} \arrow[dl, equal] \\
 & \mathbb{U}_{321}=\begin{array}{c}T_1T_2T_1\text{-}t_2T_2T_1\text{-}t_2T_1T_2\\ +t_2^2T_1+t_2^2T_2-t_2^3\end{array} &
\end{tikzcd}$$

and the basis associated to the spectral vector $[t_1^2, \text{-}t_1t_2, t_2^2]$

$$\begin{tikzcd}
 & \nabla_{123}=1 \arrow[dl, dash] \arrow[dr, equal] & \\
\nabla_{213}=T_1-t_1 \arrow[d, equal] & & \nabla_{132}=T_2-t_1 \arrow[d, dash] \\
\begin{array}{c}\nabla_{231}=T_1T_2\text{-}t_1T_2\\ -\frac{t_1^2}{t_1-t_2}T_1+\frac{t_1^3}{t_1-t_2}\end{array} \arrow[dr, dash] & & \begin{array}{c}\nabla_{312}=T_2T_1\text{-}t_1T_1\\ -\frac{t_1^2}{t_1-t_2}T_2+\frac{t_1^3}{t_1-t_2}\end{array} \arrow[dl, equal] \\
 & \nabla_{321}=\begin{array}{c}T_1T_2T_1\text{-}t_1T_2T_1\text{-}t_1T_1T_2\\ +t_1^2T_1+t_1^2T_2-t_1^3\end{array} &
\end{tikzcd}$$

One notices that $\nabla_{213}, \nabla_{132}, \nabla_{321}$, as well as $\mathbb{U}_{213}, \mathbb{U}_{132}, \mathbb{U}_{321}$ are quasi-idempotents. This is due to the choice of the spectral vectors.

As a special case of (1.9.4), one has

Corollary 1.9.1. *The bases $\{\mathbb{U}_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\nabla_\sigma : \sigma \in \mathfrak{S}_n\}$ are adjoint of each other. Precisely, one has*

$$\Big(\mathbb{U}_\sigma\,,\,\nabla_\zeta\Big)^{\mathcal{H}} = \delta_{\sigma,\omega\zeta}\,. \tag{1.9.4}$$

The preceding corollary furnishes in particular the transition between $\{\mathbb{U}_\sigma\}$ and $\{\nabla_\sigma\}$:

$$\mathbb{U}_\sigma = \sum_{\zeta\le\sigma}\Big(\mathbb{U}_\sigma, \mathbb{U}_{\omega\zeta}\Big)^{\mathcal{H}}\,\nabla_\zeta\,.$$

The inverse of the transition matrix is obtained by conjugation with the diagonal matrix $[(\text{-}1)^{\ell(\sigma)}, \sigma \in \mathfrak{S}_n]$. Non-zero entries correspond to pairs ζ, σ such that

$\zeta \le \sigma$ with respect to the Ehresmann-Bruhat order. Thus this matrix may be considered as "weighing" the order. We shall see later another weight given by the Kazhdan-Lusztig polynomials.

The case where σ is a Coxeter element is specially interesting since then the interval $[1,\sigma]$ is boolean. Let us just describe the expansion of $\mathbb{U}_\sigma$ when $\sigma = [2,\ldots,n,1]$.

Define a function φ on permutations as follows, starting from $\varphi([1]) = 1$. For $\sigma \in \mathfrak{S}_n$, if $\sigma_n \neq n$, then $\varphi(\sigma) = \varphi(\sigma \setminus n))$, else

$$\varphi(\sigma) = \varphi(\sigma \setminus n) \frac{[1]\,[2n{-}\sigma_{n-1}{-}1]}{[n{-}1]\,[n{-}\sigma_{n-1}]} .$$

For example, $\varphi([1,3,4,2,5]) = \varphi([1,3,4,2]) \frac{[1][10-2-1]}{[4][5-2]} = \varphi([1,3,2]) \frac{[1][7]}{[4][3]} = \varphi([1,2]) \frac{[1][7]}{[4][3]} = \frac{[2]}{[1]} \frac{[1][7]}{[4][3]}$.

Proposition 1.9.2. *For any integer n one has*

$$\mathbb{U}_{2\ldots n1} = \sum_{\zeta \le [2\ldots n1]} \varphi(\zeta)\, \nabla_\zeta .$$

Proof. Supposing known the expansion $\mathbb{U}_{[2,\ldots,n-1,1,n]} = \sum c_\nu \nabla_\nu$, one obtains

$$\begin{aligned} \mathbb{U}_{[2,\ldots,n,1]} = \mathbb{U}_{[2,\ldots,n-1,1,n]}\, T_{n-1}(n{-}1) &= \sum c_\nu \nabla_\nu \left(T_{n-1}(\nu_{n-1}{-}n) + \frac{[1][2n{-}1{-}\nu_{n-1}]}{[n{-}1][n{-}\nu_{n-1}]} \right) \\ &= \sum c_\nu \left(\nabla_{\nu s_{n-1}} + \frac{[1][2n{-}1{-}\nu_{n-1}]}{[n{-}1][n{-}\nu_{n-1}]} \nabla_\nu \right) , \end{aligned}$$

which is the required property. QED

For example,

$$\mathbb{U}_{231} = \nabla_{231} + \frac{[2]}{[1]}\nabla_{132} + \frac{[1][4]}{[2]^2}\nabla_{213} + \frac{[3]}{[1]}\nabla_{123} ,$$

$$\begin{aligned} \mathbb{U}_{2341} = &\left(\nabla_{2341} + \frac{[2]}{[1]}\nabla_{1342} + \frac{[1][4]}{[2]^2}\nabla_{2143} + \frac{[3]}{[1]}\nabla_{1243} \right) \\ &+ \left(\frac{[1][6]}{[3]^2}\nabla_{2314} + \frac{[1][4]^2}{[3][2]^2}\nabla_{2134} + \frac{[5]}{[3]}\nabla_{1324} + \frac{[4]}{[1]}\nabla_{1234} \right) . \end{aligned}$$

The maximal elements $\mathbb{U}_\omega$, ∇_ω can be expressed in terms of the maximal divided difference ∂_ω, according to [27] :

Theorem 1.9.3. *Given n, let $\omega = [n, \ldots, 1]$, $\omega' = [n{-}1, \ldots, 1]$. Then the maximal elements $\mathbb{U}_\omega$ and ∇_ω have the following expressions*

$$\begin{aligned}
\mathbb{U}_\omega &= \mathbb{U}_{\omega'}\, T_{n-1}(n{-}1) \ldots T_2(2)\, T_1(1) && (1.9.5)\\
&= \mathbb{U}_{\omega'}\Big(1 - t_2T_{n-1} + t_2^2T_{n-1}T_{n-2} - \cdots + (-t_2)^{n-1}T_{n-1}\ldots T_1\Big) && (1.9.6)\\
&= \sum_{w\in\mathfrak{S}_n} (-t_2)^{\ell(w\omega)}T_w && (1.9.7)\\
&= \prod_{1\le i<j\le n} (t_1x_i + t_2x_j)\, \partial_\omega && (1.9.8)
\end{aligned}$$

$$\begin{aligned}
\nabla_\omega &= \nabla_{\omega'}\, T_{n-1}(1{-}n) \ldots T_2(-2)\, T_1(-1) && (1.9.9)\\
&= \nabla_{\omega'}\Big(1 - t_1T_{n-1} + t_1^2T_{n-1}T_{n-2} - \cdots + (-t_1)^{n-1}T_{n-1}\ldots T_1\Big) && (1.9.10)\\
&= \sum_{w\in\mathfrak{S}_n} (-t_1)^{\ell(w\omega)}T_w && (1.9.11)\\
&= \partial_\omega \prod_{1\le i<j\le n} (t_2x_i + t_1x_j) && (1.9.12)
\end{aligned}$$

Proof. The first expression for $\mathbb{U}_\omega$ and ∇_ω result from the definition of a Yang-Baxter element, choosing the factorization $\omega = \omega'\, s_{n-1} \ldots s_1$.

By recursion on n, one sees the equivalence of (1.9.10), (1.9.11), products being reduced.

All the operators occurring in the above formulas commute with multiplication with symmetric functions in $\mathfrak{Sym}(n)$, one can characterize them by their action on the Schubert basis $\{X_\sigma(\mathbf{x}, \mathbf{0}),\ \sigma \in \mathfrak{S}_n\}$ (see [94]).

Since ∇_i, $i = 1, \ldots, n{-}1$, can be factorized on the left from the RHS of (1.9.11), (1.9.12), these two RHS annihilate all Schubert polynomials, except $X_\omega = x_1^{n-1} \ldots x_n^0$. Therefore ∂_ω is a left factor of them.

Every element of $\mathcal{H}_n$ can be written uniquely as a sum $\sum_{w\in\mathfrak{S}_n} \partial_w P_w$ with coefficients P_w which are polynomials in $x_1, \ldots, x_n$. The RHS of (1.9.10) and of $\nabla_{\omega'}(-t_1)^{n-1}\, T_{n-1}(-1)\frac{1}{-t_1} \ldots T_1(-1)\frac{1}{-t_1}$ have same coefficient in ∂_ω. This coefficient is obtained by mere commutation : $f\nabla_i = f\partial_i(t_2x_i + t_1x_{i+1}) \sim \partial_i f^{s_i}(t_2x_i + t_1x_{i+1})$, the extra term $(f\partial_i)(t_2x_i + t_1x_{i+1})$ imposed by Leibniz formula cannot contribute to a reduced decomposition of ∂_ω. Therefore, formula (1.9.10) is true if it is true for $n{-}1$. The same reasoning applies to the factorization $\nabla_\omega = \nabla_{\omega'}\, T_{n-1}(1{-}n) \ldots T_1(-1)$ which has the same coefficient in ∂_ω than $\nabla_{\omega'}\nabla_{n-1} \ldots \nabla_1$. By symmetry, the properties of ∇_ω imply similar properties of $\mathbb{U}_\omega$. QED

Let $\lambda \in \mathbb{N}^\ell$ be a composition. Put $v = [0, \lambda_1, \lambda_1{+}\lambda_2, \ldots, \lambda_1 + \cdots + \lambda_\ell]$,

$$\begin{aligned}
\Delta_\lambda^{t_1t_2} &= \prod_{k=1}^{\ell} \prod_{v_{k-1}+1\le i<j\le v_k} (t_1x_i + t_2x_j)\\
\Delta_\lambda^{t_2t_1} &= \prod_{k=1}^{\ell} \prod_{v_{k-1}+1\le i<j\le v_k} (t_2x_i + t_1x_j)\,.
\end{aligned}$$

Let ω_λ be the maximal element of the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$. Then, by direct product, one gets from the preceding theorem

$$\mathbb{U}_{\omega_\lambda} = \Delta_\lambda^{t_1 t_2}\, \partial_{\omega_\lambda} \tag{1.9.13}$$

$$\nabla_{\omega_\lambda} = \partial_{\omega_\lambda}\; \Delta_\lambda^{t_2 t_1}\,. \tag{1.9.14}$$

For example, for $\lambda = [3,2]$, and $\mu \in \mathbb{N}^5$, the image of x^μ under

$$\nabla_{32154} = \sum_{\sigma \in \mathfrak{S}_{32}} (-t_1)^{\ell(\sigma)} T_\sigma = \partial_{32154} \Delta_{32}^{t_2 t_1}$$

is equal to the Schur function $s_{\mu - 43210}(\mathbf{x}_5)$ times $\Delta_{32}^{t_2 t_1}$.

1.10 B, C, D action on polynomials

As for type A, one transfers operations on vectors to operations on polynomials by acting on the exponents of monomials.

Thus, $s_i^B = s_i^C$ acts on x_i only by $x_i \to x_i^{-1}$, and s_i^D acts on x_{i-1}, x_i by $x_i \to x_{i-1}^{-1}$, $x_{i-1} \to x_i^{-1}$.

We also have divided differences, this time with a difference between types B and C :

$$\partial_i^B := (1 - s_i^B)\frac{1}{x_i^{1/2} - x_i^{-1/2}},\ \pi_i^B = x_i^{1/2}\,\partial_i^B,\ \widehat{\pi}_i^B = \partial_i^C x_i^{-1/2},\ i = 1 \ldots n\,.$$

$$\partial_i^C := (1 - s_i^C)\frac{1}{x_i - x_i^{-1}},\ \pi_i^C = x_i\,\partial_i^C,\ \widehat{\pi}_i^C = \partial_i^C x_i^{-1},\ i = 1 \ldots n\,.$$

$$\partial_i^D := (1 - s_i^D)\frac{1}{x_{i-1}^{-1} - x_i},\ \pi_i^D = \left(1 - s_i^D \frac{1}{x_{i-1}x_i}\right)\frac{1}{1 - \frac{1}{x_{i-1}x_i}},$$
$$\widehat{\pi}_i^D = (1 - s_i^D)\frac{1}{x_{i-1}x_i - 1},\ i = 2 \ldots n\,.$$

As in type A, the above operators can be characterized in a simple manner, taking into account symmetries. For example, in type C, the divided differences $\partial_i^C, \pi_i^C, \widehat{\pi}_i^C$ commute with multiplication with functions symmetrical in $x_i, 1/x_i$ (which are functions of the variable $x_i^\bullet = x_i + x_i^{-1}$). It suffices to give their action on the basis $\{1, x_i\}$ of $\mathfrak{Pol}(x_i^\pm)$ as a free $\mathfrak{Pol}(x_i^\bullet)$ module :

$$\begin{array}{c|ccc} & \partial_i^C & \pi_i^C & \widehat{\pi}_i^C \\ 1 & 0 & 1 & 0 \\ x_i & 1 & x_i + x_i^{-1} & x_i^{-1} \end{array}\ .$$

For type D, say for $i = 2$, the space $\mathfrak{Pol}(x_1^\pm, x_2^\pm)$ is a free module of rank 4 over the D-invariants. One can take as a basis $x_1, x_2, x_2^{-1}, x_1^{-1}$, on which the divided differences act as follows :

$$\begin{array}{c|ccc} & \partial_2^D & \pi_2^D & \widehat{\pi}_2^D \\ x_1 & x_1x_2^{-1} & x_1 + x_2^{-1} & x_2^{-1} \\ x_2 & 1 & x_2 + x_1^{-1} & x_1^{-1} \\ x_2^{-1} & -x_1x_2^{-1} & 0 & -x_2^{-1} \\ x_1^{-1} & -1 & 0 & -x_1^{-1} \end{array}\ .$$

For type $\heartsuit = B, C$, the divided differences for two consecutive indices, say

1, 2, satisfy braid relations[12] :

$$\begin{aligned}\pi_1\,\pi_2^\heartsuit\,\pi_1\,\pi_2^\heartsuit &= \pi_2^\heartsuit\,\pi_1\,\pi_2^\heartsuit\,\pi_1\\ \widehat{\pi}_1\,\widehat{\pi}_2^\heartsuit\,\widehat{\pi}_1\,\widehat{\pi}_2^\heartsuit &= \widehat{\pi}_2^\heartsuit\,\widehat{\pi}_1\,\widehat{\pi}_2^\heartsuit\,\widehat{\pi}_1\,,\end{aligned}$$

but

$$\partial_2^C\partial_1\partial_2^C\partial_1 \neq \partial_1\partial_2^C\partial_1\partial_2^C\,.$$

In type D, for $i \neq n-2$, then π_n^D commutes with π_i, and $\widehat{\pi}_n^D$ commutes with $\widehat{\pi}_i$, and

$$\pi_n^D\,\pi_{n-1}\,\pi_n^D = \pi_{n-1}\,\pi_n^D\,\pi_{n-1} \qquad \& \qquad \widehat{\pi}_n^D\,\widehat{\pi}_{n-1}\,\widehat{\pi}_n^D = \widehat{\pi}_{n-1}\,\widehat{\pi}_n^D\,\widehat{\pi}_{n-1}\,.$$

Notice that the squares satisfy the same relations than in type A :

$$\partial_i^\heartsuit\partial_i^\heartsuit = 0 \quad \& \quad \pi_i^\heartsuit\pi_i^\heartsuit = \pi_i^\heartsuit \quad \& \quad \widehat{\pi}_i^\heartsuit\widehat{\pi}_i^\heartsuit = -\widehat{\pi}_i^\heartsuit\,,\ \heartsuit = B, C, D\,.$$

Choosing as generators $s_1,\ldots,s_{n-1}, s_n^\heartsuit$, $\heartsuit = B, C, D$, one obtains by reduced products operators $\pi_w^\heartsuit$ and $\widehat{\pi}_w^\heartsuit$ indexed by the elements of the group. Of special importance are those corresponding to $w_0^\heartsuit$.

Proposition 1.10.1. *Let n be an integer, $\rho = [n{-}1,\ldots,0]$, $x_i^\bullet = x_i + x_i^{-1}$, $i = 1,\ldots,n$. Write $\partial_i^\bullet$ for the divided differences relative to the alphabet $\mathbf{x}^\bullet = \{x_1^\bullet,\ldots,x_n^\bullet\}$. Then*

$$\begin{aligned}
\pi_{w_0}^B &= x^\rho\,\pi_1^B\cdots\pi_n^B\,\partial_\omega^\bullet = x^\rho\,\partial_\omega^\bullet\,\pi_1^B\cdots\pi_n^B && (1.10.1)\\
\widehat{\pi}_{w_0}^B &= \widehat{\pi}_1^B\cdots\widehat{\pi}_n^B\,\partial_\omega^\bullet\,x^{-\rho} = \partial_\omega^\bullet\,\widehat{\pi}_1^B\cdots\widehat{\pi}_n^B\,x^{-\rho} && (1.10.2)\\
\pi_{w_0}^C &= x^\rho\,\pi_1^C\cdots\pi_n^C\,\partial_\omega^\bullet = x^\rho\,\partial_\omega^\bullet\,\pi_1^C\cdots\pi_n^C && (1.10.3)\\
&= x^{\rho+1^n}\sum(-1)^{\ell(w)}w\cdot\prod_{1\le i<j\le n}(x_i^\bullet - x_j^\bullet)^{-1}\prod_{1\le i\le n}(x_i - x_i^{-1})^{-1} && (1.10.4)\\
\widehat{\pi}_{w_0}^C &= \widehat{\pi}_1^C\cdots\widehat{\pi}_n^C\,\partial_\omega^\bullet\,x^{-\rho} = \partial_\omega^\bullet\,\widehat{\pi}_1^C\cdots\widehat{\pi}_n^C\,x^{-\rho} && (1.10.5)
\end{aligned}$$

Notice that $\partial_\omega^\bullet = \partial_\omega\prod_{i<j\le n}(1 - x_i^{-1}x_j^{-1})^{-1}$ commutes with $\pi_1^B\cdots\pi_n^B$ and $\pi_1^C\cdots\pi_n^C$ because $x_i^\bullet$ commutes with π_i^B and π_i^C.

Consequently, images of $\pi_{w_0^B}$ and $\pi_{w_0^C}$ can be written as symmetric functions of $\mathbf{x}_n^\bullet$. For example, for $n = 3$, the image of x^{310} under $\pi_{w_0}^C$ is equal to

$$\begin{aligned}
&\left(x_1^5\pi_1^C\right)\left(x_2^2\pi_2^C\right)\left(x_3^0\pi_3^C\right)\\
&\qquad = \left((x_1^\bullet)^5 - 4(x_1^\bullet)^3 + 3x_1^\bullet\right)\left((x_2^\bullet)^2 - 1\right)\,\partial_{321}^\bullet\\
&\qquad\qquad = s_{310}(\mathbf{x}_3^\bullet) - 4s_{110}(\mathbf{x}_3^\bullet) - 3s_{000}(\mathbf{x}_3^\bullet)\,,
\end{aligned}$$

[12]One has extra relations, like

$$\begin{aligned}\partial_1^C\pi_1\partial_1^C\pi_1 &= \pi_1\partial_1^C\pi_1\partial_1^C\\ \partial_1^C\widehat{\pi}_1\partial_1^C\widehat{\pi}_1 &= \widehat{\pi}_1\partial_1^C\widehat{\pi}_1\partial_1^C\end{aligned}$$

.

since $x^{310}x^\rho = x^{520}$, and $x_1^5\pi_1^C = (x_1^\bullet)^5 - 4(x_1^\bullet)^3 + 3x_1^\bullet$, $x_2^2\pi_2^C = (x_2^\bullet)^2 - 1$.

Let
$$\Theta_n = \frac{1}{2}(1+s_1^B)\cdots(1+s_n^B) + \frac{1}{2}(1-s_1^B)\cdots(1-s_n^B)\,.$$

Proposition 1.10.2. *The maximal divided differences for type D_n satisfy*

$$\pi_{w_0}^D = x^\rho\,\Theta_n\,\partial_\omega^\bullet \tag{1.10.6}$$
$$\widehat{\pi}_{w_0}^D = \Theta_n\,\partial_\omega^\bullet\,x^{-\rho} \tag{1.10.7}$$

In type B or C, an alternating sum $\sum_{w\in W}(-1)^{\ell(w)}(x^v)^w$ may be represented as the determinant
$$\det\left(x_i^{v_j} - x_i^{-v_j}\right)_{i,j=1\ldots n}\,.$$
In type D, this alternating sum is equal to half of the sum of two determinants :
$$\det\left(x_i^{v_j} - x_i^{-v_j}\right)_{i,j=1\ldots n} + \det\left(x_i^{v_j} + x_i^{-v_j}\right)_{i,j=1\ldots n}\,,$$
the first determinant being null when some v_i is equal to 0.

The groups of type B_n or D_n can be embedded into $\mathfrak{S}_{2n}$. However, relations between type B,C,D divided differences and divided differences relative to $\mathfrak{S}_{2n}$ are not straightforward. The next proposition describe $\pi_{w_0}^C$ in terms of $\mathfrak{S}_{2n}$, using the specialization $x_{2i-1}\to x_i$, $x_{2i}\to x_i^{-1}$, $1\le i\le n$.

Proposition 1.10.3. *Given n, let $\zeta = (s_1\cdots s_{2n-1})(s_1\cdots s_{2n-3})\cdots(s_1s_2s_3)(s_1)$. Then*
$$\pi_{w_0}^C = \pi_\zeta\Big|_{\mathbf{x}\to\{x_1,x_1^{-1},x_2,x_2^{-1},\ldots\}}\,,$$
as operators on $\mathfrak{Pol}(\mathbf{x}_n)$.

Proof. The ring $\mathfrak{Pol}(\mathbf{x}_{2n})$ is a free-module over $\mathfrak{Sym}(\mathbf{x}_{2n})$, with basis $\{Y_v : [0,\ldots,0]\le v\le [2n{-}1,\ldots,0]\}$. The submodule $\mathfrak{Pol}(\mathbf{x}_n)$ has basis $\{Y_v : [0,\ldots,0]\le v\le [2n{-}1,\ldots,n,0,\ldots,0]\}$. One can as well take $\{x^v : [0^n]\le v\le[2n{-}1,\ldots,n]\}$, or, our present choice,
$$\{\,x^v : [1{-}2n,\ldots,{-}n]\le v\le[0^n]\,\}\,.$$
Specializing symmetric functions of $\mathbf{x}_{2n}$ into symmetric functions of $x_1,x_1^{-1},\ldots,x_n,x_n^{-1}$, one sees that the same set of monomials[13] span $\mathfrak{Pol}(\mathbf{x}_n)$ as a $\mathfrak{Sym}(\mathbf{x}_n^\bullet)$-module. Therefore it is sufficient to test the proposition on these monomials.

Since both $\pi_{w_0}^C$ and π_ζ admit the symmetrizer π_ω, $\omega=[n,\ldots,1]$ as a left factor, the test can be restricted to all Schur functions of $\mathbf{x}^\vee := \{x_1^{-1},\ldots,x_n^{-1}\}$ indexed by partitions contained in n^n.

[13] but they are no more independent. For example, for $n=2$, $x^{0,-2} = x^{-3,-1} - ax^{-2,-1} + bx^{-1,-1} - x^{0,0}$, with $a = x_1+x_2+x_1^{-1}+x_2^{-1}$, $b = x_1x_2 + x_1x_2^{-1} + x_2x_1^{-1} + 1 + x_1^{-1}x_2^{-1}$.

Instead of enumerating partitions, one can introduce $\mathbf{y} = \{y_1, \ldots, y_n\}$ and test the single function

$$R(\mathbf{x}^\vee, \mathbf{y}) = \prod_{i,j=1}^{n} (x_i^{-1} - y_j)\,.$$

Let us first consider $R(\mathbf{x}^\vee, \mathbf{y})\,\pi^C_{w_0}$. The monomials x^u in the expansion of $R(\mathbf{x}^\vee, \mathbf{y})$ which give a non-zero contribution are those such that $u+\rho$, with $\rho = [n, \ldots, 1]$, has all its component different in absolute value. Since $[0, \ldots, 1{-}n] \le u+\rho \le \rho$, the vector $u+\rho$ must be a signed permutation of ρ, in which case $x^u \pi^C_{w_0} = \pm 1$. Therefore, the sum $\sum_w \pm \Big(x^\rho R(\mathbf{x}^\vee, \mathbf{y})\Big)^w (\Delta^C)^{-1}$, which expresses $R(\mathbf{x}^\vee, \mathbf{y})\,\pi^C_{w_0}$, is independent of $\mathbf{x}$. Specializing $\mathbf{x} = \mathbf{y}$, only the subsum

$$\sum_{w\in\mathfrak{S}_n} \pm \Big(x^\rho R(\mathbf{y}^\vee, \mathbf{y})\Big)^w (\Delta^C(\mathbf{y}))^{-1} = \sum_{w\in\mathfrak{S}_n} \pm \Big(x^\rho\Big)^w \cdot R(\mathbf{y}^\vee, \mathbf{y})(\Delta^C(\mathbf{y}))^{-1}$$

survives. After simplification, this subsum appears to be equal to

$$y_1 \cdots y_n \prod_{i<j\le n} (1 - y_i y_j)\,.$$

Let us now treat $\pi_\zeta = \pi_\omega(\pi_n \cdots \pi_{2n-1})\pi_\eta$, with $\pi_\eta = (\pi_{n-1} \cdots \pi_{2n-3}) \cdots (\pi_2\pi_3)(\pi_1)$. The symmetrizer π_ω preserves $R(\mathbf{y}^\vee, \mathbf{y})$, the operator $(\pi_n \cdots \pi_{2n-1})$ acts only on the factor $R(x_n^{-1}, \mathbf{y})$ and sends it to $(-1)^n y_1 \cdots y_n$. One is left with the computation of

$$R(\mathbf{x}', \mathbf{y})\,\pi_\eta\Big|_{x_{2i}=x_{2i-1}^{-1}}\,,$$

with $\mathbf{x}' = \{x_1^{-1}, \ldots, x_{n-1}^{-1}\}$. Assuming by induction the validity of the proposition for $n-1$, this last function is equal to $R(\mathbf{x}', \mathbf{y})\pi_{w_0'}$, with w_0' relative to C_{n-1}.

The monomials x^u appearing in the expansion of $R(\mathbf{x}', \mathbf{y})$ being such that $[-1, \ldots, -n{+}1] \le u+\rho' \le \rho'$, with $\rho' = [n{-}1, \ldots, 1]$, then for the same reason as above, the sum

$$\sum_w \pm \Big(x^{\rho'} R(\mathbf{x}', \mathbf{y})\Big)^w (\Delta^C(x_1, \ldots, x_{n-1}))^{-1}$$

does not depend on $\mathbf{x}$. Specializing $x_1 = y_1, \ldots, x_{n-1} = y_{n-1}$, the sum reduces to

$$\sum_{w\in\mathfrak{S}_{n-1}} \Big(y^{\rho'}\Big)^w R(\mathbf{y}', \mathbf{y}) \frac{1}{\Delta^C(y_1, \ldots, y_{n-1})} = y_1 \cdots y_{n-1} \prod_{i<j\le n-1} (1 - y_i y_j) R(\mathbf{y}', y_n)\,,$$

with $\mathbf{y}' = \{y_1^{-1}, \ldots, y_{n-1}^{-1}\}$. In final, the two operators send the test function $R(\mathbf{x}', \mathbf{y})$ to the same element, and therefore are equal. QED

For example, for $n = 2$, one has

$$x^{1100}(\pi_1\pi_2\pi_3)(\pi_1) = (x_1{+}x_2)(x_3{+}x_4) + x_1x_2$$

and this polynomial is transformed, by $\mathbf{x} \to [x_1, x_1^{-1}, x_2, x_2^{-1}]$, into

$$x^{11}\pi_1\pi_2^C\pi_1\pi_2^C = (x_1{+}x_1^{-1})(x_2{+}x_2^{-1}) + 1\,,$$

which is equal, as we shall see later, to $K^C_{-1,-1}$.

The two families of divided differences $\pi_i^\heartsuit$, $\widehat{\pi}_i^\heartsuit$ are related by the equations

$$\pi_i = 1 + \widehat{\pi}_i\,,\ i = 1, \dots, n{-}1 \quad \& \quad \pi_n^\heartsuit = 1 + \widehat{\pi}_n^\heartsuit\,,\ \heartsuit = B, C, D\,.$$

For any element w of the Weyl group of type $\heartsuit$, by taking any reduced decomposition of it and the corresponding products of π_i's or $\widehat{\pi}_i$'s, one obtains an expansion of π_w in terms of $\widehat{\pi}_v$, and conversely, of $\widehat{\pi}_w$ in terms of π_v. From a simple property that followers of Bourbaki call the *exchange lemma*, which describes the growth of intervals for the Bruhat order with respect to $w \to ws_i$, one obtains the following relations between the two families of divided differences (given for type A in [85]).

Lemma 1.10.4. *For any element w of a Weyl group of type $\heartsuit = A, B, C, D$, one has the following sums over the Bruhat order :*

$$\pi_w = \sum_{v \le w} \widehat{\pi}_v \tag{1.10.8}$$

$$\widehat{\pi}_w = \sum_{v \le w} (-1)^{\ell(w)-\ell(v)} \pi_v\,. \tag{1.10.9}$$

For example, for type C, and $w = [2, \text{-}3, \text{-}1]$, then $w = s_3^C s_1 s_2 s_3^C$ and

$$\pi_w = (1 + \widehat{\pi}_3^C)(1 + \widehat{\pi}_1)(1 + \widehat{\pi}_2)(1 + \widehat{\pi}_3^C) = \widehat{\pi}_{123} + \Big(\widehat{\pi}_{213} + \widehat{\pi}_{132} + \widehat{\pi}_{12\bar{3}}\Big) + \Big(\widehat{\pi}_{231} + \widehat{\pi}_{21\bar{3}} + \widehat{\pi}_{13\bar{2}} + \widehat{\pi}_{1\bar{3}2}\Big) + \Big(\widehat{\pi}_{23\bar{1}} + \widehat{\pi}_{2\bar{3}1} + \widehat{\pi}_{1\bar{3}\bar{2}}\Big) + \widehat{\pi}_{2\bar{3}\bar{1}}\,.$$

On the other hand, for type D, $w = [2, \text{-}3, \text{-}1] = s_1 s_3^D$, and

$$\pi_w = (1 + \widehat{\pi}_1)(1 + \widehat{\pi}_3^D) = \widehat{\pi}_{123} + \widehat{\pi}_{213} + \widehat{\pi}_{1\bar{3}\bar{2}} + \widehat{\pi}_{2\bar{3}\bar{1}}\,.$$

As a matter of fact, Stembridge [**?**] shows that the 0-Hecke algebra furnishes the easiest way to compute the Möbius function relative to the Bruhat order of Coxeter groups[14].

[14] the operators π_i and $\widehat{\pi}_i$ give two realizations of the 0-Hecke algebra, since $(\pi_i - 0)(\pi_i - 1) = 0$ and $(\widehat{\pi} - 0)_i(\widehat{\pi}_i + 1) = 0$.

1.11 Operators on symmetric functions

Divided divided differences commute with multiplication with symmetric functions. They can nevertheless be used to build operators on symmetric functions, after breaking the initial symmetry, say for example, by sending x_1 to x_1^{-1}, or to qx_1, or using derivatives, then symmetrizing.

As a first example, let us use isobaric derivatives $\delta_i : f \to x_i \frac{d}{dx_i}(f)$, and more conveniently, symmetric functions in the alphabet $\daleth = \{\daleth_1 = \delta_1 - \frac{1}{2}, \daleth_2 = \delta_2 - \frac{3}{2}, \ldots, \daleth_n = \delta_n + \frac{1}{2} - n\}$.

The following lemma shows that symmetric functions in $\daleth$, followed by π_ω, act diagonally on Schur functions.

Lemma 1.11.1. *Let $g \in \mathfrak{Sym}(\mathbf{x}_n)$, $\lambda \in \mathbb{N}^n$ be a partition, $\mathcal{A}_\lambda$ be the alphabet $\{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \ldots, \lambda_n + \frac{1}{2} - n\}$. Then $s_\lambda(\mathbf{x}_n) g(\daleth)\, \pi_\omega = g(\mathcal{A}_\lambda) s_\lambda(\mathbf{x}_n)$.*

Proof. Writing $\pi_\omega = x^\rho \partial_\omega$, one can commute x^ρ with $g(\daleth)$, at the cost of changing $\daleth$ into $\daleth' = \{\lambda_1 + \frac{1}{2} - n, \lambda_2 + \frac{1}{2} - n, \ldots, \lambda_n + \frac{1}{2} - n\}$, due to the fact that $(\delta_i - a)x_i = x_i\delta_i - a - 1$. Factorizing $\partial_\omega = \left(\sum_{\sigma \in \mathfrak{S}_n} \pm\sigma\right) \Delta(\mathbf{x}_n)^{-1}$, one can commute $\sum \pm\sigma$ with the symmetric function in $\daleth'$, thus obtaining

$$s_\lambda(\mathbf{x}_n) g(\daleth)\, \pi_\omega = s_\lambda(\mathbf{x}_n) x^\rho \sum \pm\sigma g(\daleth') \Delta(\mathbf{x}_n)^{-1} = s_\lambda(\mathbf{x}_n)\Delta(\mathbf{x}_n) g(\daleth')\Delta(\mathbf{x}_n)^{-1} .$$

The action of $g(\daleth')$ on $s_\lambda(\mathbf{x}_n)\Delta(\mathbf{x}_n)$, written as a determinant of powers of $x_1, \ldots, x_n$ is immediate, furnishing the result. QED

Since $p_1(\daleth)$ acts by multiplication by $d - n^2/2$ on homogeneous symmetric functions of degree d, the first interesting operators occur in degree 2. Indeed the operator $p_2(\daleth)\pi_\omega - \frac{1}{4}\binom{2n+1}{3}$ may be found in different places, as a Hamiltonian. It can be written, in terms of derivatives with respect to power sums, as the operator

$$\mathfrak{Sym} \ni f \to \sum_{i>0} \sum_{j>0} ij p_{i+j} \frac{d}{dp_i} \frac{d}{dp_j}(f) + (i+j) p_i p_j \frac{d}{dp_{i+j}}(f) .$$

As a second example, let us introduce two parameters α, β and consider the *Sekiguchi operator*

$$\Omega = (\alpha\delta + \beta) \ldots (\alpha\delta_n + \beta - n + 1)\, \pi_\omega ,$$

on symmetric functions of $\mathbf{x} = \mathbf{x}_n$. To explicit the action of Ω, we shall take as a linear basis of $\mathfrak{Sym}(\mathbf{x})$ the Schur functions in the alphabet $\mathbf{x}^\alpha = \frac{1}{\alpha}\mathbf{x}$. Equivalently, we introduce a second alphabet $\mathbf{y}$ of cardinality n, and compute

$$\sigma(\mathbf{x}^\alpha \mathbf{y})\Omega = \prod_i \prod_j (1 - x_i y_j)^{-1/\alpha}\, \Omega .$$

Since $(1 - x_i y)^{-1/\alpha}(\alpha\delta_i + \gamma) = x_i y (1 - x_i y)^{-1/\alpha - 1} + \gamma(1 - x_i y)^{-1/\alpha}$, one sees that there exists a function $F(\mathbf{x}, \mathbf{y})$ independent of α such that $\sigma(\mathbf{x}^\alpha \mathbf{y})\Omega =$

$F(\mathbf{x},\mathbf{y})\sigma\left((1+\frac{1}{\alpha})\mathbf{x}\mathbf{y}\right)$. This function may be determined by putting $\alpha=1$, and is thus equal to $\sigma(\mathbf{x}\mathbf{y})\Omega\Big|_{\alpha=1}\sigma(-2\mathbf{x}\mathbf{y})$. We have seen just above that $\Omega\Big|_{\alpha=1}$ may be written $\Delta(\mathbf{x})(\delta_1+\gamma)\ldots(\delta_n+\gamma)$, with $\gamma=\beta+1-n$. Thanks to Cauchy, $\sigma(\mathbf{x}\mathbf{y})\Delta(\mathbf{x})=\frac{1}{\Delta(\mathbf{y})}\det\left(\frac{1}{1-x_iy_j}\right)$, and therefore

$$\sigma(\mathbf{x}\mathbf{y})\Omega\Big|_{\alpha=1}=\frac{1}{\Delta(\mathbf{y})}\det\left(\frac{\gamma+(1-\gamma)x_iy_j}{(1-x_iy_j)^2}\right)\frac{1}{\Delta(\mathbf{x})},$$

and $F(X,Y)$ is the numerator of this last function.

As in the case of Gaudin determinant $\det\left((1-x_iy_j)^{-1}(1-x_iy_j+\gamma)^{-1}\right)$, or Izergin-Korepin determinant $\det\left((1-x_iy_j)^{-1}(1-qx_iy_j)^{-1}\right)$, one can write the quotient of the numerator of $\sigma(\mathbf{x}\mathbf{y})\Omega\Big|_{\alpha=1}$ by the two Vandermonde as a product of two rectangular matrices [87, 95]. Explicitly, let $M^e(\mathbf{x}_n)$ be the matrix

$$M^e(\mathbf{x}_n)=\left[(-1)^{j-i}e_{j-i}(\mathbf{x})(\beta-n+2i-j)\right]_{\substack{i=1\ldots n\\ j=1..2n}}. \tag{1.11.1}$$

Then $F(\mathbf{x},\mathbf{y})$ is the determinant of the product of this matrix with $\left[h_{i-j}(\mathbf{y})\right]_{\substack{i=1..2n\\ j=1..n}}$.

For example, for $n=2$, $F(\mathbf{x},\mathbf{y})$ is the determinant of the product

$$\begin{bmatrix} e_0(\beta-1) & -e_1(\beta-2) & e_2(\beta-3) & 0\\ 0 & e_0\beta & -e_1(\beta-1) & e_2(\beta-2)\end{bmatrix}\begin{bmatrix} h_0 & 0\\ h_1 & h_0\\ h_2 & h_1\\ h_3 & h_2\end{bmatrix},$$

where, by symmetry between $\mathbf{x}$ and $\mathbf{y}$, the h_i are the complete functions of one alphabet, and the e_i, of the other alphabet. In terms of products of Schur functions of $\mathbf{x}_2$ and $\mathbf{y}_2$, one has

$$\begin{aligned}F(\mathbf{x}_2,\mathbf{y}_2)=\beta(\beta\text{-}1)-(\beta\text{-}1)^2s_1s_1+2s_{11}s_{11}+(\beta\text{-}1)(\beta\text{-}2)(s_2s_{11}+s_{11}s_2)\\ -(\beta\text{-}2)^2s_{21}s_{21}+(\beta\text{-}2)(\beta\text{-}3)s_{22}s_{22}\,.\end{aligned}$$

The function $\sigma(\mathbf{x}^\alpha\mathbf{y})$ expand as $\sum S_v(\mathbf{x}^\alpha)S_v(\mathbf{y})$, sum over all (increasing) partitions in $\mathbb{N}^n$. Therefore, the image of $S_v(\mathbf{x}^\alpha)$ under Ω is equal to the coefficient of $S_v(\mathbf{y})$ in $F(\mathbf{x},\mathbf{y})\sigma((1+\alpha^{-1})\mathbf{x}\mathbf{y})$, that is equal to

$$\sum_{u\uparrow}M_u^eS_{v/u}\left((1+\frac{1}{\alpha})\mathbf{x}\right)=\det\left(M^e\cdot\left[S_{v_j+j-i}((1+\frac{1}{\alpha})\mathbf{x})\right]_{\substack{i=1\ldots 2n\\ j=1\ldots n}}\right), \tag{1.11.2}$$

denoting by M_u^e the minor of M on columns $u_1+1,\ldots u_n+n$. The matrix M^e is in fact the sum of the two matrices

$$\left[(-1)^{j-i}(b-n+i)e_{j-i}(\mathbf{x})\right]\ \text{ and }\ \left[(-1)^{j-i}(i-j)e_{j-i}(\mathbf{x})\right].$$

Let $M^p(\mathbf{x}_n)$ be the $n \times \infty$ matrix of power sums

$$M^p(\mathbf{x}_n) = \begin{bmatrix} \beta{+}1{-}n & p_1(\mathbf{x}) & p_2(\mathbf{x}) & p_3(\mathbf{x}) & \cdots \\ 0 & \beta{+}2{-}n & p_1(\mathbf{x}) & p_2(\mathbf{x}) & \cdots \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & \beta & p_1(\mathbf{x}) & \cdots \end{bmatrix} .$$

Since $\sum(\text{-}1)^i e_i(\mathbf{x})\sigma((1{+}\alpha^{-1})\mathbf{x}) = \sigma(\mathbf{x}^\alpha)$, and $\sum(\text{-}1)^i i e_i(\mathbf{x})\sigma((1{+}\alpha^{-1})\mathbf{x}) = (p_1(\mathbf{x}){+}p_2(\mathbf{x}) + \dots)\sigma(\mathbf{x}^\alpha)$, the product (1.11.2) can be transformed into the product

$$M^p(\mathbf{x}_n) \cdot \Big[S_{v_j+j-i}(\mathbf{x}^\alpha) \Big]_{\substack{i=1\ldots\infty \\ j=1\ldots n}} . \tag{1.11.3}$$

Using Newton's relations $\sum_{i=1}^\infty p_i(\mathbf{x})\sigma(\mathbf{x}) = \sum_{i=0}^\infty iS_i(\mathbf{x})$, one obtains that $S_v(\mathbf{x}^\alpha)\Omega$ is equal to the determinant of

$$\Big[\Big(\alpha(v_j + j - i) + \beta - n + i\Big) S_{v_j+j-i}(\mathbf{x}^\alpha) \Big]_{i,j=1\ldots n} . \tag{1.11.4}$$

For example,

$$S_{136}(\mathbf{x}^\alpha)\Omega = \begin{vmatrix} (1\alpha+\beta-2)S_1(\mathbf{x}^\alpha) & (4\alpha+\beta-2)S_4(\mathbf{x}^\alpha) & (8\alpha+\beta-2)S_8(\mathbf{x}^\alpha) \\ (0\alpha+\beta-1)S_1(\mathbf{x}^\alpha) & (3\alpha+\beta-1)S_3(\mathbf{x}^\alpha) & (7\alpha+\beta-1)S_7(\mathbf{x}^\alpha) \\ 0 & (2\alpha+\beta)S_2(\mathbf{x}^\alpha) & (6\alpha+\beta)S_6(\mathbf{x}^\alpha) \end{vmatrix} .$$

The shifts $\beta{-}n{+}i$ in (1.11.4) are constant by rows. The expansion by rows of the determinant expressing $S_v(\mathbf{x}^\alpha)\Omega$, starting from the bottom, may be written

$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)} \varphi\Big((\lambda+\rho)\sigma - \rho\Big) S^{(\lambda+\rho)\sigma-\rho}(\mathbf{x}^\alpha) ,$$

with $\lambda = v{\downarrow}$, where, for $u \in \mathbb{N}^n$, $S^u(\mathbf{x}^\alpha)$ denotes the product of complete functions $S_{u_1}(\mathbf{x}^\alpha)\dots S_{u_n}(\mathbf{x}^\alpha)$, and $\varphi(u) = (\alpha u_1 + \beta)\dots(\alpha u_n + \beta{+}1{-}n)$.

Introduce another alphabet $\mathbf{z}$, and denote $S2z$ the linear morphim

$$\mathfrak{Sym}(x) \ni s_\lambda(x^\alpha) \to \sum \pm z^{(\lambda+\rho)\sigma-\rho} \in \mathfrak{Pol}(\mathbf{z}) ,$$

by $z2S$ the linear morphism sending z^u onto the product $S^u(\mathbf{x}^\alpha)$.

The preceding computation may be interpreted as the following factorization of the Sekiguchi operator:

$$\mathfrak{Sym} \xrightarrow{S2z} \mathfrak{Pol}(\mathbf{z}) \xrightarrow{z^v \to \varphi(v) z^v} \mathfrak{Pol}(\mathbf{z}) \xrightarrow{z2S} \mathfrak{Sym} .$$

Let $\daleth = \{\daleth_1{=}\ \alpha\delta_1 - \frac{1}{2},\ \daleth_2{=}\ \alpha\delta_2 - \frac{3}{2}, \dots, \daleth_n{=}\ \alpha\delta_n + \frac{1}{2}{-}n\}$. The Sekiguchi operator may be written $\sum(\beta + \frac{1}{2})^{n-i} e_i(\daleth)\pi_\omega$, and therefore determines the action of each elementary function $e_i(\daleth)\pi_\omega$. Since $e_1(\daleth)$ acts as a scalar on homogeneous polynomials, one more generally knows the action of any linear combination of

$e_1(\daleth)^k e_i(\daleth)$, for example $e_1(\daleth)^2 - e_2(\daleth) = p_2(\daleth)$, $e_1(\daleth)^3 - 3e_1(\daleth)e_2(\daleth) + e_3(\daleth) = p_3(\daleth)$, $e_1(\daleth)e_2(\daleth) - e_3(\daleth) = s_{21}(\daleth)$.

Explicitly, for any polynomial f in $\daleth$, any $v \in \mathbb{N}^n$, let $\varphi_f(v) = f(\alpha v_1 - \frac{1}{2}, \ldots, \alpha v_n + \frac{1}{2} - n)$. Then the description of the action of the Sekiguchi operator entails

Lemma 1.11.2. *Let $f = p_2, p_3$ or s_{21}. Then the action of $f(\daleth)\pi_\omega$ on $\mathfrak{Sym}$ factorizes as*

$$\mathfrak{Sym} \xrightarrow{S2z} \mathfrak{Pol}(\mathbf{z}) \xrightarrow{x^v \to \varphi_f(v) x^v} \mathfrak{Pol}(\mathbf{z}) \xrightarrow{z2S} \mathfrak{Sym}\,.$$

The Sekiguchi operator preserves degrees. Expression (1.11.4) shows that is triangular in the basis $\{s_\lambda(\mathbf{x}^\alpha),\, \ell(\lambda) \le n\}$. Since φ takes distinct values on $\mathbb{N}^n$, the eigenspaces of Ω are 1-dimensional, their generators being the *Jack symmetric polynomials*. Since these polynomials are specializations of Macdonald polynomials, we postpone at this point any further comments about them. The operator $\left(p_2(\daleth) - \frac{1}{4}\binom{2n+1}{3}\right)\pi_\omega$ is also diagonal in the basis of Jack polynomials, with eigenvalues $\sum(\alpha\lambda_i + 1/2 - i)^2 - \frac{1}{4}\binom{2n+1}{3} = \alpha^2 \sum \lambda_i^2 + \alpha \sum(1{-}2i)\lambda_i$. It is in fact a rewriting of the Calogero-Sutherland Hamiltonian, and has been considered by physicists [58], see also [13]. To my knowledge, the operators corresponding to $p_3(\daleth)$ and $s_{21}(\daleth)$ have not been used, though they also diagonalize in the basis of Jack polynomials. Beware that the operator $p_4(\daleth)\pi_\omega$ does not act diagonally on Jack polynomials[15].

It is easy to transform isobaric factorized operators into degree-raising operators, by introducing inside the factorization of the operator the multiplication by a fixed polynomial. For example, let us see how to transform the first operator that we saw in this section into an operator deforming the product of Schur functions.

Let λ be a partition in $\mathbb{N}^n$. Then the operator $\Omega_\lambda = x^\lambda(\delta_1 + \beta)\ldots(\delta_n + \beta{+}1{-}n)\pi_\omega$ acting on $\mathfrak{Sym}(\mathbf{x}_n)$ may be rewritten

$$x^\lambda x^\rho (\sum \pm\sigma)(\delta_1 + \beta{+}1{-}n)\ldots(\delta_n + \beta{+}1{-}n)\pi_\omega \\ = s_\lambda(\mathbf{x}_n)\Delta(\mathbf{x}_n)(\delta_1 + \beta{+}1{-}n)\ldots(\delta_n + \beta{+}1{-}n)\pi_\omega\,,$$

and therefore the image of a Schur function $s_\mu(\mathbf{x}_n)$ under Ω_λ is equal to

$$\sum_\nu (s_\lambda s_\mu, s_\nu)(\nu_1 + \beta)\ldots(\nu_n + \beta{+}1{-}n)s_\nu(\mathbf{x}_n)\,,$$

where the coefficients $(s_\lambda s_\mu, s_\nu)$ are the structure constants appearing in $s_\lambda(\mathbf{x}_n)s_\mu(\mathbf{x}_n) = \sum_\nu (s_\lambda s_\mu, s_\nu)s_\nu(\mathbf{x}_n)$. We shall meet similar operators in the case of Macdonald polynomials.

One can also use the divided differences associated to types B, C, D to define operators on $\mathfrak{Sym}$.

[15] This is compatible with the fact that $p_4 = e_1^4 - 4e_1^2 e_2 + 4e_1e_3 + 2e_{2^2} - 4e_4$, the term e_2^2 preventing to apply the preceding considerations.

Let us first consider the action of $-x_1^{-k}\pi_1^B x_1^k$, $k \in \mathbb{Z}$, on functions of x_1.

Since $S_r(x_1{+}1)x_1^{-r/2}$ is invariant under s_1^B, one has

$$-S_r(x_1{+}1)x_1^{-k}\pi_1^B x_1^k = -[r{+}1]x_1^{r/2-k}\pi_1^B x_1^k = \begin{cases} x_1[r{+}1][2k{-}r{-}1] \text{ if } k > r/2 \\ -x_1^{2k-r}[r{+}1][r{+}1{-}2k] \text{ if } k \le r/2 \end{cases},$$

with $[j] = 1{+}x_1{+}\dots{+}x_1^j$.

One notices that the same functions can be obtained by combining ∂_1 with the specialization $x_2 = 1$. More precisely, one checks that for all $r \ge 0$, all $k \in \mathbb{Z}$, one has

$$-S_r(x_1{+}1)\,x_1^{-k}\pi_1^B x_1^{k-1} = S_{r(}(x_1^{-1} + x_2)\,x_1^{2k-1}\partial_1\Big|_{x_2=1}.$$

The next proposition shows how to extend this observation to any n, and will constitute our last example for this section.

Proposition 1.11.3. *Let $\lambda \in \mathbb{N}^n$ be a partition. Then one has, for any $k \in\in \mathbb{Z}$,*

$$(-1)^n s_\lambda(\mathbf{x}_n + 1)\,x_n^{-k}\pi_n^B x_n^k \pi_{n-1}\dots\pi_1(x_1\dots x_n)^{-1} = s_\lambda\, s_n^B x_1^{2k-1}\partial_1\dots\partial_n\Big|_{x_{n+1}=1}. \tag{1.11.5}$$

Proof. By recurrence on n, one sees that, for any symmetric function $f(x_1,\dots,x_n)$, one has

$$f(x_1,\dots,x_n)\,x_n^{-k}\pi_n^B x_n^k\pi_{n-1}\dots\pi_1 = \sum_{i=1}^n f(\dots,\frac{1}{x_i},\dots)\frac{x_i^{2k-1}}{R(x_i,\mathbf{x}_n\setminus x_i)(1{-}x_i)} + f(x_1,\dots,x_n)\frac{1}{R(\mathbf{x}_n,1)}.$$

This is a Lagrange-type sum ([94, Th. 7.8.2]) which can be written

$$f(x_1,\dots,x_n)s_1^B x_i^{2k-1}(1{-}x_1)^{-1}\partial_1\dots\partial_{n-1} + f(x_1,\dots,x_n)\frac{1}{R(\mathbf{x}_n,1)},$$

but one can make this expression more symmetrical by considering the alphabet $x_1,\dots,x_{n+1}$, and by supposing[16] that f is the specialization $x_{n+1} = 1$ of a symmetric function of $x_1,\dots,x_{n+1}$, thus obtaining the stated identity. QED

For example, for $n = 3$, $\lambda = [1,0,0]$, $k = 3$, one has

$$-s_1(\mathbf{x}_3{+}1)\,x_3^{-3}\pi_3^B x_3^3 = (1{+}x_1{+}x_2)(x_3 + + \dots + x_3^5) + (x_3{+}x_3^2{+}x_3^3),$$

whose image under $\pi_2\pi_1$ is $\Big(s_1(\mathbf{x}_3{+}1) + s_{21}(\mathbf{x}_3{+}1)\Big)x_1x_2x_3$.

On the other hand,

$$(x_1^{-1} + x_2 + x_3 + x_4)x_1^5\,\partial_1\partial_2\partial_3 = s_1(\mathbf{x}_4) + s_{21}(\mathbf{x}_4),$$

and this agrees with the proposition.

[16] This is no restriction: $s_\lambda(\mathbf{x}_n) = s_\lambda(\mathbf{x}_{n+1} - 1)\big|_{x_{n+1}=1}$.

1.12 Weyl character formula

Irreducible characters for type $\heartsuit = A, B, C, D$ have been described by Weyl. For $\lambda \in \mathbb{N}^n$ dominant[17], Weyl's character formula reads

$$\chi_\lambda^\heartsuit = \frac{\sum_w (-1)^{\ell(w)} \left(x^{\lambda+\rho}\right)^w}{\sum_w (-1)^{\ell(w)} \left(x^{\rho}\right)^w}, \tag{1.12.1}$$

where $\rho = [n-1, \ldots, 0]$ in type A, D, $\rho = [n, \ldots, 1]$ in type C and $\rho = [n-\frac{1}{2}, \ldots, \frac{1}{2}]$ in type B.

Using the factorization of the alternating sum of the elements of each group, one recognizes that the characters $\chi_\lambda^\heartsuit$ are equal to the image of x^λ under $\pi_{w_0}^\heartsuit$.

Each $\pi_{w_0}^\heartsuit$ has ∂_ω as a right factor. Since, for any functions $f_1(x), \ldots, f_n(x)$, one has

$$f_1(x_1) \cdots f_n(x_n)\, \partial_\omega = \det(f_i(x_j)) / \det(x_i^{n-j})$$

one may write the numerators of Weyl character formula as the following determinants (still with $\lambda_n = 0$ for type D) :

$$\det(x_i^{\lambda_j+n-j}) \quad \text{type} \quad A \tag{1.12.2}$$

$$\det(x_i^{\lambda_j+n-j+1/2} - x_i^{-\lambda_j-n+j-1/2}) \quad \text{type} \quad B \tag{1.12.3}$$

$$\det(x_i^{\lambda_j+n-j+1} - x_i^{-\lambda_j-n+j-1}) \quad \text{type} \quad C \tag{1.12.4}$$

$$\frac{1}{2}\det(x_i^{\lambda_j+n-j} + x_i^{-\lambda_j-n+j}) \quad \text{type} \quad D \tag{1.12.5}$$

Let $\Delta(\mathbf{x}) = \prod_{1\le i<j\le n}(x_i - x_j)$. Then the denominators $\Delta^A, \Delta^B, \Delta^C, \Delta^D$ of Weyl character formula are respectively equal to

$$\Delta^A = \Delta(\mathbf{x}),\ \Delta^B = \prod_i (\sqrt{x_i} - \frac{1}{\sqrt{x_i}})\Delta(\mathbf{x}^\bullet),\ \Delta^C = \prod_i (x_i - \frac{1}{x_i})\Delta(\mathbf{x}^\bullet),\ \Delta^D = \Delta(\mathbf{x}^\bullet),$$

still using the notation $\mathbf{x}^\bullet = \{x_1^\bullet, \ldots, x_n^\bullet\}$, with $x_i^\bullet = x_i + x_i^{-1}$.

The numerators of Weyl's formula may also be written as determinants, so that the right hand side of Weyl's formula for type A, B, C, D, say in the case $\lambda = [3, 1, 0]$, would look like

$$\frac{\begin{vmatrix} x_1^5 & x_1^2 & 1 \\ x_2^5 & x_2^2 & 1 \\ x_3^5 & x_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}}, \quad \frac{\begin{vmatrix} x_1^{11/2} - x_1^{-11/2} & x_1^{5/2} - x_1^{-5/2} & x_1^{1/2} - x_1^{-1/2} \\ x_2^{11/2} - x_2^{-11/2} & x_2^{5/2} - x_2^{-5/2} & x_2^{1/2} - x_2^{-1/2} \\ x_3^{11/2} - x_3^{-11/2} & x_3^{5/2} - x_3^{-5/2} & x_3^{1/2} - x_3^{-1/2} \end{vmatrix}}{\begin{vmatrix} x_1^{5/2} - x_1^{-5/2} & x_1^{3/2} - x_1^{-3/2} & x_1^{1/2} - x_1^{-1/2} \\ x_2^{5/2} - x_2^{-5/2} & x_2^{3/2} - x_2^{-3/2} & x_2^{1/2} - x_2^{-1/2} \\ x_3^{5/2} - x_3^{-5/2} & x_3^{3/2} - x_3^{-3/2} & x_3^{1/2} - x_3^{-1/2} \end{vmatrix}},$$

[17] For simplicity, we impose $\lambda_n = 0$ in type D, but we shall lift this restriction later.

$$\frac{\begin{vmatrix} {x_1}^6 - {x_1}^{-6} & {x_1}^3 - {x_1}^{-3} & x_1 - {x_1}^{-1} \\ {x_2}^6 - {x_2}^{-6} & {x_2}^3 - {x_2}^{-3} & x_2 - {x_2}^{-1} \\ {x_3}^6 - {x_3}^{-6} & {x_3}^3 - {x_3}^{-3} & x_3 - {x_3}^{-1} \end{vmatrix}}{\begin{vmatrix} {x_1}^3 - {x_1}^{-3} & {x_1}^2 - {x_1}^{-2} & x_1 - {x_1}^{-1} \\ {x_2}^3 - {x_2}^{-3} & {x_2}^2 - {x_2}^{-2} & x_2 - {x_2}^{-1} \\ {x_3}^3 - {x_3}^{-3} & {x_3}^2 - {x_3}^{-2} & x_3 - {x_3}^{-1} \end{vmatrix}} , \frac{\begin{vmatrix} {x_1}^5 + {x_1}^{-5} & {x_1}^2 + {x_1}^{-2} & 1 \\ {x_2}^5 + {x_2}^{-5} & {x_2}^2 + {x_2}^{-2} & 1 \\ {x_3}^5 + {x_3}^{-5} & {x_3}^2 + {x_3}^{-2} & 1 \end{vmatrix}}{\begin{vmatrix} {x_1}^2 + {x_1}^{-2} & x_1 + {x_1}^{-1} & 1 \\ {x_2}^2 + {x_2}^{-2} & x_2 + {x_2}^{-1} & 1 \\ {x_3}^2 + {x_3}^{-2} & x_3 + {x_3}^{-1} & 1 \end{vmatrix}} .$$

When λ is an integral multiple of ρ, the numerator in Weyl's character formula is the image of the denominator under raising the variables to some power. Writing $k(\rho)$ for $[k\rho_1, k\rho_2, \ldots, k\rho_n]$, and $h_k(a+b)$ for the complete function of degree k in the variables a, b, one has

$$\begin{aligned}
\chi^A_{k(\rho)} &= \frac{\prod_{1\le i<j\le n}(x_i^{k+1} - x_j^{k+1})}{\prod_{1\le i<j\le n}(x_i - x_j)} = \prod_{1\le i<j\le n} h_k(x_i+x_j)\,, \\
\chi^D_{k(\rho)} &= \chi^A_{k(\rho)} \frac{\prod_{1\le i<j\le n}\left(1 - x_i^{-k-1}x_j^{-k-1}\right)}{\prod_{1\le i<j\le n} 1 - x_i^{-1}x_j^{-1}} \\
&= \prod_{1\le i<j\le n} h_k(x_i+x_j)h_k(1+x_i^{-1}x_j^{-1})\,, \\
\chi^B_{k(\rho)} &= \chi^D_{k(\rho)} \prod_{i=1}^n \frac{x_i^{(k+1)/2} - x_i^{-(k+1)/2}}{x_i^{1/2} - x_i^{-1/2}} \\
&= \prod_{i=1}^n h_k\left(\sqrt{x_i} + \frac{1}{\sqrt{x_i}}\right) \prod_{1\le i<j\le n} h_k(x_i+x_j)h_k(1+x_i^{-1}x_j^{-1})\,, \\
\chi^C_{k(\rho)} &= \chi^D_{k(\rho)} \prod_{i=1}^n \frac{x_i^{k+1} - x_i^{-k-1}}{x_i - x_i^{-1}} \\
&= \prod_{i=1}^n h_k\left(x_i + \frac{1}{x_i}\right) \prod_{1\le i<j\le n} h_k(x_i+x_j)h_k(1+x_i^{-1}x_j^{-1})\,.
\end{aligned}$$

For example, for $n = 2$, $k = 2$, one has

$$\chi^C_{42} = \left(x_1^2 + 1 + \frac{1}{x_1^2}\right)\left(x_2^2 + 1 + \frac{1}{x_2^2}\right)(x_1^2 + x_1x_2 + x_2^2)\left(1 + \frac{1}{x_1x_2} + \frac{1}{x_1^2x_2^2}\right) .$$

1.13 Macdonald Poincaré polynomial

The *length* of a reduced decomposition of an element w of a Weyl group is equal, using standard notions from the theory of *root systems*, to the number of roots in the intersection of $\mathcal{R}^+$ and $-w\mathcal{R}^+$.

Instead of enumerating inversions, let us define an *inversion weight* as follows. Embed the Weyl group of type B_n, C_n, D_n into $\mathfrak{S}_{2n}$. Given $w \in W$ and the corresponding $\sigma \in \mathfrak{S}_{2n}$, to a pair $(i,j) : 1 \le i < j \le n$, such that $\sigma_i > \sigma_j$ associate a factor h_{ji}. To a pair such that $\sigma_i > \sigma_{2n+1-j}$ associate a factor h_{ij}. Moreover, to all $i : 1 \le i \le n$ such that $w_i < 0$ associate a factor h_i in type B, and a factor h_{ii} in type C. The *inversion weight* $\mathcal{I}(w)$ of $w \in W$ is the product of these factors.

One can also define $\mathcal{I}(w)$ recursively by left multiplication by simple transpositions. Given w, s_k such that $\ell(s_k w) > \ell(w)$, then w and $s_k w$ either differ in two positions i,j or $(s_k w)_i = -w_i$. In that last case (which do not occur for type A or D), one has $\mathcal{I}(s_k w)/\mathcal{I}(w) = h_i$ in type B and $= h_{ii}$ in type C. In the first case, if $w_i w_j > 0$ and $[\ldots w_i \ldots w_j \ldots] \to [\ldots w_j \ldots w_i \ldots]$, then $\mathcal{I}(s_k w)/\mathcal{I}(w) = h_{ji}$. Otherwise, if $w_i w_j < 0$, then $[\ldots w_i \ldots w_j \ldots] \to [\ldots - w_j \ldots - w_i \ldots]$ and $\mathcal{I}(s_k w)/\mathcal{I}(w) = h_{ij}$.

For example, for type C_4, one has the following chain of inversion factors :

$$[2,\overline{4},\overline{1},3] \xleftarrow{s_4^C\, h_{22}} [2,4,\overline{1},3] \xleftarrow{s_3\, h_{42}} [2,3,\overline{1},4] \xleftarrow{s_1\, h_{13}} [1,3,\overline{2},4]$$
$$\xleftarrow{s_2\, h_{23}} [1,2,\overline{3},4] \xleftarrow{s_3\, h_{34}} [1,2,\overline{4},3] \xleftarrow{s_4^C\, h_{33}} [1,2,4,3] \xleftarrow{s_3\, h_{43}} [1,2,3,4]$$

The inversions are more straightforward to read when writing the inverse elements :

$$[\overline{3},1,4,\overline{2}]^{-1} \xleftarrow{s_4^C\, h_{22}} [\overline{3},1,4,2]^{-1} \xleftarrow{s_3\, h_{42}} [\overline{3},1,2,4]^{-1} \xleftarrow{s_1\, h_{13}} [1,\overline{3},2,4]^{-1}$$
$$\xleftarrow{s_2\, h_{23}} [1,2,\overline{3},4]^{-1} \xleftarrow{s_3\, h_{34}} [1,2,4,\overline{3}]^{-1} \xleftarrow{s_4^C\, h_{33}} [1,2,4,3]^{-1} \xleftarrow{s_3\, h_{43}} [1,2,3,4]^{-1}$$

For each Weyl group of type $\heartsuit = A_{n-1}, B_n, C_n, D_n$, Macdonald defined the following kernel[18] $\mathcal{M}^\heartsuit$, introducing formal parameters h_{ji} :

$$\begin{aligned}
\mathcal{M}^A &= \prod_{1\le i<j\le n} (1 - h_{ji} x_j x_i^{-1}) \\
\mathcal{M}^D &= \mathcal{M}^A \prod_{1\le i<j\le n} (1 - h_{ij} x_i^{-1} x_j^{-1}) \\
\mathcal{M}^B &= \mathcal{M}^D \prod_{1\le i\le n} (1 - h_i x_i^{-1}) \\
\mathcal{M}^C &= \mathcal{M}^D \prod_{1\le i\le n} (1 - h_{ii} x_i^{-2})
\end{aligned}$$

[18] Of course, Macdonald does not mix types, but taking a pure combinatorial point of view leaves us more freedom.

For example, for A_2 and D_3, one has

$$\mathcal{M}^A = \left(1 - \frac{h_{21}\, x_2}{x_1}\right)\left(1 - \frac{h_{31}\, x_3}{x_1}\right)\left(1 - \frac{h_{32}\, x_3}{x_2}\right),$$

$$\mathcal{M}^D = \mathcal{M}^A \left(1 - \frac{h_{12}}{x_1\, x_2}\right)\left(1 - \frac{h_{13}}{x_1\, x_3}\right)\left(1 - \frac{h_{23}}{x_2\, x_3}\right).$$

The following theorem, due to Macdonald [124, Th.2.8], generalizes the enumeration of elements of a Weyl group according to their length.

Theorem 1.13.1. *For a Weyl group of type $\heartsuit = A, B, C, D$, with maximal element w_0, one has*

$$\mathcal{M}\, \pi^{\heartsuit}_{w_0} = \sum_{w \in W} \mathcal{I}(w)\,.$$

Proof. Each kernel, multiplied by $x^{\rho^{\heartsuit}}$ is a sum of monomials x^v, where the exponents respectively satisfy the conditions (componentwise comparison) : for type A, $[0, \ldots, 0] \le v \le [n{-}1, \ldots, n{-}1]$,
for type B, $[1{-}n, \ldots, 1{-}n] \le v + [\frac12, \ldots, \frac12] \le [n, \ldots, n]$,
for type C, $[-n, \ldots, -n] \le v \le [n, \ldots, n]$,
for type D, $[1{-}n, \ldots, 1{-}n] \le v \le [n{-}1, \ldots, n{-}1]$.

Under the operator $\sum_w (-1)^{\ell(w)} \frac{1}{\Delta^{\heartsuit}}$, such monomials are sent to 0, or to ± 1 if they appear in the expansion of $\Delta^{\heartsuit}$. One checks that in that last case, the coefficient is indeed the inversion weight $\mathcal{I}(w)$. QED

For example, for type C_2, the contributing terms are

$$x^{2,1} - x^{2,-1} h_{22} + x^{1,-2} h_{12} h_{22} - x^{-1,-2} h_{11}\, h_{12}\, h_{22} - x^{1,2} h_{21} \\ + x^{-1,2} h_{21}\, h_{11} - x^{-2,1} h_{21}\, h_{11}\, h_{12} + x^{-2,-1} h_{21}\, h_{11}\, h_{12}\, h_{22}\,.$$

One could have decided[19] to denote the elements of the group by the element of the orbit of $\rho^{\heartsuit}$. In type A, one would have permutations of $[n{-}1, \ldots, 0]$, in type B, signed permutations of $[n{-}\frac12, \ldots, \frac12]$, in type C, signed permutations of $[n, \ldots, 1]$, and finally, in type D, signed permutations of $[n{-}1, \ldots, 0]$.

The usual Poincaré polynomial is obtained by specializing all h_i, h_{ij} to q and thus is obtained by symmetrizing the "q-Vandermonde".

One could have taken an arbitrary subsum of the expansion of $\mathcal{M}^{\heartsuit}$. Macdonald's theorem states that the only terms surviving after symmetrization are those having for coefficient the inversion weight of an element of the group. The following theorem shows how to apply this property to generate intervals for the weak order.

For $v, w \in W$, write $w \ge_L v$ if the product $(wv^{-1})\, v$ is reduced, i.e. if $\ell(w) = \ell(wv^{-1}) + \ell(v)$. In that case $\mathcal{I}(v)$ is a factor of $\mathcal{I}(w)$. In the following statement,

[19] In type A, Cauchy considered the Vandermonde determinant, that he in fact introduced, as the generating function of permutations together with their signs, and consequently, the Vandermonde determinant as the "generic" determinant.

we shall use the same notation $\mathcal{I}(w)$ for the set of inversions and the inversion weight of $w \in W$.

Let $h^x_{ji} = h_{ji} x_j x_i^{-1}$, $j > i$, and $h^x_{ij} = h_{ij} x_i^{-1} x_j^{-1}$, $i \le j$, $h^x_i = h_i x_i^{-1}$.

Theorem 1.13.2. *Given a pair w, v such that $w \ge_L v$, then*

$$\prod_{\alpha \in \mathcal{I}(w) \setminus \mathcal{I}(v)} (1 - h^x_\alpha) \prod_{\alpha \in \mathcal{I}(v)} (-h^x_\alpha)\, \pi^{\heartsuit}_{w_0} = \sum_{u:\, w \ge_L u \ge_L v} \mathcal{I}(v)\,. \tag{1.13.1}$$

Proof. We already remarked that we have only to extract the products of h_{ji} which are inversion weights of elements of W. But $u \in W$ is such that $w \ge_L u$ if and only if $\mathcal{I}(u)$ divides $\mathcal{I}(w)$, thus u in the RHS if and only if it belongs to the left-order interval $[w, v]$. QED

It is interesting to notice that the interval $[1, w]$ for the Bruhat order can be obtained, thanks to Lemma 1.10.4, by taking any reduced decomposition $w = s_i \cdots s_j$ and evaluating the product $(1 + \widehat{\pi}_i) \cdots (1 + \widehat{\pi}_j)$. On the other hand, the preceding theorem gives the interval $[1, w]_L$ for the weak order by symmetrizing a factor of degree $\ell(w)$.

For example, for $w = [3, 4, 1, 2] \in \mathfrak{S}_4$, the initial interval for the Bruhat order is given by

$$\begin{aligned}\pi_{3412} = \pi_2\pi_3\pi_1\pi_2 &= (1+\widehat{\pi}_2)(1+\widehat{\pi}_3)(1+\widehat{\pi}_1)(1+\widehat{\pi}_3)\\ &= \widehat{\pi}_{3412} + \widehat{\pi}_{3214} + \widehat{\pi}_{3142} + \widehat{\pi}_{3124} + \widehat{\pi}_{2413} + \widehat{\pi}_{2314} + \widehat{\pi}_{2143}\\ &\qquad + \widehat{\pi}_{2134} + \widehat{\pi}_{1432} + \widehat{\pi}_{1423} + \widehat{\pi}_{1342} + \widehat{\pi}_{1324} + \widehat{\pi}_{1243} + \widehat{\pi}_{1234}\,,\end{aligned}$$

while the initial interval for the left order is obtained by computing

$$\begin{aligned}&\left(1 - h_{31}\frac{x_3}{x_1}\right)\left(1 - h_{32}\frac{x_3}{x_2}\right)\left(1 - h_{41}\frac{x_4}{x_1}\right)\left(1 - h_{42}\frac{x_4}{x_2}\right)\pi_{4321}\\ &\qquad = 1 + h_{32} + h_{31}h_{32} + h_{32}h_{42} + h_{31}h_{32}h_{42} + h_{31}h_{41}h_{32}h_{42}\,,\end{aligned}$$

which translates, passing from the inversion weights to the permutations, into

$$[1,2,3,4], [1,3,2,4], [1,4,2,3], [2,3,1,4], [2,4,1,3], [3,4,1,2]\,.$$

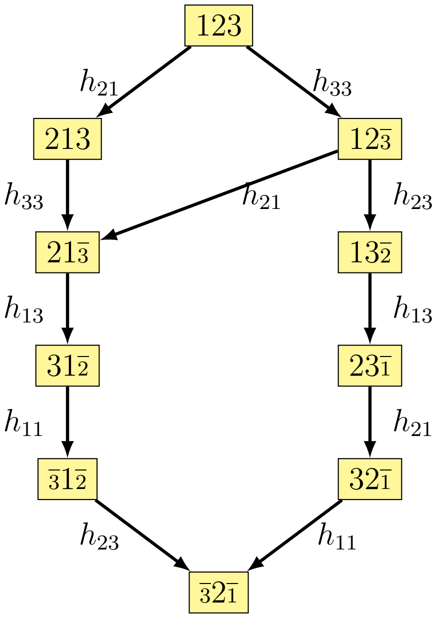
The Poincaré polynomial is obtained by specializing all h_α to q. For example, let $w = [5, 2, 4, 6, 1, 3]$, $v = [3, 1, 2, 5, 4, 6]$ in $\mathfrak{S}_6$. Then

$$\mathcal{I}([5,2,4,6,1,3])/\mathcal{I}([3,1,2,5,4,6] = h_{51}h_{52}h_{53}h_{61}h_{63}h_{64}\,,\ \mathcal{I}([3,1,2,5,4,6]) = h_{21}h_{31}h_{54}$$

and the polynomial of the interval is equal to

$$\begin{aligned}&\left(1 - h_{51}\frac{x_5}{x_1}\right)\left(1 - h_{52}\frac{x_5}{x_2}\right)\left(1 - h_{53}\frac{x_5}{x_3}\right)\left(1 - h_{61}\frac{x_6}{x_1}\right)\left(1 - h_{63}\frac{x_6}{x_3}\right)\left(1 - h_{64}\frac{x_6}{x_4}\right)\\ &\times \left(-\frac{x_2}{x_1}\right)\left(-\frac{x_3}{x_1}\right)\left(-\frac{x_5}{x_4}\right)\pi_{654321}\Big|_{h_{ji}=q} = q^6 + 2q^5 + 2q^4 + 3q^3 + 2q^2 + 2q + 1\,.\end{aligned}$$

We end by giving an example in type C, for $n = 3$, writing the interval and the inversions in the order they are created.



Thus, the Poincaré polynomial for this interval is equal to $1 + h_{21} + h_{33} + h_{23}h_{33} + h_{21}h_{33} + h_{13}h_{23}h_{33} + h_{21}h_{13}h_{33} + h_{21}h_{13}h_{11}h_{33} + h_{21}h_{13}h_{23}h_{33} + h_{21}h_{13}h_{23}h_{11}h_{33}$.

Chapter 2

Linear Bases for type A

2.1 Schubert, Grothendieck and Demazure

To interpolate a function $f(x_1)$ at points $y_1, y_2, \ldots,$, Newton [130] chose the basic polynomials $Y_0 = 1$, $Y_1 = (x_1 - y_1)$, $Y_2 = (x_1 - y_1)(x_1 - y_2), \ldots$ and found that the coefficients of $f(x_1)$ in this basis could be obtained by divided differences.

One can add the remark to Newton's computations that the Newton basis $Y_0, Y_1, Y_2, \ldots$ is invariant under the divided differences $\partial_i^{\mathbf{y}}$. Indeed, $Y_k \partial_k^{\mathbf{y}} = -Y_{k-1}$, and $Y_k \partial_i^{\mathbf{y}} = 0$ for $i \neq k$. It is therefore natural to generate bases of polynomials using the different operators $\partial_i, \pi_i, \widehat{\pi}_i, T_i$ that are at our disposal. However, we also need starting points, i.e. polynomials such that them together with their descent will constitute a basis. In the case of non symmetric Macdonald polynomials, because one also has "raising operators" which increase degree, we need only one starting point, which is 1. For the other families of polynomials, the starting points will be associated to the diagrams of partitions, to the cost of having to check compatibility conditions between the different starting points.

Given $\lambda \in \mathbb{N}^n$ a partition (i.e. $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$), then

$$Y_\lambda := \prod_{i=1..n, j=1..\lambda_i} (x_i - y_j) \quad \& \quad G_\lambda := \prod_{i=1..n, j=1..\lambda_i} (1 - y_j x_i^{-1})$$

are the *dominant Schubert polynomials* and the *dominant Grothendieck polynomial* respectively, of index λ, and

$$K_\lambda = x^\lambda = \widehat{K}_\lambda$$

are the *dominant Demazure characters for type A*. We shall rather say *key polynomials* instead of Demazure characters [21] in reference to their combinatorial interpretation in terms of keys.

x_3-y_1	x_3-y_2	
x_2-y_1	x_2-y_2	
x_1-y_1	x_1-y_2	x_1-y_3

Y_{322}

x_3	x_3	
x_2	x_2	
x_1	x_1	x_1

K_{322}

$1-\frac{y_1}{x_3}$	$1-\frac{y_2}{x_3}$	
$1-\frac{y_1}{x_2}$	$1-\frac{y_2}{x_2}$	
$1-\frac{y_1}{x_1}$	$1-\frac{y_2}{x_1}$	$1-\frac{y_3}{x_1}$

G_{322}

We define Schubert polynomials to be all[1] the non-zero images of the dominant Schubert polynomials under products of ∂_i's and Grothendieck polynomials[2] to be all the images of the dominant Grothendieck polynomials under products of π_i's. Similarly, the two types of key polynomials are defined by taking all images under products of π_i's or of $\widehat{\pi}_i$'s respectively.

Since the operators satisfy relations, we cannot index the polynomials by the choice of the starting point and the sequence of operators used. In fact, all these polynomials can be indexed by weights in $\mathbb{N}^n$, the recursive definition being

$$Y_{\dots,v_{i+1},v_i-1,\dots} = Y_v\,\partial_i \quad \& \quad G_{\dots,v_{i+1},v_i-1,\dots} = G_v\,\pi_i \text{ when } v_i > v_{i+1} \tag{2.1.1}$$

$$K_v\,\pi_i = K_{v\,s_i} \;\&\; \widehat{K}_v\,\widehat{\pi}_i = \widehat{K}_{v\,s_i},\text{ when } v_i > v_{i+1}\,. \tag{2.1.2}$$

Thus, the operators act on the indices just by sorting increasingly in the case of key polynomials, and by sorting and decreasing the biggest of the two components exchanged, in the case of Schubert and Grothendieck polynomials[3].

It is clear that these four families constitute linear bases of $\mathfrak{Pol}(n)$, because Y_v, K_v, $\widehat{K}_v$ have leading term[4] x^v, and G_v has leading term x^{-v}. However, it is unsatisfactory to have mere bases, one must be able to express a general polynomial

[1] There are dominant polynomials in the images of a dominant polynomial, in the Schubert and Grothendieck cases; therefore, one has to check consistency, as we already mentioned, but this easy.

[2] As a natural continuation of my work about syzygies of determinantal varieties, I had determined the classes, as polynomials, of the structure sheaves of the Schubert subvarieties of a flag manifold. It was a time where Grothendieck had some complaints about the world of mathematicians. I proposed to M.P. Schützenberger to call these classes *Grothendieck polynomials*, to which suggestion he readily agreed. They appear under the label G-polynomials in the paper[104] introducing them, the referee having disagreed with the terminology. The said referee fortunately forgot to extend his ban to future work. Moreover, Alexandre Grothendieck did not protest against this appellation.

[3] Choosing permutations as indexing sets, then the action is simply sorting. We did not give the case $v_i \le v_{i+1}$ because it is determined by the relations $\partial_i^2 = 0$, $\pi_i^2 = \pi_i$, $\widehat{\pi}_i^2 = -\widehat{\pi}_i$. Thus in that case,

$$Y_v\partial_i = 0,\; G_v\pi_i = G_v,\; K_v\pi = K_v,\; \widehat{K}_v\widehat{\pi}_i = -\widehat{K}_v\,.$$

[4] Notice that $x^{ji}\partial_1 = x^{j-1,i} + x^{j-2,i+1} + \cdots + x^{i,j-1}$ and that $x^{ji}\pi_1 = x^{j,i} + x^{j-1,i+1} + \cdots + x^{i,j}$. From this, it is easy to prove by induction that the monomials x^u appearing in Y_v, K_v are such that $u_n \le v_n$, $u_n + u_{n-1} \le v_n + v_{n-1}, \dots$. In particular, $u \le v$ for the *right lexicographic order*,

in term of these bases. We shall see how to do it in the next section, by defining a scalar product.

As examples of Schubert and Grothendieck polynomials, one obtains the following polynomials starting from the dominant ones Y_{210} and G_{210}.

$$Y_{210} = \begin{array}{ll} (x_2-y_1) & \\ (x_1-y_1) & (x_1-y_2) \end{array}$$

$$Y_{110} = \begin{array}{ll} (x_2-y_1) & \\ (x_1-y_1) & \bullet \end{array} \qquad\qquad Y_{200} = \begin{array}{ll} \bullet & \\ (x_1-y_1) & (x_1-y_2) \end{array}$$

$$Y_{100} = \begin{array}{ll} \bullet & \\ x_1-y_1 & \bullet \end{array} \qquad\qquad Y_{010} = x_1+x_2-y_1-y_2$$

$$Y_{000} = 1$$

$$G_{210} = \begin{array}{ll} (1-\frac{y_1}{x_2}) & \\ (1-\frac{y_1}{x_1}) & (1-\frac{y_2}{x_1}) \end{array}$$

$$G_{110} = \begin{array}{ll} (1-\frac{y_1}{x_2}) & \\ (1-\frac{y_1}{x_1}) & \bullet \end{array} \qquad\qquad G_{200} = \begin{array}{ll} \bullet & \\ (1-\frac{y_1}{x_1}) & (1-\frac{y_2}{x_1}) \end{array}$$

$$G_{100} = \begin{array}{ll} \bullet & \\ 1-\frac{y_1}{x_1} & \bullet \end{array} \qquad\qquad G_{010} = 1-\frac{y_1y_2}{x_1x_2}$$

$$G_{000} = 1$$

For these two families, only the polynomial indexed by 010 is not dominant. However, in general Schubert and Grothendieck polynomials do not factorize, though they still have the same type of vanishing properties than the dominant ones.

Our starting Schubert polynomials are products of linear factors $x_i - y_j$. We shall be able to express general Schubert or Grothendieck polynomials as sums of

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i.e. the order such that if $u < v$ then there exist k such that $u_i = v_i$ for $i = k{+}1, \ldots, n$ and $u_k < v_k$. Similarly, all monomials x^u appearing in the expansion of G_v are such that $-u_n \le -v_n$, $-u_n - u_{n-1} \le -v_n - v_{n-1}, \ldots$.

products of linear factors[5]. For example, using Leibnitz' formula, one obtains the sequence of polynomials

$$Y_{320} = \begin{array}{|c|c|c}\hline x_2{-}y_1 & x_2{-}y_2 & \multicolumn{1}{c}{} \\ \hline x_1{-}y_1 & x_1{-}y_2 & \multicolumn{1}{c|}{x_1{-}y_3} \\ \hline \end{array}\,, \quad Y_{301} = \begin{array}{|c|c|c|}\cline{1-1} x_3{-}y_2 & \multicolumn{2}{c}{} \\ \hline x_1{-}y_1 & x_1{-}y_2 & x_1{-}y_3 \\ \hline \end{array} + \begin{array}{|c|c|c|}\cline{1-1} x_2{-}y_1 & \multicolumn{2}{c}{} \\ \hline x_1{-}y_1 & x_1{-}y_2 & x_1{-}y_3 \\ \hline \end{array}\,,$$

$$Y_{021} = \begin{array}{|c|c|}\cline{1-1} x_3{-}y_2 & \multicolumn{1}{c}{} \\ \hline x_2{-}y_2 & x_2{-}y_3 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} x_3{-}y_2 & \multicolumn{1}{c}{} \\ \hline x_1{-}y_1 & x_2{-}y_3 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} x_2{-}y_1 & \multicolumn{1}{c}{} \\ \hline x_1{-}y_1 & x_2{-}y_3 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} x_3{-}y_2 & \multicolumn{1}{c}{} \\ \hline x_1{-}y_1 & x_1{-}y_2 \\ \hline \end{array} + \begin{array}{|c|c|}\cline{1-1} x_2{-}y_1 & \multicolumn{1}{c}{} \\ \hline x_1{-}y_1 & x_1{-}y_2 \\ \hline \end{array}\,,$$

and the last polynomial, Y_{021}, does not factorize anymore.

2.2 Using the y-variables

Some properties of Schubert and Grothendieck polynomials are easier to follow using permutations for the indexing. Given a permutation σ of code v, then one uses both notations $Y_v(\mathbf{x},\mathbf{y})$ and $X_\sigma(\mathbf{x},\mathbf{y})$ for the same Schubert polynomial, as well as $G_v(\mathbf{x},\mathbf{y})$ and $G_{(\sigma)}(\mathbf{x},\mathbf{y})$ for the same Grothendieck polynomial.

Both families satisfy a fundamental symmetry in $\mathbf{x},\mathbf{y}$. Indeed, given $i \le n{-}1$, denoting as usual $\omega = [n,\dots,1]$, then it is immediate, because the statement reduces to compute the image of $(x_i{-}y_{n-i})$ or $(1{-}y_{n-i}x_i^{-1})$, that

$$X_\omega(\mathbf{x},\mathbf{y})\,\partial_i^{\mathbf{x}} = -X_\omega(\mathbf{x},\mathbf{y})\,\partial_{n-i}^{\mathbf{y}} \qquad (2.2.1)$$

$$G_{(\omega)}(\mathbf{x},\mathbf{y})\,\pi_i^{\mathbf{x}} = G_{(\omega)}(\mathbf{x},\mathbf{y})\,\pi_{n-i}^{\mathbf{1/y}}\,, \qquad (2.2.2)$$

where $\pi_{n-i}^{\mathbf{1/y}}$ denotes the isobaric divided differences relative to $\mathbf{y}^\vee = \{y_1^{-1}, y_2^{-1}, \dots\}$.

By iteration, noticing that the symmetry is valid for $X_\omega(\mathbf{x},\mathbf{y})$ and $G_{(\omega)}(\mathbf{x},\mathbf{y})$, one obtains the following proposition.

Proposition 2.2.1. *The Schubert and Grothendieck polynomials satisfy the recursion*

$$X_{s_i\sigma}(\mathbf{x},\mathbf{y}) = -X_\sigma(\mathbf{x},\mathbf{y})\,\partial_i^{\mathbf{y}} \quad \& \quad G_{(s_i\sigma)}(\mathbf{x},\mathbf{y}) = G_{(\sigma)}(\mathbf{x},\mathbf{y})\,\pi_i^{\mathbf{1/y}}\,, \qquad (2.2.3)$$

for i such that $\ell(s_i\sigma) \le \ell(\sigma)$, as well as the symmetry

$$X_\sigma(\mathbf{x},\mathbf{y}) = (\text{-}1)^{\ell(\sigma)} X_{\sigma^{-1}}(\mathbf{y},\mathbf{x}) \quad \& \quad G_{(\sigma)}(\mathbf{x},\mathbf{y}) = G_{(\sigma^{-1})}(\mathbf{y}^\vee,\mathbf{x}^\vee)\,. \qquad (2.2.4)$$

Symmetry in consecutive variables can be seen on the indexing. Indeed, if i and v are such that $v_i \le v_{i+1}$, then Y_v and G_v are symmetrical in x_i, x_{i+1}, because they are equal to $Y_u\partial_i$ and $G_u\pi_i$ respectively, with $u = [\dots, v_{i+1}+1, v_i, \dots]$. Consequently, one has the following lemma.

[5] these expressions are not unique.

Lemma 2.2.2. *Let i, j, v be such that $v_i \le v_{i+1} \le \cdots \le v_j$. Then Y_v, G_v, K_v are symmetric in $x_i, \ldots, x_j$.*

In the case where $v \in \mathbb{N}^n$ is antidominant (i.e. $v = v\!\uparrow$), then Y_v, G_v, K_v are therefore symmetric in $x_1, \ldots, x_n$. In fact, let $\lambda = v\!\downarrow$ be the decreasing reordering of v. Then $K_v = x^\lambda \pi_\omega = x^{\lambda+\rho}\partial_\omega$ is equal to the Schur function $s_\lambda(\mathbf{x}_n)$, and $Y_v = Y_{\lambda+\rho}\partial_\omega$ specializes to $s_\lambda(\mathbf{x}_n)$ for $\mathbf{y} = \mathbf{0}$, because $Y_{\lambda+\rho}$ specializes to $x^{\lambda+\rho}$. The polynomial G_v, v antidominant, can also be considered as a deformation of a Schur function. It still possesses a determinantal expression. Geometrically, it is interpreted as the class of the structure sheaf of a Schubert variety in the Grothendieck ring of a Graßmannian and I described it in [81] by pure manipulation of determinants without using divided differences.

Let us call *Graßmannian* Schubert (resp. Grothendieck) polynomials. the polynomials indexed by antidominant v.

2.3 Flag complete and elementary functions

Both Schubert, Demazure and Grothendieck polynomials are non symmetric generalizations of the fundamental basis of symmetric functions that are Schur functions. In fact, the present notes will illustrate that many properties of the Schur basis can be extended to properties of the Y_v, K_v, G_v bases. But there are other bases of $\mathfrak{Sym}(\mathbf{x})$, particularly the products of elementary functions $e_i(\mathbf{x})$ and the products of complete functions $h_i(\mathbf{x})$. Let us generalize these into *flag elementary functions* and *flag complete functions*.

Definition 2.3.1. *For any r, any $v \in \mathbb{N}^r$, $v \le [r{-}1, \ldots, 0]$, let*

$$P_v = e_{v_1}(\mathbf{x}_{r-1}) \cdots e_{v_r}(\mathbf{x}_0)$$

and, for any n, any $v \in \mathbb{N}^n$, let

$$H_v = h_{v_1}(\mathbf{x}_1) \cdots h_{v_n}(\mathbf{x}_n)\,.$$

It is clear that $\{H_v : v \in \mathbb{N}^n\}$ is a linear basis of $\mathfrak{Pol}(\mathbf{x}_n)$, which is triangular in the basis of monomials. Identifying v and $0v$, one checks that $\cup_r \{P_v : v \in \mathbb{N}^r\}$ is also a linear basis of the space of polynomials in $x_1, x_2, \ldots$. Notice that the restriction on v eliminates the elementary functions which are null because of degree strictly higher than the cardinality of the alphabet. Beware that P_{v0} is different from P_v, because of the order we write the flag of alphabets. This change of convention for the indexing of the basis of flag elementary functions will be justified by the non-commutative extension of P_v.

It is not straightforward to express monomials in these two bases. For example,

$$\begin{aligned} x_2^2 &= P_{1,1,0,0} - P_{2,0,0,0} - P_{1,1,0} \\ &= (x_1 + x_2 + x_3)(x_1 + x_2) - (x_1x_3 + x_1x_2 + x_2x_3) - (x_1 + x_2)x_1 \end{aligned}$$

$$x_2^2 = H_{0,2} - H_{1,1} = (x_1^2 + x_1x_2 + x_2^2) - x_1(x_1 + x_2)\,.$$

We shall obtain such expansions by using a scalar product on polynomials.

More generally, monomials can be written as flag Schur functions. Let $v \in \mathbb{N}^n$, $u = [v_n, \ldots, v_1]$. Then [94, 1.4.10]

$$x^v = S_u(\mathbf{x}_n, \ldots, x_1) = \left| h_{u_j+j-i}(\mathbf{x}_{n+1-j}) \right|. \tag{2.3.1}$$

For example,

$$x^{0,3,1,2} = S_{2,1,3,0}(\mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) = \begin{vmatrix} h_2(\mathbf{x}_4) & h_2(\mathbf{x}_3) & h_5(\mathbf{x}_2) & h_3(\mathbf{x}_1) \\ h_1(\mathbf{x}_4) & h_1(\mathbf{x}_3) & h_4(\mathbf{x}_2) & h_2(\mathbf{x}_1) \\ h_0(\mathbf{x}_4) & h_0(\mathbf{x}_3) & h_3(\mathbf{x}_2) & h_1(\mathbf{x}_1) \\ 0 & 0 & h_2(\mathbf{x}_2) & h_0(\mathbf{x}_1) \end{vmatrix}.$$

Expanding by columns (but from the right!), one finds the expression of the monomial in the H-basis :

$$\begin{aligned} x^{0,3,1,2} = {} & H_{0,3,1,2} - H_{1,2,1,2} - H_{0,4,0,2} + H_{2,2,0,2} - H_{0,3,2,1} \\ & + H_{1,2,2,1} + H_{0,5,0,1} - H_{3,2,0,1} + H_{0,4,2,0} - H_{2,2,2,0} - H_{0,5,1,0} + H_{3,2,1,0}\,. \end{aligned}$$

The following proposition illustrates that Schur functions in $\mathbf{x}_n$ can also be easily expressed in these two bases, using flags of alphabets[6] in the Jacobi-Trudi determinants.

Proposition 2.3.2. *Let v be the increasing reordering of a partition λ, $u \in \mathbb{N}^r$ be the reordering of the conjugate $\lambda^\sim$. Then the Schur function $s_\lambda(\mathbf{x}_n)$, also denoted $S_v(\mathbf{x}_n)$, is equal to both determinants*

$$S_v(\mathbf{x}_1/\mathbf{x}_2/\ldots/\mathbf{x}_n) = \left| h_{v_j+j-i}(\mathbf{x}_i) \right|$$
$$\text{and} \quad \Lambda_u(\mathbf{x}_{n+r-1}/\mathbf{x}_{n+r-2}/\ldots/\mathbf{x}_n) = \left| e_{u_j+j-i}(\mathbf{x}_{n+r-i}) \right|. \tag{2.3.2}$$

The expansions of these determinants furnishes the required expressions of $s_\lambda(\mathbf{x}_n)$. For example, for $n = 3$, $\lambda = [4, 2]$, one has $\lambda^\sim = [2, 2, 1, 1]$ and

$$s_{42}(\mathbf{x}_3) = S_{024}(\mathbf{x}_1/\mathbf{x}_2/\mathbf{x}_3) = \begin{vmatrix} h_0(\mathbf{x}_1) & h_3(\mathbf{x}_1) & h_6(\mathbf{x}_1) \\ 0 & h_2(\mathbf{x}_2) & h_5(\mathbf{x}_2) \\ 0 & h_1(\mathbf{x}_3) & h_4(\mathbf{x}_3) \end{vmatrix}$$

$$\begin{vmatrix} e_1(\mathbf{x}_6) & e_2(\mathbf{x}_6) & e_4(\mathbf{x}_6) & e_5(\mathbf{x}_6) \\ e_0(\mathbf{x}_5) & e_1(\mathbf{x}_5) & e_3(\mathbf{x}_5) & e_4(\mathbf{x}_5) \\ 0 & e_0(\mathbf{x}_4) & e_2(\mathbf{x}_4) & e_3(\mathbf{x}_4) \\ 0 & 0 & e_1(\mathbf{x}_3) & e_2(\mathbf{x}_3) \end{vmatrix} = \Lambda_{1122}(\mathbf{x}_6/\mathbf{x}_5/\mathbf{x}_4/\mathbf{x}_3)\,,$$

[6] but this time, flags are constant by rows.

which entails

$$s_{4,2}(\mathbf{x}_3) = H_{0,2,4} - H_{0,5,1} = P_{1,1,2,2,0,0,0} - P_{1,1,3,1,0,0,0} - P_{2,0,2,2,0,0,0} + P_{4,0,0,2,0,0,0} + P_{2,0,3,1,0,0,0} + P_{1,4,0,1,0,0,0} - P_{1,3,0,2,0,0,0} - P_{5,0,0,1,0,0,0}\,.$$

Given i, there is at most one component of the function P_v and of the function H_v which is not symmetrical in x_i, x_{i+1}. Since

$$e_k(\mathbf{x}_i)\partial_i = \Big(e_k(\mathbf{x}_{i-1}) + x_i e_{k-1}(\mathbf{x}_{i-1})\Big)\,\partial_i = e_{k-1}(\mathbf{x}_{i-1})$$

and

$$h_k(\mathbf{x}_i)\,\pi_i = h_k(\mathbf{x}_{i+1})\,,$$

the image of $P_v = \cdots e_k(\mathbf{x}_i)e_\ell(\mathbf{x}_{i-1})\cdots$ under ∂_i is a flag $\cdots e_{k-1}(\mathbf{x}_{i-1})e_\ell(\mathbf{x}_{i-1})\cdots$ which is not permitted if $(k{-}1)\ell \neq 0$. Similarly, the image of $H_v = \cdots h_k(\mathbf{x}_i)h_\ell(\mathbf{x}_{i+1})\cdots$ under π_i,which is $\cdots h_k(\mathbf{x}_{i+1})h_\ell(\mathbf{x}_{i+1})\cdots$, is also illegal if $k\ell \neq 0$.

But, from the case of order 2 of (2.3.2), one has, with $\alpha = \min(k\text{-}1, \ell)$ and $\beta = \max(k\text{-}1, \ell)$,

$$e_{k-1}(\mathbf{x}_{i-1})e_\ell(\mathbf{x}_{i-1}) = \Big(e_\alpha(\mathbf{x}_i)e_\beta(\mathbf{x}_{i-1}) + e_{\alpha-1}(\mathbf{x}_i)e_{\beta+1}(\mathbf{x}_{i-1}) + \cdots + e_0(\mathbf{x}_i)e_{\beta+\alpha}(\mathbf{x}_{i-1})\Big) - \Big(e_{\beta+1}(\mathbf{x}_i)e_{\alpha-1}(\mathbf{x}_{i-1}) + \cdots + e_{\beta+\alpha}(\mathbf{x}_i)e_0(\mathbf{x}_{i-1})\Big)\,,$$

and, with $\alpha = \min(k, \ell)$, $\beta = \max(k, \ell)$,

$$h_k(\mathbf{x}_{i+1})h_\ell(\mathbf{x}_{i+1}) = \Big(h_\alpha(\mathbf{x}_i)h_\beta(\mathbf{x}_{i+1}) + \cdots + h_0(\mathbf{x}_i)h_{\beta+\alpha}(\mathbf{x}_{i+1})\Big) - \Big(h_{\beta+1}(\mathbf{x}_i)h_{\alpha-1}(\mathbf{x}_{i+1}) + \cdots + h_{\beta+\alpha}(\mathbf{x}_i)h_0(\mathbf{x}_{i+1})\Big)\,.$$

This entails the following actions of ∂_i and π_i.

Lemma 2.3.3. *Let n, i be two positive integers, $0 < i < n$, $v \in \mathbb{N}^n$ being such that $v \leq [n\text{-}1, \ldots, 0]$, $\alpha = \min(v_{n-i} - 1, v_{n-i+1})$, $\beta = \max(v_{n-i} - 1, v_{n-i+1})$. Then*

$$P_{\bullet\bullet v_{n-i}, v_{n-i+1}\bullet\bullet}\,\partial_i = \sum_{j=0}^{\alpha} P_{\bullet\bullet\alpha-j,\beta+j\bullet\bullet} - \sum_{j=1}^{\alpha} P_{\bullet\bullet\beta+j,\alpha-j\bullet\bullet}\,. \tag{2.3.3}$$

For $v \in \mathbb{N}^n$, $\alpha = \min(v_i, v_{i+1})$, $\beta = \max(v_i, v_{i+1})$, one has

$$H_{\bullet\bullet v_i, v_{i+1}\bullet\bullet}\,\pi_i = \sum_{j=0}^{\alpha} H_{\bullet\bullet\alpha-j,\beta+j\bullet\bullet} - \sum_{j=1}^{\alpha} H_{\bullet\bullet\beta+j,\alpha-j\bullet\bullet}\,. \tag{2.3.4}$$

For example,

$$\begin{aligned} P_{52\,03210}\,\partial_6 &= P_{24\,03210} + P_{15\,03210} + (P_{06\,03210}) - P_{51\,03210} - P_{60\,03210}\,, \\ H_{9\,26\,99}\,\pi_2 &= H_{9\,26\,99} + H_{9\,17\,99} + H_{9\,08\,99} - H_{9\,71\,99} - H_{9\,80\,99}\,, \end{aligned}$$

the term $P_{0603210}$ being null because $e_6(\mathbf{x}_5) = 0$.

2.4 Three scalar products

Let us first look for a scalar product on $\mathfrak{Pol}(n)$ compatible with the product structure and with degree.

When $n = 1$,

$$(f(x_1), g(x_1)) = CT\left(f(x_1), g(\frac{1}{x_1})\right) ,$$

where CT means "constant term", is a good candidate. Generalizing to $(f, g) = CT\left(f(x_1, \ldots, x_n), g(\frac{1}{x_1}), \ldots, \frac{1}{x_n})\right)$ means considering the ring of polynomials as a tensor product of rings of polynomials in 1 variable, a rather poor structure. Reversing the order of variables in the function g is not enough, one needs a kernel to link the variables.

We define

$$(f, g) = CT\Big(f(x_1, \ldots, x_n) g(x_n^{-1}, \ldots, x_1^{-1}) \prod_{1\le i<j\le n} (1 - x_i x_j^{-1})\Big), \qquad (2.4.1)$$

and write $\Omega_n = \prod_{1\le i<j\le n}(1 - x_i x_j^{-1})$ for the kernel.

Explicitely, for two monomials, $(x^u, x^v) = (x^{u_1-v_n,\ldots,u_n-v_1}, 1)$ and $(x^v, 1) \neq 0$ only when x^{-v} appears in the expansion of Ω_n. In that case $(x^v, 1) = \pm 1$ according to the sign x^{-v} has in Ω_n.

Similar definitions and properties hold for the root systems of type B, C, D (see later sections) with appropriate kernels $\Omega_n^B, \Omega_n^C, \Omega_n^D$.

For $n = 3$, one has

$$\Omega_3 = x^{000} - x^{1,-1,0} - x^{0,1,-1} + x^{2,-1,-1} + x^{1,1,-2} - x^{2,0,-2}$$

and therefore

$$(x^{000}, 1) = 1 = (x^{-2,1,1}, 1) = (x^{-1,-1,2}, 1) \;\&\; (x^{-1,1,0}, 1) = -1 = (x^{0,-1,1}, 1) = (x^{-2,0,2}, 1),$$

the other monomials being orthogonal to 1 (one has enumerated the positive and negative roots for type A_2).

Notice that, for symmetric functions, Weyl has defined the scalar product

$$(f, g)^{Weyl} = \frac{1}{n!} CT\Big(f(x_1, \ldots, x_n) g(x_1^{-1}, \ldots, x_n^{-1})\, \Omega_n^2\Big) .$$

We shall see that in the case of Schur functions

$$(s_\lambda, s_\mu) = (s_\lambda, s_\mu)^{Weyl} = \delta_{\lambda,\mu} ,$$

so that the restriction of all these scalar products to symmetric functions coincides with the usual scalar product with respect to which Schur functions constitute an orthonormal basis.

However, we have also to use the structure of $\mathfrak{Pol}(n)$ as a free $\mathfrak{Sym}(n)$-module. We define for $f, g \in \mathfrak{Pol}(n)$,

$$(f,g)^{\partial} := fg\,\partial_\omega \quad \& \quad (f,g)^{\pi} := fg\,\pi_\omega \,.$$

These quadratic forms take values in $\mathfrak{Sym}(n)$ and are $\mathfrak{Sym}(n)$-linear.

The main properties of all these quadratic forms is the compatibility with the operators used to define the different bases.

Proposition 2.4.1. *For $i:\ 1 \le i \le n{-}1$,*

- *π_i is adjoint to π_{n-i} with respect to $(\ ,\ )$,*
- *∂_i is self-adjoint with respect to $(\ ,\ )^{\partial}$,*
- *π_i is self-adjoint with respect to $(\ ,\ )^{\pi}$.*

Proof. Let us check that all these statements reduce to the case $n = 2$.

$$(f\partial_i, g)^{\partial} = \Big((f\partial_i)g\Big)\,\partial_\omega = (f\partial_i\, g)\partial_i\partial_{s_i\omega} = \Big((f\partial_i)(g\partial_i)\Big)\partial_{s_i\omega}\,.$$

The last expression being symmetrical in f, g, one has, indeed, $(f\partial_i, g)^{\partial} = (f, g\partial_i)^{\partial}$. The same computation applies to the case $(\ ,\ )^{\pi}$.

The kernel Ω_n can be written $\Omega'\,(1 - x_i x_{i+1}^{-1})$, with Ω' symmetrical in x_i, x_{i+1}, and one can first compute the constant term in x_i, x_{i+1}. Let us write $f = f_1 + x_{i+1}f_2$, $g(x_n^{-1}, \dots, x_1^{-1}) = h(x_1, \dots, x_n) = g_1 + x_{i+1}g_2$, with f_1, f_2, g_1, g_2 invariant under s_i. The difference $f\pi_i h - h\pi_i f = f\widehat{\pi}_i h - h\widehat{\pi}_i f$ is equal to $(f_1g_2 - g_1f_2)x_{i+1}$. Therefore the constant term

$$\begin{aligned} CT_{x_i,x_{i+1}}&\Big((f\pi_i h - h\pi_i f)\,(1 - x_i/x_{i+1})\Omega'\Big) \\ &= CT_{x_i,x_{i+1}}\Big((f\widehat{\pi}_i h - h\widehat{\pi}_i f)\,(1 - x_i/x_{i+1})\Omega'\Big) \\ &= CT_{x_i,x_{i+1}}\Big((x_{i+1} - x_i)\,(f_1g_2 - g_1f_2)\Omega'\Big) \end{aligned}$$

is null, because the function inside parentheses is antisymmetrical in x_i, x_{i+1}. Taking into account the transformation $x_i \to x_{n+1-i}^{-1}$, this nullity proves that π_i is adjoint to π_{n-i}. QED

Thanks to Proposition 2.4.1, the scalar products $(f, s_\lambda(\mathbf{x}_n))$ can be rewritten as scalar products with dominant monomials. Indeed $s_\lambda(\mathbf{x}_n) = x^\lambda\pi_\omega$, and therefore

$$\Big(f, s_\lambda(\mathbf{x}_n)\Big) = \Big(f, x^\lambda\pi_\omega\Big) = \Big(f\pi_\omega, x^\lambda\pi_\omega\Big) = \Big(f\pi_\omega, x^\lambda\Big)\,.$$

On the other hand,

$$\Big(f, s_\lambda(\mathbf{x}_n)\Big)^{\partial} = \Big(f, 1\Big)^{\partial} s_\lambda(\mathbf{x}_n) \quad \& \quad \Big(f, s_\lambda(\mathbf{x}_n)\Big)^{\pi} = \Big(f, 1\Big)^{\pi} s_\lambda(\mathbf{x}_n)\,,$$

since these last two scalar products are $\mathfrak{Sym}(\mathbf{x}_n)$-linear.

2.5 Kernels

With a scalar product and a basis defined by self-adjoint operators, it is easy to find the adjoint basis. Once more, it is sufficient to understand the case $n = 2$.

Lemma 2.5.1. *Let $i \in \{1, \dots, n\text{–}1\}$, $D_i = \pi_i$, $\widehat{D_i} = \widehat{\pi}_i$ (resp. $D_i = \partial_i = \widehat{D_i}$). Let $f, g \in \mathfrak{Pol}(n)$, $f' = fD_i$, $g' = g\widehat{D_i}$. Then the two equalities $(f, g)^D = 0$, $(f', g)^D = 1$ imply that $(f', g')^D = 0$ and that $(f, g')^D = 1$.*

Proof. Consider first the case $D_i = \pi_i$ and write $f = f_1 + x_{i+1}f_2$, $g = g_1 + x_{i+1}g_2$. Then $f' = f_1$, $g' = g(\pi_i - 1) = g_1 - g$. Consequently,

$$(f, g')^\pi = (f_1, g_1)^\pi - (f, g)^\pi = (f', g)^\pi = 1 \;\&\; (f', g')^\pi = (f_1, g_1)^\pi - (f_1, g)^\pi = 0\,.$$

The computation is similar for $D_i = \partial_i$. QED

This lemmma will allow propagating orthogonality relations. But to produce a hen, we need an egg, or conversely.

Let

$$\Theta_n^Y := \prod_{1 \le i < j \le n} (y_i - x_j) \qquad \& \qquad \Theta_n^G := \prod_{1 \le i < j \le n} (1 - x_j y_i^{-1})\,.$$

Lemma 2.5.2. *Let $v : \mathbf{0} \le \rho = [n\text{–}1, \dots, 0]$. Then*

$$(Y_v, \Theta_n^Y)^\partial = 0 = (G_v, \Theta_n^G)^\pi\,,$$

except for $v = \mathbf{0}$, in which case

$$(Y_\mathbf{0}, \Theta_n^Y)^\partial = 1 = (G_\mathbf{0}, \Theta_n^G)^\pi\,.$$

Proof. By definition, $(f(\mathbf{x}), \Theta_n^Y)^\partial = f(\mathbf{x})\Theta_n^Y\,\partial_\omega$ for any polynomial $f(\mathbf{x})$. If this polynomial belong to the span of $x^v : v \le \rho$, then $f(\mathbf{x})\Theta_n^Y$ belong to the span of $x^v : v \le [n\text{–}1, \dots, n\text{–}1]$ and its image under ∂_ω is a symmetric polynomial of degree 0 (only the monomials which are a permutation of x^ρ have a non-zero image). On the other hand, the scalar product can also be written as a sum :

$$(Y_v, \Theta_n^Y)^\partial = \sum_\sigma (\text{-}1)^{\ell(\sigma)} \Big(Y_v \Theta_n^Y\Big)^\sigma \frac{1}{\Delta(\mathbf{x})}\,.$$

Since this is a function of degree 0 in $\mathbf{x}$, one can specialize $\mathbf{x} = \mathbf{y}$ without changing its value. However, all $(\Theta_n^Y)^\sigma$ then vanish, except for the identity, in which case Θ_n^Y specializes to Δ. Therefore,[7] $(Y_v, \Theta_n^Y)^\partial = Y_v(\mathbf{y}, \mathbf{y}) = \delta_{v,\mathbf{0}}$.

The proof is similar for Grothendieck polynomials. QED

[7] The vanishing of $Y_v(\mathbf{y}, \mathbf{y})$, which is evident for dominant v, is proved following an induction which in fact furnishes more specializations. Thus we do not prove it at this point, but refer to Corollary 3.1.3 below.

2.6 Adjoint Schubert and Grothendieck polynomials

The ring $\mathfrak{Pol}(n)$ is a free $\mathfrak{Sym}(n)$-module with bases $\{x^v : v \le \rho\}$ and $\{x^{-v} : v \le \rho\}$ (one takes Laurent polynomials in the second case). Therefore $\{Y_v : v \le \rho\}$ and $\{G_v : v \le \rho\}$ are two linear bases. Starting with $\widehat{Y}_\rho := \Theta_n^Y$ and $\widehat{G}_\rho := \Theta_n^G$, instead of Y_ρ and G_ρ, one generates recursively two other bases

$$\widehat{Y}_{\dots,v_{i+1},v_i-1,\dots} = \widehat{Y}_v\, \partial_i \quad \& \quad \widehat{G}_{\dots,v_{i+1},v_i-1,\dots} = \widehat{G}_v\, \widehat{\pi}_i \text{ when } v_i > v_{i+1} \,. \tag{2.6.1}$$

Here are these bases for $n = 3$.

$$\widehat{Y}_{210} = \begin{array}{ll} (y_1 - x_3) & (y_2 - x_3) \\ (y_1 - x_2) & \end{array}$$

$$\widehat{Y}_{110} = (y_1 - x_3)(y_2 - x_3) \qquad \widehat{Y}_{200} = \begin{array}{l} (y_1 - x_3) \\ (y_1 - x_2) \end{array}$$

$$\widehat{Y}_{100} = y_1 + y_2 - x_2 - x_3 \qquad \widehat{Y}_{010} = y_1 - x_3$$

$$\widehat{Y}_{000} = 1$$

$$\widehat{G}_{210} = \begin{array}{ll} (1 - \frac{x_3}{y_1}) & (1 - \frac{x_3}{y_2}) \\ (1 - \frac{x_2}{y_1}) & \end{array}$$

$$\widehat{G}_{110} = \frac{x_2}{y_2}(1 - \frac{x_3}{y_1})(1 - \frac{x_3}{y_2}) \qquad \widehat{G}_{200} = \frac{x_3}{y_2}\begin{array}{l} (1 - \frac{x_3}{y_1}) \\ (1 - \frac{x_2}{y_1}) \end{array}$$

$$\widehat{G}_{100} = \frac{x_3}{y_1}\left(1 - \frac{x_2 x_3}{y_1 y_2}\right) \qquad \widehat{G}_{010} = \frac{x_2 x_3}{y_1 y_2}\left(1 - \frac{x_3}{y_1}\right)$$

$$\widehat{G}_{000} = \frac{x_2 x_3^2}{y_1^2 y_2}$$

Lemmas 2.5.1, 2.5.2 give the following pairs of adjoint bases.

Theorem 2.6.1. *The bases $\{Y_v : v \le \rho\}$ and $\{\widehat{Y}_v : v \le \rho\}$ are adjoint with respect to $(\ ,\ )^\partial$. The bases $\{G_v : v \le \rho\}$ and $\{\widehat{G}_v : v \le \rho\}$ are adjoint with respect to $(\ ,\ )^\pi$.*

More precisely, the pairing is

$$(Y_v, \widehat{Y}_u)^\partial = \delta_{v,\rho-u} = (G_v, \widehat{G}_u)^\pi \,. \tag{2.6.2}$$

The two bases $\{\widehat{Y}_v\}$ and $\{\widehat{G}_v\}$ can in fact be easily obtained as images of $\{Y_v\}$ and $\{G_v\}$ respectively. Indeed, Ω is obtained from Y_ρ by reversing the alphabet $\mathbf{x}$, but divided differences satisfy

$$\omega\, \partial_i\, \omega = -\partial_{n-i}\,. \tag{2.6.3}$$

Similarly, let $\clubsuit$ be the involution $x_i \to x_{n+1-i}^{-1}$. Then

$$\clubsuit\, \pi_i\, \clubsuit = \pi_{n-i} \quad \& \quad \omega x^{-\rho}\, \pi_i\, x^\rho \omega = -\widehat{\pi}_{n-i}\,. \tag{2.6.4}$$

Extend the involution to codes of permutations : $u\clubsuit = v$ if and only iff the corresponding permutations σ, ζ, are such that $\omega\sigma\omega = \zeta$. Then, the relations (2.6.3, 2.6.4) induce

Lemma 2.6.2. *The adjoint polynomials $\widehat{Y}_v$ and $\widehat{G}_v$ are related to the original ones by*

$$\left(\widehat{Y}_v\right)^\omega = (-1)^{|v|}\, Y_{v\clubsuit} \quad \& \quad \left(\widehat{G}_v\right)^\omega = (-1)^{|v|}\, G_{v\clubsuit} \frac{x^\rho}{y^\rho}\,. \tag{2.6.5}$$

As a consequence, for any $\sigma, \zeta \in \mathfrak{S}_n$, one has

$$\Big(X_\sigma(\mathbf{x}, \mathbf{y})\,,\, X_\zeta(\mathbf{x}^\omega, \mathbf{y})\Big)^\partial = (-1)^{\ell(\zeta)} \delta_{\sigma, \zeta\omega} \tag{2.6.6}$$

$$\left(G_{(\sigma)}(\mathbf{x}, \mathbf{y})\,,\, \left(\frac{x^\rho}{y^\rho} G_{(\zeta)}(\mathbf{x}, \mathbf{y})\right)^\omega\right)^\pi = (-1)^{\ell(\zeta)} \delta_{\sigma, \zeta\omega} \tag{2.6.7}$$

The decomposition of any polynomial in the Schubert or Grothendieck basis can easily be computed using the scalar products with their adjoint bases. Here is the matrix of change of basis between monomials x^v : $0 \geq v \geq [-2, -1, 0]$ and Grothendieck polynomials :

	000	100	010	200	110	210
$1/x^{000}$	1	0	0	0	0	0
$1/x^{100}$	$\frac{1}{y_1}$	$-\frac{1}{y_1}$	0	0	0	0
$1/x^{010}$	$\frac{1}{y_2}$	$\frac{1}{y_1}$	$-\frac{1}{y_2}$	0	$-\frac{1}{y_1}$	0
$1/x^{200}$	$\frac{1}{y_1^2}$	$-\frac{y_2+y_1}{y_1{}^2y_2}$	0	$\frac{1}{y_1y_2}$	0	0
$1/x^{110}$	$\frac{1}{y_1y_2}$	0	$-\frac{1}{y_1y_2}$	0	0	0
$1/x^{210}$	$\frac{1}{y_1{}^2y_2}$	$-\frac{1}{y_1{}^2y_2}$	$-\frac{1}{y_1{}^2y_2}$	$\frac{1}{y_1{}^2y_2}$	$\frac{1}{y_1{}^2y_2}$	$-\frac{1}{y_1{}^2y_2}$

2.7 Bases adjoint to elementary and complete functions

Expanding the kernels Θ_n^Y and Θ_n^G, one finds the bases adjoint to monomials, for the two scalar products $(\ ,\ )^\partial$ and $(\ ,\ )^\pi$.

Proposition 2.7.1. *Given n, let $\mathbf{x}^\vee = \{x_1^{-1}, \dots, x_n^{-1}\}$. Then for any $u, v : u \le \rho$, $v \le \rho$ one has*

$$\Big(P_{\rho-v}(\mathbf{x})\,,\,x^{u\omega}\Big)^\partial = (-1)^{|v|}\delta_{v,u} = \Big(P_v(\mathbf{x}^\vee)\,,\,x^{u\omega}\Big)^\pi\,. \tag{2.7.1}$$

The basis adjoint to $\{H_v : v \le \rho\}$ requires a little more work, because the monomials appearing in the expansion of H_v do not respect the condition that their exponent be majorized by ρ. We first some technical properties of divided differences.

Lemma 2.7.2. *Let $a, b, k, n \in \mathbb{N}$ be such that $1 \le k < n$, $0 \le a, b \le n{-}k$. Then*

$$S_{1^b}(\mathbf{x}_n - x_k)S_a(\mathbf{x}_k)\partial_k \dots \partial_1 = \begin{cases} (-1)^b \;\textit{if}\quad a{+}b = n{-}k \\ 0 \quad \textit{otherwise} \end{cases} \tag{2.7.2}$$

Proof. One expands $S_{1^b}(\mathbf{x}_n - x_k) = S_{1^b}(\mathbf{x}_n) - x_k S_{1^b}(\mathbf{x}_n) + \dots + (-x_k)^b$. On the other hand, $x_k^i S_a(\mathbf{x}_k) = S_{a+i}(\mathbf{x}_k) - \sum x^u$, sum over monomials x^u, $u \in \mathbb{N}^k$ such that $u_k \le i{-}1$. The initial function is therefore equal to

$$\Big(S_{1^b}(\mathbf{x}_n)S_a(\mathbf{x}_k) - S_{1^{b-1}}(\mathbf{x}_n)S_{a+1}(\mathbf{x}_k) - \dots + (-1)^b S_0(\mathbf{x}_n)S_{a+b}(\mathbf{x}_k)\Big) - \sum c_u x^u\,,$$

with $c_u \in \mathfrak{Sym}(\mathbf{x}_n)$ and $u\mathbb{N}^k$ such that $u_k \le b{-}1 < n{-}k$. The extra monomials x^u are sent to 0 by $\partial_k \dots \partial_{n-1}$ for degree reason. The sum inside parentheses is sent to

$$\begin{aligned} S_{1^b}(\mathbf{x}_n)S_{a-n+k}(\mathbf{x}_k) - S_{1^{b-1}}(\mathbf{x}_n)S_{a+1-n+k}(\mathbf{x}_k) - \dots \\ + (-1)^b S_0(\mathbf{x}_n)S_{a+b-n+k}(\mathbf{x}_k) = (-1)^b S_{a+b-n+k}(\mathbf{x}_n - \mathbf{x}_n)\,. \end{aligned}$$

This last function is different from 0 only in the case $S_0(\mathbf{x}_n{-}\mathbf{x}_n) = 1$, that is only for $a{+}b = n{-}k$. QED

Proposition 2.7.3. *Given n, for any $v \le \rho$, let $\widehat{H}_v = S_{1^{v_1}}(\mathbf{x}_n - x_1)S_{1^{v_2}}(\mathbf{x}_n - x_2)\dots S_{1^{v_{n-1}}}(\mathbf{x}_n - x_{n-1})$. Then*

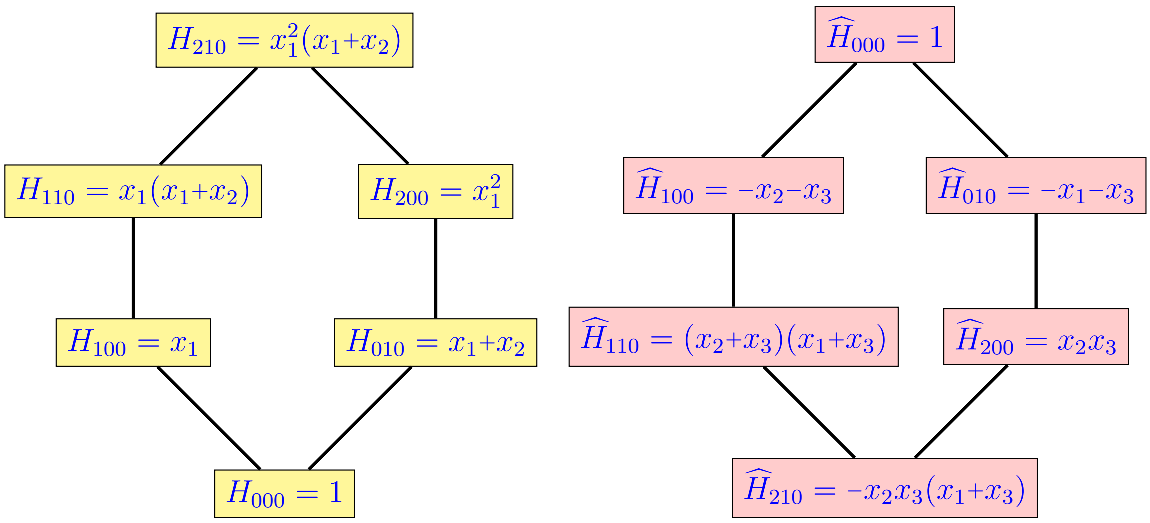
$$\Big(\widehat{H}_v\,,\,H_u\Big)^\partial = (-1)^{|v|}\delta_{v,\rho-u}\,,\;\; u, v \le \rho\,. \tag{2.7.3}$$

Proof. Factorize $\partial_\omega = (\partial_{n-1})(\partial_{n-2}\partial_{n-1})\dots(\partial_1 \dots \partial_{n-1})$. By decreasing induction on k, one has to compute

$$\begin{aligned} \Big(S_{1^{v_1}}(\mathbf{x}_n - x_1)\dots S_{1^{v_k}}(\mathbf{x}_n - x_k)\Big)\Big(S_{v_1}(\mathbf{x}_1)\dots S_{v_k}(\mathbf{x}_k)\Big)\partial_k \dots \partial_{n-1} \\ = f\,\big(S_{1^{v_k}}(\mathbf{x}_n - x_k)S_{v_k}(\mathbf{x}_k)\big)\,\partial_k \dots \partial_{n-1}\,, \end{aligned}$$

with f symmetrical in $x_k, \dots, x_n$, and therefore commuting with $\partial_k \dots \partial_{n-1}$. Eq. 2.7.2 forces the equality $v_k{+}u_k = n{-}k$, to have non nullity, and we can proceed with $k{-}1$. QED

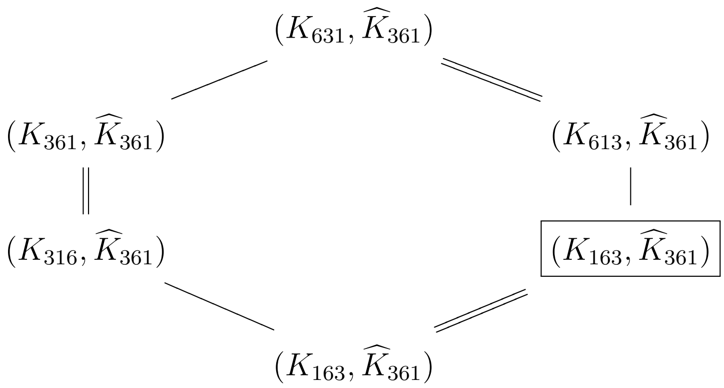
For example, for $n=3$, one has the following pair of adjoint bases.



2.8 Adjoint key polynomials

The two families $\{Y_v : v\in\mathbb{N}^n\}$, $\{G_v : v\in\mathbb{N}^n\}$ are bases of $\mathfrak{Pol}(n)$ (as a vector space). We have also given two other bases, $\{K_v : v\in\mathbb{N}^n\}$ and $\{\widehat{K}_v : v\in\mathbb{N}^n\}$, that are in fact adjoint with respect to $(\ ,\ )$, as states the next theorem.

First, one checks that for any partition λ, then $(K_v, x^\lambda)=0$, except when $v=\lambda\omega=[\lambda_n,\dots,\lambda_1]$, in which case $(K_{\lambda\omega}, x^\lambda)=1$ (cf. [37, Cor 12]). Using that π_i is adjoint to π_{n-i}, this allows to compute any $(K_v,\widehat{K}_u)$. For example, writing in a box the non-zero scalar products, the knowledge of all $(K_v,\widehat{K}_{361})$



determines all $(K_v, \widehat{K}_{316})$

$$\begin{tikzcd}
& (K_{631}, \widehat{K}_{316}) \arrow[dl, dash] \arrow[dr, equal] & \\
(K_{316}, \widehat{K}_{316}) \arrow[d, equal] & & \boxed{(K_{613}, \widehat{K}_{316})} \arrow[d, dash] \\
(K_{316}, \widehat{K}_{316}) \arrow[dr, dash] & & (K_{163}, \widehat{K}_{316}) \arrow[dl, equal] \\
& (K_{163}, \widehat{K}_{316}) &
\end{tikzcd}$$

In conclusion, one has the following property (cf. [37, Th 15]) :

Theorem 2.8.1. *Given $u, v \in \mathbb{N}^n$, then $(K_v, \widehat{K}_u) = 0$, except $(K_v, \widehat{K}_{v\omega}) = 1$.*

In particular, if λ is dominant, then $(K_v, x^\lambda) = 0$, except if $v = \lambda\omega$, in which case K_v is a Schur function.

Notice that the pairing, for Schubert and Grothendieck polynomials, is also the reversing $\sigma \to \sigma\omega$, when indexing these polynomials by permutations, but not when using codes.

2.9 Reproducing kernels for Schubert and Grothendieck

In the theory of orthogonal polynomials in one variable one finds it convenient to make use of *reproducing kernels* $K_n(x,y) = P_0(x)P_0(y) + \cdots + P_n(x)P_n(y)$, associated to a family of polynomials $P_0(x), P_1(x), \ldots$ of degree $0, 1, \ldots$, which are orthonormal with respect to a linear functional $f \to \int f$. The name "reproducing" comes from the property that

$$\int f(x) K_n(x,y) = f(y)$$

whenever f is a polynomial of degree $\leq n$.

The *Cauchy kernel* $\prod_{x\in\mathbf{x},y\in\mathbf{y}}(1-xy)^{-1}$ plays a similar role in the theory of symmetric polynomials. It does not require much effort nor imagination to deduce from the preceding section kernels corresponding to the bases $\{Y_v\}$, $\{G_v\}$ or $\{K_v\}$. Write $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$ for the identification of any symmetric function of x_n with the same symmetric function of $\mathbf{y}_n$.

Theorem 2.9.1. *For any $v : \mathbf{0} \leq v \leq \rho$, one has*

$$\left(\Theta_n^Y , x^v\right)^{\partial} = y^v \quad \& \quad \left(\Theta_n^G , x^{-v}\right)^{\pi} = y^{-v} . \tag{2.9.1}$$

For any Laurent polynomial f in x_n, one has, modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$,

$$\left(\Theta_n^Y , f(\mathbf{x})\right)^{\partial} \equiv f(\mathbf{y}) \quad \& \quad \left(\Theta_n^G , f(\mathbf{x})\right)^{\pi} \equiv f(\mathbf{y}) . \tag{2.9.2}$$

The two kernels expand as follows

$$\Theta_n^Y(\mathbf{x},\mathbf{z}) = \prod_{1\leq i<j\leq n} (z_i - x_j) \quad = \quad \sum_{v\leq\rho} Y_v(\mathbf{z},\mathbf{y})\, \widehat{Y}_{\rho-v}(\mathbf{x},\mathbf{y}) \tag{2.9.3}$$

$$\Theta_n^G(\mathbf{x},\mathbf{z}) = \prod_{1\leq i<j\leq n} (1 - x_j z_i^{-1}) \quad = \quad \sum_{v\leq\rho} G_v(\mathbf{z},\mathbf{y})\, \widehat{G}_{\rho-v}(\mathbf{x},\mathbf{y}) \tag{2.9.4}$$

There is no real need of a proof. The reproducing property has been obtained in the course of proving Lemma 2.5.2. Taking coefficients in $\mathfrak{Sym}(\mathbf{x}_n)$, one obtains (2.9.2) from (2.9.1). The function $\Theta_n^Y(\mathbf{x},\mathbf{z})$ belongs to the span of $\{z^u x^{v\omega} : u, v \leq \rho\}$, and therefore can be written

$$\Theta_n^Y(\mathbf{x},\mathbf{z}) = \sum_{u,v} c_{u,v}(\mathbf{y})\, Y_u(\mathbf{z},\mathbf{y}) \widehat{Y}_{\rho-v}(\mathbf{x},\mathbf{y}) .$$

Therefore, for any $v \leq \rho$, one has $\left(\Theta_n^Y(\mathbf{x},\mathbf{z}), Y_v(\mathbf{x},\mathbf{y})\right)^{\partial} = \sum_u c_{u,v}(\mathbf{y})\, Y_u(\mathbf{z},\mathbf{y})$. However, the reproducing property shows that this is also equal to $Y_v(\mathbf{z},\mathbf{y})$ and this proves (2.9.3), the case of Grothendieck polynomials being similar. QED

For example, for $n=2$, one has

$$\begin{aligned}\Theta_2^G(\mathbf{x},\mathbf{z}) = 1 - x_2/z_1 \quad &= \quad G_{00}(\mathbf{z},\mathbf{y})\widehat{G}_{10}(\mathbf{x},\mathbf{y}) + G_{10}(\mathbf{z},\mathbf{y})\widehat{G}_{00}(\mathbf{x},\mathbf{y}) \\ &= \quad 1 \cdot \left(1 - \frac{x_2}{y_1}\right) \qquad + \qquad \left(1 - \frac{y_1}{z_1}\right) \cdot \frac{x_2}{y_1} .\end{aligned}$$

For $n=3$, Maple computes

$$\Theta_3^Y(\mathbf{x},\mathbf{z}) = (z_1-x_2)(z_1-x_3)(z_2-x_3) = -(-y_1+x_2)(-y_1+x_3)(-y_2+x_3) \\ +(z_2-y_2+z_1-y_1)(-y_1+x_3)(-y_1+x_2) - (-z_1+y_2)(z_1-y_1)(y_1-x_3) \\ +(z_1-y_1)(-y_2+x_3)(-y_1+x_3) + (-y_1+z_2)(z_1-y_1)(-x_2+y_2+y_1-x_3) \\ -(-z_1+y_2)(z_1-y_1)(-y_1+z_2)\,.$$

The essential property of $\Theta_n^Y(\mathbf{x},\mathbf{y})$ and $\Theta_n^G(\mathbf{x},\mathbf{y})$ is that $\Theta_n^Y(\mathbf{y}^\sigma,\mathbf{y})$ and $\Theta_n^G(\mathbf{y}^\sigma,\mathbf{y})$ both vanish when σ is different from the identity. Along the same lines as for Θ_n^Y and Θ_n^G, one sees that the kernels $Y_\rho(\mathbf{x},\mathbf{y})$ and $G_\rho(\mathbf{x},\mathbf{y})$ satisfy a twisted reproduction property :

$$\Big(Y_\rho(\mathbf{x},\mathbf{y})\,,\, f(\mathbf{x})\Big)^\partial \equiv f(\mathbf{y}^\omega) \quad \& \quad \Big(G_\rho(\mathbf{x},\mathbf{y}) \;,\, f(\mathbf{x})\Big)^\pi \equiv f(\mathbf{y}^\omega)\,, \tag{2.9.5}$$

modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$, the equivalence being replaced by an equality when f belongs to the span of $\{x^v : [0,\dots,0] \le v \le [0,\dots,n{-}1]\}$. For example,

$$(G_{210}(\mathbf{x},\mathbf{y}), x_3^2)^\pi = \left(1-\frac{y_1}{x_1}\right)\left(1-\frac{y_2}{x_1}\right)\left(1-\frac{y_1}{x_2}\right) x_3^2\,\pi_{321} = y_1^2\,.$$

Notice that, using (2.2.4) and (2.6.5), exchanging the role of $\mathbf{y}$ and $\mathbf{x}$, one can rewrite (2.9.4) into

$$\sum_{v\le\rho} (-1)^{|v|} G_v(\mathbf{x},\mathbf{z})\, G_{\rho-v}(\mathbf{x},\mathbf{y}) = Y_\rho(\mathbf{z},\mathbf{y})\, x^{-\rho}\,. \tag{2.9.6}$$

By taking the image of (2.9.3) under products of ∂_i's and the image of (2.9.4) under products of $\widehat{\pi}_i$'s, one obtains decompositions of general $\widehat{Y}_v$ or general $\widehat{G}_v$, and by involution, of general Y_v and G_v. Let us detail these decompositions in the next sections.

2.10 Cauchy formula for Schubert

Given $u,v,w\in\mathbb{N}^n$, majorized by ρ, write $w=u\odot v$ iff and only the permutations $\sigma(w),\sigma(u),\sigma(v)$ of which they are the codes, are such that $\sigma(w)=\sigma(u)\sigma(v)$ and the product is reduced[8]. With this notation one has the following Cauchy formula for Schubert polynomials (given in [84] for $\mathbf{y}=\mathbf{0}$).

Theorem 2.10.1. *Let σ be a permutation in $\mathfrak{S}_n$, $w\in\mathbb{N}^n$ be its code. Then*

$$Y_w(\mathbf{x},\mathbf{z}) = \sum_{u,v:\, u\odot v = w} Y_u(\mathbf{y},\mathbf{z}) Y_v(\mathbf{x},\mathbf{y}) \tag{2.10.1}$$

$$X_\sigma(\mathbf{x},\mathbf{z}) = \sum_{\eta,\nu:\, \partial_\eta\partial_\nu=\partial_\sigma} X_\eta(\mathbf{y},\mathbf{z})\, X_\nu(\mathbf{x},\mathbf{y})\,. \tag{2.10.2}$$

[8] i.e. such that lengths add: $\ell(\sigma(w)) = \ell(\sigma(u)) + \ell(\sigma(v))$. Notice that the product of two permutations η,ν is *reduced* if and only if $\partial_\eta\partial_\nu = \partial_{\eta\nu}$.

Proof. One starts from the formula in the case $\sigma = \omega$, which is a rewriting of (2.9.3) using (2.6.5). Supposing (2.10.2) to be true for σ, let i be such that $\ell(\sigma s_i) < \ell(\sigma)$. The terms in the RHS are of two types: either $\ell(\nu s_i) < \ell(\nu)$, or not. These last terms are such that $X_\nu(\mathbf{x},\mathbf{y})\partial_i = 0$. Therefore the image of (2.10.2) under ∂_i is

$$X_{\sigma s_i}(\mathbf{x},\mathbf{z}) = \sum_{\eta,\zeta:\, \partial_\eta \partial_\zeta = \partial_{\sigma s_i}} X_\eta(\mathbf{y},\mathbf{z})\, X_\zeta(\mathbf{x},\mathbf{y})\,,$$

with $\zeta = \nu s_i$. QED

For example, for $w = [0,3,1]$, one has the following expansion of $Y_{031}(\mathbf{x},\mathbf{z})$, writing $Y_u\, Y_v$ for $Y_u(\mathbf{y},\mathbf{z})Y_v(\mathbf{x},\mathbf{y})$

$$Y_{031}(\mathbf{x},\mathbf{z}) = \begin{tikzcd}[column sep=small]
 & Y_{0000}Y_{031} \arrow[dl, dash] \arrow[dr, dash, dashed] & & \\
Y_{0100}Y_{030} \arrow[dr, dash, dashed] & & Y_{0001}Y_{021} \arrow[dl, dash] \arrow[dr, equal] & \\
 & Y_{0101}Y_{020} \arrow[d, equal] & & Y_{002}Y_{011} \arrow[d, dash] \\
 & Y_{012}Y_{010} \arrow[dr, dash] & & Y_{03}Y_{001} \arrow[dl, equal] \\
 & & Y_{031}Y_{000} &
\end{tikzcd} ,$$

or, indexing by permutations,

$$X_{15324}(\mathbf{x},\mathbf{z}) = \begin{tikzcd}[column sep=small]
 & X_{12345}X_{15324} \arrow[dl, dash] \arrow[dr, dash, dashed] & & \\
X_{13245}X_{15234} \arrow[dr, dash, dashed] & & X_{12354}X_{14325} \arrow[dl, dash] \arrow[dr, equal] & \\
 & X_{13254}X_{14235} \arrow[d, equal] & & X_{12534}X_{13425} \arrow[d, dash] \\
 & X_{13524}X_{13245} \arrow[dr, dash] & & X_{15234}X_{12435} \arrow[dl, equal] \\
 & & X_{15324}X_{12345} &
\end{tikzcd} .$$

In these last conventions, the edges are simple transpositions: $X_\eta X_{s_i\zeta} \to X_{\eta s_i} X_\zeta$.

Notice that the above decomposition of $Y_\rho(\mathbf{x},\mathbf{z}) = \prod_{i+j\le n}(x_i - z_j)$, becomes similar, when specializing $\mathbf{y} = \mathbf{0}$, to the Cauchy expansion of the resultant $\prod_{i,j\le n}(x_i - z_j)$ in terms of Schur functions in $\mathbf{x}$ and in $\mathbf{z}$. In fact, let m, r be two integers such that $r + m < n$. Then the special case of (2.10.1) for $w = r^m$, $\mathbf{y} = \mathbf{0}$ is

$$Y_{r^m}(\mathbf{x},\mathbf{z}) = \sum_{u,v:\, u\odot v = w} Y_u(\mathbf{0},\mathbf{z})Y_v(\mathbf{x},\mathbf{0}) = \sum_{\lambda \le r^m} (-1)^{|\mu|} s_\mu(\mathbf{z}_r) s_\lambda(\mathbf{x}_m)\,, \qquad (2.10.3)$$

sum over all pairs of partitions λ, μ such that the conjugate of μ is $[r-\lambda_m, \dots, r-\lambda_1]$.

2.11 Cauchy formula for Grothendieck

The analogous formula for Grothendieck polynomials is not more complicated. Instead of taking reduced products, i.e. products $\partial_\eta \partial_\nu \neq 0$, one has to use products in the 0-Hecke algebra, of the type $\pi_\eta \pi_\nu$.

Theorem 2.11.1. *Let σ be a permutation in $\mathfrak{S}_n$, $\omega = [n, \dots, 1]$.*

$$\widehat{G}_{(\sigma)}(\mathbf{x}, \mathbf{z}) = \sum_{\zeta \in \mathfrak{S}_n} G_{(\zeta)}(\mathbf{z}, \mathbf{y}) \widehat{G}_{(\omega\zeta)}(\mathbf{x}, \mathbf{y}) \, \widehat{\pi}_{\omega\sigma} \tag{2.11.1}$$

$$\frac{y^\rho}{z^\rho} G_{(\sigma)}(\mathbf{x}, \mathbf{z}) = \sum_{\zeta} (\text{-}1)^{\ell(\zeta)} G_{(\zeta\omega)}(\mathbf{z}, \mathbf{y}) \left(G_{(\zeta)}(\mathbf{x}, \mathbf{y}) \pi_{(\omega\sigma)} \right) \tag{2.11.2}$$

Proof. The first formula is the image of (2.9.4) under $\widehat{\pi}_{\omega\sigma}$, the second is the image of the case $\sigma = \omega$, which is a rewriting of (2.9.6), under $\pi_{\omega\sigma}$. QED

For example, for $n = 3$, writing G_v for $G_v(\mathbf{z}, \mathbf{y})$ and $\widehat{G}_v$ for $\widehat{G}_v(\mathbf{x}, \mathbf{y})$, the image of $\widehat{G}_{210}(\mathbf{x}, \mathbf{z}) = \sum_v G_v \widehat{G}_{210-v}$ under $\widehat{\pi}_1$ is

$$\widehat{G}_{110}(\mathbf{x}, \mathbf{z}) = (G_{110} - G_{210}) \, \widehat{G}_{000} + (G_{010} - G_{200}) \, \widehat{G}_{010} + (G_{000} - G_{100}) \, \widehat{G}_{110} \, ,$$

then under $\widehat{\pi}_2$,

$$\widehat{G}_{100}(\mathbf{x}, \mathbf{z}) = (G_{010} - G_{200} - G_{110} + G_{210}) \, \widehat{G}_{000} + (G_{000} - G_{100}) \, \widehat{G}_{100} \, .$$

2.12 Divided differences as scalar products

Since the ∂_i's are self-adjoint with respect to $(\ ,\ )^{\partial}$, and the π_i's are self-adjoint with respect to $(\ ,\ )^{\pi}$, one can use (2.9.1) to express any $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma$.

Proposition 2.12.1. *Let $f \in \mathfrak{Pol}(\mathbf{x}_n, \mathbf{y}_n)$, $\sigma \in \mathfrak{S}_n$, and $\mathbf{z} = \mathbf{z}_n$ be an extra alphabet. Then*

$$f\,\partial_\sigma = \Big(f\,,\, X_{\omega\sigma}(\mathbf{z}, \mathbf{x}^\omega)\Big)^{\partial}\Big|_{\mathbf{z}=\mathbf{x}} \tag{2.12.1}$$

$$f\,\pi_\sigma = \Big(f\,,\, G_{(\omega\sigma^{-1})}(\mathbf{x}, \mathbf{z})\Big)^{\pi}\Big|_{\mathbf{z}=\mathbf{x}^\omega} \tag{2.12.2}$$

$$f\,\widehat{\pi}_\sigma = \Big(f\,,\, \widehat{G}_{(\omega\sigma^{-1})}(\mathbf{x}, \mathbf{z})\Big)^{\pi}\Big|_{\mathbf{z}=\mathbf{x}} \tag{2.12.3}$$

Proof. The proofs of the three assertions are similar, let us consider only the first one.

$$(-1)^{\ell(\omega\sigma)} X_{\omega\sigma}(\mathbf{z}, \mathbf{x}^\omega) = X_{\sigma^{-1}\omega}(\mathbf{x}, \mathbf{z})\,\omega = X_\omega(\mathbf{x}, \mathbf{z})\partial_{\omega\sigma^{-1}\omega}\,\omega$$
$$= X_\omega(\mathbf{x}, \mathbf{z})\omega\Big(\omega\partial_{\omega\sigma^{-1}\omega}\omega\Big) = X_\omega(\mathbf{x}^\omega, \mathbf{z})\partial_{\sigma^{-1}}(-1)^{\ell(\sigma)}$$

and therefore one has

$$\Big(f\,,\, X_{\omega\sigma}(\mathbf{z}, \mathbf{x}^\omega)\Big)^{\partial} = \Big(f\,,\, (-1)^{\ell(\omega)} X_\omega(\mathbf{x}^\omega, \mathbf{z})\partial_{\sigma^{-1}}\Big)^{\partial} = \Big(f\partial_\sigma,,\, X_\omega(\mathbf{z}, \mathbf{x}^\omega)\Big)^{\partial}.$$

Specializing $\mathbf{z} = \mathbf{x}$ and using the reproducing property (2.9.1), one gets (2.12.1). QED

For example, for $n = 3$, $\sigma = [2, 3, 1]$, one has $\omega\sigma = [2, 1, 3]$, $\omega\sigma^{-1} = [1, 3, 2]$, and

$$f\partial_{231} = f\partial_1\partial_2 = \Big(f\,,\, X_{213}(\mathbf{z}, \mathbf{x}^\omega)\Big)^{\partial}\Big|_{\mathbf{z}=\mathbf{x}} = \Big(f\,,\, z_1 - x_3\Big)^{\partial}\Big|_{\mathbf{z}=\mathbf{x}}$$
$$f\pi_{231} = f\pi_1\pi_2 = \Big(f\,,\, G_{(132)}(\mathbf{x}, \mathbf{z})\Big)^{\pi}\Big|_{\mathbf{z}=\mathbf{x}^\omega} = \Big(f\,,\, 1 - \frac{z_1 z_2}{x_1 x_2}\Big)^{\pi}\Big|_{\mathbf{z}=\mathbf{x}^\omega}$$
$$f\widehat{\pi}_{231} = f\widehat{\pi}_1\widehat{\pi}_2 = \Big(f\,,\, \widehat{G}_{(132)}(\mathbf{x}, \mathbf{z})\Big)^{\pi}\Big|_{\mathbf{z}=\mathbf{x}} = \Big(f\,,\, \frac{x_2 x_3}{z_1 z_2}\Big(1 - \frac{x_3}{y_1}\Big)\Big)^{\pi}\Big|_{\mathbf{z}=\mathbf{x}}.$$

2.13 Divided differences in terms of permutations

Let $D = \sum_{\zeta\in\mathfrak{S}_n} \zeta c_\zeta(\mathbf{x}_n)$ be a sum of permutations with coefficients which are rational functions in $\mathbf{x}_n$. Any function $f(\mathbf{x}_n, \mathbf{y}_n)$ which vanish in all specializations $\mathbf{x}_n^\sigma = \mathbf{y}_n$, except in $\mathbf{x}_n = \mathbf{y}_n$, can be used to determine the coefficients $c_\zeta(\mathbf{x}_n)$. Indeed, putting $g(\mathbf{x}_n, \mathbf{y}_n) = f(\mathbf{x}_n, \mathbf{y}_n)\, D$, one has $g(\mathbf{x}_n, \mathbf{y}_n) = \sum_\zeta f(\mathbf{x}_n^\zeta, \mathbf{y}_n)\, c_\zeta(\mathbf{x}_n)$, and therefore

$$g(\mathbf{x}_n, \mathbf{x}_n^\zeta) = f(\mathbf{x}_n^\zeta, \mathbf{x}_n^\zeta)\, c_\zeta(\mathbf{x}_n)\,. \qquad (2.13.1)$$

The kernels Θ_n^Y, Θ_n^G have the required vanishing properties. In consequence the operators $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma$ can be expressed in terms of specializations of Schubert or Grothendieck polynomials, and one obtains the following expansions (the expression of the coefficients are not unique, due to the many symmetries of Schubert and Grothendieck polynomials).

Proposition 2.13.1. *Given $\sigma \in \mathfrak{S}_n$, the divided differences $\partial_\sigma, \pi_\sigma, \widehat{\pi}_\sigma$ are equal to the following sums of permutations :*

$$\begin{aligned}
\partial_\sigma \prod_{i<j\le n} (x_i - x_j) &= \sum_{\zeta\le\sigma} (-1)^{\ell(\zeta)} \zeta\, X_{\omega\sigma}(\mathbf{x}_n, \mathbf{x}_n^{\zeta^{-1}\omega}) && (2.13.2)\\
\pi_\sigma &= \sum_{\zeta\le\sigma} \zeta\, f_\sigma(\mathbf{x}_n^{\zeta^{-1}}, \mathbf{x}_n^{\omega}) && (2.13.3)\\
\widehat{\pi}_\sigma \prod_{i<j\le n} \left(1 - \frac{x_i}{x_j}\right) &= \sum_{\zeta\le\sigma} \zeta\, G_{(\sigma\omega)}(\mathbf{x}_n^{\omega}, \mathbf{x}_n^{\zeta^{-1}})\,, && (2.13.4)
\end{aligned}$$

with $f_\sigma(\mathbf{x}_n, \mathbf{y}_n) = G_{(\omega\sigma^{-1})}(\mathbf{x}_n, \mathbf{y}_n) \prod_{i<j\le n} (1 - x_j x_i^{-1})^{-1}$.

For example

$$\begin{aligned}
\partial_1\partial_2 &= \Big(s_1 s_2 (x_1{-}x_2) - s_2 (x_1{-}x_2) - (x_1{-}x_3)\, s_1 + (x_1{-}x_3)\Big) \frac{1}{\Delta(\mathbf{x}_3)}\\
\pi_1\pi_2 &= s_1 s_2 \frac{x_3^2}{(x_1{-}x_3)(x_2{-}x_3)} - s_2 \frac{x_1 x_3}{(x_1{-}x_3)(x_2{-}x_3)} - s_1 \frac{x_2^2}{(x_1{-}x_2)(x_2{-}x_3)}\\
&\qquad + \frac{x_1 x_2}{(x_1{-}x_2)(x_2{-}x_3)}\\
\widehat{\pi}_1\widehat{\pi}_2 &= (s_1 s_2 - s_2) \frac{x_3^2}{(x_1{-}x_3)(x_2{-}x_3)} + (1 - s_1) \frac{x_2 x_3}{(x_1{-}x_2)(x_2{-}x_3)}\,.
\end{aligned}$$

One can compare these expressions to those given in the preceding section. In fact, they can be obtained by mere expansion of

$$\begin{aligned}
\partial_1\partial_2 &= (1 - s_1)\frac{1}{x_1 - x_2}\,(1 - s_2)\frac{1}{x_2 - x_3}\\
\pi_1\pi_2 &= \left(s_1 \frac{x_2}{x_2 - x_1} + \frac{x_1}{x_1 - x_2}\right)\left(s_2 \frac{x_3}{x_3 - x_2} + \frac{x_2}{x_2 - x_3}\right)\\
\widehat{\pi}_1\widehat{\pi}_2 &= (s_1 - 1)\frac{1}{1 - x_1 x_2^{-1}}\,(s_2 - 1)\frac{1}{1 - x_2 x_3^{-1}}\,.
\end{aligned}$$

This is essentially the method followed by Kostant and Kumar [71, 72], but with this method properties of the resulting coefficients are more difficult to extract than when specializing polynomials in two sets of variables. For example we shall see later that the inverse transition matrices, from permutations to the different types of divided differences, involve the same coefficients as the transition matrices, and this fact can easily be obtained from properties of Schubert and Grothendieck polynomials.

The leading term of π_σ and $\widehat{\pi}_\sigma$, i.e. the coefficient of σ, is obtained by mere commutation. Taking a reduced decomposition $\sigma = s_i s_j s_h \cdots s_k$, then this leading term is

$$s_i \frac{1}{1 - x_i x_{i+1}^{-1}} \, s_j \frac{1}{1 - x_j x_{j+1}^{-1}} \, s_h \cdots s_k \frac{1}{1 - x_k x_{k+1}^{-1}}$$
$$= s_i \cdots s_k \left(\frac{1}{1 - x_i x_{i+1}^{-1}} \right)^{s_j s_h \cdots s_k} \left(\frac{1}{1 - x_j x_{j+1}^{-1}} \right)^{s_h \cdots s_k} \cdots \frac{1}{1 - x_k x_{k+1}^{-1}} \, .$$

In the language of root systems, this property reads as follows.

Lemma 2.13.2. *Let Φ^+, Φ^- be the positive (resp. negative) roots of the root system of type A_{n-1}. Then, in the basis of permutations, π_σ and $\widehat{\pi}_\sigma$ have leading term*

$$F(\sigma) := \prod_{\alpha \in \Phi^+ \cap \sigma \Phi^-} \frac{1}{1 - e^\alpha} \, .$$

This leading term intervenes in geometry, for what concerns the postulation of Schubert varieties.

Let $\lambda \in \mathbb{N}^n$ be dominant weight , v be a permutation of λ, $\sigma \in \mathfrak{S}_n$ be of minimum length such that $v = \lambda\sigma$. One defines the limit $m \to \infty$ of $K_{mv} x^{-mv}$ to be

$$(1 - zx^\lambda)^{-1} \pi_\sigma (1 - zx^v) \Big|_{z = x^{-v}} \, .$$

Expanding π_σ in terms of permutations, one has

$$(1 - zx^\lambda)^{-1} \pi_\sigma (1 - zx^v) = F(\sigma) + \sum_{\zeta < \sigma} \frac{1 - zx^v}{1 - zx^{\lambda\zeta}} \, c_\sigma^\zeta \, ,$$

with coefficients c_σ^ζ obtained in (2.13.3). The hypothesis on the pair λ, σ insures that all terms, but the first one, vanish under the specialization $z = x^{-v}$. One thus recovers in the special case of type A a property due to Peterson and Kumar in the more general context of Kac-Moody algebras.

Corollary 2.13.3. *Let $\lambda \in \mathbb{N}^n$ be dominant, $\sigma \in \mathfrak{S}_n$ be of minimum length modulo the stabilizer of λ. Then the common limit $m \to \infty$ of $x^{m\lambda} \pi_\sigma x^{-m\lambda\sigma}$ and $x^{m\lambda} \widehat{\pi}_\sigma x^{-m\lambda\sigma}$ is equal to*

$$\prod_{\alpha \in \Phi^+ \cap \sigma \Phi^-} \frac{1}{1 - e^\alpha} \, .$$

For example, for $\lambda = [2, 1, 0]$, $v = [1, 0, 2]$, one has $\sigma = s_1 s_2$ and the limit of $K_{m,0,2m} x^{-m,0,-2m}$ and $\widehat{K}_{m,0,2m} x^{-m,0,-2m}$ is equal to $\big((1 - x_1 x_3^{-1})(1 - x_2 x_3^{-1})\big)^{-1}$. The limit of $K_{0,0,m} x^{0,0,-m} = S_m(x_1 + x_2 + x_3) x_3^{-m} = S_m(x_1 x_3^{-1} + x_2 x_3^{-1} + 1)$ is also $\big((1 - x_1 x_3^{-1})(1 - x_2 x_3^{-1})\big)^{-1}$, in accordance with the fact that σ is still equal to $s_1 s_2$.

2.14 Schubert, Grothendieck and Demazure as commutation factors

One could obtain the expression of permutations in terms of divided differences by iterating Leibnitz formula, starting with expressions like

$$s_2 s_1 s_2 = \Big(1 + \partial_2 (x_3 {-} x_2)\Big)\Big(1 + \partial_1 (x_2 {-} x_1)\Big)\Big(1 + \partial_2 (x_3 {-} x_2)\Big)\,.$$

Let us specially examine the commutation with ∂_ω or π_ω. For example,

$$\begin{aligned}
&\partial_1 x_2 = x_1 \partial_1 - 1 \\
&\qquad \partial_2\partial_1\partial_2 x_2 x_3^2 = \partial_2 x_3 \partial_1 x_2 \partial_2 x_3 = (x_2\partial_2 - 1)(x_1\partial_1 - 1)(x_2\partial_2 - 1) = \dots \\
&\qquad\qquad = x^{210}\partial_2\partial_1\partial_2 - x^{200}\partial_1\partial_2 - x^{110}\partial_2\partial_1 + x^{100}\partial_1 + (x^{100} + x^{010})\partial_2 - 1\,.
\end{aligned}$$

This case shows a disymmetry which can be cured by using Schubert polynomials instead of monomials :

$$\begin{aligned}
\partial_2\partial_1\partial_2 x_2 x_3^2 = {}& Y_{210}(\mathbf{x}, \mathbf{0})\partial_1\partial_2 - Y_{200}(\mathbf{x}, \mathbf{0})\partial_2\partial_1\partial_2 - Y_{110}(\mathbf{x}, \mathbf{0})\partial_2\partial_1 \\
& + Y_{100}(\mathbf{x}, \mathbf{0})\partial_1 + Y_{010}(\mathbf{x}, \mathbf{0})\partial_2 - Y_{000}(\mathbf{x}, \mathbf{0})\,.
\end{aligned}$$

The following theorem states that Schubert and Grothendieck polynomials do occur in the commutation of some element with ∂_ω or π_ω. Notice that this gives a generation which does not require division.

Theorem 2.14.1. *Fixing n, with $\rho = [n{-}1, \dots, 0]$, one has*

$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)} X_\sigma(\mathbf{x}, \mathbf{y})\, \partial_{\sigma^{-1}} = \partial_\omega X_\omega(\mathbf{y}, \mathbf{x}^\omega) \tag{2.14.1}$$

$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma\omega)} \partial_\sigma\, X_\sigma(\mathbf{x}, \mathbf{y}) = X_\omega(\mathbf{y}, \mathbf{x}^\omega)\partial_\omega \tag{2.14.2}$$

$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)} x^\rho G_{(\sigma)}(\mathbf{x}, \mathbf{y})\, \pi_{\sigma^{-1}} = \pi_\omega\, X_\omega(\mathbf{y}, \mathbf{x}^\omega) \tag{2.14.3}$$

$$\sum_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)} \pi_\sigma\, G_{(\sigma)}(\mathbf{x}, \mathbf{y}) x^\rho = X_\omega(\mathbf{y}, \mathbf{x}^\omega)\, \pi_\omega\,. \tag{2.14.4}$$

Proof. (2.14.1) and (2.14.2) are equivalent, by left-right symmetry of the Leibnitz relations. Let us prove (2.14.2). The factor $X_\omega(\mathbf{y}, \mathbf{x}^\omega)$ is the reproducing kernel

Θ_n^Y, and therefore (2.14.2) can be proved by checking that, for any $f(\mathbf{x})$ in the linear span of $\langle x : 0 \le v \le \rho \rangle$, one has

$$\sum (-1)^{\ell(\sigma)} f(x) \partial_\sigma X_\sigma(\mathbf{x},\mathbf{y}) = f(y)\,.$$

Introducing an extra alphabet $\mathbf{z}$, one needs a single check,

$$X_\omega(\mathbf{y},\mathbf{z}) = \sum_\sigma (-1)^{\ell(\sigma)} X_\omega(\mathbf{x},\mathbf{z}) \partial_\sigma X_\sigma(\mathbf{x},\mathbf{y}) = \sum_\sigma (-1)^{\ell(\sigma)} X_{\omega\sigma}(\mathbf{x},\mathbf{z}) X_\sigma(\mathbf{x},\mathbf{y})\,.$$

But this is the Cauchy formula

$$X_\omega(\mathbf{y},\mathbf{z}) = \sum_\sigma X_{\omega\sigma}(\mathbf{x},\mathbf{z}) X_{\sigma^{-1}}(\mathbf{y},\mathbf{x})\,.$$

Similarly, (2.14.4) is proved by checking the action on $G_{(\omega)}(\mathbf{x},\mathbf{z})$. Thanks to (2.9.6), one has

$$\begin{aligned} G_{(\omega)}(\mathbf{x},\mathbf{z}) \sum (-1)^{\ell(\sigma)} \pi_\sigma G_{(\sigma)}(\mathbf{x},\mathbf{y})\, x^\rho &= \sum (-1)^{\ell(\sigma)} G_{(\omega\sigma)}(\mathbf{x},\mathbf{z}) G_{(\sigma)}(\mathbf{x},\mathbf{y})\, x^\rho \\ &= X_\omega(\mathbf{y},\mathbf{z})\,. \end{aligned}$$

On the other hand, $X_\omega(\mathbf{y},\mathbf{x}^\omega) y^{-\rho} = \Theta_n^G$ is a reproducing kernel with respect to π_ω, and therefore, one has

$$G_{(\omega)}(\mathbf{x},\mathbf{z}) X_\omega(\mathbf{y},\mathbf{x}^\omega)\, \pi_\omega = G_{(\omega)}(\mathbf{y},\mathbf{z}) y^\rho\,.$$

In final, the images of $G_{(\omega)}(\mathbf{x},\mathbf{z})$ under the two sides of (2.14.4) are equal. QED

By specialisation of $\mathbf{y}$, one obtains the following commutations :

$$\sum (-1)^{\ell(\sigma)} \partial_\sigma\, X_\sigma(\mathbf{x},\mathbf{0}) = x^{01\ldots n-1}\, \partial_\omega \tag{2.14.5}$$

$$\sum (-1)^{\ell(\sigma)} x^\rho G_{(\sigma)}(\mathbf{x},\mathbf{1})\, \pi_{\sigma^{-1}} = \pi_\omega\, (1-x_2)\ldots(1-x_n)^{n-1}\,. \tag{2.14.6}$$

For example, for $n = 3$, one has

$$\pi_1\pi_2\pi_1\,(1-x_2)(1-x_3)^2 = \left\{ \begin{array}{c} -(x_1-1)^2(x_2-1)\pi_1\pi_2\pi_1 \\ -x^{210} G_{210}\pi_1\pi_2\pi_1 \end{array} \right.$$

$$\left\{ \begin{array}{c} x_1(x_1-1)(x_2-1)\pi_2\pi_1 \\ x^{210} G_{110}\,\pi_2\pi_1 \end{array} \right. \qquad \left\{ \begin{array}{c} x_2(x_1-1)^2\pi_1\pi_2 \\ x^{210} G_{200}\,\pi_1\pi_2 \end{array} \right.$$

$$\left\{ \begin{array}{c} -x_1x_2(x_1-1)\pi_1 \\ -x^{210} G_{100}\,\pi_1 \end{array} \right. \qquad \left\{ \begin{array}{c} -x_1(x_1x_2-1)\pi_2 \\ -x^{210} G_{010}\,\pi_2 \end{array} \right.$$

$$x_1^2 x_2$$

Given n, using on products of divided differences and rational functions in $\mathbf{x}$ the *double reversal*

$$\partial_i P \partial_j \ldots \partial_k Q \to Q^\omega \partial_{n-k} \ldots P^\omega \partial_{n-i}\,,$$

one transforms (2.14.3) into

$$X_\omega(\mathbf{x},\mathbf{y})\,\partial_\omega = \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_\sigma G_{(\omega\sigma\omega)}(\mathbf{x}^\omega,\mathbf{y})\,. \tag{2.14.7}$$

For example,

$$\begin{aligned}
X_{321}(\mathbf{x},\mathbf{y})\partial_{321} &= \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_3(1-y_1x_3^{-1})(1-y_1x_2^{-1})(1-y_2x_3^{-1})+\widehat{\pi}_1\widehat{\pi}_2(1-y_1x_3^{-1})(1-y_2x_3^{-1})\\
&\quad+\widehat{\pi}_2\widehat{\pi}_1(1-y_1x_3^{-1})(1-y_1x_2^{-1})+\widehat{\pi}_1(1-y_1y_2x_3^{-1}x_2^{-1})+\widehat{\pi}_2(1-y_1x_3^{-1})+1\\
&= \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 G_{(321)}(\mathbf{x}^\omega,\mathbf{y})+\widehat{\pi}_1\widehat{\pi}_2 G_{(312)}(\mathbf{x}^\omega,\mathbf{y})\\
&\quad+\widehat{\pi}_2\widehat{\pi}_1 G_{(231)}(\mathbf{x}^\omega,\mathbf{y})+\widehat{\pi}_1 G_{(132)}(\mathbf{x}^\omega,\mathbf{y})+\widehat{\pi}_2 G_{(213)}(\mathbf{x}^\omega,\mathbf{y})+1\,.
\end{aligned}$$

Notice that pushing the coefficients on the right in $X_\zeta(\mathbf{0},\mathbf{x}^\omega)\,\partial_\omega$, for any $\zeta\in\mathfrak{S}_n$, can be obtained by expanding $X_\omega(\mathbf{y},\mathbf{x}^\omega)$ in (2.14.2).

In fact, $X_\omega(\mathbf{y},\mathbf{x}^\omega)$ may be thought as the generating function of a linear basis of $\mathfrak{Pol}(\mathbf{x}_n)$ as a $\mathfrak{Sym}(\mathbf{x}_n)$-free module. Hence Formula 2.14.2 implies that for any function $g(\mathbf{x}_n)$, one has

$$g(\mathbf{x}_n^\omega)\,\partial_\omega = \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\ell(\sigma)}\partial_\sigma\left(g(\mathbf{x}_n)\,\partial_{\omega\sigma}\right). \tag{2.14.8}$$

When restricting the action of $g(\mathbf{x}_n^\omega)\,\partial_\omega$ to functions having partial symmetries, one reduces summation (2.14.8), as in the next case.

Corollary 2.14.2. *Let $m\le n$, $r=n-m$, $k\ge 0$. For any partition $\lambda\le r^m$, denote*

$$\partial^\lambda = \left(\partial_m\ldots\partial_{m+\lambda_1-1}\right)\ldots\left(\partial_1\ldots\partial_{\lambda_m-1}\right).$$

Then the restriction of the action of $Y_{k^r}(\mathbf{x}^\omega,\mathbf{y})\,\partial^{r^m}$ to $\mathfrak{Sym}(m,r)$ is equal to

$$Y_{k^r}(\mathbf{x}^\omega,\mathbf{y})\,\partial^{r^m} = \sum_{\lambda\le r^m}(-1)^{|\mu|}\partial^\lambda\, Y_{0^{\mu_r},k-\mu_r,0^{\mu_{r-1}-\mu_r},k-\mu_{r-1},\ldots,0^{\mu_1-\mu_2},k-\mu_1}(\mathbf{x},\mathbf{y})\,, \tag{2.14.9}$$

denoting by μ the partition which is conjugate to $[r-\lambda_m,\ldots,r-\lambda_1]$.

Proof. The operators $X_\omega(\mathbf{x}^\omega,\mathbf{y})\partial_\omega$ and $Y_{k^r}(\mathbf{x}^\omega,\mathbf{y})\,\partial^{r^m}$ have the same action on $\mathfrak{Sym}(m,r)$, up to sign. Moreover, the permutations σ which are not minimal in their coset $(\mathfrak{S}_m\times\mathfrak{S}_r)\,\sigma$ annihilate elements of $\mathfrak{Sym}(m,r)$, and therefore disappear from summation (2.14.8). QED

For example, for $n=5, m=2$, writing $\begin{array}{|c|c|c|}\hline 2&3&\cdot\cdot\\\hline 1&\cdot\cdot\\\cline{1-2}\end{array}$ for $(\partial_2\partial_3\ldots)(\partial_1\ldots)$, one has

$$
\begin{aligned}
Y_{666}(\mathbf{x}^\omega,\mathbf{y})\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2&3\\\hline\end{array} &= \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2&3\\\hline\end{array}Y_{666} - \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&2\\\cline{1-2}\end{array}Y_{6605} + \begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array}Y_{66004}\\
&+\begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1\\\cline{1-1}\end{array}Y_{6055} - \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}Y_{60504} - \begin{array}{|c|c|c|}\hline 2&3&4\\\hline\end{array}Y_{0555} + \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}Y_{60044}\\
&+\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}Y_{05504} - \begin{array}{|c|}\hline 2\\\hline\end{array}Y_{05044} + \begin{array}{|c|}\hline \;\\\hline\end{array}Y_{00444}\,.
\end{aligned}
$$

Formula 2.14.4 :

$$\sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\pi_\sigma\, G_{(\sigma)}(\mathbf{x},\mathbf{y}) = X_\omega(\mathbf{y},\mathbf{x}^\omega)\,\pi_\omega x^{-\rho}$$

can be rewritten

$$\sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\pi_\sigma\left(G_{(\omega)}(\mathbf{x},\mathbf{y})\pi_{\omega\sigma}\right) = (\text{-}1)^{\ell(\omega)}x^{\rho\omega}G_{(\omega)}(\mathbf{x}^\omega,\mathbf{y})\pi_\omega x^{-\rho}\,,$$

and implies that, for any function $g(\mathbf{x}_n)$, one has

$$\sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\omega\sigma)}\pi_\sigma\left(g(\mathbf{x}_n)\pi_{\omega\sigma}\right) = x^{\rho\omega}g(\mathbf{x}_n^\omega)\,\pi_\omega x^{-\rho} = g(\mathbf{x}_n^\omega)\,\widehat{\pi}_\omega\,. \tag{2.14.10}$$

Using, thanks to (2.6.4), that $\pi_\sigma = (\text{-}1)^{\ell(\sigma)}x^\rho\omega\,\widehat{\pi}_{\omega\sigma\omega}\,\omega x^{-\rho}$, putting $\zeta=\omega\sigma\omega$, $h=(x^\rho g)^\omega$, this last equation can be transformed into

$$\sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\omega\sigma)}\widehat{\pi}_\sigma\left(h(\mathbf{x}_n)\widehat{\pi}_{\omega\sigma}\right) = h(\mathbf{x}_n^\omega)\,\widehat{\pi}_\omega\,. \tag{2.14.11}$$

Taking $g(\mathbf{x}_n)=x^\lambda=h(\mathbf{x}_n)$, with λ dominant, one obtains key polynomials by commutation :

Theorem 2.14.3. *Given an integer n and a partition $\lambda\in\mathbb{N}^n$, then one has*

$$\textstyle\sum_{\sigma\in\mathfrak{S}_n}(\text{-}1)^{\ell(\sigma)}\pi_{\omega\sigma}\left(K_\lambda\pi_\sigma\right) = \left(x^{\rho+\lambda}\right)^\omega\pi_\omega x^{-\rho} \tag{2.14.12}$$

$$\sum_{\sigma\in\mathfrak{S}_n,\,\sigma\, min}(\text{-}1)^{\ell(\sigma)}\widehat{\pi}_{\omega\sigma}\,\widehat{K}_{\lambda\sigma} = x^{\lambda\omega}\,\widehat{\pi}_\omega\,, \tag{2.14.13}$$

the sum being limited, in the second expression, to the permutations minimum in their coset modulo the stabilizer of λ.

For example, for $\lambda=[3,1,0]$, one has

$$\pi_2\pi_1\pi_2K_{310} - \pi_1\pi_2K_{130} - \pi_2\pi_1K_{301} + \pi_1K_{103} + \pi_2K_{031} - K_{013} = x^{025}\,\pi_{321}/x^{210}\,,$$

and for $\lambda = [1,0,0]$, one has

$$\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\,\widehat{K}_{100} - \widehat{\pi}_1\widehat{\pi}_2\,\widehat{K}_{010} + \widehat{\pi}_1\widehat{K}_{001} = x^{001}\,\widehat{\pi}_{321}\,.$$

Using (1.4.8), one rewrites (2.14.13) into the following commutation of π_ω with a dominant monomial :

$$\pi_\omega\, x^\lambda = \sum_{\sigma\in\mathfrak{S}_n,\,\sigma\, min} \widehat{K}_{\lambda\sigma}(\mathbf{x}^\omega)\pi_{\omega\sigma}\,, \tag{2.14.14}$$

sum over all permutations σ which are of minimum length in their coset modulo the stabilizer of λ.

For example,

$$\begin{aligned}\pi_1\pi_2\pi_1\,x_1^2 &= x^{002}\pi_1\pi_2\pi_1 + \left(x^{020}+x^{011}\right)\pi_1\pi_2 + \left(x^{200}+x^{110}+x^{101}\right)\pi_2\\ &= \widehat{K}_2(\mathbf{x}^\omega)\pi_1\pi_2\pi_1 + \widehat{K}_{02}(\mathbf{x}^\omega)\pi_1\pi_2 + \widehat{K}_{002}(\mathbf{x}^\omega)\pi_2\,.\end{aligned}$$

Taking in (2.14.10) $g(\mathbf{x}_n) = G_\lambda(\mathbf{x},\mathbf{y})$, with λ dominant, one obtains again Grothendieck polynomials by commutation :

$$G_\lambda(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_\omega = \sum\nolimits_{\sigma\in\mathfrak{S}_n} (\text{-}1)^{\ell(\sigma)}\pi_{\sigma^{-1}\omega}\left(G_\lambda(\mathbf{x},\mathbf{y})\pi_\sigma\right). \tag{2.14.15}$$

For example, for $\lambda = [1,1,0]$, one has

$$\begin{aligned}\left(1-y_1x_2^{-1}\right)\left(1-y_1x_3^{-1}\right)\widehat{\pi}_{321} &= (\pi_2\pi_1\pi_2 - \pi_1\pi_2)\left(1-y_1x_1^{-1}\right)\left(1-y_1x_2^{-1}\right)\\ &\quad + (-\pi_2\pi_1+\pi_1)\left(1-y_1x_1^{-1}\right) + (\pi_2-1)\\ &= (\pi_2\pi_1\pi_2-\pi_1\pi_2)G_{110} + (-\pi_2\pi_1+\pi_1)G_{100} + (\pi_2-1)G_{000}\,.\end{aligned}$$

Thanks to the symmetry (1.4.8), one deduces from the preceding formula the expression of the product of π_ω with a dominant Grothendieck polynomial in terms of $\widehat{\pi}_\sigma$:

$$\pi_\omega\, G_{(\lambda}(\mathbf{x},\mathbf{y}) = \sum\nolimits_{\sigma\in\mathfrak{S}_n}\left(G_\lambda(\mathbf{x},\mathbf{y})\pi_\sigma\right)^\omega\widehat{\pi}_{\omega\sigma^{-1}}\,. \tag{2.14.16}$$

For example, for $n=3$, one has

$$\begin{aligned}\pi_{321}G_{210}(\mathbf{x},\mathbf{y}) &= G_{210}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 + G_{200}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_2\widehat{\pi}_1 + G_{110}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1\widehat{\pi}_2\\ &\quad + G_{010}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1 + G_{100}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_2 + G_{000}(\mathbf{x}^\omega,\mathbf{y})\,.\end{aligned}$$

The expression of $\pi_\omega G_\lambda(\mathbf{x},y)$ can be reduced when λ has repeated parts, i.e. when there exists i such that $G_\lambda(\mathbf{x},y)\pi_i = G_\lambda(\mathbf{x},y)$. Thus

$$\begin{aligned}\pi_{321}G_{110}(\mathbf{x},\mathbf{y}) &= G_{110}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 + G_{100}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_2\widehat{\pi}_1 + G_{110}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1\widehat{\pi}_2\\ &\quad + G_{000}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1 + G_{100}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_2 + G_{000}(\mathbf{x}^\omega,\mathbf{y})\end{aligned}$$

can be written, by right multiplication with π_1, as

$$\pi_{321}G_{110}(\mathbf{x},\mathbf{y}) = G_{110}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_1\widehat{\pi}_2\pi_1 + G_{100}(\mathbf{x}^\omega,\mathbf{y})\,\widehat{\pi}_2\pi_1 + G_{000}(\mathbf{x}^\omega,\mathbf{y})\,\pi_1\,.$$

2.15 Cauchy formula for key polynomials

The usual Cauchy formula is the expansion of $\prod_{i,j\le n}(1-x_iy_j)^{-1}$ in terms of Schur functions. We are going to see that "half" the Cauchy kernel $\prod_{i+j\le n+1}(1-x_iy_j)^{-1}$ expands in terms of key polynomials.

Notice first that

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)\cdots(1-x_1\cdots x_ny_1\cdots y_n)} = \sum_\lambda x^\lambda y^\lambda$$

is the generating function of dominant monomials $x^\lambda y^\lambda$ in $\mathbf{x}$ and $\mathbf{y}$. Its image under the product of the two symmetrizers $\pi_\omega^{\mathbf{x}}\,\pi_\omega^{\mathbf{y}}$ transforms this equality into

$$\prod_{i,j\le n}(1-x_iy_j)^{-1} = \sum_\lambda S_\lambda(\mathbf{x})S_\lambda(\mathbf{y})\,.$$

We can use the same starting point, but symmetrize partially in $\mathbf{x}$ and $\mathbf{y}$. Let $\Xi_n := \sum_{\sigma\in\mathfrak{S}_n}\widehat{\pi}_\sigma^x\,\pi_{\sigma\omega}^y$. Filtering the set of permutations according to the position of n, one gets the following factorization (we refer to [37] for more details).

Lemma 2.15.1. *We have*

$$\Xi_n = \Xi_{n-1}\left(\sum_{i=0}^{n-1}\widehat{\pi}_{[n-1:i]}^x\,\pi_{[n-1:n-1-i]}^y\right), \tag{2.15.1}$$

where $\pi_{[n-1:i]} := \pi_{n-1}\,\pi_{n-2}\cdots\pi_{n-i}$.

For example, the element Ξ_4 factorizes as

$$\Xi_4 = \Xi_3\left(\pi_3^y\pi_2^y\pi_1^y + \widehat{\pi}_3^x\pi_3^y\pi_2^y + \widehat{\pi}_3^x\widehat{\pi}_2^x\pi_3^y + \widehat{\pi}_3^x\widehat{\pi}_2^x\widehat{\pi}_1^x\right).$$

From the definition of key polynomials, the image under Ξ_n of $\sum_\lambda x^\lambda y^\lambda$ is equal to a sum of products of $K_v(\mathbf{y})$, $\widehat{K}_u(\mathbf{x})$. More precisely

$$\sum_\lambda x^\lambda y^\lambda\,\Xi_n = \sum_v K_v(\mathbf{y})\widehat{K}_{v\omega}(\mathbf{x})\,.$$

Using no more, but repeatedly, that

$$f(1-x_ig)^{-1}\pi_i^{\mathbf{x}} = f(1-x_ig)^{-1}(1-x_{i+1}g)^{-1}$$

when f,g belong to $\mathfrak{Sym}(x_i,x_{i+1})$, one checks that the image of $(1-x_1y_1)^{-1}(1-x_1x_2y_1y_2)^{-1}\cdots$ under Ξ_n is equal to $\prod_{i+j\le n+1}(1-x_iy_j)^{-1}$ [37, Prop 3]. Hence the following kernel.

Theorem 2.15.2. *For every n one has*

$$\prod_{i+j\le n+1}(1-x_iy_j)^{-1} = \sum_{v\in\mathbb{N}^n} K_v(\mathbf{y})\widehat{K}_{v\omega}(\mathbf{x})\,.$$

For example, for $n = 2$, one has

$$\frac{1}{(1-x_1y_1)(1-x_1x_2y_1y_2)}(\pi_1^y+\widehat{\pi}_1^x) = \frac{1}{(1-x_1y_1)(1-x_1y_2)}+\frac{y_1x_2}{(1-x_1y_1)(1-x_2y_1)}$$
$$= \frac{1}{(1-x_1y_1)(1-x_1y_2)(1-x_2y_1)} = 1+\sum_{i\le j} K_{ij}(\mathbf{y})x^{ji} + \sum_{j>i} y^{ji}\widehat{K}_{ij}(\mathbf{x})\,,$$

the key polynomials $K_{ij}(\mathbf{y})$ being Schur functions in y_1, y_2, while $\widehat{K}_{ij}(\mathbf{x}) = K_{ij}(\mathbf{x}) - x^{ji}$, when $i \le j$.

2.16 π and $\widehat{\pi}$-reproducing kernels

We have shown in (2.9.2) a reproducing property of the operator $f \to (f\,,\,\Theta_n^G)^\pi$. Let us rewrite it without using the scalar product $(\,,\,)^\pi$. Let

$$
\begin{aligned}
{}^{\widehat{\pi}}\Theta_n^G &= \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_{\sigma^{-1}}\, G_{(\sigma)}(\mathbf{z},\mathbf{x}) && (2.16.1)\\
{}^{\pi}\Theta_n^G &= \sum_{\sigma\in\mathfrak{S}_n} \pi_{\sigma^{-1}}\, \widehat{G}_{(\sigma)}(\mathbf{z},\mathbf{x}^\omega) && (2.16.2)
\end{aligned}
$$

For example, for $n=3$, one has

$$
\begin{aligned}
{}^{\widehat{\pi}}\Theta_3^G = 1 &+ \widehat{\pi}_1\left(1-\frac{x_1}{z_1}\right) + \widehat{\pi}_2\left(1-\frac{x_1x_2}{z_1z_2}\right) + \widehat{\pi}_1\widehat{\pi}_2\left(1-\frac{x_1}{z_1}\right)\left(1-\frac{x_2}{z_1}\right)\\
&+ \widehat{\pi}_2\widehat{\pi}_1\left(1-\frac{x_1}{z_1}\right)\left(1-\frac{x_1}{z_2}\right) + \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\left(1-\frac{x_1}{z_1}\right)\left(1-\frac{x_2}{z_1}\right)\left(1-\frac{x_1}{z_2}\right).
\end{aligned}
$$

With the alphabets $\mathbf{z},\mathbf{x}^\omega,\mathbf{y}$ instead of $\mathbf{x},\mathbf{y},\mathbf{z}$, Formula 2.9.4 reads

$$
\Theta_n^G = \sum_{v\le\rho} G_v(\mathbf{y},\mathbf{x}^\omega)\widehat{G}_{\rho-v}(\mathbf{z},\mathbf{x}^\omega)\,.
$$

Indexing by permutations, using the symmetry $G_{(\sigma)}(\mathbf{x},y^\vee)\clubsuit = G_{(\sigma^{-1})}(\mathbf{y},\mathbf{x}^\omega)$ given in (2.2.4), and the conjugation $\clubsuit\pi_i\clubsuit = \pi_{n-i}$, one rewrites this last formula as

$$
\begin{aligned}
\Theta_n^G(\mathbf{z},\mathbf{y}) &= \sum_{v\le\rho} \Theta_n^G(\mathbf{x},\mathbf{y})\pi_{\sigma^{-1}}\, \widehat{G}_{(\sigma)}(\mathbf{z},\mathbf{x}^\omega) && (2.16.3)\\
&= \Theta_n^G(\mathbf{x},\mathbf{y})\,{}^{\pi}\Theta_n^G\,, && (2.16.4)
\end{aligned}
$$

In other words, for any $v\,:\,[0,\dots,0]\le v\le[0,\dots,n{-}1]=\rho^\omega$, one has the reproducing property $x^v\,{}^{\pi}\Theta_n^G = z^v$. Equivalently, (2.16.4) rewrites as

$$
\widehat{G}_\rho(\mathbf{x},\mathbf{y})\,{}^{\pi}\Theta_n^G = \widehat{G}_\rho(\mathbf{z},\mathbf{y})\,. \tag{2.16.5}
$$

A similar computation shows that for $0\le v\le\rho$, one has $x^{-v}\,{}^{\widehat{\pi}}\Theta_n^G = z^{-v}$, or, equivalently,

$$
G_\rho(\mathbf{x},\mathbf{y})\,{}^{\widehat{\pi}}\Theta_n^G = G_\rho(\mathbf{z},\mathbf{y})\,. \tag{2.16.6}
$$

These two sets of monomials are bases of $\mathfrak{Pol}(\mathbf{x}_n)$ as a free $\mathfrak{Sym}(\mathbf{x}_n)$-module, and therefore the reproducing property extends to the full space, after identifying $\mathfrak{Sym}(\mathbf{x}_n)$ and $\mathfrak{Sym}(\mathbf{z}_n)$. In final, one has

Proposition 2.16.1. *For any $f\in\mathfrak{Pol}(\mathbf{x}_n)$ one has*

$$
f(\mathbf{x}_n)^{\pi}\Theta_n^G \equiv f(\mathbf{z}_n) \equiv f(\mathbf{x}_n)^{\widehat{\pi}}\Theta_n^G\,, \tag{2.16.7}
$$

modulo $\mathfrak{Sym}(\mathbf{x}_n)=\mathfrak{Sym}(\mathbf{z}_n)$.

Notice that the two operators ${}^{\pi}\Theta_n^G$ and ${}^{\widehat{\pi}}\Theta_n^G$ are not equal. Thus

$$\begin{aligned} x_2{}^{\widehat{\pi}}\Theta_2^G &= x_2\left(1+\widehat{\pi}_1(1-\frac{x_1}{z_1})\right) = x_1x_2z_1^{-1}\,, \\ x_2{}^{\pi}\Theta_2^G &= x_2\left(\frac{z_2}{x_2}+\pi_1(1-\frac{z_2}{x_2})\right) = z_2\,, \end{aligned}$$

evaluating modulo $\mathfrak{Sym}(\mathbf{x}_2) = \mathfrak{Sym}(\mathbf{z}_2)$ being necessary to insure equality.

Notice also that the two formulas $x^v\,{}^{\pi}\Theta_n^G = z^v$ for $0 \le v \le \rho^\omega$ and $x^{-v}\,{}^{\widehat{\pi}}\Theta_n^G = z^{-v}$ for $0 \le v \le \rho$ show that both operators ${}^{\pi}\Theta_n^G$ and ${}^{\widehat{\pi}}\Theta_n^G$ take values in $\mathfrak{Sym}(\mathbf{x}_n)\otimes \mathfrak{Pol}(\mathbf{z}_n)$.

In the case $n=2$, one can rewrite ${}^{\widehat{\pi}}\Theta_2^G = \pi_1 - \partial_1 \frac{x_1x_2}{z_1}$, ${}^{\pi}\Theta_2^G = \pi_1 - \partial_1 z_2$. This prompts us to define, for any i,

$$\widehat{\theta}_i = \pi_i - \partial_i \frac{x_ix_{i+1}}{z_i} \quad \& \quad \theta_i = \pi_i - \partial_i z_{i+1}\,.$$

These operators do not satisfy the braid relations if the parameters z_i are not all equal. Let us show however, that one can use them to factorize ${}^{\widehat{\pi}}\Theta_n^G$ and ${}^{\pi}\Theta_n^G$.

The action of $\theta_2\theta_1\theta_2$ on $\widehat{G}_{210}(\mathbf{x},\mathbf{y})$ is such that each step is of the type $(1 - x_{i+1}y_j^{-1})f\theta_i = (1-z_{i+1}y_j^{-1})f$, with f symmetrical in x_i, x_{i+1}. Therefore one has $\widehat{G}_{210}(\mathbf{x},\mathbf{y})\theta_2\theta_1\theta_2 = \widehat{G}_{210}(\mathbf{z},\mathbf{y})$, and, more generally,

$$\widehat{G}_\rho(\mathbf{x},\mathbf{y})\,(\theta_{n-1})(\theta_{n-2}\theta_{n-1})\ldots(\theta_1\ldots\theta_{n-1}) = \widehat{G}_\rho(\mathbf{z},\mathbf{y})\,.$$

One checks similarly that

$$G_\rho(\mathbf{x},\mathbf{y})\,(\widehat{\theta}_1)(\widehat{\theta}_2\widehat{\theta}_1)\ldots(\widehat{\theta}_{n-1}\ldots\widehat{\theta}_1) = G_\rho(\mathbf{z},\mathbf{y})\,.$$

Hence, these two products of operators have the same action on $\mathfrak{Pol}(\mathbf{x}_n)$ than ${}^{\pi}\Theta_n^G$ and ${}^{\widehat{\pi}}\Theta_n^G$ respectively, and one has the following proposition.

Proposition 2.16.2. *Given n, one has the factorizations*

$$\begin{aligned} {}^{\pi}\Theta_n^G &= (\theta_{n-1})(\theta_{n-2}\theta_{n-1})\ldots(\theta_1\ldots\theta_{n-1}) && (2.16.8)\\ {}^{\widehat{\pi}}\Theta_n^G &= (\widehat{\theta}_1)(\widehat{\theta}_2\widehat{\theta}_1)\ldots(\widehat{\theta}_{n-1}\ldots\widehat{\theta}_1)\,. && (2.16.9) \end{aligned}$$

2.17 Decompositions in the affine Hecke algebra

The elementary constituents of all the operators that we have used so far in type A are divided differences, together with "multiplication by elements of $\mathfrak{Rat}(\mathbf{x})$", the ring of rational functions in $\mathbf{x}$. One could as well take permutations and elements of $\mathfrak{Rat}(\mathbf{x})$. Indeed, the algebras generated by $\{\partial_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$, or $\{s_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$, or $\{\pi_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$, or $\{T_i, i = 1 \ldots n-\} \cup \mathfrak{Rat}(\mathbf{x}_n)$ all coincide. With M.P. Schützenberger, we call it *algebra of divided differences*, Bourbaki prefers *produit croisé de l'algèbre du groupe symétrique et de* $\mathfrak{Rat}(\mathbf{x})$, Kostant and Kumar use the expression *smash product*, and finally, the terminology *affine Hecke algebra for type A* puts the emphasis on the elements T_i.

Every element of this algebra is uniquely written as a sum $\sum_{\sigma\in\mathfrak{S}_n} \partial_\sigma R^\partial_\sigma$, $\sum_{\sigma\in\mathfrak{S}_n} \sigma R^s_\sigma$, $\sum_{\sigma\in\mathfrak{S}_n} \pi_\sigma R^\pi_\sigma$, $\sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_\sigma R^{\widehat{\pi}}_\sigma$, or $\sum_{\sigma\in\mathfrak{S}_n} T_\sigma R^T_\sigma$ respectively, choosing to put the coefficients on the right. Symmetry properties like (1.4.8) allow to pass from the right module structure to the left one.

We show in (3.3.1), as a consequence of the multivariate Newton interpolation formula, how to pass from divided differences to permutations using Schubert polynomials, or conversely in (3.3.3). In fact, this type of expansions uses only the obvious fact that the kernel $\Theta^Y(\mathbf{x},\mathbf{y})$ vanish for all specializations $\mathbf{y} = \mathbf{x}^\zeta$, except when ζ is the identity. Instead of $\Theta^Y(\mathbf{x},\mathbf{y})$, one could as well use as a kernel $Y_\rho(\mathbf{x},\mathbf{y})$, $G_\rho(\mathbf{x},\mathbf{y})$, or $\widehat{G}_\rho(\mathbf{x},\mathbf{y})$, the non vanishing being obtained for the identity or for the maximal permutation according to the choice of the kernel.

More generally, given any $f(\mathbf{x}_n) \in \mathfrak{Pol}(\mathbf{x}_n)$, let $\Theta^f(\mathbf{x},\mathbf{y}) = f(\mathbf{x}_n)\Theta^Y_n$. Then for any element $\nabla = \sum_\sigma \sigma R^s_\sigma$, one has $\Theta^f(\mathbf{x},\mathbf{y})\nabla = \sum_\sigma \Theta^f(\mathbf{x}^\sigma,\mathbf{y})R^s_\sigma$, and therefore the coefficients are such that

$$R^s_\sigma = \Theta^f(\mathbf{x},\mathbf{y})\sigma^{-1}\nabla\Big|_{\mathbf{y}=\mathbf{x}} \frac{1}{f(\mathbf{x}_n)\Delta(\mathbf{x}_n)}\ .$$

Similar expressions hold for the other coefficients $R^\partial_\sigma, R^\pi_\sigma, R^{\widehat{\pi}}_\sigma$.

As a matter of fact, some of the formulas in preceding sections may be interpreted as identities in the affine Hecke algebra. For example, taking $\mathbf{z} = \mathbf{x}^\zeta$ in (2.16.7), one obtains the expansion of any permutation in the basis $\{\pi_\sigma\}$ or $\{\widehat{\pi}_\sigma\}$.

Let us summarize the main expansions, that will be needed later, of any element ∇ of the affine Hecke algebra.

$$\nabla = \sum_{\sigma\in\mathfrak{S}_n} \partial_\sigma \left(X_{\sigma^{-1}}(\mathbf{x},\mathbf{y})\nabla\Big|_{\mathbf{y}=\mathbf{x}}\right) \tag{2.17.1}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \sigma \left(\prod_{1\le i<\le j\le n} (x_i - y_j)^{\sigma^{-1}} \nabla\Big|_{\mathbf{y}=\mathbf{x}}\right) \tag{2.17.2}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \pi_\sigma \left(\widehat{G}_{(\sigma^{-1})}(\mathbf{x},\mathbf{y}^\omega)\nabla\Big|_{\mathbf{y}=\mathbf{x}}\right) \tag{2.17.3}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} \widehat{\pi}_\sigma \left(G_{(\sigma^{-1})}(\mathbf{x},\mathbf{y})\nabla\Big|_{\mathbf{y}=\mathbf{x}}\right)\ . \tag{2.17.4}$$

For example,

$$
\begin{aligned}
s_1s_2 = \sum_{\sigma\in\mathfrak{S}_3} \widehat{\pi}_{\sigma^{-1}}\, G_\sigma(\mathbf{x}^{s_1s_2},\mathbf{y})\Big|_{\mathbf{y}=\mathbf{x}} &= 1 + \widehat{\pi}_2 \frac{(x_3x_1 - y_1y_2)}{x_3x_1} + \widehat{\pi}_1\left(1-\frac{y_1}{x_3}\right) \\
&\quad + \widehat{\pi}_2\widehat{\pi}_1\left(1-\frac{y_1}{x_1}\right)\left(1-\frac{y_1}{x_3}\right) + \widehat{\pi}_1\widehat{\pi}_2\left(1-\frac{y_1}{x_3}\right)\left(1-\frac{y_2}{x_3}\right) \\
&\quad + \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\left(1-\frac{y_1}{x_3}\right)\left(1-\frac{y_2}{x_3}\right)\left(1-\frac{y_1}{x_1}\right)\Big|_{\mathbf{y}=\mathbf{x}} \\
&= 1 + \widehat{\pi}_2\left(1-\frac{x_2}{x_3}\right) + \widehat{\pi}_1\left(1-\frac{x_1}{x_3}\right) + \widehat{\pi}_1\widehat{\pi}_2\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right).
\end{aligned}
$$

Specific cases of the above expansions appear in the literature. Kostant and Kumar [71] consider the transition matrices $\{\sigma\} \leftrightarrow \{\partial_\sigma\}$. Berline and Vergne [6], Arabia [1], Kostant and Kumar [72] consider the transition matrices $\{\sigma\} \leftrightarrow \{\pi_\sigma\}$. Kumar shows in [77] how to relate the entries of these last matrices (which are specializations of Grothendieck polynomials) to the singularities of Schubert varieties.

Notice that the above expansions are obtained by specializing polynomials in $\mathbf{x},\mathbf{y}$. These polynomials are not unique. For example, instead of (2.17.3), one could use as well

$$
\nabla = \sum_{\sigma\in\mathfrak{S}_n} \pi_\sigma \left(G_{(\omega)}(\mathbf{x},\mathbf{y})\widehat{\pi}_{\omega\sigma^{-1}}\nabla\Big|_{\mathbf{y}=\mathbf{x}}\right).
$$

Let us mention in final the interest of expressing the basis of the usual Hecke algebra (with normalization $(T_i-t_1)(T_i-t_2)=0$) in terms of the basis $\{\widehat{\pi}_\sigma\}$. For example, for $n=3$, one has

$$
T_1 = \widehat{\pi}_1\frac{(x_2t_1+x_1t_2)}{x_2} + t_1 \quad \& \quad T_2 = \widehat{\pi}_2\frac{(x_3t_1+x_2t_2)}{x_3} + t_1
$$

$$
\begin{aligned}
T_1T_2 &= \widehat{\pi}_1\widehat{\pi}_2\tfrac{(x_3t_1+x_2t_2)(x_1t_2+x_3t_1)}{x_3{}^2} + \widehat{\pi}_2\tfrac{(x_3t_1+x_2t_2)t_1}{x_3} + \widehat{\pi}_1\tfrac{\left(-x_1t_2{}^2+x_3t_1{}^2\right)}{x_3} + t_1{}^2 \\
T_2T_1 &= \widehat{\pi}_2\widehat{\pi}_1\tfrac{(x_2t_1+x_1t_2)(x_1t_2+x_3t_1)}{x_3x_2} + \widehat{\pi}_2\tfrac{\left(-x_1t_2{}^2+x_3t_1{}^2\right)}{x_3} + \widehat{\pi}_1\tfrac{t_1(x_2t_1+x_1t_2)}{x_2} + t_1{}^2
\end{aligned}
$$

$$
\begin{aligned}
T_1T_2T_1 &= \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1\frac{(x_3t_1+x_2t_2)\,(x_1t_2+x_3t_1)\,(x_2t_1+x_1t_2)}{x_2x_3{}^2} \\
&\quad + \widehat{\pi}_1\widehat{\pi}_2\frac{(x_3t_1+x_2t_2)\,t_1\,(x_1t_2+x_3t_1)}{x_3{}^2} + \widehat{\pi}_2\widehat{\pi}_1\frac{(x_2t_1+x_1t_2)\,(x_1t_2+x_3t_1)\,t_1}{x_3x_2} \\
&\quad + \widehat{\pi}_2\frac{(-x_1t_2{}^2+x_3t_1{}^2)\,t_1}{x_3} + \widehat{\pi}_1\frac{(-x_1t_2{}^2+x_3t_1{}^2)\,t_1}{x_3} + t_1^3.
\end{aligned}
$$

and these expansions specialize to the expression of permutations in the basis $\{\widehat{\pi}_\sigma\}$ for $t_1=1$, $t_2=-1$, the coefficients being then specializations of Grothendieck polynomials.

Chapter 3

Properties of Schubert polynomials

3.1 Schubert by vanishing properties

To have linear bases, we could have considered only the special case where $\mathbf{y} = \mathbf{0}$ in the case of Schubert polynomials, and $\mathbf{y} = \mathbf{1}$ in the case of Grothendieck polynomials. But doing so, we would lose many interesting specialization properties that these polynomials possess, and that can be used to characterize them easily, as we are going to see in this section for Schubert polynomials.

Given a permutation σ (considered as an element of $\mathfrak{S}_\infty$, whose code is v), let $\langle v\rangle = \mathbf{y}^\sigma = [y_{\sigma_1}, \ldots, y_{\sigma_n}]$.

We call $\langle v\rangle$ a *spectral vector*[1] and write $f\Big(\langle v\rangle\Big)$ for the specialisation of $f \in \mathfrak{Pol}(\mathbf{x}_n, \mathbf{y})$ in $x_1 = y_{\sigma_1}, \ldots, x_n = y_{\sigma_n}$.

Theorem 3.1.1. *Given $v \in \mathbb{N}^n$, and σ such that $v = \text{\c{}}(\sigma)$, then the Schubert polynomial $Y_v(\mathbf{x}, \mathbf{y})$ is the only polynomial in the space of degree $\leq |v|$ in $\mathbf{x}_n$ such that*

$$Y_v(\langle u\rangle, \mathbf{y}) \;=\; 0\,,\;\; u \neq v,\; |u| \leq |v| \tag{3.1.1}$$

$$Y_v\Big(\langle v\rangle, \mathbf{y}\Big) \;=\; ⋒(v) := \prod_{i<j,\,\sigma_i>\sigma_j} (y_{\sigma_i} - y_{\sigma_j}) \tag{3.1.2}$$

The specialization $⋒(v)$ is called the *inversion polynomial* of σ. We shall also denote it $⋒(\sigma)$ when no ambiguity is to be feared.

Proof. First, it is straightforward that the dominant Schubert polynomials, which are products of linear factors, satisfy both (3.1.1, 3.1.2).

[1] We use the same term as for the Yang-Baxter equation, because these two uses are related in several ways. Notice that $\mathbf{x}^{s_1} = [x_2, x_1, x_3, \ldots]$, $\mathbf{x}^{s_1 s_2} = [x_2, x_3, x_1, \ldots] = [x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}]$, with $\sigma = s_1 s_2 = [2,3,1]$. We are acting on the components of the vector $[x_1, x_2, \ldots]$. On the other hand, the action on the right on exponents of monomials: $x_1^\sigma = x^{[100]s_1 s_2} = x^{001} = x_3$, $x_2^\sigma = x^{[010]s_1 s_2} = x^{100} = x_1$, $x_3^\sigma = x^{[001]s_1 s_2} = x^{010} = x_2$ involves the inverse permutation $[3,1,2]$.

Therefore, we have just to check the behaviour of these conditions with respect to divided differences.

Lemma 3.1.2. *Let $v \in \mathbb{N}^n$, $\sigma = \langle v\rangle$, i be such that $v_i > v_{i+1}$. Suppose that Y_v satisfies (3.1.1, 3.1.2). Then $Y_v\,\partial_i$ also satisfies (3.1.1, 3.1.2) for the index $v' = [v_1, \ldots, v_{i-1}, v_{i+1}, v_i{-}1, v_{i+2}, \ldots, v_n]$, which is the code of σs_i.*

Proof. Write $Y_v = f(x_i, x_{i+1}) - x_{i+1}g(x_i, x_{i+1})$, with $f, g \in \mathfrak{Sym}(x_i, x_{i+1})$. Let us check that g is the polynomial defined by (3.1.1, 3.1.2) for the index index v'.

If Y_v vanishes in $[x_i, x_{i+1}] = [a, b]$ and $[x_i, x_{i+1}] = [b, a]$, with $a \neq b$, then g inherits these vanishings: $g(a, b) = g(b, a) = 0$. On the other hand, in the points $\langle v\rangle$ and $\langle v'\rangle$, one has

$$\begin{aligned} Y_v\Big(\langle v\rangle, \mathbf{y}\Big) = \Cap(v) \;&=\; f(y_{\sigma_i}, y_{\sigma_{i+1}}) - y_{\sigma_{i+1}}g(y_{\sigma_i}, y_{\sigma_{i+1}}) \\ Y_v\Big(\langle v'\rangle, \mathbf{y}\Big) = 0 \;&=\; f(y_{\sigma_i}, y_{\sigma_{i+1}}) - y_{\sigma_i}g(y_{\sigma_i}, y_{\sigma_{i+1}})\,. \end{aligned}$$

Therefore $g(y_{\sigma_{i+1}}, y_{\sigma_i}) = \Cap(v)\Big(y_{\sigma_i} - y_{\sigma_{i+1}}\Big)^{-1}$ is the inversion polynomial of σs_i, and g satisfies the conditions (3.1.1, 3.1.2). This proves the lemma. But $Y_v\partial_i = -x_{i+1}g\partial_i = g$, and therefore g is the Schubert polynomial of index v'. This proves the theorem. QED

For example,

$$Y_{2010}(\mathbf{x}, \mathbf{y}) = (x_1 - y_1)(x_1 - y_2)(x_2 + x_3 - y_1 - y_2)$$

is characterized, among all polynomials in x_1, x_2, x_3, x_4 of degree no more than 3, by the vanishing in all $\mathbf{x}_4 = y^\zeta$, $\zeta \in \mathfrak{S}_4$, $\ell(\zeta) \leq 3$, $\zeta \neq \sigma = [3, 1, 4, 2]$, and by the normalization

$$Y_{2010}(\mathbf{y}^\sigma, \mathbf{y}) = (y_3 - y_1)(y_3 - y_2)(y_1 + y_4 - y_1 - y_2) = \Cap([2, 0, 1, 0])\,.$$

A consequence of the theorem is the following vanishing property (which evident only for dominant polynomials), corresponding to $\langle \mathbf{0}\rangle = [y_1, y_2, \ldots, y_n]$.

Corollary 3.1.3. *For any $v \neq [0, \ldots, 0]$, one has $Y_v(\mathbf{y}, \mathbf{y}) = 0$.*

3.2 Multivariate interpolation

We have already used several times the vanishing in $\mathbf{x} = \mathbf{y} = \langle \mathbf{0}\rangle$, this property is better understood as a special case of (3.1.1).

Notice that the polynomials $Y_k = (x_1 - y_1)\cdots(x_1 - y_k)$ are the interpolation polynomials that Newton used in his famous interpolation formula. The next theorem states that the Schubert polynomials are precisely the universal coefficients in the generalization of Newton's formula to several variables (this theorem could be deduced from the Cauchy formula that we gave in Th. 2.10.2.

Given $v \in \mathbb{N}^n$, let ∂^v be any product of divided differences[2] such that $Y_v \partial^v = Y_{0\ldots 0}$. It is easy to see that for any $u \neq v$, then $Y_u \partial^v$ is either 0 or a Schubert polynomial of index $\neq [0, \ldots, 0]$.

Theorem 3.2.1 (MultivariateNewton)**.** *For any $f \in \mathfrak{Pol}(\mathbf{x}, \mathbf{y})$, one has the expansion*

$$f(\mathbf{x}) = \sum_{v \in \mathbb{N}^n} f(\mathbf{x})\partial^v \Big|_{\mathbf{x}=\mathbf{y}} Y_v(\mathbf{x}, \mathbf{y}) . \tag{3.2.1}$$

Proof. Test the statement on the Schubert basis. In that case, $f(\mathbf{x})\partial^v$ is either 0 or a Schubert polynomial, whose specialization in $\mathbf{x} = \mathbf{y}$ (i.e. in the point $\langle 0 \ldots 0 \rangle$) is $\neq 0$ (and equal to 1) iff $f(\mathbf{x}) = Y_v$. QED

The preceding theorem gives the expansion of any polynomial in the Schubert basis, the coefficients being all the non-zero images under divided differences. In particular, one can take the key polynomials, or the Grothendieck polynomials[3]. For example, the polynomial K_{021} has only 6 non-zero images under divided differences, the images under $1, \partial_2, \partial_3, \partial_2\partial_3, \partial_3\partial_2, \partial_3\partial_2\partial_2$. Writing the coefficients in y as key polynomials, one has

$$K_{021}(\mathbf{x}) = K_0(\mathbf{y})Y_{0,2,1} + K_{0,1}(\mathbf{y})Y_{0,2} + K_{0,1,1}(\mathbf{y})Y_{0,1} \\ + K_{0,0,1}(\mathbf{y})Y_{0,1,1} + K_{0,2}(\mathbf{y})Y_{0,0,1} + K_{0,2,1}(\mathbf{y})Y_0 .$$

In the case where f is a polynomial in x_1 (and $\mathbf{y}$) only, the only non-zero divided differences are $f\partial_1$, $f\partial_1\partial_2$, $f\partial_1\partial_2\partial_3, \ldots$, and the theorem is the original theorem of Newton, apart from notations :

$$\begin{aligned} f(x_1) &= f(y_1) + f\partial_1 Y_1 + f\partial_1\partial_2 Y_2 + f\partial_1\partial_2\partial_3 Y_3 + \cdots \\ &= f(y_1) + f\partial_1(x_1 - y_1) + f\partial_1\partial_2(x_1 - y_1)(x_1 - y_2) + \cdots \end{aligned} \tag{3.2.2}$$

The interpolation of functions $f(x_1, x_2)$ of two variables reads

$$f(x_1, x_2) = f(y_1, y_2)Y_{00} + f\partial_2 Y_{01} + f\partial_1 Y_{10} + f\partial_2\partial_3 Y_{02} + f\partial_2\partial_1 Y_{11} \\ + f\partial_1\partial_2 Y_{20} + f\partial_2\partial_3\partial_4 Y_{03} + f\partial_2\partial_3\partial_1 Y_{12} + f\partial_2\partial_1\partial_2 Y_{21} + f\partial_1\partial_2\partial_3 Y_{30} + \ldots$$

In the case that $f(x_1, x_2)$ is symmetrical, then $f\partial_1 = 0$, and only the terms $Y_{i,j}$, $i \leq j$, which are those symmetrical in x_1, x_2, survive in the preceding formula:

$$f(x_1, x_2) = f(y_1, y_2)Y_{00} + f\partial_2 Y_{01} + f\partial_2\partial_3 Y_{02} + f\partial_2\partial_1 Y_{11} + f\partial_2\partial_3\partial_4 Y_{03} \\ + f\partial_2\partial_3\partial_1 Y_{12} + f\partial_2\partial_3\partial_4\partial_5 Y_{04} + f\partial_2\partial_3\partial_4\partial_1 Y_{13} + f\partial_2\partial_3\partial_1\partial_2 Y_{22} + \ldots$$

[2] Take any reduced decomposition $s_i s_j \cdots s_k$ of σ, with σ of code v. Then $\partial_k \cdots \partial_j \partial_i$ is such product.

[3] after some change of variables, like $x_i \to 1/x_i$ or $x_i \to 1/(1-x_i)$, to transform Grothendieck polynomials into polynomials in $\mathbf{x}$, and not in $x_1^{-1}, x_2^{-1}, \ldots$.

Interpolation methods can also be used in the theory of symmetric polynomials. If $f(\mathbf{x}_n)$ belongs to $\mathfrak{Sym}(\mathbf{x}_n)$, then only anti-dominant indices occur in the RHS of (3.2.1). In other words, Newton's interpolation give an expansion of symmetric polynomials in terms of Graßmannian Schubert polynomials.

For example, the Schur function $s_{32}(\mathbf{x}_3)$, which is equal to $Y_{023}(\mathbf{x},\mathbf{0})$, has the following expansion in terms of Graßmannian Schubert polynomials (writing Y_uY_v for $Y_u(\mathbf{y},\mathbf{0})Y_v(\mathbf{x},\mathbf{y})$) :

$$\begin{aligned} s_{32}(\mathbf{x}_3) = Y_{023}(\mathbf{x},\mathbf{0}) &= Y_{000}Y_{023} - Y_{00001}Y_{022} - Y_{001}Y_{013} + Y_{00101}Y_{012} \\ &+ Y_{011}Y_{003} - Y_{01101}Y_{002} - Y_{00201}Y_{011} + Y_{01201}Y_{001} - Y_{02201}Y_{000}\,. \end{aligned}$$

Such expansions have been considered by Chen and Louck [17] and by Olshanski and Okounkov [137], in the case where $\mathbf{y} = \{0,1,2,\dots\}$ or $\mathbf{y} = \{q^0, q^1, q^2, \dots\}$ (in which case the polynomials are called *factorial Schur functions*).

Newton interpolation is compatible with symmetry by blocks. Indeed, let $f(\mathbf{x}) \in \mathfrak{Sym}(m,n,p,\dots)$, i.e. $f(\mathbf{x})$ is a function which is symmetrical in $x_1,\dots,x_m$, symmetrical in $x_{m+1},\dots,x_{m+n}$, &c. Then $f(\mathbf{x}) = \sum c_v Y_v(\mathbf{x},\mathbf{y})$, the set of indices v being restricted to those such that $v_1 \le \dots \le v_m$, $v_{m+1} \le \dots \le v_{m+n}$, &c., i.e. to those v for which $Y_v(\mathbf{x},\mathbf{y})$ belongs to $\mathfrak{Sym}(m,n,p,\dots)$. Otherwise, there would exist a divided difference ∂_i annihilating $f(\mathbf{x})$ and not $\sum c_v Y_v$. For example, if $f \in \mathfrak{Sym}(3,4,2)$, then the interpolation

$$f(\mathbf{x}) = \sum f(\mathbf{x})\partial^v \Big|_{\mathbf{x}=\mathbf{y}} Y_v(\mathbf{x},\mathbf{y})$$

involves only the $v \in \mathbb{N}^9$ such that $v_1 \le v_2 \le v_3$, $v_4 \le v_5 \le v_6 \le v_7$, $v_8 \le v_9$.

3.3 Permutations versus divided differences

Fashion has changed since Newton, and it may seem of little interest to interpolate functions by polynomials. In fact, classical interpolation theory may be thought as a way of producing algebraic identities involving polynomials or rational functions in several variables. In this interpretation, it still begs the right to exist, even to expand. Moreover, one can disguise interpolation under a more sophisticated terminology.

For example, consider the problem of expressing a permutation $\sigma \in \mathfrak{S}_n$, considered as an operator on $\mathfrak{Pol}(\mathbf{x}_n)$, in terms of divided differences. The image of (3.2.1) under σ is

$$f(\mathbf{x}^\sigma) = \sum_{v\in\mathbb{N}^n} f(\mathbf{x})\partial^v \Big|_{\mathbf{x}=\mathbf{y}} Y_v(\mathbf{x}^\sigma,\mathbf{y})\,.$$

Putting $\mathbf{y} = \mathbf{x}$ gives the following property obtained by Kostant and Kumar [71] in the more general context of Kac-Moody groups (they call the algebra of divided differences the *nil Hecke ring*).

Proposition 3.3.1. *Any permutation $\sigma \in \mathfrak{S}_n$ expands, in terms of divided differences, as*

$$\sigma = \sum_{v \le \rho} \partial^v \, Y_v(\mathbf{x}^\sigma, \mathbf{x}) \,. \tag{3.3.1}$$

For example,

$$s_2 s_1 = 1 + \partial_2(x_3 - x_1) + \partial_1(x_2 - x_1) + \partial_2\partial_1(x_3 - x_1)(x_2 - x_1)\,,$$

$$s_2 s_1 s_3 = 1 + \partial_1(x_2 - x_1) + \partial_2(x_4 - x_1) + \partial_3(x_4 - x_3) + \partial_2\partial_3(x_4 - x_3)(x_4 - x_1)$$
$$+\partial_1\partial_3(x_2-x_1)(x_4-x_3)+\partial_2\partial_1(x_2-x_1)(x_4-x_1)+\partial_2\partial_1\partial_3(x_2-x_1)(x_4-x_1)(x_4-x_3)\,.$$

Conversely, one may express divided differences in terms of permutations, and more generally, any linear combination with rational coefficients in $\mathbf{x}$.

Lemma 3.3.2. *Let n be an integer, $\Theta^Y(\mathbf{x}, \mathbf{y}) := \prod_{1 \le i < j \le n}(y_i - x_j)$ as before, and $\hbar = \sum_{\sigma \in \mathfrak{S}_n} \sigma \, h_\sigma$ be a sum with rational coefficients h_σ in $\mathbf{x}$. Then*

$$\Theta^Y(\mathbf{x}, \mathbf{y}) \, \hbar\Big|_{\mathbf{y}=\mathbf{x}^\sigma} = (-1)^{\ell(\sigma)} \, h_\sigma \prod_{1 \le i < j \le n} (x_i - x_j) \ . \tag{3.3.2}$$

Proof. We have already used that $\Theta^Y(\mathbf{x}, \mathbf{x}^\zeta)$ vanishes for all permutations ζ different from the identity. Therefore $\Theta^Y(\mathbf{y}^\sigma, \mathbf{y}^\zeta)$ vanishes except for $\zeta = \sigma$, and the sum $\Theta^Y(\mathbf{x}, \mathbf{y}) \, \hbar = \sum \Theta^Y(\mathbf{x}^\sigma, \mathbf{y}) h_\sigma$ reduces to a single term when specializing $\mathbf{y}$ to a permutation of $\mathbf{x}$. QED

We can take now $\hbar = \partial_\tau$. Then

$$\Theta^Y(\mathbf{x}, \mathbf{y})\partial_\tau = X_\omega(\mathbf{x}^\omega, \mathbf{y})\partial_\tau = X_\omega(\mathbf{x}, \mathbf{y}) \, \omega \partial_\tau \omega \, \omega$$
$$= (-1)^{\ell(\tau)} X_\omega(\mathbf{x}, \mathbf{y}) \, \partial_{\omega\tau^{-1}\omega} \omega = (-1)^{\ell(\tau)} X_{\tau^{-1}\omega}(\mathbf{x}^\omega, \mathbf{y})\,.$$

In final, one has the following expression of ∂_τ [94, Prop. 10.2.5] :

Proposition 3.3.3. *Let $\tau \in \mathfrak{S}_n$. Let $\partial_\tau = \sum \zeta \, c^\tau_\zeta$ be the expression of ∂_τ in terms of permutations. Then*

$$(-1)^{\ell(\zeta)} \, c^\tau_\zeta = (-1)^{\ell(\omega\tau)} \, X_{\tau^{-1}\omega}(\mathbf{x}^{\omega\zeta}, \mathbf{x}) \frac{1}{\Delta(\mathbf{x})} = X_{\omega\tau}(\mathbf{x}, \mathbf{x}^{\omega\zeta}) \frac{1}{\Delta(\mathbf{x})} \ . \tag{3.3.3}$$

Notice that, apart from signs and the factor $\Delta(\mathbf{x})$, the entries of the transition matrix from permutations to divided differences, and its inverse, are the same.

Here are the two transition matrices for $n = 3$, to be read by rows, coding $x_1 - x_2 = 12$, $x_1 - x_3 = 13$, $x_2 - x_3 = 23$:

	1	∂_2	∂_1	$\partial_1\partial_2$	$\partial_2\partial_1$	$\partial_1\partial_2\partial_1$
1	1	0	0	0	0	0
s_2	1	23	0	0	0	0
s_1	1	0	12	0	0	0
s_2s_1	1	23	13	0	$23 \cdot 13$	0
s_1s_2	1	13	12	$13 \cdot 12$	0	0
$s_1s_2s_1$	1	13	13	$12 \cdot 13$	$13 \cdot 23$	$12 \cdot 13 \cdot 23$

	1	s_2	s_1	s_1s_2	s_2s_1	$s_1s_2s_1$
Δ	$12\cdot 13\cdot 23$	0	0	0	0	0
$\partial_2\Delta$	$-12\cdot 13$	$12\cdot 13$	0	0	0	0
$\partial_1\Delta$	$-13\cdot 23$	0	$13\cdot 23$	0	0	0
$\partial_2\partial_1\Delta$	13	-13	-23	0	23	0
$\partial_1\partial_2\Delta$	13	-12	-13	12	0	0
$\partial_1\partial_2\partial_3\Delta$	-1	1	1	-1	-1	1

Pairs of permutations τ, σ such that the specialisation $X_\tau(\mathbf{x}^\sigma, \mathbf{x})$ is not a divisor of the Vandermonde correspond singularities of Schubert varieties. There are only two singularities when $n = 4$. One of them occurs in the expansion of $\partial_2\partial_3\partial_1\partial_2$, which involves the specializations of $X_{2143} = (x_1 - x_2)(x_1 + x_2 + y_3 - y_1 - y_2 - y_3)$, among which one finds $(x_1 - x_4)^2$.

The full expansion of $\partial_2\partial_3\partial_1\partial_2$ is

$$(1-s_2)\Bigg(\frac{x_1-x_4}{(x_3-x_4)(x_2-x_4)(x_2-x_3)(-x_3+x_1)(x_1-x_2)}$$
$$-s_1\frac{1}{(x_3-x_4)(x_2-x_3)(-x_3+x_1)(x_1-x_2)}-s_3\frac{1}{(x_3-x_4)(x_2-x_4)(x_2-x_3)(x_1-x_2)}$$
$$+s_3s_2\frac{1}{(x_3-x_4)(x_2-x_4)(x_2-x_3)(x_1-x_3)}+s_1s_2\frac{1}{(x_2-x_4)(x_2-x_3)(x_1-x_3)(x_1-x_2)}$$
$$+s_1s_3\frac{1}{(x_3-x_4)(x_2-x_3)(x_1-x_4)(x_1-x_2)}-s_1s_3s_2\frac{1}{(x_2-x_4)(x_2-x_3)(x_1-x_4)(-x_3+x_1)}\Bigg)$$

The other singularity, when $n = 4$, occurs for $\partial_3\partial_2\partial_1\partial_2\partial_3$, which requires specializing $X_{1324} = x_1 + x_2 - y_1 - y_2$:

$$\partial_3\partial_2\partial_1\partial_2\partial_3\,\Delta = (1-s_1)(1-s_3)\Big((x_1+x_2-x_3-x_4) - s_2(x_1-x_4) + s_2s_3(x_1-x_3)$$
$$+ s_2s_1(x_2-x_4) - s_2s_1s_3(x_2-x_3)\Big)\,.$$

On could obtain the expansion of a reduced product $\partial_i \cdots \partial_j$ by writing it as $(1-s_i)(x_i - x_{i+1})^{-1} \cdots (1-s_j)(x_j - x_{j+1})^{-1}$ and enumerating all subwords of $s_i \cdots s_j$. This is the method followed by Kostant and Kumar [71]. We prefer relating the coefficients to Schubert polynomials, in particular because the number of subwords of a reduced decomposition of a permutation σ is far greater than the number of permutations in the interval $[1, \sigma]$.

Since the coefficients c_ζ^τ in (3.3.3) must vanish when ζ does not belong to the interval $[1, \tau]$, one obtains the following characterization of the Ehresmann-Bruhat by vanishing properties of Schubert polynomials, which generalizes (3.1.1).

Proposition 3.3.4. *Given n and two permutations $\sigma, \zeta \in \mathfrak{S}_n$, then $X_\sigma(\mathbf{x}^\zeta, \mathbf{x}) \neq 0$ if and only if $\sigma \le \zeta$ with respect to the Ehresmann-Bruhat order.*

Graßmannian Schubert polynomials $Y_v : v \in \mathbb{N}^n$, $v = v\!\uparrow$ are symmetrical in $x_1, \ldots, x_n$. One does not need to specialize them in all permutations of $y_1, y_2, \ldots$, but, by symmetry, only in $\langle u \rangle = [y_{\sigma_1}, \ldots, y_{\sigma_n}]$ with σ of code $u0 \ldots 0$ such that $u = u\!\uparrow$. In that case, the last proposition becomes :

Corollary 3.3.5. *Let $u, v \in \mathbb{N}^n$ be anti-dominant. Then $Y_v(\langle u \rangle, \mathbf{y}) \neq 0$ if and only if $v \le u$ (componentwise).*

This property is given by Okounkov [133] in the case where $\mathbf{y} = \{0, 1, 2, \ldots\}$.

3.4 Wronskian of symmetric functions

Given a positive integer r, and r functions f_i of a single variable, the determinant $|f_i(x_j|$ is divisible by the Vandermonde in $x_1, x_2, \ldots$, and the quotient may be thought as a discrete analogue of the Wronskian [94, Prop. 9.3.1].

Writing $f_i(x_j) = f_i(x_1)s_1 \ldots s_{j-1}$, and using (3.3.1), one sees that

$$\Big|f_i(x_j)\Big|_{i,j=1,\ldots,r} \prod_{r \ge j > i \ge 1} (x_j - x_i)^{-1} = \Big|f_i x_1\, \partial_1 \ldots \partial_{j-1}\Big|_{i,j=1,\ldots,r} .$$

The same formula (3.3.1) may be applied to symmetric functions, replacing the integer r by a partition. Let $\lambda \in \mathbb{N}^n$ be a partition. To a family of symmetric functions $f_1(\mathbf{x}_n), f_2(\mathbf{x}_n), \ldots$ of cardinality the number of partitions contained in λ, we shall associate a Wronskian $W_\lambda(f_i)$.

For each $\mu \subseteq \lambda$, let σ^μ be the Graßmannian permutation of code $\mu\!\uparrow$. Thanks to (3.3.1), every symmetric function $f(\mathbf{x}_n)$ satisfies

$$f\left(x_{\sigma_1^\mu}, \ldots, x_{\sigma_n^\mu}\right) = f(\mathbf{x}_n) + \cdots + f\partial^{\mu\uparrow} \Cap (\sigma^\mu) .$$

Therefore, a determinant $\left|f_i\left(\mathbf{x}_n^{\sigma^\mu}\right)\right|$ may be transformed, by multiplication by a unitriangular matrix, into the determinant $\left|f_i(\mathbf{x}_n)\partial^{\mu\uparrow} \Cap (\sigma^\mu)\right|$.

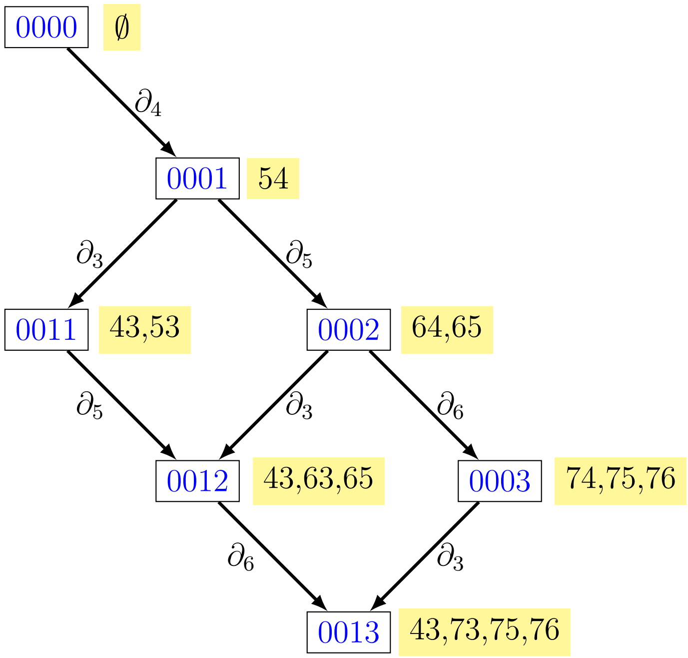
Definition 3.4.1. *Given a partition $\lambda \in \mathbb{N}^n$, and a family of symmetric functions $f_i(\mathbf{x}_n)$ of cardinality the number N of partitions contained in λ, then the* Wronskian *is*

$$W_\lambda(f_i(\mathbf{x}_n)) = \Big|f_i \partial^{\mu\uparrow}\Big|_{\substack{i=1\ldots N \\ \mu \subseteq \lambda}} .$$

The preceding analysis has shown that the Wronskian is equal to

$$\left|f_i\left(\mathbf{x}_n^{\sigma^\mu}\right)\right| \frac{1}{\prod_{\mu \subseteq \lambda} \Cap(\sigma^\mu)} .$$

For example, let $n = 4$, $\lambda = [3, 1, 0, 0]$. Then the family $\{\mu\!\uparrow\}$, as well as the inversion polynomials $\Cap(\sigma^\mu)$, are displayed on the next figure (writing ji instead of $x_j - x_i$). The family $\{\partial^{\mu\uparrow}\}$ is the set of paths from the origin.



In the case where the family $\{f_{i(\mathbf{x}_n)}\}$ is the set of Schur functions $\{s_\mu(\mathbf{x}_n) : \mu \subseteq \lambda\}$, the Wronskian is unitriangular, and thus its determinant is equal to 1.

In the case of a rectangular partition $\lambda \subseteq r^n$, the sets $\{\sigma^\mu(\mathbf{x}_n)\}$ are all the subsets of cardinality n of $\{x_1, \dots, x_{n+r}\}$. Given any $f \in \mathfrak{Sym}(\mathbf{x}_n)$, and $i : 1 \le i \le n{+}r{-}1$, then the set $\{f^{\mu\uparrow}\}$ is such that, either $f^{\mu\uparrow}$ and $f^{\mu\uparrow}\partial_i$ occur simultaneously, or $f^{\mu\uparrow}\partial_i = 0$. Thanks to the Leibnitz formula, this forces the Wronskian $W_{r^n}(f_1, f_2, \dots)$ to be annihilated by all ∂_i, $i = 1, \dots, n{+}r{-}1$. In other words, the Wronskian is a symmetric function when λ is a rectangular partition. Moreover, any inversion (j, i), $n{+}r \ge j > i \ge 1$, occurs $\binom{n+r-2}{n-1}$ times in the set of Graßmannian permutations $\{\sigma^\mu\}$.

In summary, one has the following lemma.

Lemma 3.4.2. *Let n, r be two positive integers, let $f_1, \dots, f_N$, with $N = \binom{n+r}{n}$, belong to $\mathfrak{Sym}(\mathbf{x}_{n+r})$. Then*

$$\frac{1}{\prod_{n+r\ge j>i\ge 1}(x_j - x_i)^{\binom{n+r-2}{n-1}}} \Big| f_i(X) \Big|_{\substack{i=1\dots N \\ X\subset\{x_1,\dots,x_{n+r}\}}} = W_{r^n}(f_1, \dots, f_N)$$

is a symmetric function of $x_1, \dots, x_{n+r}$.

For example, for $n = r = 2$, the Wronskian

$$
\begin{array}{l}
W_{22}\Big(Y_0(\mathbf{x},\mathbf{0}), Y_{01}(\mathbf{x},\mathbf{0}), Y_{11}(\mathbf{x},\mathbf{0}), Y_{03}(\mathbf{x},\mathbf{0}), Y_{23}(\mathbf{x},\mathbf{0}), Y_{34}(\mathbf{x},\mathbf{0})\Big) \\
\qquad = \begin{array}{c|cccccc}
 & 1 & \partial_2 & \partial_2\partial_1 & \partial_2\partial_3 & \partial_2\partial_3\partial_1 & \partial_2\partial_3\partial_1\partial_2 \\
\hline
Y_0 & Y_0 & 0 & 0 & 0 & 0 & 0 \\
Y_{01} & Y_{01} & Y_0 & 0 & 0 & 0 & 0 \\
Y_{11} & Y_{11} & Y_1 & Y_0 & 0 & 0 & 0 \\
Y_{03} & Y_{03} & Y_{002} & 0 & Y_{0001} & 0 & 0 \\
Y_{23} & Y_{23} & Y_{202} & Y_{012} & Y_{2001} & Y_{0101} & Y_{0001} \\
Y_{35} & Y_{35} & Y_{304} & Y_{024} & Y_{3003} & Y_{0203} & Y_{0013}
\end{array}
\end{array}
$$

is equal to

$$
Y_{0001}\Big(Y_{0101}Y_{0013} - Y_{0203}Y_{0001}\Big) = Y_{0001}^2 Y_{0113}\,.
$$

3.5 Yang-Baxter and Schubert

One can degenerate Yang-Baxter bases of Hecke algebras into bases of the algebra of divided differences. However, instead of taking products of factors of the type $\partial_i{+}1/c$, let us take factors $1+c\partial_i$. Accordingly, given a spectral vector $[y_1,\ldots,y_n]$, one defines recursively a Yang-Baxter basis $\mho^\partial_\sigma$, starting from 1 for the identity permutation, by

$$\mho^\partial_{\sigma s_i} = \mho^\partial_\sigma \Big(1+\partial_i\,(y_{\sigma_{i+1}} - y_{\sigma_i})\Big) \text{ for } \sigma_i < \sigma_{i+1}\,. \tag{3.5.1}$$

For example,

$$\begin{aligned}\mho^\partial_{321} = (1+\partial_1(y_2{-}y_1))&(1+\partial_2(y_3{-}y_1))(1+\partial_1(y_3{-}y_2))\\ &= 1+\partial_1(y_3{-}y_1)+\partial_2(y_3{-}y_1)+\partial_1\partial_2(y_2{-}y_1)(y_3{-}y_1)\\ &\qquad + \partial_2\partial_1(y_3{-}y_2)(y_3{-}y_1)+\partial_1\partial_2\partial_2(y_2{-}y_1)(y_3{-}y_1)(y_3{-}y_2)\end{aligned}$$

One remarks that the coefficients are the same as in the expression of $\sigma = [3,2,1]$ in terms of divided differences.

The following proposition shows that this property is true in general, and that the coefficients are still specialisations of Schubert polynomials.

Theorem 3.5.1. *The matrix of change of basis between $\{\mho^\partial_\sigma\}$ and $\{\partial_\sigma\Delta(\mathbf{y})\}$, and its inverse, have entries which are specializations of Schubert polynomials :*

$$\mho^\partial_\sigma = \sum_{\nu\le\sigma} \partial_\nu\, X_\nu(\mathbf{y}^\sigma,\mathbf{y})\,, \tag{3.5.2}$$

$$\partial_\nu\,\Delta(\mathbf{y}) = \sum \mho^\partial_\sigma\, X_{\omega\nu}(\mathbf{y},\mathbf{y}^{\omega\sigma})\ . \tag{3.5.3}$$

Proof. Let σ and i be such that $\ell(\sigma)<\ell(\sigma s_i)$. Suppose known the expansion

$$\mho^\partial_\sigma = \sum_\nu \partial_\nu\, X_\nu(\mathbf{y}^\sigma,\mathbf{y}) + \partial_{\nu s_i}\, X_{\nu s_i}(\mathbf{y}^\sigma,\mathbf{y})\ ,$$

with $\nu:\ \ell(\nu)<\ell(\nu s_i)$. Then its product by $1{+}(y_{\sigma_{i+1}}{-}y_{\sigma_i})\partial_i$ is

$$\sum_\nu \partial_\nu\, X_\nu(\mathbf{y}^\sigma,\mathbf{y}) + \partial_{\nu s_i}\Big(X_{\nu s_i}(\mathbf{y}^\sigma,\mathbf{y}) + X_\nu(\mathbf{y}^\sigma,\mathbf{y})(y_{\sigma_{i+1}}{-}y_{\sigma_i})\Big)\ ,$$

and the identities

$$X_\nu(\mathbf{y}^{\sigma s_i},\mathbf{y}) \qquad \& \qquad X_{\nu s_i}(\mathbf{y}^{\sigma s_i},\mathbf{y}) = X_{\nu s_i}(\mathbf{y}^\sigma,\mathbf{y}) + X_\nu(\mathbf{y}^\sigma,\mathbf{y})(y_{\sigma_{i+1}}{-}y_{\sigma_i})$$

give a similar expansion for $\mathcal{Y}_{\sigma s_i}$. QED

Notice that to expand products of factors $1+\partial_i(x_{i+1}-x_i)$, one has used the Leibnitz relations while in the present case the coefficients (in $\mathbf{y}$) commute with the operators acting on $\mathbf{x}$.

The analogy between Yang-Baxter elements and permutations can be materialised by acting on a proper element, as shows the following proposition.

Proposition 3.5.2. *For any $\sigma \in \mathfrak{S}_n$, one has*

$$X_\omega(\mathbf{x}, \mathbf{y}^\omega)\, \mho^\partial_\sigma = X_\omega(\mathbf{x}, \mathbf{y}^{\sigma\omega}) \tag{3.5.4}$$

Proof. In the step by step action of the factorised element $\mho^\partial_\sigma$, each step is of the type, $f(x_i - y_k)(1 + \partial_i(y_k - y_j) = f(x_i - y_j)$, $f \in \mathfrak{Sym}(x_i, x_{i+1})$. QED

For example, for $\sigma = [3, 4, 1, 2]$, writing the non-symmetric factor in a box, one has $\mho^\partial_{3412} = \big(1 + \partial_2(y_3 - y_2)\big)\big(1 + \partial_1(y_3 - y_1)\big)\big(1 + \partial_3(y_4 - y_2)\big)\big(1 + \partial_2(y_4 - y_1)\big)$ and

$$\begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_3 & \boxed{x_2{-}y_3} & \\ x_1{-}y_4 & x_2{-}y_4 & x_3{-}y_4 \end{array} \xrightarrow{1+\partial_2(y_3-y_2)} \begin{array}{lll} x_1{-}y_2 & & \\ \boxed{x_1{-}y_3} & x_2{-}y_2 & \\ x_1{-}y_4 & x_2{-}y_4 & x_3{-}y_4 \end{array} \xrightarrow{1+\partial_1(y_3-y_1)} \begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_1 & x_2{-}y_2 & \\ x_1{-}y_4 & x_2{-}y_4 & \boxed{x_3{-}y_4} \end{array}$$

$$\xrightarrow{1+\partial_3(y_4-y_2)} \begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_1 & x_2{-}y_2 & \\ x_1{-}y_4 & \boxed{x_2{-}y_4} & x_3{-}y_2 \end{array} \xrightarrow{1+\partial_2(y_4-y_1)} \begin{array}{lll} x_1{-}y_2 & & \\ x_1{-}y_1 & x_2{-}y_2 & \\ x_1{-}y_4 & x_2{-}y_1 & x_3{-}y_2 \end{array}$$

$$= X_{4321}(\mathbf{x}, \mathbf{x}^{2143})\,.$$

The general properties of Yang-Baxter bases induce properties of specialisations of Schubert polynomials.

The symmetry (1.8.4) entails

$$(\text{-}1)^{\ell(\nu)} X_\nu(\mathbf{y}^\sigma, \mathbf{y}) = X_{\omega\nu\omega}(y^{\omega\sigma\omega}, \mathbf{y}^\omega)\,. \tag{3.5.5}$$

Each of the equations (1.8.9) and (1.8.10) gives in turn

$$\sum_\nu (\text{-}1)^{\ell(\nu)} X_\nu(\mathbf{y}^\sigma, \mathbf{y}) X_{\nu\omega}(\mathbf{y}^\zeta, \mathbf{y}) = \Delta(\mathbf{y})\, \delta_{\sigma,\zeta\omega}\,, \tag{3.5.6}$$

but this is a special case of Cauchy formula

$$\sum_\nu (\text{-}1)^{\ell(\nu)} X_\nu(\mathbf{y}^\sigma, \mathbf{y}) X_{\nu\omega}(\mathbf{y}^\zeta, \mathbf{y}) \;=\; \sum_\nu X_{\nu^{-1}}(\mathbf{y}, \mathbf{y}^\sigma) X_{\nu\omega}(\mathbf{y}^\zeta, \mathbf{y}) \;=\; X_\omega(\mathbf{y}^\zeta, \mathbf{y}^\sigma)\,.$$

The quadratic form $(\ ,\ )^{\mathcal{H}}$ defined in (1.8.5) degenerates into the form

$$(f\,,\, g)^{\mathcal{H}00} = f\, g^\vee \cap \partial_\omega\,, \tag{3.5.7}$$

still denoting $f \to f^\vee$ be the anti-automorphism of the algebra of divided differences induced by $(\partial_\sigma)^\vee = \partial_{\sigma^{-1}}$.

Property (1.9.4) becomes

Proposition 3.5.3. *The Yang-Baxter bases associated to the spectral vectors $[y_1, \ldots, y_n]$ and $[y_n, \ldots, y_1]$ satisfy the relations*

$$\Big(\mho^{\partial,\mathbf{y}}_\sigma\,,\, \mho^{\partial,\mathbf{y}\omega}_\zeta\Big)^{\mathcal{H}00} = \delta_{\sigma,\omega\zeta}\, \Delta(\mathbf{y}^\sigma)\,. \tag{3.5.8}$$

For example, for $\sigma = \zeta = [2, 3, 1]$, one has to take the product of

$$\mho_{231}^{\partial,\mathbf{y}} = 1 + \partial_1(y_2 - y_1) + \partial_2(y_3 - y_1) + \partial_1\partial_2(y_2 - y_1)(y_3 - y_1)$$

and

$$\left(\mho_{231}^{\partial,\mathbf{y}\omega}\right)^{\vee} = 1 + \partial_1(y_2 - y_3) + \partial_2(y_1 - y_3) + \partial_2\partial_1(y_2 - y_3)(y_1 - y_3)\,.$$

The coefficient of ∂_{321} in this product is equal to $(y_2-y_1)(y_3-y_1)(y_2-y_3)+(y_2-y_1)(y_2-y_3)(y_1-y_3) = 0$, and this proves that $\left(\mho_{231}^{\partial,\mathbf{y}}\,,\,\mho_{231}^{\partial,\mathbf{y}\omega}\right)^{\mathcal{H}00} = 0$.

3.6 Distance 1 and multiplication

The ring $\mathfrak{Sym}(\mathbf{x})$ has a linear basis consisting of Schur functions. Its multiplicative structure is determined by the *Pieri formulas*, i.e. by the products of Schur functions by the elementary (or complete) symmetric functions. In the non-symmetric case, the requirement to recover the ring structure is easier. Polynomials being sums of monomials, and monomials being products of variables, we need only describe the images of the different bases under multiplication by $x_1, x_2, \ldots$.

Our bases being obtained by the use of ∂_i's or π_i's, we could use the commutation properties of these operators with multiplication by a single variable.

In the case of Schubert polynomials, let us rather use interpolation methods. This time, it will be more convenient to index polynomials by permutations, passing from the notation Y_v to the notation X_σ, where v is the code $\mathfrak{c}(\sigma)$ of σ.

Definition 3.6.1. *$v \in \mathbb{N}^n$ is a* successor *of u if $|v| = |u| + 1$ & $Y_u(\langle v\rangle, \mathbf{y}) \neq 0$. Given two permutations ζ, σ, then ζ is a* successor *of σ iff this is so for their codes.*

Theorem 3.6.2. *A permutation ζ, of code v, is a successor of σ iff $\zeta\sigma^{-1}$ is a transposition (a, b), and $\ell(\zeta) = \ell(\sigma) + 1$. In that case,*

$$X_\sigma(\langle v\rangle, \mathbf{y}) = ⋒(v)\,(y_{\zeta_b} - y_{\zeta_a})^{-1}\,.$$

Proof. If $u = \mathfrak{c}(\sigma)$ is dominant, then it is immediate to write the specializations of Y_u and check the proposition in that case. Let us therefore suppose that there exists i such that $u_i < u_{i+1}$, and let η be such that $\mathfrak{c}(\eta) = [u_1, \ldots, u_{i-1}, u_{i+1}{+}1, u_i, u_{i+2}, \ldots, u_n]$. Since for any permutation ζ of code v, one has

$$\Big(X_\eta(\langle v\rangle, \mathbf{y}) - (X_\eta(\langle v\rangle), \mathbf{y})^{s_i}\Big)\,(y_{\zeta_i} - y_{\zeta_{i+1}})^{-1} = X_\sigma(\langle v\rangle, \mathbf{y})\,,$$

ζ can be a successor of σ only if $\zeta = \eta$, or if ζs_i is a successor of η. In the first case,

$$X_\sigma(\langle v\rangle, \mathbf{y}) = X_\eta(\langle v\rangle, \mathbf{y})(y_{\eta_i} - y_{\eta_{i+1}})^{-1} = ⋒(v))\,,$$

while in the second,

$$\frac{-X_\eta(\langle v\rangle^{s_i}, \mathbf{y})}{y_{\zeta_i} - y_{\zeta_{i+1}}} = \frac{⋒(\mathfrak{c}(\zeta s_i))}{(y_{\zeta_{i+1}} - y_{\zeta_i})(y_{\zeta_b} - y_{\zeta_a})} = \frac{⋒(\mathfrak{c}(\zeta))}{y_{\zeta_b} - y_{\zeta_a}}\,,$$

and this proves the proposition. QED

Corollary 3.6.3 (Monk formula [69])**.** *Given $v \in \mathbb{N}^n$, $\sigma = \langle v\rangle$, $k \in \{1, \ldots, n\}$, then*

$$(x_k - y_{\sigma_k})X_\sigma(\mathbf{x}, \mathbf{y}) = \sum_{j>k} X_{\sigma\tau_{k,j}}(\mathbf{x}, \mathbf{y}) - \sum_{j<k} X_{\sigma\tau_{k,j}}(\mathbf{x}, \mathbf{y})\,, \qquad (3.6.1)$$

summed over all transpositions $\tau_{k,j}$ such that $\ell(\sigma\tau_{k,j}) = \ell(\sigma) + 1$.

Proof. The polynomial $(x_k - y_{\sigma_k})X_\sigma(\mathbf{x},\mathbf{y})$ belongs to the linear span of $Y_w : |w| = |v|+1$, because it is of degree $|v|+1$ and vanishes in all $\mathbf{y}^{\langle w\rangle} : |w| \le |v|$. Writing it $\sum c_\zeta X_\zeta(\mathbf{x},\mathbf{y})$, and testing all the specializations $\mathbf{y}^\zeta$, one finds that the permutations appearing in the sum are exactly the successors of σ such that $y_{\zeta_k} \neq y_{\sigma_k}$. QED

Instead of multiplying by x_k, on can equivalently multiply by $x_1 + \cdots + x_k$ at once, obtaining the following Pieri formula generalizing the product of a Schur function by the elementary symmetric function of degree 1.

Corollary 3.6.4 (Degree 1 Pieri formula)**.** *Given $n, k : k \le n$, $v \in \mathbb{N}^n$, $\sigma = \langle v\rangle$, $i \in \{1,\ldots,n\}$, then*

$$(x_1{+}\cdots{+}x_k - y_{\sigma_1}{-}\cdots{-}y_{\sigma_k})\, X_\sigma(\mathbf{x},\mathbf{y}) = \sum_{1\le i\le k<j} X_{\sigma\tau_{i,j}}(\mathbf{x},\mathbf{y})\,, \tag{3.6.2}$$

summed over transpositions $\tau_{i,j}$ such that $\ell(\sigma\tau_{i,j}) = \ell(\sigma)+1$.

One can iterate Monk formula. Let us call *k-path of length r* a sequence of permutations $\sigma^0, \sigma^1, \ldots, \sigma^r$ such that $\ell(\sigma^{i+1}) = \ell(\sigma^i)+1$ and $(\sigma^{i+1})^{-1}\sigma^i)$ is a transposition (k,j).

A k-path can be denoted by the sequence $[a_r,\ldots,a_0]$ of values permuted, with

$$a_0 = (\sigma^0)_k,\ a_1 = (\sigma^1)_k,\ \ldots, a_r = (\sigma^r)_k\,.$$

For $i = 1,\ldots,r$, each permutation $\sigma^i(\sigma^0)^{-1}$ is a cycle $(a_i \ldots a_1 a_0)$. The following proposition shows that the multiplication by a power of x_k can be described in terms of k-paths, the coefficients being complete functions $S_j(\,)$ of the variables y_i indexed by the values permuted.

Proposition 3.6.5. *Let $\sigma \in \mathfrak{S}_n$, $k \le n$, $m \in \mathbb{N}$. Then, modulo $\mathfrak{Sym}(\mathbf{x}_n) = \mathfrak{Sym}(\mathbf{y}_n)$, one has*

$$x_k^m\, X_\sigma(\mathbf{x},\mathbf{y}) = \sum \epsilon\, S_{m-1-r}(y_{a_0},\ldots,y_{a_r})\, X_{\tau_{a_r,a_{r-1}}\cdots\tau_{a_1a_0}\sigma}(\mathbf{x},\mathbf{y})\,, \tag{3.6.3}$$

sum over the k-paths of length $\le m$, the sign being given by the number of times $\tau_{a_i,a_{i-1}}$ transposes a value at position smaller than k.

Proof. Multiplying by x_k^m, using (3.6.1), involves enumerating paths with possible loops $\sigma^i = \sigma^{i+1}$ having weight y_j, with $j = (\sigma^i)_k$. The proposition results from grouping all the paths differing only by their loops, this explaining that the coefficient be a complete function. Each application of Monk formula possibly involves increasing the size of the symmetric group. One avoids that by using the ideal generated by the identification of symmetric functions in $\mathbf{x}_n$ with the same symmetric functions in $\mathbf{y}_n$. QED

The following tree describes the product $x_2^3\, X_{31425}(\mathbf{x},\mathbf{y})$, writing each permutation ζ above the coefficient of $X_\zeta(\mathbf{x},\mathbf{y})$.

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c} 3\boxed{1}425 \\ y_1^3 \end{array} \arrow[dl, dash] \arrow[dr, dash] & & \\
\begin{array}{c} 3\boxed{2}415 \\ S_2(y_1,y_2) \end{array} \arrow[d, dash] & & \begin{array}{c} 3\boxed{4}125 \\ S_2(y_1,y_4) \end{array} \arrow[d, dash] \arrow[dr, dash] & \\
\begin{array}{c} 3\boxed{4}215 \\ y_1+y_2+y_3 \end{array} \arrow[d, dash] \arrow[dr, dash] & & \begin{array}{c} 4\boxed{3}125 \\ -y_1-y_3-y_4 \end{array} \arrow[d, dash] & \begin{array}{c} 3\boxed{5}124 \\ y_1+y_4+y_5 \end{array} \arrow[d, dash] \\
\begin{array}{c} 4\boxed{3}215 \\ -1 \end{array} & \begin{array}{c} 3\boxed{5}214 \\ 1 \end{array} & \begin{array}{c} 4\boxed{5}123 \\ -1 \end{array} & \begin{array}{c} 5\boxed{3}124 \\ -1 \end{array}
\end{tikzcd}
$$

or, for the readers who prefer one-dimensional formulas,

$$
x_2^3 X_{31425} = y_1^3 X_{3142} + (y_1^2+y_4^2+y_1y_4)X_{3412} + (y_1^2+y_1y_2+y_2^2)X_{3241} + (y_1+y_5+y_4)X_{35124}
$$
$$
-(y_3+y_1+y_4)X_{4312} + (y_4+y_1+y_2)X_{3421} - X_{45123} + X_{361245} - X_{53124} + X_{35214} - X_{4321}\,.
$$

3.7 Pieri formula for Schubert polynomials

The Italian geometer Pieri described the intersection of a Schubert cycle by a "special" one in the cohomology ring of the Grassmannian. In modern terms, he described the product of a Schur function by an elementary or complete function, the remarkable property being that there is no multiplicity in his formula.

Let us generalize Pieri's result to Schubert polynomials, the presence of extra variables $\mathbf{y}$ allowing to interpret the intersection numbers 1 as complete functions of degree 0.

Our starting point will be the following case.

Lemma 3.7.1. *Let* $n, k, r \in \mathbb{N}$, $\rho = [n{-}1, \ldots, 0]$, $m = \max(n{-}k, 0)$ *and* $\mathbf{y}^{\heartsuit} = \{y_{m+1}, y_{m+2}, y_{m+3}, \ldots\}$. *Then*

$$Y_\rho(\mathbf{x},\mathbf{y})Y_{0^{k-1}r}(\mathbf{x},\mathbf{z}) = Y_\rho(\mathbf{x},\mathbf{y})Y_{0^{k-1}r}(\mathbf{y}^{\heartsuit},\mathbf{z}) + \sum_{i=1}^{k}\sum_{j=1}^{r} Y_{\rho+[0^{i-1}j0^{n-k}]}(\mathbf{x},\mathbf{y})Y_{0^{k-1+j}r-j}(\mathbf{y}^{\heartsuit},\mathbf{z})\,. \tag{3.7.1}$$

Proof. One uses Newton's interpolation (3.2.1) on the product fg, with $f = Y_\rho(\mathbf{x},\mathbf{y})$, $g = Y_{0^{k-1}r}(\mathbf{x},\mathbf{z})$, using Leibnitz' formula (1.4.2). The images of f under products of divided differences are 0 or Schubert polynomials that one has to specialize in $\mathbf{x} = \mathbf{y}$. Only $Y_{0\ldots 0}$ subsists. Let us first suppose that $n \leq k$. In a sum $\sum_{\epsilon_i,\ldots\epsilon_h\in\{0,1\}}\Big(f\partial_i^{\epsilon_i}\partial_j^{\epsilon_j}\cdots\partial_h^{\epsilon_h}\Big)\Big(g s_i^{\epsilon_i}\partial_i^{1-\epsilon_i}s_j^{\epsilon_j}\partial_j^{1-\epsilon_j}\cdots s_h^{\epsilon_h}\partial_h^{1-\epsilon_h}\Big)$ there remains only divided differences ∂_i, $i < n$ acting on f, s_i preserving g, and products $\partial_k\partial_{k+1}\cdots\partial_{k+j-1}$ acting on g and sending it to $Y_{0^{k-1+j}r-j}(\mathbf{x},\mathbf{z})$.

In final, for $n = 3 = k$ for example, the only non-zero contributions in Newton's formula are for $\partial_2\partial_1\partial_2(\partial_3\partial_4\cdots)$, $\partial_2(\partial_3\partial_4\cdots)\partial_1\partial_2$ and $(\partial_3\partial_4\cdots)\partial_2\partial_1\partial_2$, and this corresponds indeed to the RHS of (3.7.1).

In the case where $n > k$, writing $\mathbf{y}^{\heartsuit} = \{y_{n-k}, y_{k+1}, \ldots\}$, one factors $Y_\rho(x,y) = Y_{(n-k)^k,n-k-1,\ldots,0}(\mathbf{x},\mathbf{y})\,Y_{k-1,\ldots,0}(\mathbf{x},\mathbf{y}^{\heartsuit})$, and write the interpolation for the product $Y_{k-1,\ldots,0}(\mathbf{x},\mathbf{y}^{\heartsuit})Y_{0^{k-1}r}(\mathbf{x},\mathbf{z})$. QED

For example, for $n = 5$, $k = 3$, $r = 2$, one has $\mathbf{y}^{\heartsuit} = \{y_3, y_4, \ldots\}$ and

$$Y_{43210}(\mathbf{x},\mathbf{y})Y_{002}(\mathbf{x},\mathbf{z}) = Y_{43210}(\mathbf{x},\mathbf{y})Y_{002}(\mathbf{y}^{\heartsuit},\mathbf{z}) + \Big(Y_{53210}(\mathbf{x},\mathbf{y}) + Y_{44210}(\mathbf{x},\mathbf{y}) + Y_{43310}(\mathbf{x},\mathbf{y})\Big)\times Y_{0001}(\mathbf{y}^{\heartsuit},\mathbf{z}) + \Big(Y_{63210}(\mathbf{x},\mathbf{y}) + Y_{45210}(\mathbf{x},\mathbf{y}) + Y_{43410}(\mathbf{x},\mathbf{y})\Big)\,.$$

To describe the general Pieri formula, it is convenient to index Schubert polynomials by permutations, and generalize consecutivity in the Bruhat order.

Given an integer k, a pair of permutations $\sigma, \eta :\ \sigma \leq \eta$ is called a k-*soulèvement* of degree $\ell(\eta){-}\ell(\sigma)$ if each cycle ζ_i in the cycle-decomposition $\eta\sigma^{-1} = \zeta_1\cdots\zeta_m$ is of the type $\zeta_i = (\alpha, \delta, \gamma, \ldots, \beta)$ with $\delta > \gamma > \cdots > \beta > \alpha$, $\{\delta, \ldots, \alpha\}\cap\{\sigma_1, \ldots, \sigma_k\} = \{\alpha\}$ and $\ell(\eta) = \ell(\sigma) + (\#\zeta_1 - 1) + \cdots + (\#\zeta_m - 1)$. Denote furthermore $\mathbf{y}^{\sigma,\eta} = \{y_{\sigma_1}, \ldots, y_{\sigma_k}\} \cup \{y_i :\ i \in \{\zeta_1\}\cup\cdots\cup\{\zeta_m\}\}$.

For example the pair $\sigma = [5,2,7,4,1,6,8,3,9]$, $\eta = [6,2,9,4,3,5,7,1,8]$) is a 5-soulèvement of degree $1+1+2 = \ell(\eta) - \ell(\sigma)$, because $\eta\sigma^{-1} = (1,3)(5,6)(7,9,8)$, and $\mathbf{y}^{\sigma,\eta} = \{y_5, y_2, y_7, y_4, y_1\} \cup \{y_1, y_3\} \cup \{y_5, y_6\} \cup \{y_7, y_9, y_8\}$
$= \{y_5, y_2, y_7, y_4, y_1, y_6, y_8, y_9\}$.

Theorem 3.7.2. *Let $n, k, r \in \mathbb{N}$, $\sigma \in \mathfrak{S}_n$. Then*

$$X_\sigma(\mathbf{x}, \mathbf{y}) Y_{0^{k-1}r}(\mathbf{x}, \mathbf{z}) = \sum_\eta X_\eta(\mathbf{x}, \mathbf{y}) Y_{0^{k-1+j}r-j}(\mathbf{y}^{\sigma,\eta}, \mathbf{z}) \,, \tag{3.7.2}$$

sum over all k-soulèvements (σ, η) of degree $j = 0, \ldots, r$.

Proof. The divided differences in $\mathbf{y}$ send $X_{n\ldots1}(\mathbf{x}, \mathbf{y})$ onto any $X_\sigma(\mathbf{x}, \mathbf{y})$, up to sign. Thus, the theorem can be proved by decreasing induction on $\ell(\sigma)$, checking the evolution of the RHS of (3.7.2) under a simple divided difference in $\mathbf{y}$, starting from (3.7.1). QED

For example of the recursion, the term $X_{3471256}(\mathbf{x}, \mathbf{y}) Y_{0^5 2}(y_3, y_1, y_5, y_4, y_7, y_6)$ occur in the expansion of $X_{31542}(\mathbf{x}, \mathbf{y}) Y_{005}(\mathbf{x}, \mathbf{z})$, and the permutation $[3,4,7,1,2,5,6][3,1,5,4,2]^{-1}$ is equal to the product of cycles $(1,4)(5,7,6)$. Under $-\partial_2^y$, this term gives, in the expansion of $X_{21543}(\mathbf{x}, \mathbf{y}) Y_{005}(\mathbf{x}, \mathbf{z})$ the two terms $X_{3471256}(\mathbf{x}, \mathbf{y}) Y_{0^6 1}(y_2, y_1, y_5, y_4, y_3, y_7, y_6)$ and $X_{2471356}(\mathbf{x}, \mathbf{y}) Y_{0^5 2}(y_2, y_1, y_5, y_4, y_7, y_6)$, in accordance with

$$[3,4,7,1,2,5,6][2,1,5,4,3]^{-1} = (1,4)(2,3)(5,7,6) \,,$$
$$[2,4,7,1,3,5,6][3,1,5,4,2]^{-1} = (1,4)(5,7,6) \,.$$

3.8 Transition for Schubert polynomials

The right-hand side of Monk formula (3.6.1) involves two sets W_+, W_- of permutations:

$$(x_k - y_{\sigma_k})X_\sigma(\mathbf{x},\mathbf{y}) = \sum_{\zeta\in W_+} X_\zeta(\mathbf{x},\mathbf{y}) - \sum_{\nu\in W_-} X_\nu(\mathbf{x},\mathbf{y}) \,,$$

Let us call *transition* the case where W_+ is a singleton, rewriting the equation

$$X_\zeta(\mathbf{x},\mathbf{y}) = (x_k - y_{\sigma_k})X_\sigma(\mathbf{x},\mathbf{y}) + \sum_{\nu\in W_-} X_\nu(\mathbf{x},\mathbf{y}) \,, \tag{3.8.1}$$

the set W_- depending on the pair (k,ζ), or equivalently, the pair (k,σ) as described in (3.6.1).

For example,

$$\begin{aligned} X_{52186347}(\mathbf{x},\mathbf{y}) &= (x_2{-}y_1)X_{51286347}(\mathbf{x},\mathbf{y}) \\ &= (x_4{-}y_7)X_{5217634}(\mathbf{x},\mathbf{y}) + X_{5271634}(\mathbf{x},\mathbf{y}) \\ &\qquad + X_{5712634}(\mathbf{x},\mathbf{y}) + X_{7215634}(\mathbf{x},\mathbf{y}) \\ &= (x_5{-}y_4)X_{52184367}(\mathbf{x},\mathbf{y}) + X_{52481367}(\mathbf{x},\mathbf{y}) + X_{54182367}(\mathbf{x},\mathbf{y}) \end{aligned}$$

Transitions are compatible with Young subgroups. Indeed, let ζ belong to $\mathfrak{S}_{r|n-r}$. Then $\zeta = \zeta'\zeta''$, where ζ' fixes $r{+}1,\ldots,n$ and ζ'' fixes $1,\ldots,r$. Any transition for ζ' induces a transition for ζ. A transition

$$X_{\zeta'}(\mathbf{x},\mathbf{y}) = (x_k - y_{\sigma_k})X_\sigma(\mathbf{x},\mathbf{y}) + \sum_{\nu\in W_-} X_\nu(\mathbf{x},\mathbf{y}) \,,$$

all the permutations ν fix $r{+}1,\ldots,n$, and therefore one has the transition

$$X_\zeta(\mathbf{x},\mathbf{y}) = (x_k - y_{\sigma_k})X_{\sigma\zeta''}(\mathbf{x},\mathbf{y}) + \sum_{\nu\in W_-} X_{\nu\zeta''}(\mathbf{x},\mathbf{y}) \,. \tag{3.8.2}$$

By recurrence on the length of ζ', one obtains the following factorisation property of Schubert polynomials.

Corollary 3.8.1. *Let ζ belong to a Young subgroup, and $\zeta = \zeta'\zeta''$ its corresponding factorisation. Then*

$$X_\zeta(\mathbf{x},\mathbf{y}) = X_{\zeta'}(\mathbf{x},\mathbf{y})\, X_{\zeta''}(\mathbf{x},\mathbf{y}) \,. \tag{3.8.3}$$

Transitions may be used recursively to decompose Schubert polynomials into sums of "shifted monomials" $\prod(x_i - y_j)$, stopping the process when arriving at dominant polynomials.

Among all transitions for a given ζ, let us choose the one for which k is maximum, and call it *maximal transition*. For this transition, let us rather index polynomials by codes instead of permutations. Let $v \in \mathbb{N}^n$ be the code of ζ, and k be such that that $v_k > 0$, $v_{k+1} = 0 = \cdots = v_n$. Let $v' = v - [0^{k-1}10^{n-k}]$

and $\sigma = \langle v' \rangle$. In other words, $x^v = x^{v'} x_k$, with k maximal. Then the maximal transition rewrites as

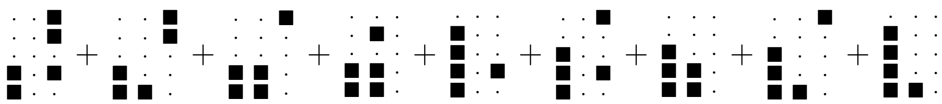
$$Y_v(\mathbf{x}, \mathbf{y}) = (x_k - y_{\sigma_k}) Y_{v'}(\mathbf{x}, \mathbf{y}) + \sum\nolimits_u Y_u(\mathbf{x}, \mathbf{y}) \ , \tag{3.8.4}$$

summed over all u such that $|u| = |v|$ and $\langle u \rangle \sigma^{-1}$ is a transposition τ_{ik} with $i < k$.

For example, starting with $v = [2, 0, 3]$, $\langle v' \rangle = \sigma = [3, 1, 5, 2, 4]$, one has the following sequence of transitions :

$$\begin{array}{rcl} Y_{203}(\mathbf{x}, \mathbf{y}) & = & (x_3 - y_5) Y_{202}(\mathbf{x}, \mathbf{y}) + Y_{230}(\mathbf{x}, \mathbf{y}) + Y_{401}(\mathbf{x}, \mathbf{y}) \, , \\ Y_{230}(\mathbf{x}, \mathbf{y}) & = & (x_2 - y_4) Y_{220}(\mathbf{x}, \mathbf{y}) + Y_{320}(\mathbf{x}, \mathbf{y}) \, , \\ Y_{401}(\mathbf{x}, \mathbf{y}) & = & (x_3 - y_2) Y_{400}(\mathbf{x}, \mathbf{y}) + Y_{410}(\mathbf{x}, \mathbf{y}) \, , \\ \cdots & & \cdots \end{array}$$

that one terminates when attaining dominant indices. Finally, writing each shifted monomial as a diagram of black squares in the Cartesian plane $\Big($ a square in column i, row j corresponds to a factor $(x_i - y_j)$ $\Big)$, the polynomial $Y_{203}(\mathbf{x}, \mathbf{y})$ reads



the first diagram, for example, coding the product

$$\begin{array}{ccc} \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \blacksquare \\ \cdot & \cdot & \cdot \\ \blacksquare & \cdot & \blacksquare \\ \blacksquare & \cdot & \cdot \end{array} \quad \Rightarrow \quad \begin{array}{cccc} \cdot & \cdot & (x_3 - y_5) & \\ \cdot & \cdot & (x_3 - y_4) & \\ \cdot & \cdot & \cdot & . \\ (x_1 - y_2) & \cdot & (x_3 - y_2) & \\ (x_1 - y_1 \cdot & \cdot & & \end{array}$$

We shall give in the sequel a different combinatorial description of Schubert polynomials in terms of tableaux.

Fomin and Kirillov [32] give configurations from which one reads a different decomposition of Schubert polynomials into shifted monomials.

3.9 Branching rules

Let us ignore the term $(x_k - y_{\sigma_k}) Y_{v'}(\mathbf{x}, \mathbf{y})$ in the maximal transition formula (3.8.4) and write

$$Y_v \to \sum_u \quad \text{or} \quad X_\sigma \to \sum_\zeta X_\zeta \, , \tag{3.9.1}$$

where the u's or ζ's are described in (3.8.4).

However, if v is dominant, then $Y_v = (x_k - y_{\sigma_k}) Y_{v'}$ and it would not be very informative to write $Y_v \to 0$. Let us introduce the equivalence $v \sim [0, v]$, allowing the concatanation of $0's$ on the left, which corresponds to identify $\mathfrak{S}_n$ and its image $\mathfrak{S}_1 \times \mathfrak{S}_n$ in $\mathfrak{S}_{n+1}$.

We can now iterate (3.9.1), producing an infinite graph.

Let us examine more closely the case where a permutation σ has only one successor. Write this permutation $\sigma = A\,2\,B\,4\,C\,3\,D$, with $2 < 3 < 4$, A, B, C, D being factors[4] such that $C\,3\,D$ is increasing, $D > 4$ and $B \cap [2, \ldots, 3] = \emptyset$. The successors of σ are all the permutations obtained by exchanging 3 in $A\,2\,B\,3\,C\,4\,D$ with a letter on its left such that length increases by 1 only. The permutation $\zeta = A\,3\,B\,2\,C\,4\,D$ fulfills this requirement, and if B does not contain any letter smaller than 2, then it is the unique successor of σ.

This indicates that permutations avoiding the pattern 2143 play a special role. Let us say that σ is *vexillary*[5] if there does not exist $i, j, k, l : \sigma_j < \sigma_i < \sigma_l < \sigma_k$. A *vexillary code* is the code of a vexillary permutation.

We have just seen that if σ is vexillary, then it has only one successor in a transition. In terms of codes, transition for vexillary codes reads as follows (eventually transforming v into $[0, v]$).

Lemma 3.9.1. *Let $v = [A\,b\,D\,c] \in \mathbb{N}^n$ be a vexillary code, with $c \neq 0$, the letter b being the rightmost occurence of the maximal value in $\{AbD\} \cap \{0, 1, \ldots, c{-}1\}$. Let $v' = [A\,b\,D\,c{-}1]$, $u = [A\,c\,D\,b]$, $\sigma = \langle v' \rangle$, $k = \sigma_n$. Then v' and u are vexillary codes, and*

$$Y_v(\mathbf{x}, \mathbf{y}) = (x_n - y_k) Y_{v'}(\mathbf{x}, \mathbf{y}) + Y_u(\mathbf{x}, \mathbf{y})\,. \tag{3.9.2}$$

With this rule, here is the graph originating from the vexillary code $[0, 1, 2, 8, 2, 7, 6, 4]$:

$$\begin{aligned}
&[0,1,2,8,2,7,6,4] \to [0,1,2,8,4,7,6,2] \to [0,2,2,8,4,7,6,1] \\
&\qquad \to [1,2,2,8,4,7,6] \to [1,2,2,8,6,7,4] \to [1,2,4,8,6,7,2] \\
&\qquad\qquad \to [2,2,4,8,6,7,1] \sim [0,2,2,4,8,6,7,1] \\
&\qquad \to [1,2,2,4,8,6,7] \to [1,2,2,4,8,7,6] \to [1,2,2,6,8,7,4] \\
&\qquad\qquad\qquad \to [1,2,4,6,8,7,2] \to [2,2,4,6,8,7,1] \to \ldots
\end{aligned}$$

Since a vexillary code has only one successor, one can truncate any transition graph, stopping at each vexillary code. For example, for $v = [0, 3, 1, 2, 0, 2]$, the transition graph is :

[4] σ is considered as a word, and the letters $2, 3, 4$ are not necessarily consecutive in the alphabet. One requires only that $2 < 3 < 4$.

[5] There are a lot of flags in a flag variety, but M.P. Schützenberger and I needed still more, to describe the properties of certain permutations. This is why we introduced the latin root “vexillum”, which survived a first period of drought and flourished afterwards.

$$
\begin{tikzcd}[arrows=dash]
 & Y_{031202} \arrow[dl] \arrow[dr] & & \\
Y_{03122} & & Y_{031301} \arrow[dl] \arrow[dr] & \\
 & Y_{03131} \arrow[d] & & Y_{0323} \arrow[dl] \arrow[d] \\
 & Y_{1313} \arrow[dl] \arrow[dr] & Y_{0332} & Y_{0422} \\
Y_{1331} & & Y_{1412} &
\end{tikzcd}
$$

Garsia [43] studies in detail this transition tree.

3.10 Vexillary Schubert polynomials

To a permutation σ, with code $v \in \mathbb{N}^n$, one associates two partitions $\mu, \lambda \in \mathbb{N}^n$ as follows. Let $w \in \mathbb{N}^n$ be such that $w_i = \max(j : j \geq i, v_j \geq v_i)$. Then μ, is the decreasing reordering of w and λ be the minimum dominant weight such that Y_v is the image of Y_λ under a product of divided differences.

The next property shows that vexillary Schubert polynomials can be expressed as a multi-Schur function.

Proposition 3.10.1. *Let v be a vexillary code, μ and λ be the associated partitions defined just above. Then*

$$Y_v(\mathbf{x}, \mathbf{y}) = S_{v\uparrow}(\mathbf{x}_{\mu_1} - \mathbf{y}_{\lambda_n}, \ldots, \mathbf{x}_{\mu_n} - \mathbf{y}_{\lambda_1})\,. \tag{3.10.1}$$

Proof. Normalize v by suppressing terminal 0's, so that one may suppose $r = v_n \neq 0$. Then the transition formula (3.9.2) states that

$$Y_v(\mathbf{x}, \mathbf{y}) = (x_n - y_k)Y_{v'}(\mathbf{x}, \mathbf{y}) + Y_u(\mathbf{x}, \mathbf{y})$$

Suppose the proposition to be true for v', by induction on weight, and u. The two Schur functions differ in only one column the sum being

$$(x_n - y_k)S_{\bullet, r-1, \bullet}(\bullet, \mathbf{x}_n - \mathbf{y}_{k-1}, \bullet) + S_{\bullet, r, \bullet}(\bullet, \mathbf{x}_{n-1} - \mathbf{y}_{k-1}, \bullet)\,.$$

Since for any j, any A (here, $A = \mathbf{x}_{n-1} - \mathbf{y}_{k-1}$), one has

$$(x_n - y_k)S_{j-1}(A + x_n) + S_j(A) = S_j(A + x_n - y_k)$$

this sum is equal to the expected multiSchur function $S_{\bullet, r, \bullet}(\bullet, \mathbf{x}_n - \mathbf{y}_k, \bullet)$. One initiates the proposition by the Grasmannian case, where the determinant is obtained as the image of $Y_\lambda(\mathbf{x}, \mathbf{y})$ under ∂_ω. QED

For example, for $v = [0, 2, 7, 2, 4, 5, 5, 4]$ one has

$$\begin{aligned}
&Y_{02724554} \\
&\quad = S_{02244557}(\mathbf{x}_8{-}\mathbf{y}_0, \mathbf{x}_8{-}\mathbf{y}_3, \mathbf{x}_8{-}\mathbf{y}_3, \boxed{\mathbf{x}_8{-}\mathbf{y}_7}, \mathbf{x}_8{-}\mathbf{y}_7, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9) \\
&\qquad = (x_8{-}y_7)Y_{027245530000} + Y_{027445520000} \\
&= (x_8{-}y_7)S_{02234557}(\mathbf{x}_8{-}\mathbf{y}_0, \mathbf{x}_8{-}\mathbf{y}_3, \mathbf{x}_8{-}\mathbf{y}_3, \boxed{\mathbf{x}_8{-}\mathbf{y}_6}, \mathbf{x}_8{-}\mathbf{y}_7, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9) \\
&\qquad + S_{02244557}(\mathbf{x}_8{-}\mathbf{y}_0, \mathbf{x}_8{-}\mathbf{y}_3, \mathbf{x}_8{-}\mathbf{y}_3, \boxed{\mathbf{x}_7{-}\mathbf{y}_6}, \mathbf{x}_8{-}\mathbf{y}_7, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9, \mathbf{x}_7{-}\mathbf{y}_9)\,.
\end{aligned}$$

3.11 Stable part of Schubert polynomials

In the theory of symmetric functions, one usually prefers to eliminate variables by taking the projective limit $\mathfrak{Sym}(\mathbf{x}_\infty)$ of the ring $\mathfrak{Sym}(x_1,\ldots,x_n)$, which amounts to using infinite alphabets.

In terms of Schubert polynomials, the embedding $\mathfrak{Sym}(\mathbf{x}_n) \hookrightarrow \mathfrak{Sym}(\mathbf{x}_{n+1})$ translates into the transformation $Y_v(\mathbf{x},\mathbf{0}) \to Y_{0v}(\mathbf{x},\mathbf{0})$ for v antidominant. This leads to define the *stable part* $\mathcal{S}t(Y_v)$ of a Schubert polynomial $Y_v(\mathbf{x},\mathbf{y})$, as

$$\mathcal{S}t(Y_v) = Y_{0^N v}(\mathbf{x},\mathbf{y})\Big|_{x_j=0=y_j,\, j>N},$$

with N big enough, and consider it as an element of $\mathfrak{Sym}(\mathbf{x}_\infty) \otimes \mathfrak{Sym}(\mathbf{y}_\infty)$.

We first need to analyze the transformation $Y_v(\mathbf{x},\mathbf{y}) \to Y_{0v}(\mathbf{x},\mathbf{y})$ to compare $Y_{0^N v}(\mathbf{x},\mathbf{y})$ and $Y_{0^{N+1} v}(\mathbf{x},\mathbf{y})$ and precise what "N big enough" means.

Lemma 3.11.1. *Let $v \in \mathbb{N}^n$, $v \le [n,\ldots,1]$. Then*

$$\begin{aligned} Y_v(\mathbf{x},\mathbf{0})\pi^x_n \ldots \pi^x_1 &= Y_{0v}(\mathbf{x},\mathbf{0}) &(3.11.1)\\ Y_v(\mathbf{x},\mathbf{y})\pi^x_n \ldots \pi^x_1 \pi^y_n \ldots \pi^y_1 &= Y_{0v}(\mathbf{x},\mathbf{y})\,. &(3.11.2)\end{aligned}$$

Proof. By trivial commutation, one writes $\pi^x_n \ldots \pi^x_1 = x_n \ldots x_1 \partial^x_n \ldots \partial^x_1$, and one uses that $Y_v(\mathbf{x},\mathbf{0}) x_n \ldots x_1 = Y_{v+1^n}(\mathbf{x},\mathbf{0})$ when $v \in \mathbb{N}^n$. This proves the first statement. Writing $Y_v(\mathbf{x},\mathbf{y})$ as a sum $\sum c_{u,u'} Y_u(\mathbf{x},\mathbf{0}) Y_{u'}(\mathbf{y},\mathbf{0})$, one obtains that $Y_v(\mathbf{x},\mathbf{y})\pi^x_n \ldots \pi^y_1$ is equal to $\sum c_{u,u'} Y_{0u}(\mathbf{x},\mathbf{0}) Y_{0u'}(\mathbf{y},\mathbf{0})$, that is, to $Y_{0v}(\mathbf{x},\mathbf{y})$. QED

Lemma 3.11.2. *Let $f \in \mathfrak{Pol}(\mathbf{x}_n) \otimes \mathfrak{Pol}(\mathbf{y}_m)$, $\omega_n = [n,\ldots,1]$, $\omega_m = [m,\ldots,1]$, $\pi_{n\times n} = (\pi_n \ldots \pi_{2n-1}) \ldots (\pi_1 \ldots \pi_n)$. Then*

$$f\, \pi^x_{\omega_n} \pi^y_{\omega_m} = f\, \pi^x_{n\times n} \pi^y_{m\times m}\Big|_{x_i=0, i>n,\, y_j=0, j>m}. \tag{3.11.3}$$

Proof. Any monomial x^v, $v \in \mathbb{N}^n$, can be written $x^v = S_{v\omega}(\mathbf{x}_n, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_1)$, and its image under $\pi_n \ldots \pi_{2n-1}$ is equal to $S_{v\omega}(\mathbf{x}_{2n}, \mathbf{x}_{n-1}, \ldots, \mathbf{x}_1)$, which is sent to $S_{v\omega}(\mathbf{x}_{2n}, \mathbf{x}_{2n-1}, \mathbf{x}_{n-2}, \ldots)$ under $\pi_{n-1} \ldots \pi_{2n-2}$. In final, $x^v \pi_{n\times n}$ is equal to $S_{v\omega}(\mathbf{x}_{2n}, \mathbf{x}_{2n-1}, \ldots, \mathbf{x}_{n+1})$, and this function restricts to $S_{v\omega}(\mathbf{x}_n) = x^v \pi_{\omega_n}$. QED

For $v \le [n,\ldots,1]$, the stable part of $Y_v(\mathbf{x},\mathbf{y})$ is obtained by computing $Y_{0^n v}(\mathbf{x},\mathbf{y})$, which is the image of $Y_v(\mathbf{x},\mathbf{y})$ under $(\pi^x_n \ldots \pi^x_1) \ldots (\pi^x_{2n-1} \ldots \pi^x_1)\, (\pi^y_n \ldots \pi^y_1) \ldots (\pi^y_{2n-1} \ldots \pi^y_1)$ according to (3.11.2). But the product of divided differences can be rewritten $\pi^x_{1,\ldots,n,2n,\ldots,n+1} \pi^y_{1,\ldots,n,2n,\ldots,n+1} \pi^x_{n\times n} \pi^y_{n\times n}$. The first two factors preserve functions of $\mathbf{x}_n$ and $\mathbf{y}_n$. Therefore,

$$Y_{0^n v}(\mathbf{x},\mathbf{y}) = Y_v(\mathbf{x},\mathbf{y})\, \pi^x_{n\times n} \pi^y_{n\times n}\,.$$

Using (3.11.3), one sees that

$$\mathcal{S}t(Y_v(\mathbf{x},\mathbf{y})) = Y_v(\mathbf{x},\mathbf{y})\, \pi^x_{n\times n} \pi^y_{n\times n}\,. \tag{3.11.4}$$

A transition

$$Y_v(\mathbf{x},\mathbf{y}) = (x_k - y_j) Y_{v'}(\mathbf{x},\mathbf{y}) + \sum_{u\in\mathcal{U}} Y_u(\mathbf{x},\mathbf{y})$$

entails a transition

$$Y_{0^n v}(\mathbf{x},\mathbf{y}) = (x_{k+n} - y_{j+n}) Y_{0^n v'}(\mathbf{x},\mathbf{y}) + \sum_{u\in\mathcal{U}} Y_{0^n u}(\mathbf{x},\mathbf{y})\,.$$

Therefore transitions may be used to compute stable parts :

$$\mathcal{S}t(Y_v(\mathbf{x},\mathbf{y})) = \mathcal{S}t(Y_{0^n v}(\mathbf{x},\mathbf{y})) = \sum_{u\in\mathcal{U}} \mathcal{S}t(Y_u(\mathbf{x},\mathbf{y}))\,. \tag{3.11.5}$$

The determinantal expression of a vexillary polynomial, for $v \le [n,\ldots,1]$, shows that its stable part is equal to

$$\mathcal{S}t(Y_{0^n v}(\mathbf{x},\mathbf{y})) = S_{v\uparrow}(\mathbf{x}_n - \mathbf{y}_n)\,.$$

One can in fact relax the condition on v. If $Y_\lambda(\mathbf{x},\mathbf{y})$ is a dominant ancestor of $Y_v(\mathbf{x},\mathbf{y})$, with $v\in\mathbb{N}^n$ and $m=\lambda_1$, then $Y_v(\mathbf{x},\mathbf{y})$ is a polynomial in $x_1,\ldots,x_n$ and $y_1,\ldots y_m$. Using (3.11.2) and (3.11.3), one sees[6] that

$$Y_v(\mathbf{x},\mathbf{y})\,\pi^x_{\omega_n}\pi^y_{\omega_m} = S_{v\uparrow}(\mathbf{x}_n - \mathbf{y}_m)\,. \tag{3.11.6}$$

In summary, one has the following three ways of determining the stable part of a Schubert polynomial.

Theorem 3.11.3. *Let $v\in\mathbb{N}^n$, Y_λ be a dominant ancestor of Y_v, $m=\lambda_1$. Let $Y_{0v}(\mathbf{x},\mathbf{y}) = (x_k - y_j)Y_{0v'}(\mathbf{x},\mathbf{y}) + \sum_{u\in\mathcal{U}} Y_u(\mathbf{x},\mathbf{y})$ be a transition. Then*

$$\begin{aligned}
\mathcal{S}t(Y_v(\mathbf{x},\mathbf{y})) &= Y_v(\mathbf{x},\mathbf{y})\,\pi^x_{\omega_n}\pi^y_{\omega_m} && (3.11.7)\\
&= Y_{0^{n+m}v}(\mathbf{x},\mathbf{y})\Big|_{x_i=0,i>n,\,y_j=0,j>m} && (3.11.8)\\
&= \sum_{u\in\mathcal{U}} \mathcal{S}t(Y_u(\mathbf{x},\mathbf{y}))\,. && (3.11.9)
\end{aligned}$$

For example, the transition graph for $v=[0,3,1,2,0,2]$ given above has five terminal vertices: $Y_{03122}, Y_{1331}, Y_{1412}, Y_{0332}, Y_{0422}$, and this implies that

$$\mathcal{S}t(Y_{031202}(\mathbf{x},\mathbf{y})) = s_{3221}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{3311}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{4211}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{332}(\mathbf{x}_\infty - \mathbf{y}_\infty) + s_{422}(\mathbf{x}_\infty - \mathbf{y}_\infty)\ .$$

[6] The action of $\pi^x_{\omega_n}$ on the determinant of complete functions of $\mathbf{x}_k - \mathbf{y}_j$ expressing $Y_v(\mathbf{x},\mathbf{y})$ consists in replacing all $\mathbf{x}_k$ by $\mathbf{x}_n$. The action of $\pi^y_{\omega_m}$ is much more delicate, one has to use that some determinants of complete functions in $\mathbf{x}_k - \mathbf{y}_j$ can be written as determinants of complete functions in $\mathbf{y}_j - \mathbf{x}_k$ (cf. [81]). For example, the equality $X_\sigma(\mathbf{x},\mathbf{y}) = (-1)^{\ell(\sigma)} X_{\sigma^{-1}}(\mathbf{y},\mathbf{x})$ gives such a transformation of determinants in the vexllary case. We have bypassed this transformation by using $Y_v(\mathbf{x},\mathbf{y}) \to Y_{0^N v}(\mathbf{x},\mathbf{y})$.

We shall see later that

$$Y_{031202}(\mathbf{x}, \mathbf{0}) = K_{31202} + K_{31301} + K_{41201} + K_{323} + K_{422} \,.$$

Since evidently the image under π_ω of a key polynomial is a Schur function, the decomposition of a Schubert polynomial (specialized in $\mathbf{y} = \mathbf{0}$) into key polynomials is still another way of computing its stable part.

A special case of the determination of the stable part of a vexillary Schubert polynomial is the *Sergeev-Pragacz formula* showing that a Schur function of a difference of alphabets $\mathbf{x}_n - \mathbf{y}_m$ can be obtained by symmetrization of a product of differences $x_i - y_j$. Indeed, let $\lambda \in \mathbb{N}^n$ be dominant, $m \geq \lambda_1$. Then

$$Y_\lambda(\mathbf{x}, \mathbf{y}) \pi^x_{\omega_n} \pi^y_{\omega_m} = S_{\lambda\uparrow}(\mathbf{x}_n, \mathbf{y}_m) \,. \tag{3.11.10}$$

For example, writing the explicit expression of π_ω a a sum over the symmetric group, one has

$$\begin{aligned} S_{024}(\mathbf{x}_3 - \mathbf{y}_4) &= Y_{420}\, \pi^x_{321} \pi^y_{4321} \\ &= \frac{1}{\Delta(x_1, x_2, x_3)\Delta(y_1, y_2, y_3, y_4)} \sum_{\sigma \in \mathfrak{S}^x_3, \zeta \in \mathfrak{S}^y_4} (-1)^{\ell(\sigma)+\ell(\zeta)} \Big(x^{210} y^{3210}\, Y_\lambda \Big)^{\sigma\zeta} \,. \end{aligned}$$

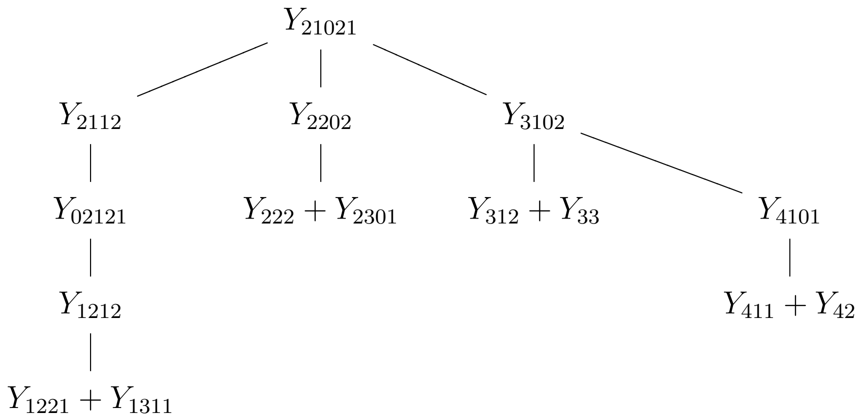
3.12 Schubert and the Littlewood-Richardson rule

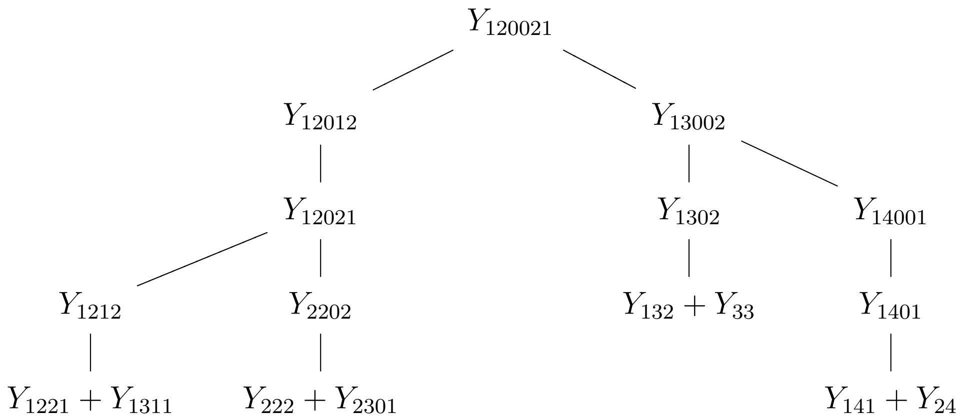
When a permutation $\sigma \in \mathfrak{S}_n$ belongs to a Young subgroup $\mathfrak{S}_{n'} \times \mathfrak{S}_{n''}$, the Schubert polynomial $X_\sigma(\mathbf{x}, \mathbf{y}) = Y_{v',v''}(\mathbf{x}, \mathbf{y})$ factorizes. This factorization is compatible with the restriction[7] of $Y_{0^N,v',v''}(\mathbf{x}, \mathbf{y})$ to $\mathbf{x}_N, \mathbf{y}_N$, and therefore in that case

$$\mathcal{S}t(Y_v(\mathbf{x}, \mathbf{y})) = \mathcal{S}t(Y_{v'}(\mathbf{x}, \mathbf{y}))\, \mathcal{S}t(Y_{v''}(\mathbf{x}, \mathbf{y}))\,.$$

In particular, when the Schubert polynomial factorizes into two vexillary Schubert polynomials, then its stable part is the product of two Schur functions. Since the stable part can be computed by transition, this observation furnishes many ways, different from the usual *Littlewood-Richardson rule*, of computing the product of Schur functions.

For example, to compute the square of s_{21}, one can start with any $v = v'v''$, with $v', v'' \in \{[2,1,0], [2,0,1,0], [1,2,0,0]\}$. Here are two possible transition graphs, starting with $[2,1,0,2,1,0]$ or $[2,1,0,1,2,0,0]$, which are the codes of the permutations $[3,2,1,6,5,4] \in \mathfrak{S}_3 \times \mathfrak{S}_3$ and $[3,2,1,5,7,4,6] \in \mathfrak{S}_3 \times \mathfrak{S}_4$, and stopping at vexillary codes.





[7]using symmetization is more delicate, since symmetrization does not commute vith product in general.

Both graphs imply that

$$s_{21}s_{21} = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{3111} + s_{222} + s_{2211} \ .$$

Chapter 4

Products and transitions for Grothendieck and Keys

4.1 Monk formula for type A key polynomials

Instead of considering the multiplication by each x_i in the key basis, let us describe the multiplication by

$$\xi = \xi_n^A = y_1 x_1 + \cdots + y_n x_n \,.$$

This element is invariant under the symmetric group acting on x_i and y_i simultaneously, and therefore, for any permutation σ, one has $(\xi)^{\sigma^x} = (\xi)^{(\sigma^y)^{-1}}$.

Since key polynomials are obtained by applying on dominant monomials the operators π_σ, $\sigma \in \mathfrak{S}_n$, we essentially need to describe the products $\pi_\sigma \xi$, that we shall write

$$\pi_\sigma \xi = x_1 \varphi_\sigma^1 + \cdots + x_n \varphi_\sigma^n \,.$$

The commutation relations $\pi_i x_i = x_{i+1}\pi_i + x_i$, $\pi_i x_{i+1} = x_i \pi_i - x_i = x_i \widehat{\pi}_i$, $\pi_1 \ldots \pi_i x_{i+1} = x_1 \widehat{\pi}_1 \ldots \widehat{\pi}_i$ imply

$$\begin{aligned} \pi_1 \ldots \pi_{k-1}\xi = \pi_1 \ldots \pi_{k-2}\, (\xi)^{s_{k-1}^y} \pi_{k-1} + \pi_1 \ldots \pi_{k-2} x_{k-1}(y_{k-1} - y_k) \\ = \pi_1 \ldots \pi_{k-3}\, (\xi)^{s_{k-2}^y s_{k-1}^y} \pi_{k-2}\pi_{k-1} \\ + \pi_1 \ldots \pi_{k-3} x_{k-2} \pi_{k-1} (y_{k-2} - y_k) + x_1 \widehat{\pi}_1 \ldots \widehat{\pi}_{k-3} (y_{k-1} - y_k)\,. \end{aligned}$$

Iterating and grouping the coefficients of y_k, one obtains

$$\begin{aligned} \pi_1 \ldots \pi_{k-1}\xi = (\xi)^{s_1^y \ldots s_{k-1}^y} \pi_1 \ldots \pi_{k-1} + x_1 \Big(\widehat{\pi}_1 \ldots \widehat{\pi}_{k-1} y_k + \widehat{\pi}_1 \ldots \widehat{\pi}_{k-2} y_{k-1} \\ + \widehat{\pi}_1 \ldots \widehat{\pi}_{k-3} y_{k-2} \pi_{k-1} + \widehat{\pi}_1 \ldots \widehat{\pi}_{k-4} y_{k-3} \pi_{k-2}\pi_{k-1} + \cdots + y_1 \pi_2 \ldots \pi_k \Big)\,. \quad (4.1.1) \end{aligned}$$

Given a permutation $\sigma \in \mathfrak{S}_n$, let us write it $\sigma = \zeta s_1 \ldots s_{k-1}$, with $\zeta \in \mathfrak{S}_{1\times n-1}$. Relation 4.1.1 entails

$$\varphi_\sigma^i = \left(\varphi_\zeta^i\right)^{s_1^y \ldots s_{k-1}^y} \pi_1 \ldots \pi_{k-1} \;,\; i \geq 2$$

$$\varphi_\sigma^1 = \pi_\zeta\Big(y_1\pi_2\dots\pi_k + \cdots + \widehat{\pi}_1\dots\widehat{\pi}_{k-1}y_k\Big)\,,$$

while $\varphi_\zeta^1 = \pi_\zeta y_1$.

These recursions furnish an induction on n for the products $K_v\xi$.

Proposition 4.1.1. *Let $v \in \mathbb{N}^n$, $\lambda = v\downarrow$, $\sigma \in \mathfrak{S}_n$, $\zeta \in \mathfrak{S}_{1\times n-1}$ be such that $K_v\pi_\sigma = x^\lambda\pi_\zeta\pi_1\dots\pi_{k-1}$. Then*

$$K_v\xi = \left(x^\lambda\pi_\zeta\xi\Big|_{y_1=0}\right)^{s_1^y\dots s_{k-1}^y} + \\ x^\lambda x_1\pi_\zeta\left(y_1\pi_2\dots\pi_k + \widehat{\pi}_1 y_2\pi_3\dots\pi_k + \cdots + \widehat{\pi}_1\dots\widehat{\pi}_{k-1}y_k\right). \quad (4.1.2)$$

For example, when $v = [1,3,5,7]$, one has $\lambda = [7,5,3,1]$, $\sigma = [4,3,2,1]$, $\zeta = [1,4,3,2]$. Supposing known that

$$K_{7135}\,\xi - y_1K_{8135} = \Big(y_4K_{7136} + (y_3-y_4)K_{7163} + (y_2-y_3)K_{7613}\Big) \\ + \Big(y_3K_{7145} + (y_2-y_3)K_{7415}\Big) + y_2K_{7235}\,,$$

one obtains

$$x^{7531}\Big(x_2\varphi_{4321}^2 + x_3\varphi_{4321}^3 + x_4\varphi_{4321}^4\Big) = \Big(y_3K_{1367} + (y_2-y_3)K_{1637} \\ + (y_1-y_2)K_{6137}\Big) + \Big(y_2K_{1457} + (y_1-y_2)K_{4157}\Big) + y_1K_{2357}\,,$$

while

$$x^{7531}x_1\varphi_{4321}^1 = K_{7135}x_1\Big(y_1\pi_2\pi_3 + \widehat{\pi}_1y_2\pi_3 + \widehat{\pi}_1\widehat{\pi}_2y_3 + \widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_3y_4\Big) \\ = y_4K_{1358} + (y_3-y_4)K_{1385} + (y_2-y_3)K_{1835} + (y_1-y_2)K_{8135}\,,$$

the sum of these two terms being equal to $K_{1357}\xi$.

A fully explicit Monk formula would require finding combinatorial objects compatible with the above recursion, as well as a justification of the fact that the coefficients seem to be of the type y_i or $(y_i - y_j)$ only. For example,

$$K_{20424}\,\xi = y_5K_{20425} + (y_3-y_5)K_{20524} + (y_2-y_3)K_{25024} + (y_1-y_3)K_{50224} \\ + (y_4-y_2)K_{32404} + (y_3-y_2)K_{52024} + y_4K_{20434} + y_2K_{21424} \\ + (y_1-y_4)K_{30424} + (y_4-y_5)K_{20452} + (y_5-y_4)K_{20542} + (y_2-y_4)K_{23404}\,.$$

4.2 Product $G_v\,x_1\dots x_k$

We first need to extend the Ehresmann-Bruhat order to weights. Let $u, v \in \mathbb{N}^n$ be permuted of each other. Then $u \geq v$ if and only if for $k = 1,\dots,n$ one has $[u_1,\dots,u_k]\uparrow \geq [v_1,\dots,v_k]\uparrow$ componentwise.

Given $v \in \mathbb{N}^n$, $k \le n$, let

$$\mathcal{C}(v,k) = \{u :\ u \ge v\ \&\ \big(\forall i \ne k, us_i \ge v \text{ implies } us_i \ge u\big)\}\,.$$

In other words, $\mathcal{C}(v,k)$ is the set of weights above v which are minimum in the intersection of their coset modulo $\mathfrak{S}_{k\times n-k}$ with the interval $[v, [n \ldots 1]]$.

Using these sets, we define two operations $\circledast, \circledcirc$. Given $v \in \mathbb{N}^n$, $k \le n$, $z \in \mathbb{N}^k$, let $u \in \mathcal{C}(v,k)$ be such that $[u_1, \ldots, u_k] \uparrow = [z_1, \ldots, z_k] \uparrow$ if it exists. In that case, define

$$v \circledcirc z = u \quad \& \quad v \circledast z = u + [1^k 0^{n-k}]\,.$$

Otherwise put $v \circledcirc z = \emptyset = v \circledast z$.

For example, for $v = [3,5,1,6,2,4]$, $z = [6,3,2]$, one has

$$v \circledcirc z = [3,6,2,5,1,4] \quad \& \quad v \circledast z = [4,7,3,5,1,4]\,.$$

We have given in Lemma 1.4.2 the normal reordering of products of the type $\pi_\sigma x_1 \cdots x_k$. These reorderings provide the decomposition of $G_v x_1 \cdots x_k$ and $K_v x_1 \cdots x_k$ in the Grothendieck or key basis respectively, in terms of punched diagrams.

Let us index Grothendieck polynomials by permutations, putting $G_\emptyset = 0$, and let us introduce the ideal $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$ generated by $e_i(\mathbf{x}_n) - e_i(\mathbf{y}_n)$, $i = 1 \ldots n$.

Theorem 4.2.1. *Let $\sigma \in \mathfrak{S}_n$, $k \le n$. Then, modulo the ideal $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$, one has*

$$G_{(\sigma)} x_1 \cdots x_k \equiv \sum_{\tau \in \mathcal{C}(\sigma,k)} y_{\tau_1} \cdots y_{\tau_k}\, G_{(\tau)} = \sum_{z \in \mathbb{N}^k :\, n \ge z_1 > \cdots z_k} y_{z_1} \cdots y_{z_k}\, G_{(\sigma \circledcirc z)}\,. \tag{4.2.1}$$

Proof. Let ζ be the maximal permutation in the coset $\sigma\mathfrak{S}_{k\times(n-k)}$. Then

$$G_{(\sigma)} x_1 \cdots x_k = G_{(\omega)} \pi_{(\omega\zeta)} \pi_{(\zeta^{-1}\omega\sigma)}\, x_1 \cdots x_k = G_{(\omega)} \pi_{(\omega\zeta)}\, x_1 \cdots x_k\, \pi_{(\zeta^{-1}\omega\sigma)}\,.$$

Thanks to (1.4.7), the product $\pi_{(\omega\zeta)} x_1 \cdots x_k$ is equal to a sum $\sum x^{\mathcal{U}} \pi^{\mathcal{U}}$ over some punched diagrams. However, for any i, one has[1] $\prod_{j=1}^{i} \prod_{h=1}^{n-i} (x_i - y_j) \equiv 0$, hence $G_{(\omega)}(1 - y_{n+1-i} x_i^{-1}) \equiv 0$, that is, $G_{(\omega)} x_i \equiv G_{(\omega)} y_{n+1-i}$. Therefore $G_{(\sigma)} x_1 \cdots x_k$ is congruent to a sum $\sum_\tau c_\tau G_{(\tau)}$, with c_τ a monomial in $\mathbf{y}_n$ of degree k. It remains, but we shall not do it, to check the equivalence between enumerating punched diagrams and permutations in $\mathcal{C}(\sigma,k)$. QED

[1] For every $i \le n$, one has $\prod_{j=1}^{i} \prod_{h=1}^{n-i} (x_i - y_j) = S_{(n+1-i))^i}(\mathbf{x}_i - \mathbf{y}_{n-i}) = S_{(n+1-i))^i}\big((\mathbf{y}_n - \mathbf{y}_{n+1-i}) - (\mathbf{x}_n - \mathbf{x}_i) + (\mathbf{x}_n - \mathbf{y}_n)\big) \equiv S_{(n+1-i))^i}\big((\mathbf{y}_n - \mathbf{y}_{n+1-i}) - (\mathbf{x}_n - \mathbf{x}_i)\big)$. his last function is null because the cardinality of $\mathbf{y}_n - \mathbf{y}_{n+1-i}$ is $< i$ and the cardinality of $\mathbf{x}_n - \mathbf{x}_i$ is $< n+1-i$. For example, for $n = 5$, $i = 2$, $S_{44}(\mathbf{x}_2 - \mathbf{y}_4) \equiv S_{44}\big(y_5 - (x_3 + x_4 + x_5)\big) = 0$.

For example, for $\sigma = [4, 2, 1, 5, 3]$, and $k = 3$, then $G_{(42153)} = G_{(54321)}\pi_1\pi_3\pi_2\pi_4\pi_3$ and one has to enumerate the punched 122-diagrams to describe the product $G_{(42153)}\, x_1x_2x_3 = G_{31010}\, x_1x_2x_3 =$

$$
\begin{aligned}
&\left(x_2x_4x_5 \begin{array}{c|c|c|} \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{1} & 2 & 3 \\ \hline \end{array} \to y_4y_2y_1G_{(42153)} \right) + \left(x_1x_4x_5 \begin{array}{c|c|c|} \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline \end{array} \to y_5y_2y_1G_{(52143)} \right) \\
&+ \left(x_2x_3x_5 \begin{array}{c|c|c|} \cline{2-3} & \bullet & 4 \\ \hline \multicolumn{1}{|c|}{1} & 2 & 3 \\ \hline \end{array} \to y_4y_3y_1G_{(43142)} \right) + \left(x_1x_2x_5 \begin{array}{c|c|c|} \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{1} & \bullet & 3 \\ \hline \end{array} \to y_5y_4y_1G_{(45123)} \right) \\
&+ \left(x_2x_3x_4 \begin{array}{c|c|c|} \cline{2-3} & 3 & \bullet \\ \hline \multicolumn{1}{|c|}{1} & 2 & 3 \\ \hline \end{array} \to y_4y_3y_2G_{(42351)} \right) + \left(x_1x_2x_4 \begin{array}{c|c|c|} \cline{2-3} & 3 & 4 \\ \hline \multicolumn{1}{|c|}{1} & 2 & \bullet \\ \hline \end{array} \to y_5y_4y_2G_{(42513)} \right) \\
&+ \left(x_1x_3x_5 \begin{array}{c|c|c|} \cline{2-3} & \bullet & 4 \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline \end{array} \to y_5y_3y_1G_{(53142)} \right) + \left(x_1x_3x_4 \begin{array}{c|c|c|} \cline{2-3} & 3 & \bullet \\ \hline \multicolumn{1}{|c|}{\bullet} & 2 & 3 \\ \hline \end{array} \to y_5y_3y_2G_{(52341)} \right) \\
&\qquad\qquad\qquad + \left(x_1x_2x_3 \begin{array}{c|c|c|} \cline{2-3} & \bullet & 4 \\ \hline \multicolumn{1}{|c|}{1} & 2 & \bullet \\ \hline \end{array} \to y_5y_4y_3G_{(43512)} \right) .
\end{aligned}
$$

One obtains the products $G_{(\eta)}\, x_1x_2x_3$, for any η in the coset $\sigma\mathfrak{S}_{3\times 2}$, by taking the image of the preceding expansion under products of π_i's, $i \neq 3$. For example, $G_{(24153)}\, x_1x_2x_3 = G_{(42153)}\, x_1x_2x_3\pi_1$ results from sorting each permutation τ in the preceding sum into $[[\tau_1, \tau_2] \uparrow, \tau_3, \tau_4, \tau_5]$.

The number of terms in (4.2.1) is equal to the number of strict partitions $z \in \mathbb{N}^k$ between u and $[n, \ldots, n{+}1{-}k]$, where $u = [\sigma_1, \ldots, \sigma_k] \downarrow$, or, equivalently, the number of partitions containing $[u_1{-}n, \ldots, u_n{-}1]$ and contained in $[(n{-}k)^k]$.

The original Schubert calculus involved Graßmannians, and, in our terms, Schubert and Grothendieck polynomials indexed by Graßmannian permutations. For any Graßmannian permutation σ, corresponding to the partition $\mu = [\sigma_k{-}k, \ldots, \sigma_1{-}1]$, any r, the number of terms in the expansion of $G_{(\sigma)}\, (x_1 \cdots x_k)^r$ is the dimension of some space of sections, and is called a *postulation number*. From what precedes, it is equal to the number of increasing chains of partitions $\mu^0 = \mu \leq \mu^1 \leq \cdots \leq \mu^k \leq \mu^{k+1} = [(n{-}k)^k]$. This number has a determinantal formula proved by Hodge, with some help from Littlewood.

For example, the product $G_{(145236)}(x_1x_2x_3)^2$ involves 46 chains of strict partitions $[541] \le \mu^1 \le \mu^2 \le [654]$ (represented as two-columns Young tableaux) :

$$
\begin{aligned}
&\left(\begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 1&1\\\hline\end{array}\right) G_{(145236)} + \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 1&1\\\hline\end{array}\right) G_{(146235)} + \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&1\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&1\\\hline\end{array}\right) G_{(156234)} \\
&+ \left(\begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 2&2\\\hline\end{array}\right) G_{(245136)} + \left(\begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&5\\\hline 4&4\\\hline 2&3\\\hline\end{array}\right) G_{(345126)} \\
&+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 2&2\\\hline\end{array}\right) G_{(246135)} \\
&+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 2&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&2\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 2&2\\\hline\end{array}\right) G_{(256134)} \\
&+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 2&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&4\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&4\\\hline 2&3\\\hline\end{array}\right) G_{(346125)} \\
&+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 2&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 2&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 3&3\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 2&3\\\hline\end{array}\right) G_{(356124)} \\
&+ \left(\begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 1&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 2&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 2&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 2&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 1&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 5&6\\\hline 4&5\\\hline 1&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 4&5\\\hline 3&4\\\hline\end{array} + \begin{array}{|c|c|}\hline 6&6\\\hline 5&5\\\hline 4&4\\\hline\end{array}\right) G_{(456123)}
\end{aligned}
$$

4.3 Product $K_v\, x_1 \ldots x_k$

The computations of $K_v\, x_1 \cdots x_k$ and $G_v\, x_1 \cdots x_k$ are similar, and use the same equivalence, detailed in the appendix, between enumerating punched diagrams and describing sets $\mathcal{C}(v,k)$. It translates into the following theorem for what concerns key polynomials.

Theorem 4.3.1. *Let $v \in \mathbb{N}^n$, $k \le n$. Then*

$$K_v\, x_1 \cdots x_k = \sum_{u \in \mathcal{C}(v,k)} K_{u+[1^k,0^{n-k}]} = \sum_z K_{v \circledast z}\,, \tag{4.3.1}$$

sum over all $z \in \mathbb{N}^k$, $z = z\!\uparrow$, z subword of $v\!\uparrow$.

For example, for v=[2132], $k=2$, we frame the elements of $\mathcal{C}([2132])$ inside the interval $[2132, 3221]$, and figure the intersection of this interval with cosets modulo $\mathfrak{S}_{2\times 2}$.

$$
\begin{tikzcd}[column sep=small]
 & & [3221] \arrow[dl, dash] \arrow[dr, dash] & & \\
 & {[2321]} \arrow[dr, dash] & & {[3212]} \arrow[dl, dash] & \\
 \boxed{[2231]} & & \boxed{[2312]} & & \boxed{[3122]} \\
 & & \boxed{[2132]} & &
\end{tikzcd}
$$

On the other side, the subwords of length 2 of $v{\uparrow}= [1223]$ are 12, 13, 22, 23 and one has $v \circledast 12 = [2132]+[1100]$, $v \circledast 22 = [2231]+[1100]$, $v \circledast 13 = [3122]+[1100]$, $v \circledast 23 = [2312] + [1100]$, so that

$$
\begin{aligned}
K_{2132}\, x_1x_2 &= K_{2132+1100} + K_{2231+1100} + K_{3122+1100} + K_{2312+1100} \\
&= K_{3232} + K_{3331} + K_{4222} + K_{3412}\, .
\end{aligned}
$$

Notice that

$$
\begin{aligned}
K_{2132}x_1x_2 = K_{3221}\pi_1\pi_3\pi_2\, x_1x_2 = x^{3221}x_2x_4 \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\!\!\begin{array}{c}\boxed{3}\\ \end{array} + x^{3221}x_1x_3 \begin{array}{cc} & \boxed{3} \\ \boxed{1} & \boxed{\bullet}\end{array} \\
+ x^{3221}x_1x_2 \begin{array}{cc} & \boxed{\bullet} \\ \boxed{1} & \boxed{2}\end{array} + x^{3221}x_1x_4 \begin{array}{cc} & \boxed{3} \\ \boxed{\bullet} & \boxed{2}\end{array} + x^{3221}x_1x_3 \begin{array}{cc} & \boxed{\bullet} \\ \boxed{\bullet} & \boxed{2}\end{array} ,
\end{aligned}
$$

but that the term $x^{3221}x_1x_3 \begin{array}{cc} & \boxed{\bullet} \\ \boxed{\bullet} & \boxed{2}\end{array} = x^{4231}\pi_2 = 0$ disappears.

Dominant monomials can be written as products of fundamental weights $x_1 \cdots x_k$. Iterating (4.2.1) and (4.3.1), one obtains the product of a Grothendieck or a key polynomial by any dominant monomial. The rule will however take (later) a more satisfactory formulation when stated in terms of the plactic monoid.

4.4 Relating the two products

Let us show how to relate the products $G_{(\sigma)}x^\lambda$ and $K_u x^\lambda$.

Proposition 4.4.1. *Let $\sigma \in \mathfrak{S}_n$, $\lambda \in \mathbb{N}^n$ be a partition, $r \geq \lambda_1$, and $u = [r\sigma_1, \ldots, r\sigma_n]$. Then $K_u x^\lambda = \sum_w K_w$ is a sum without multiplicities and $G_{(\sigma)}x^\lambda$ is a sum over the same weights :*

$$
G_{(\sigma)}x^\lambda = \sum_w y^{\langle w\rangle} G_{\zeta(w)}\, ,
$$

with $\zeta(w) = \Big[\lfloor w_1/r\rfloor, \ldots, \lfloor w_n/r\rfloor\Big]$, $z = w\uparrow$, $\langle w\rangle = [z_1{-}r, \ldots, z_n{-}r]$.

Proof. The product by x^λ is a chain of λ_1 multiplications by monomials of the type $x_1 \cdots x_k$. From the preceding theorems, it can be written in terms of the operators $x^t \pi_\eta$, with $t \le [\lambda_1, \dots, \lambda_1]$. The hypothesis on u is such that each $u\!\downarrow +t$ is dominant, and therefore, gives the key polynomial indexed by $[u\!\downarrow +t]\,\eta$. On the other hand, the same operator $x^t \pi_\eta$ contributes to a Grothendieck polynomial multiplied by the monomial in y of exponent $[t_n \dots, t_1]$. QED

The following table describes the product $G_{3142} x^{2200}$ as the same time, taking $r = 3$, as the product $K_{9,3,12,6} x^{2200}$.

G_{4321}	y^{0112}	$K_{14,10,7,3}$
G_{3142}	y^{2020}	$K_{11,5,12,6}$
G_{3421}	y^{0121}	$K_{11,13,7,3}$
G_{4312}	y^{1012}	$K_{14,10,4,6}$
G_{3241}	$y^{1120} + y^{0220}$	$K_{11,8,12,3} + K_{11,7,12,4}$
G_{4132}	$y^{2011} + y^{2002}$	$K_{14,5,9,6} + K_{13,5,10,6}$
G_{3412}	$y^{1021} + y^{0022}$	$K_{11,14,3,6} + K_{11,13,4,6}$
G_{4231}	$y^{0211}+y^{0202}+y^{1102}+y^{1111}$	$K_{13,8,10,3} + K_{14,8,9,3} + K_{14,7,9,4} + K_{13,7,10,4}$

Of special importance is the case of multiplication by $x^{k\dots 1}$. Let us show in the next lemma a case where it is of interest to mix bases.

Lemma 4.4.2. *Let $k \le n$, $u \in \mathbb{N}^n$ be such that $u_1 \ge \cdots \ge u_k$, $u_{k+1} \ge \cdots \ge u_n$. Then*

$$\widehat{K}_u\, x_1^k \cdots x_{k-1}^2 x_k = Y_{u+[k,\dots,1,0^{n-k}]}(\mathbf{x}, \mathbf{0})\,.$$

Proof. The hypothesis on u implies that, with $\lambda = u\downarrow$, there exists a strictly increasing $v \in \mathbb{N}^k$ such that

$$\begin{aligned}\widehat{K}_u &= \widehat{K}_\lambda\,(\widehat{\pi}_{v_1} \cdots \widehat{\pi}_1)(\widehat{\pi}_{v_2} \cdots \widehat{\pi}_2) \cdots (\widehat{\pi}_{v_k} \cdots \widehat{\pi}_k)\\ &= \widehat{K}_\lambda\,(\partial_{v_1} \cdots \partial_1 x_2 \cdots x_{v_1+1})(\partial_{v_2} \cdots \partial_2 x_3 \cdots x_{v_1+1}) \cdots (\partial_{v_k} \cdots \partial_k x_{k+1} \cdots x_{v_k+1})\end{aligned}$$

Using repeatedly that $(\partial_j \cdots \partial_i x_{i+1} \cdots x_{j+1}) x_1 \cdots x_i = x_1 \cdots x_{j+1 \_} j \cdots \partial_i$, one can transfer all monomials to the left and obtain

$$\widehat{K}_u\, x_1^k \cdots x_k = x^\lambda (x_1 \cdots x_{v_1+1}) \cdots (x_1 \cdots x_{v_k+1})\,(\partial_{v_1} \cdots \partial_1) \cdots (\partial_{v_k} \cdots \partial_k)\,.$$

This is the image of a dominant monomial under a product of divided differences, hence the lemma after identifying the index of the Schubert polynomial. QED

4.5 Product with $(x_1 \dots x_k)^{-1}$

The original formulas of Pieri involved intersection of Schubert varieties with special Schubert varieties corresponding to elementary symmetric functions. At the level of Grothendieck polynomials, one has to consider products of Grothendieck polynomials with some special ones, for example with $G_{0^{k-1},1} = 1-y_1\cdots y_k x_1^{-1}\cdots x_k^{-1}$. This is not what we have done in (4.2.1), having taken $x_1\cdots x_k$ intead of its inverse. Let us repair this in the next theorem, which can be found in [85, Th 6.4].

Theorem 4.5.1. *Let $\sigma \in \mathfrak{S}_n$, $k \le n$. Let $\zeta \in \mathfrak{S}_n$ be such that $[\zeta_1,\ldots,\zeta_k] = [\sigma_1,\ldots,\sigma_k]\downarrow$, $[\zeta_{k+1},\ldots,\zeta_n] = [\sigma_{k+1},\ldots,\sigma_n]\downarrow$, and $\omega = [n,\ldots,1]$. Then, modulo the ideal $\mathfrak{Sym}(\mathbf{x}_n = \mathbf{y}_n)$, one has*

$$G_{(\sigma)} \frac{y_{\sigma_1}\cdots y_{\sigma_k}}{x_1\cdots x_k} \equiv G_{(\omega)}\,\widehat{\pi}_{\omega\zeta}\,\pi_{\zeta^{-1}\sigma}\,. \tag{4.5.1}$$

Proof. The hypothesis on ζ implies that, with $\mathcal{V}$ the diagram of $v = [n-\zeta_1,\ldots,n-k+1-\zeta_k]$, one has $\pi_{\omega\zeta} = \pi^{\mathcal{V}}$. Thanks to (1.4.4), one has $\pi^{\mathcal{V}}(x_1\cdots x_k)^{-1} = (x_{v_1+1}\cdots x_{v_k+k})^{-1}\,\widehat{\pi}^{\mathcal{V}}$. Since the factor $(x_1\cdots x_k)^{-1}$ commutes with $\pi_{\zeta^{-1}\omega\sigma}$ because $\zeta^{-1}\sigma$ belongs to $\mathfrak{S}_{k\times n-k}$, the theorem follows. QED

For example, for $k=3$, $\sigma = [4,3,6,7,8,2,1,5]$, one has $\zeta = [6,4,3,8,7,5,2,1]$, $v = [8,7,6]-[6,4,3] = [2,3,3]$, $\mathcal{V} = \begin{array}{|c|c|c|}\hline & 4 & 5\\ \hline 2 & 3 & 4\\ \hline 1 & 2 & 3\\ \hline\end{array}$, and $\zeta^{-1}\sigma = [2,3,1,5,4,7,8,6]$ has reduced decomposition $s_1s_2s_4s_6s_7$. Altogether,

$$\begin{aligned} G_{(\sigma)}\frac{y_4y_3y_6}{x_1x_2x_3} &\equiv G_{(\omega)}\Big(\widehat{\pi}_2\widehat{\pi}_1\ \widehat{\pi}_4\widehat{\pi}_3\widehat{\pi}_2\ \widehat{\pi}_5\widehat{\pi}_4\widehat{\pi}_3\Big)\Big(\pi_1\pi_2\pi_4\pi_6\pi_7\Big)\\ &= G_{(4,3,6,7,8,2,1,5)} - G_{(4,3,7,6,8,2,1,5)} - G_{(5,3,6,7,8,2,1,4)} + G_{(5,3,7,6,8,2,1,4)}\\ &\qquad - G_{(4,5,6,7,8,2,1,3)}G_{(5,4,6,7,8,2,1,3)} + G_{(4,5,7,6,8,2,1,3)} - G_{(5,4,7,6,8,2,1,3)}\,. \end{aligned}$$

V. Pons [140] shows that the expansion of the right hand side of (4.5.1) in the Grothendieck basis is a signed interval. Lenart and Postnikov [120] give a more general *equivariant K-Chevalley formula* valid for any Weyl group.

The preceding theorem involves products of π_i's and $\widehat{\pi}_j$'s, that one can study using key polynomials rather than Grothendieck polynomials. Let ∇ be an arbitrary product of π_i's and $\widehat{\pi}_j$'s, $i,j<n$. If $G_{(\omega)}\nabla = \sum c_\sigma G_{(\sigma)}$, then $K_\omega\nabla = \sum c_\sigma K_\sigma$, with the same coefficients, since every π_i acts in the same manner on the indices of both families of polynomials. This will allow us to reformulate (4.5.1) in the next statement.

Proposition 4.5.2. *Let $k\le n$, $v\in\mathbb{N}^k$ be antidominant, $\mathcal{V}$ be the v-diagram and σ be a permutation in $\mathfrak{S}_{k\times n-k}$. Then*

$$K_\omega\,\widehat{\pi}^{\mathcal{V}}\,\pi_\sigma = \sum \widehat{K}_\tau\,, \tag{4.5.2}$$

sum over all weights τ in the interval $[\eta,\eta\sigma]$, with $\eta\in\mathbb{N}^n$ permutation of $\omega = [n,\ldots,1]$ such that $\eta_1 = v_k+k,\ldots,\eta_k = v_1+1$, $\eta_{k+1} > \cdots, \eta_n$.

Proof. The weight η is such that $K_\omega\,\widehat{\pi}^{\mathcal{V}} = \widehat{K}_\eta$. The operator π_σ is equal to a sum $\sum_{\nu\le\sigma}\widehat{\pi}_\nu$, where all ν belong to $\mathfrak{S}_{k\times n-k}$. Hence products are reduced and $K_\omega\,\widehat{\pi}^{\mathcal{V}}\,\pi_\sigma = \sum_n u\widehat{K}_{\eta\nu}$. QED

For example, let $k=3$, $v=[1,2,2]$, $\sigma=[3,1,2,5,4]$. Then $\eta=[4,2,1,5,3]$ and

$$\widehat{K}_{54321}\,\widehat{\pi}^{\mathcal{V}}\,\pi_\sigma = \widehat{K}_{54321}\,(\widehat{\pi}_1\widehat{\pi}_3\widehat{\pi}_2\widehat{\pi}_4\widehat{\pi}_3)\,(\pi_2\pi_1\pi_4) = \widehat{K}_{42153}\,(1+\widehat{\pi}_2)(1+\widehat{\pi}_1)(1+\widehat{\pi}_4)$$
$$= \widehat{K}_{42153} + \left(\widehat{K}_{41253}+\widehat{K}_{24153}+\widehat{K}_{42135}\right) + \left(\widehat{K}_{14253}+\widehat{K}_{41235}+\widehat{K}_{24135}\right) + \widehat{K}_{14235}\,.$$

This is also equal to $K_{14235}-K_{15234}-K_{14325}+K_{15324}$, in accordance with

$$G_{(14235)}\frac{y_1y_2y_4}{x_1x_2x_3} = G_{(14235)} - G_{(15234)} - G_{(14325)} + G_{(15324)}\,.$$

4.6 More keys: K^G polynomials

Stability properties of Schubert polynomials can be analyzed by using the isobaric divided differences π_i. Let us show that the operators

$$D_i = (1-x_i^{-1})\pi_i = (x_i-1)\partial_i \tag{4.6.1}$$

play a similar role for what concerns the Grothendieck polynomials.

These operators satisfy the braid relations, being the images of the π_i under the transformation $x_i \to x_i{-}1$. As an operator commuting with multiplication by elements of $\mathfrak{Sym}(x_i, x_{i+1})$, D_i is characterized by

$$1D_i = 1 \quad \& \quad x_{i+1}D_i = 1\,.$$

More generally, $D_\omega = (x_1{-}1)^{n-1}\dots(x_{n-1}{-}1)\partial_\omega = G_\rho(\mathbf{x},\mathbf{1})\,\pi_\omega$ is characterized by the fact that it commutes with multiplication by elements of $\mathfrak{Sym}(\mathbf{x}_n)$ and sends any $x^v : \mathbf{0}\le v\le[0,\dots,n{-}1]$ to 1. Indeed, x^vD_ω may be written $(x^v, G_\rho(\mathbf{x},\mathbf{1}))^\pi$, and Formula 2.9.5 tells that $(x^v, G_\rho(\mathbf{x},\mathbf{y})) = y^{v\omega}$.

Taking the same starting points as for $G_v(\mathbf{x},\mathbf{1})$, one defines recursively K_v^G polynomials by

$$K_\lambda^G = G_\lambda(\mathbf{x},\mathbf{1}) \text{ when } \lambda \text{ dominant } \& \; K_{vs_i}^G = K_v^G\,D_i \text{ when } v_i\ge v_{i+1}\,. \tag{4.6.2}$$

The operators D_i, combined with multiplication by $G_{1^k}(\mathbf{x},\mathbf{1})$, can be used to generate recursively the Grothendieck polynomials $G_v(\mathbf{x},\mathbf{1})$.

Proposition 4.6.1. *Given $v\in\mathbb{N}^n$. If $0\notin v$, then*

$$G_v(\mathbf{x},\mathbf{1}) = (1{-}x_1^{-1})\dots(1{-}x_n^{-1})G_{v-1^n}(\mathbf{x},\mathbf{1})\,.$$

Otherwise, let k be such that $v_k=0$ and $v_i>0$ for $i<k$, let $u=[v_1{-}1,\dots,v_{k-1}{-}1,v_{k+1},\dots,v_n]$. Then

$$G_v(\mathbf{x},\mathbf{1}) = G_u(\mathbf{x},\mathbf{1})\,(1{-}x_{k-1}^{-1})\cdots(1{-}x_1^{-1})\,D_{n-1}\cdots D_k$$
$$= G_u(\mathbf{x},\mathbf{1})\,D_{n-1}\cdots D_k\,(1{-}x_{k-1}^{-1})\cdots(1{-}x_1^{-1})\,. \tag{4.6.3}$$

Proof. By trivial commutation, one can transform $D_{n-1}\cdots D_k$ $= (1\text{-}x_{n-1}^{-1})\pi_{n-1}\ldots(1\text{-}x_k^{-1})\pi_k$ into $(1\text{-}x_{n-1}^{-1})\ldots(1\text{-}x_k^{-1})\pi_{n-1}\ldots\pi_k$. Therefore

$$\begin{aligned}G_u(\mathbf{x},\mathbf{1})\,(1\text{-}x_{n-1}^{-1})\ldots(1\text{-}x_k^{-1})\,\pi_{n-1}\ldots\pi_k\,(1\text{-}x_{k-1}^{-1})\cdots(1\text{-}x_1^{-1})&\\ = G_u(\mathbf{x},\mathbf{1})\,(1\text{-}x_{n-1}^{-1})\ldots(1\text{-}x_1^{-1})\,\pi_{n-1}\ldots\pi_k&\\ = G_{u+1^{n-1}}(\mathbf{x},\mathbf{1})\,\pi_{n-1}\ldots\pi_k = G_v(\mathbf{x},\mathbf{1})\,,&\end{aligned}$$

as claimed. QED

With the same notations than in (**??**), if v is vexillary, then u is also vexillary, as well as $u' = u + [1^{k-1}, 0^{n-k}]$. Suppose that $G_{u'}(\mathbf{x},\mathbf{1}) = K^G_{u'}$. Then

$$\begin{aligned}G_v(\mathbf{x},\mathbf{1}) = G_{u+1^{n-1}}(\mathbf{x},\mathbf{1})\pi_{n-1}\ldots\pi_k&\\ = G_{u'}(\mathbf{x},\mathbf{1})(1\text{-}x_{n-1}^{-1})\ldots(1\text{-}x_k^{-1})\pi_{n-1}\ldots\pi_k&\\ = G_{u'}(\mathbf{x},\mathbf{1})D_{n-1}\ldots D_k = K^G_{u'}D_{n-1}\ldots D_k = K^G_v\,.&\end{aligned}$$

By recursion on n this proves

Corollary 4.6.2. *If v is vexillary code, then $G_v(\mathbf{x},\mathbf{1}) = K^G_v$.*

Notice that the shift of indices $G_v(\mathbf{x},\mathbf{1}) \to G_{0v}(\mathbf{x},\mathbf{1})$ may be obtained with the D_i. Indeed, if $v \in \mathbb{N}^n$, then

$$\begin{aligned}G_v(\mathbf{x},\mathbf{1})D_n\ldots D_1 = G_v(\mathbf{x},\mathbf{1})(1\text{-}x_n^{-1})\ldots(1\text{-}x_1^{-1})\pi_n\ldots\pi_1&\\ = G_{v+1^n}(\mathbf{x},\mathbf{1})\pi_n\ldots\pi_1 = G_{0v}(\mathbf{x},\mathbf{1})\,.&\end{aligned}$$

4.7 Transitions for Grothendieck polynomials

We have seen that multiplication by x_i, in the case of Schubert polynomials, can be used to provide a recursive definition of these polynomials. We are going to show that one still has a transition formula for Grothendieck and key polynomials (and later also Macdonald polynomials).

The case of Grothendieck polynomials is an extension of the case of Schubert polynomials, and is described in [90, Prop. 3]. Since it is proved by a straighforward recursion, let us state the property without proof (caution: in reference [90], one uses the variables $1 - 1/x_i$ instead of x_i).

It is more convenient to use indexing by permutations and write $G_{(\sigma)}$ instead of G_v, if v is the code of σ. In terms of permutations, the maximal transition formula for Schubert polynomials (3.8.4) reads as follows.

Given ζ and its code v, let k be such that $v_i = 0$ for $i > k$ and $v_k > 0$. Let σ be the permutation whose code is $v - [0^{k-1}10^{n-k}]$. Then

$$X_\zeta = (x_k - y_j)X_\sigma + \sum_i X_{\tau_{ji}\sigma}\,, \tag{4.7.1}$$

sum over all transpositions τ_{ji} such that $\sigma = [\ldots i \ldots j \ldots]$, $\tau_{ji}\sigma = [\ldots j \ldots i \ldots]$ and $\ell(\tau_{ji}\sigma) = \ell(\tau) + 1$.

Order decreasingly the integers i occuring in (4.7.1): $i_m > \cdots > i_1$, and write $(1 - \tau_{ji}) \star G_{(\sigma)}$ for $G_{(\sigma)} - G_{(\tau_{ji}\sigma)}$. With these conventions, one has

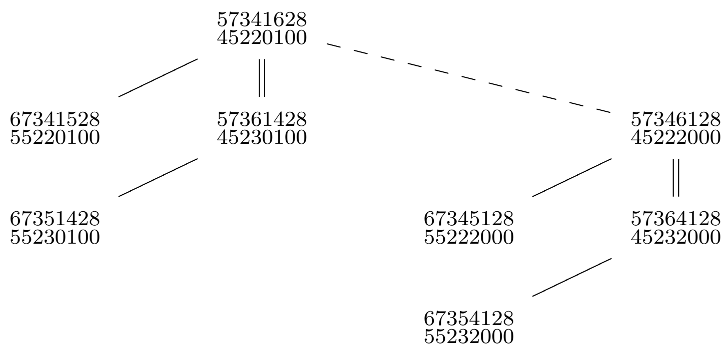
Theorem 4.7.1. *With the conventions of (4.7.1), one has the following transition formula*

$$\Big(G_{(\sigma)} - G_{(\zeta)}\Big)\,\frac{x_k}{y_j} = (1 - \tau_{ji_m}) \star \cdots (1 - \tau_{ji_1}) \star G_{(\sigma)}\,. \tag{4.7.2}$$

For example, for $\zeta = [5, 7, 3, 4, 1, 8, 2, 6]$, one has $\sigma = [5, 7, 3, 4, 1, 6, 2, 8]$, $k = 6$, $j = 6$, and

$$\Big(G_{(57341628)} - G_{(57341826)}\Big)\,\frac{x_6}{y_6} = (1 - \tau_{65}) \star (1 - \tau_{64}) \star (1 - \tau_{61}) \star G_{(57341628)}$$

is equal to the alternating sum of Grothendieck polynomials displayed below (with both indexings) :



Relation (2.6.5) allows to transform transition for G-polynomials to transition for $\widehat{G}$-polynomials.

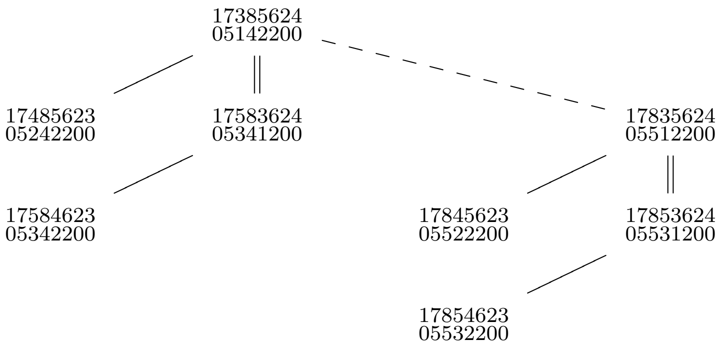
Corollary 4.7.2. *With the conventions of (4.7.1), writing i' for $n+1-i$, $i = 1, \ldots, n$, one has the following transition formula*

$$\left(\widehat{G}_{(\omega\sigma\omega)} + \widehat{G}_{(\omega\zeta\omega)}\right) \frac{x_{k'}}{y_j} = \left(1 + \tau_{j'i'_m}\right) \star \cdots \left(1 + \tau_{j'i'_1}\right) \star \widehat{G}_{(\omega\sigma\omega)} \,. \tag{4.7.3}$$

For example, the transition for $\widehat{G}_{(\omega\zeta\omega)} = \widehat{G}_{[37185624)}$ is the image of the transition for $G_{(\zeta)}$ given above :

$$\left(\widehat{G}_{(17385624)} + \widehat{G}_{(37185624)}\right) \frac{x_3}{y_6} = (1 + \tau_{34}) \star (1 + \tau_{35}) \star (1 + \tau_{38}) \star \widehat{G}_{(17385624)} \,,$$

and can be displayed as



One could in fact extend all transitions of Schubert polynomials, and not only maximal transitions, to transitions of Grothendieck polynomials. This is useful in the case of a permutation $\zeta = \zeta'\zeta''$ belonging to a Young subgroup as in (3.8.3). One has the same property as in (3.8.2). A transition

$$\left(G_{(\sigma)} - G_{(\zeta')}\right) \frac{x_k}{y_j} = (1 - \tau_{ji_m}) \star \cdots (1 - \tau_{ji_1}) \star G_{(\sigma)}$$

entails the relation

$$\left(G_{(\sigma\zeta'')} - G_{(\zeta)}\right) \frac{x_k}{y_j} = (1 - \tau_{ji_m}) \star \cdots (1 - \tau_{ji_1}) \star G_{(\sigma\zeta'')} \,. \tag{4.7.4}$$

As a consequence, Grothendieck polynomials satisfy the following factorization property (shown in [85, Prop. 6.7] for the polynomials $G_{(\zeta)}(\mathbf{x}, \mathbf{1})$).

Corollary 4.7.3. *Let ζ belong to a Young subgroup, and $\zeta = \zeta'\zeta''$ its corresponding factorisation. Then*

$$G_{(\zeta)}(\mathbf{x}, \mathbf{y}) = G_{(\zeta')}(\mathbf{x}, \mathbf{y})\, G_{(\zeta'')}(\mathbf{x}, \mathbf{y}) \,. \tag{4.7.5}$$

Using the recursive definition of Grothendieck polynomials to prove factorization would be delicate. For example, $G_{0120}(\mathbf{x},\mathbf{y})$ is a sum of 12 monomials which does not factorize[2]. Its image under π_3 is equal to

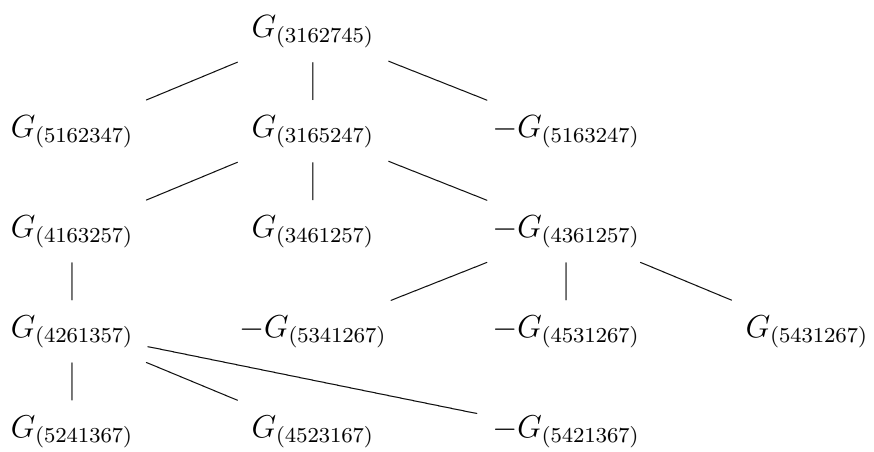
$$G_{0101}(\mathbf{x},\mathbf{y}) = G_{01}(\mathbf{x},\mathbf{y})G_{0001}(\mathbf{x},\mathbf{y}) = \left(1-\frac{y_1y_2}{x_1x_2}\right)\left(1-\frac{y_1y_2y_3y_4}{x_1x_2x_3x_4}\right).$$

[2] We shall see in (**??**) that it is equal to $S_{222}(\mathbf{x}_3, \mathbf{x}_3-\mathbf{y}_2, \mathbf{x}_3-\mathbf{y}_4)/x^{222}$.

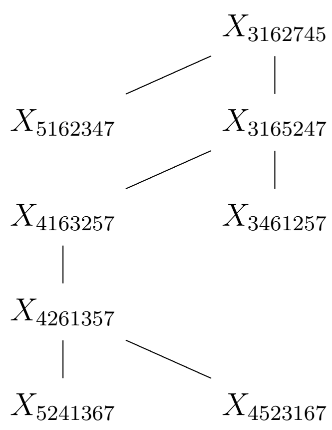
4.8 Branching and stable G-polynomials

As in the case of Schubert polynomials, one can use the transition formula (4.7.2) to obtain a transition graph with root a Grothendieck polynomial (indexed by a permutation), vertices being $\pm$ a Grothendieck polynomial, stopping at vexillary permutations.

For example, for $\sigma = [3, 1, 6, 2, 7, 4, 5]$, one has



The corresponding tree for $X_{3162745}$ is



If $v \in \mathbb{N}^n$ is antidominant, then K_v^G is symmetrical in $x_1, \ldots, x_n$, and one has the stability property $K_{0v}^G\Big|_{x_{n+1}=1} = K_v^G$. As for Schubert polynomials, this leads to define the *stable part* of a Grothendieck polynomial[3], for $v \in \mathbb{N}^n$ and $\omega = [n, \ldots, 1]$.

$$\mathcal{S}t(G_v) = G_v(\mathbf{x}, \mathbf{1}) D_\omega = G_{0^n v}(\mathbf{x}, \mathbf{1})\Big|_{x_{n+1}=1=\cdots=x_{2n}}. \tag{4.8.1}$$

[3]Contrary to the Schubert case, we eliminate for simplicity the alphabet $\mathbf{y}$.

A transition

$$G_{0^n v}(\mathbf{x}, \mathbf{1}) = (1 - x_k^{-1}) G_{0^n v'}(\mathbf{x}, \mathbf{1}) + x_k^{-1} \sum G_{0^n u}(\mathbf{x}, \mathbf{1})$$

induces the equality

$$\mathcal{S}t(G_v) = \sum \mathcal{S}t(G_u)\,,$$

and therefore, the transition graph is a convenient way of obtaining the stable part of a Grothendieck polynomial.

For example, the above graph shows that the stable part of $G_{(3162745)}$ is equal to

$$\begin{aligned} &\mathcal{S}t(G_{(5162347)}) + \mathcal{S}t(G_{(5241367)}) + \mathcal{S}t(G_{(4521367)}) - \mathcal{S}t(G_{(5421367)}) + \mathcal{S}t(G_{(3461257)}) \\ &\quad - \mathcal{S}t(G_{(5341267)}) - \mathcal{S}t(G_{(4531267)}) + \mathcal{S}t(G_{(5431267)}) - \mathcal{S}t(G_{(5163247)}) \\ &\quad = K^G_{0000124} + K^G_{0000234} + K^G_{0000034} - 2K^G_{0000134} \\ &\qquad + K^G_{0000223} - K^G_{0000224} + K^G_{0000133} - K^G_{0000233}\,. \end{aligned}$$

The terms $\mathcal{S}t(G_{(5421367)})$ and $\mathcal{S}t(G_{(5163247)})$ are both equal to $K^G_{0000134}$, hence a multiplicity 2.

4.9 Transitions for Key polynomials

Key polynomials satisfy a similar transition formula, exhibiting a boolean lattice, except that now one uses weights instead of permutations. The following considerations are drawn from an unpublished manuscript with Lin Hui and Arthur L.B. Yang.

Let $v \in \mathbb{N}^n$, let k be such that $v_i = 0$ for $i > k$ and $v_k > 0$. The leading term x^v of K_v is equal to $x^u x_k$, and we want to describe the difference $K_v - x_k K_u$ as a sum of key polynomials. We can suppose that $v_1 \geq \cdots \geq v_{k-1}$, because $\pi_1, \dots, \pi_{k-2}$ commute with multiplication by x_k.

Let us compute an example :

$$\begin{tikzcd}
x_6 K_{543103} = & & K_{543104} \arrow[dl, dash] \arrow[d, equal] \arrow[drr, dashed, dash] & & \\
& -K_{544103} & -K_{543401} \arrow[dl, dash] & & -K_{543140} \arrow[dl, dash] \arrow[d, equal] \\
& K_{544301} & & K_{544130} & K_{543410} \arrow[dl, dash] \\
& & & -K_{544310} &
\end{tikzcd}$$

Using the same notation as above for operations on indices, one may rewrite the preceding identity into

$$x_6 K_{543103} = (1 - \tau_{43}) \star (1 - \tau_{41}) \star (1 - \tau_{40}) \star K_{543104} \,.$$

We have used transpositions of values τ_{4i}, ignoring the leftmost 4. However, this example is not generic enough. What to do when values i are repeated?

Let us take a bigger example, which, this time, will pass the test of genericity. Let $v = [5, 4, 3, 3, 1, 1, 1, 0, 5]$. We have to compute

$$K_{5,4,3,3,1,1,1,0,4}\, x_9 = K_{5,4,4,3,3,1,1,1,0}\, \pi_3 \dots \pi_8\, x_9 \,.$$

Noticing, by the Leibnitz' commutations (1.4.3), that

$$\pi_3 \dots \pi_8\, x_9 = x_3\partial_3 x_4\partial_4 x_5\partial_5 x_6\partial_6 x_7\partial_7 x_8\partial_8\, x_9 = x_3\, \widehat{\pi}_3\widehat{\pi}_4\widehat{\pi}_5\widehat{\pi}_6\widehat{\pi}_7\widehat{\pi}_8 \,,$$

one obtains that $K_{5,4,3,3,1,1,1,0,4}\, x_9 = \widehat{K}_{5,4,3,3,1,1,1,0,4}$. The general case is similar and given in the following statement.

Lemma 4.9.1. *Let $v \in \mathbb{N}^n$ be such that $v_1 \geq \cdots \geq v_{n-1}$, $v_n \neq 0$, and let $u = [\dots, v_{n-1}, v_n{-}1]$. Then*

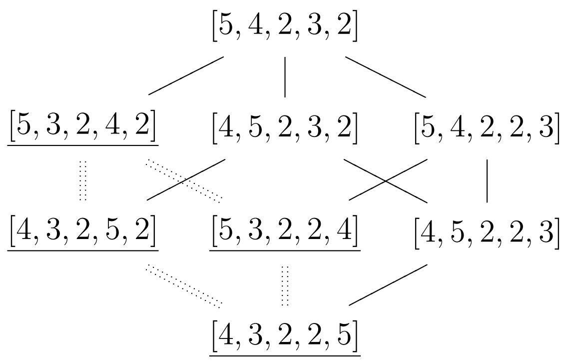
$$K_u\, x_n = \widehat{K}_v \,. \tag{4.9.1}$$

Expanding $\widehat{K}_v$ in terms of K_u (which means taking the Ehresmann-Bruhat interval), one obtains the transition for key polynomials in that case. Let us show the evolution of the transition under successive applications of π_i, $i \neq n-1$.

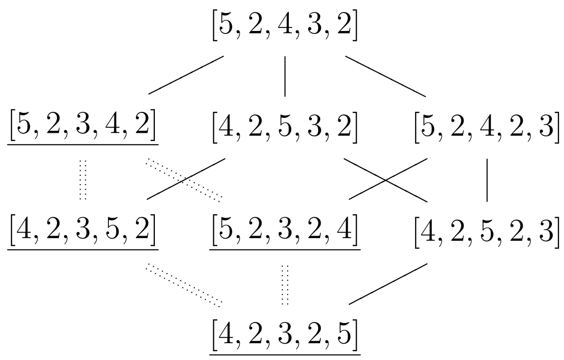
We begin with the transition for $K_{4,3,2,2,5}$:

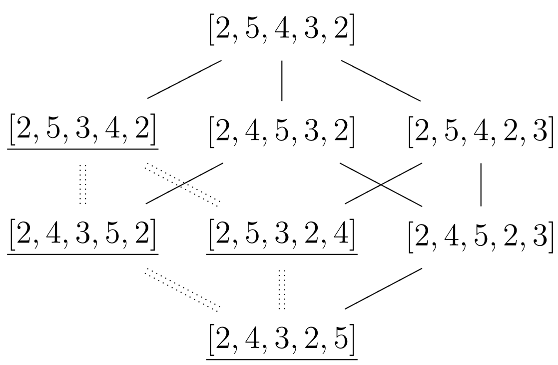
$$K_{4,3,2,2,5} - K_{4,3,2,2,4}\, x_5 = K_{4,3,2,2,5} - \widehat{K}_{4,3,2,2,5}$$
$$= \Big(K_{5,3,2,2,4} + K_{4,5,2,2,3} + K_{4,3,2,5,2}\Big) - \Big(K_{5,4,2,2,3} + K_{5,3,2,4,2} + K_{4,5,2,3,2}\Big) + \Big(K_{5,4,2,3,2}\Big)\,,$$

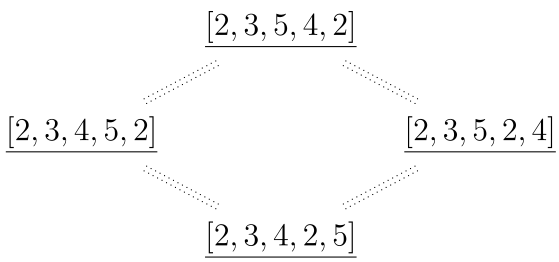
that we display as a boolean lattice (forgetting signs), writing the starting element as the bottom element



Applying π_2, then π_1, then again π_2, one obtains the transitions for $K_{2,4,3,2,5}$ and $K_{2,3,4,2,5}$:







The terms which are not underlined cancel two by two at the last stage, because $(K_{\bullet ji\bullet\bullet} - K_{\bullet ji\bullet\bullet})\,\pi_2 = 0$.

To write the general transition, we need to introduce, for each pair of integers i, j, an operator $\tau_{i,j}$ on linear combinations of K_u, defined[4] by

$$K_{\ldots u_i \ldots u_j \ldots} \star \tau_{i,j} = K_{\ldots u_j \ldots u_i \ldots}\,.$$

Then, one has the following transition formula, similar to the one for Grothendieck polynomials.

Theorem 4.9.2. *Let $v \in \mathbb{N}^n$, such that $v_n > 0$, and $u = [v_1, \ldots, v_{n-1}, v_n - 1]$. Let $i_1 < \cdots < i_r < n$ be the places i such that v_i is strictly maximal among the values $\{v_j : i \le j < n,\, v_j < v_n\}$. Then*

$$K_u\, x_n = K_v \star (1 - \tau_{i_1 n}) \cdots (1 - \tau_{i_r n})\,. \tag{4.9.2}$$

Proof. When $v_1 \ge \cdots \ge v_{n-1}$, the statement comes from rewriting the expansion of $\widehat{K_v}$ in (4.9.1) in terms of the operators τ_{in}.

Given any k such that $v_k > v_{k+1}$, one has $K_u\, x_n\, \pi_k = K_{us_k}\, x_n$. On the other hand, the product of the RHS of (4.9.2) is obtained by replacing v by vs_k and exchanging k and $k{+}1$ in the indices of the operators $\tau_{i,n}$, except one has the double factor $(1 - \tau_{k,n})(1 - \tau_{k+1,n})$. In that case the factor $(1 - \tau_{k,n})$ disappears, and this corresponds to the pairs $K_w - K_{ws_k}$ which vanish under π_k. QED

The four examples above must be rewritten

$$\begin{aligned}
K_{43224}\, x_5 &= K_{43225} \star (1 - \tau_{15})(1 - \tau_{25})(1 - \tau_{45}) \\
K_{42324}\, x_5 = K_{43224}\, x_5\, \pi_2 &= K_{42325} \star (1 - \tau_{15})(1 - \tau_{35})(1 - \tau_{45}) \\
K_{24324}\, x_5 = K_{42324}\, x_5\, \pi_1 &= K_{24325} \star (1 - \tau_{25})(1 - \tau_{35})(1 - \tau_{45}) \\
K_{23424}\, x_5 = K_{42324}\, x_5\, \pi_2 &= K_{23425} \star (1 - \tau_{35})(1 - \tau_{45})\,.
\end{aligned}$$

If $v \in \mathbb{N}^n$ is a vexillary code such that $v_n \neq 0$ and there exists $i:\ v_i < v_n$, then $Y_v(\mathbf{x}, \mathbf{0})$ and K_v satisfy the same transition :

$$Y_v(\mathbf{x}, \mathbf{0}) = x_k Y_{v'}(\mathbf{x}, \mathbf{0}) + Y_u(\mathbf{x}, \mathbf{0}) \quad \& \quad K_v = x_k K_{v'} + K_u\,,$$

with v' and u vexillary (cf. [105, Lemma 3.10]). Therefore, one has the following property, which is a special case of the expansion of a Schubert polynomial in terms of keys given in (7.3.2).

Lemma 4.9.3. *If v is a vexillary code, then*

$$Y_v(\mathbf{x}, \mathbf{0}) = K_v\,. \tag{4.9.3}$$

For example, there are 23 Schubert polynomials $Y_v(\mathbf{x}, \mathbf{0})$, $v \le [3, 2, 1, 0]$, which coincide with the key polynomial of the same index, while $Y_{1010}(\mathbf{x}, \mathbf{0}) = x_1(x_1{+}x_2{+}x_3)$ is different from $K_{1010} = x_1(x_2{+}x_3)$.

[4] If needed, u is transformed into $u, 0, 0, \ldots$

4.10 Vexillary polynomials

We have already stated that vexillary Schubert and key polynomials have a determinantal expression. This property is also satisfied by Grothendieck polynomials, and we collect together these three families in the next theorem.

First, dominant polynomials can be written as *multi-Schur* functions. Let v be dominant, $u = v\omega$, $k = v_1$. Then

$$\begin{aligned} Y_v &= S_u(\mathbf{x}_n - \mathbf{y}_{v_n}, \dots, \mathbf{x}_1 - \mathbf{y}_{v_1}) \\ G_v &= (x_1 \cdots x_n)^{-k} S_{k^n}(\mathbf{x}_n - \mathbf{y}_{v_n}, \dots, \mathbf{x}_1 - \mathbf{y}_{v_1}) \\ K_v &= S_u(\mathbf{x}_n, \dots, x_1) \end{aligned}$$

For example, for $v = [6, 3, 1]$, one has

$$Y_{631} = S_{136}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_2 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6) = \begin{vmatrix} S_1(\mathbf{x}_3 - \mathbf{y}_1) & S_4(\mathbf{x}_2 - \mathbf{y}_3) & S_8(\mathbf{x}_1 - \mathbf{y}_6) \\ S_0(\mathbf{x}_3 - \mathbf{y}_1) & S_3(\mathbf{x}_2 - \mathbf{y}_3) & S_7(\mathbf{x}_1 - \mathbf{y}_6) \\ 0 & S_2(\mathbf{x}_2 - \mathbf{y}_3) & S_6(\mathbf{x}_1 - \mathbf{y}_6) \end{vmatrix},$$

$$G_{631} = (x_1 x_2 x_3)^{-6} S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_2 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6),$$
$$K_{631} = S_{136}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1).$$

As we already saw, the action of ∂_i or π_i on a determinant of complete functions $S_k(\mathbf{x}_p - \mathbf{y}_q)$ is straightforward if only one column or one row is not invariant under the transposition of x_i, x_{i+1}. In that case, one has to transform this row or column, following the rules $S_k(\mathbf{x}_i - \mathbf{y})\partial_i = S_{k-1}(\mathbf{x}_{i+1} - \mathbf{y})$, $S_k(\mathbf{x}_i - \mathbf{y})\pi_i = S_k(\mathbf{x}_{i+1} - \mathbf{y})$.

For example,

$$\begin{aligned} &Y_{631} \xrightarrow{\partial_2} Y_{612} = S_{126}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6) \xrightarrow{\partial_1} Y_{152} \\ &\quad = S_{125}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_2 - \mathbf{y}_6) \xrightarrow{\partial_2} Y_{124} = S_{124}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_3 - \mathbf{y}_6), \end{aligned}$$

$$\begin{aligned} &G_{631}(x_1x_2x_3)^6 \xrightarrow{\pi_2} = S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_1 - \mathbf{y}_6) \xrightarrow{\pi_1} \\ &\quad = S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_2 - \mathbf{y}_6) \xrightarrow{\pi_2} = S_{666}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_3 - \mathbf{y}_3, \mathbf{x}_3 - \mathbf{y}_6). \end{aligned}$$

On the other hand, $Y_{631}\partial_1 = S_{135}(\mathbf{x}_3 - \mathbf{y}_1, \mathbf{x}_2 - \mathbf{y}_3, \mathbf{x}_2 - \mathbf{y}_6)$ and we cannot proceed so easily with ∂_2, since two columns involve x_2 and not x_3.

When v is vexillary, we have already used the property that there exists at least one sequence of operators ∂_i or π_i respectively, starting from a dominant case, such that at each step, only one column is transformed by the operator

To describe the missing determinants in the Grothendieck case, we have to follow the same recursion than for Schubert, but with different flags. To any $v \in \mathbb{N}^n$, let us associate the two following flags of alphabets. Let w be the sequence $w_i := \max(j : j \geq i, v_j \geq v_i$. Then v^x is the decreasing reordering of w. Let now u be the element of $\mathbb{N}^n$ obtained by decreasingly reordering v according to the rule $[\dots i, j \dots] \to [\dots j{+}1, i \dots]$ whenever $i < j$. Then v^y is set to be the increasing reordering of u.

Theorem 4.10.1. *Let $v \in \mathbb{N}^n$ be vexillary, v^x, v^y be the two vectors defined above, $k = \max(v^y)$. Then*

$$Y_v = S_{v\uparrow}(\mathbf{x}_{v^x_1} - \mathbf{y}_{v^y_1}, \ldots, \mathbf{x}_{v^x_n} - \mathbf{y}_{v^y_n}), \tag{4.10.1}$$

$$G_v = S_{k^n}(\mathbf{x}_{v^x_1} - \mathbf{y}_{v^y_1}, \ldots, \mathbf{x}_{v^x_n} - \mathbf{y}_{v^y_n})\,(x_1 \cdots x_n)^{-k}, \tag{4.10.2}$$

$$K_v = S_{v\uparrow}(\mathbf{x}_{v^x_1}, \ldots, \mathbf{x}_{v^x_n}). \tag{4.10.3}$$

In particular, when v is vexillary, then $K_v = Y_v(\mathbf{x}, \mathbf{0})$.

For example, for $v = [3,5,4,0,2]$, one has $w = [3,2,3,5,5]$, which reorders into $v^x = [5,5,3,3,2]$. On the other hand, the chain $v = [3,5,4,0,2] \to [6,3,4,0,2] \to [6,5,3,0,2] \to [6,5,3,3,0]$ gives the second flag $v^y = [0,3,3,5,6]$. Hence, one has

$$\begin{aligned} Y_{35402} &= S_{02345}(\mathbf{x}_5{-}\mathbf{y}_0, \mathbf{x}_5{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_5, \mathbf{x}_2{-}\mathbf{y}_6) \\ G_{35402} &= S_{66666}(\mathbf{x}_5{-}\mathbf{y}_0, \mathbf{x}_5{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_3, \mathbf{x}_3{-}\mathbf{y}_5, \mathbf{x}_2{-}\mathbf{y}_6)(x_1 \ldots x_5)^{-6} \\ K_{35402} &= S_{02345}(\mathbf{x}_5, \mathbf{x}_5, \mathbf{x}_3, \mathbf{x}_3, \mathbf{x}_2). \end{aligned}$$

Property (2.6.5) allows to write from (4.10.2) a determinantal formula for $\widehat{G}_v$ polynomials such that $v♣$ be vexillary. This condition is in fact equivalent to requiring that v be vexillary, since if a permutation σ avoids the pattern 2143, then $\omega\sigma\omega$ also avoids this pattern, and conversely.

4.11 Grothendieck and Yang-Baxter

One can degenerate Yang-Baxter bases of Hecke algebras into bases of the 0-Hecke algebra, i.e. the algebra generated by $\widehat{\pi}_1, \widehat{\pi}_2, \ldots$. But as in the case of divided differences, instead of taking products of factors of the type $\widehat{\pi}_i + 1/c$, let us take factors $1 + c\widehat{\pi}_i$. Accordingly, given a spectral vector $[y_1, \ldots, y_n]$, one defines recursively a Yang-Baxter basis $\mho^{\widehat{\pi}}_\sigma$, starting from 1 for the identity permutation, by

$$\mho^{\widehat{\pi}}_{\sigma s_i} = \mho^{\widehat{\pi}}_\sigma \left(1 + \left(1 - \frac{y_{\sigma_i}}{y_{\sigma_{i+1}}}\right) \widehat{\pi}_i\right) \qquad \text{for } \sigma_i < \sigma_{i+1} \,. \tag{4.11.1}$$

For example,

$$\begin{aligned}\mho^{\widehat{\pi}}_{321} &= \left(1 + \left(1 - \frac{y_1}{y_2}\right)\widehat{\pi}_1\right)\left(1 + \left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_2\right)\left(1 + \left(1 - \frac{y_2}{y_3}\right)\widehat{\pi}_1\right) \\ &= 1 + \left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_1 + \left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_2 + \left(1 - \frac{y_1}{y_2}\right)\left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_1\widehat{\pi}_2 \\ &\quad + \left(1 - \frac{y_2}{y_3}\right)\left(1 - \frac{y_1}{y_3}\right)\widehat{\pi}_2\widehat{\pi}_1 + \left(1 - \frac{y_1}{y_2}\right)\left(1 - \frac{y_1}{y_3}\right)\left(1 - \frac{y_2}{y_3}\right)\widehat{\pi}_1\widehat{\pi}_2\widehat{\pi}_1 \,.\end{aligned}$$

s As in the case of divided differences, the Yang-Baxter coefficients are specialisations of known polynomials. The proof of the next properties is similar to the proof of Theorem 3.5.1, and we can avoid repeating it.

Theorem 4.11.1. *The matrix of change of basis between $\{\mho^{\widehat{\pi}}_\sigma\}$ and $\{\widehat{\pi}_\sigma\}$, and its inverse, have entries which are specializations of Grothendieck polynomials :*

$$\mho^{\widehat{\pi}}_\sigma = \sum_{\nu \le \sigma} \widehat{\pi}_\nu \, G_{(\nu)}(\mathbf{y}^\sigma, \mathbf{y}) \,, \tag{4.11.2}$$

$$\widehat{\pi}_\nu \prod_{i<j}\left(1 - \frac{y_i}{y_j}\right) = \sum_{\sigma \le \nu} (-1)^{\ell(\sigma)-\ell(\nu)} \mho^{\widehat{\pi}}_\sigma \, G_{(\nu^{-1}\omega)}(\mathbf{y}^\omega, \mathbf{y}^\sigma) \,. \tag{4.11.3}$$

For example, for $\nu = [2,3,1]$, one has $\nu^{-1}\omega = [2,1,3]$, and the coefficients of the expansion of $\widehat{\pi}_{231}$ are specialisations of the polynomial $G_{(213} = 1 - y_1 x_1^{-1}$. One has

$$\begin{aligned}\widehat{\pi}_{231} \prod_{i<j\le 3}(1 - y_i y_j^{-1}) &= \mho^{\widehat{\pi}}_{123} G_{(213}(\mathbf{y}^{321}, \mathbf{y}) - \mho^{\widehat{\pi}}_{213} G_{(213}(\mathbf{y}^{321}, \mathbf{y}^{213}) \\ &\quad - \mho^{\widehat{\pi}}_{132} G_{(213}(\mathbf{y}^{321}, \mathbf{y}^{132}) + \mho^{\widehat{\pi}}_{231} G_{(213}(\mathbf{y}^{321}, \mathbf{y}^{231}) \\ &= \left(1 - \frac{y_1}{y_3}\right)\mho^{\widehat{\pi}}_{123} - \left(1 - \frac{y_2}{y_3}\right)\mho^{\widehat{\pi}}_{213} - \left(1 - \frac{y_1}{y_3}\right)\mho^{\widehat{\pi}}_{132} + \left(1 - \frac{y_2}{y_3}\right)\mho^{\widehat{\pi}}_{231} \,.\end{aligned}$$

The general properties of Yang-Baxter bases induce properties of specializations of Grothendieck polynomials.

The symmetry (1.8.4) entails

$$\Big(G_{(\nu)}(\mathbf{y}^{\sigma}, \mathbf{y})\Big)^{\clubsuit} = G_{(\omega\nu\omega)}(\mathbf{y}^{\omega\sigma\omega}, \mathbf{y}^{\omega})\,, \tag{4.11.4}$$

using the involution $\clubsuit : y_i \to y_{n+1-i}^{-1}$, $i = 1, \ldots, n$ introduced in (2.6.4).

Each of the equations (1.8.9) and (1.8.10) gives, after some rewriting,

$$\sum_{\nu} (-1)^{\ell(\nu)+\ell(\sigma)} G_{(\nu)}(\mathbf{y}^{\sigma}, \mathbf{y}) G_{(\nu\omega)}(\mathbf{y}^{\zeta}, \mathbf{y}) = \delta_{\sigma,\zeta\omega} \prod_{i<j} \left(1 - \frac{y_i}{y_j}\right)\,, \tag{4.11.5}$$

which is a special instance of formula (2.9.4).

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