Doppelgänger posets and K-theory

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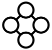
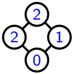
University of Michigan

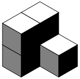
Capsule Research Talk
August 2017

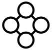
Based on joint work with
Zach Hamaker, Becky Patrias, and Nathan Williams

Plane partitions

- Consider the poset $\mathcal{P} =$ 
- A **plane partition** (of height ℓ) over $\mathcal{P}$ is a weakly order-preserving map $\mathcal{P} \to \{0, 1, \ldots, \ell\}$





- Consider the poset $\mathcal{P} =$ 
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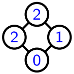
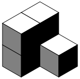
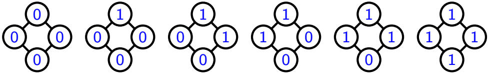




- **Ex:** Plane partitions of height 1 over $\mathcal{P}$:

$\mathrm{PP}^{[1]}(\mathcal{P}) =$ 

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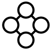
- Consider the poset $\mathcal{P} =$ 
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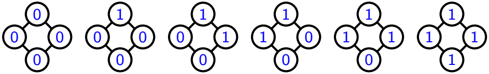
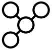
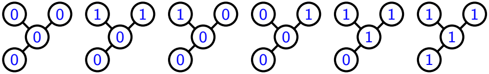
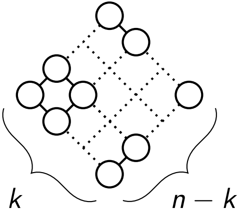
$\mathrm{PP}^{[1]}(\mathcal{P}) =$ 

- **Ex:** Plane partitions of height 1 over $\mathcal{Q} =$ :

$\mathrm{PP}^{[1]}(\mathcal{Q}) =$ 

Doppelgängers

- Let $\Lambda_{\mathrm{Gr}(k,n)} =$ 

(rectangle)

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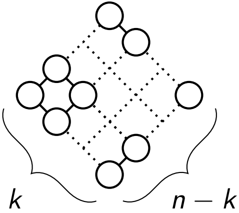

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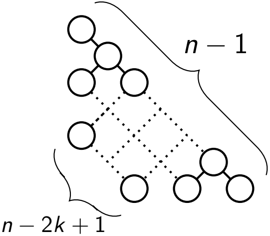

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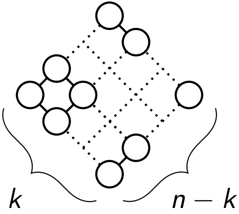

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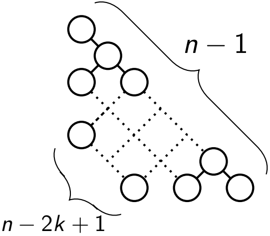

(trapezoid)

Theorem (Proctor, 1983)

For all ℓ, $\mathrm{PP}^{[\ell]}\left(\Lambda_{\mathrm{Gr}(k,n)}\right) \cong \mathrm{PP}^{[\ell]}\left(\Phi^+_{B_{k,n}}\right)$

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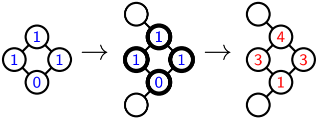
Theorem (HPPW, 2016)

For all ℓ, explicit bijections $\mathrm{PP}^{[\ell]}\left(\Lambda_{\mathrm{Gr}(k,n)}\right) \cong \mathrm{PP}^{[\ell]}\left(\Phi^{+}_{B_{k,n}}\right)$ are given via the combinatorics of K-theoretic Schubert calculus.

Example bijection

Convert to increasing tableau: 

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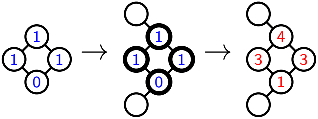
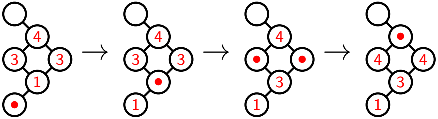
Convert to increasing tableau: 

K**-jeu de taquin:** 

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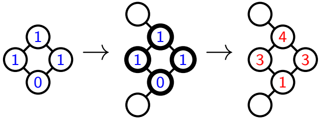

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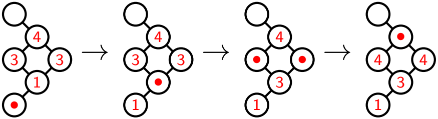
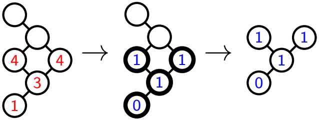
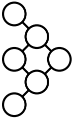
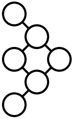
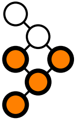


Convert back to PP:



- The ambient poset  is $\Lambda_{\mathrm{OG}(n,2n)}$, which describes the Schubert decomposition of the **orthogonal Grassmannian** $\mathrm{OG}(n, 2n)$ parametrizing isotropic n-planes in $\mathbb{C}^{2n}$.

The secret geometry

- The ambient poset  is $\Lambda_{OG(n,2n)}$, which describes the Schubert decomposition of the **orthogonal Grassmannian** $OG(n, 2n)$ parametrizing isotropic n-planes in $\mathbb{C}^{2n}$.
- The embedded trapezoid  indexes a particular Schubert variety $X_w \hookrightarrow OG(n, 2n)$

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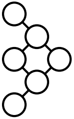
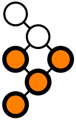
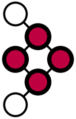
- The ambient poset  is $\Lambda_{OG(n,2n)}$, which describes the Schubert decomposition of the **orthogonal Grassmannian** $OG(n, 2n)$ parametrizing isotropic n-planes in $\mathbb{C}^{2n}$.
- The embedded trapezoid  indexes a particular Schubert variety $X_w \hookrightarrow OG(n, 2n)$
- The embedded rectangle  indexes a particular Richardson variety $X_u^v = X_u \cap X^v \hookrightarrow OG(n, 2n)$

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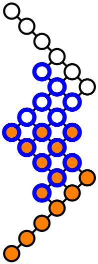
- Generalizes to other spaces...

- Let $\Lambda_{OG(6,12)}$ be the thick blue-circled nodes of 

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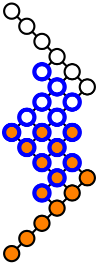
- Let $\Lambda_{OG(6,12)}$ be the thick blue-circled nodes of 
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Corollary (HPPW, 2016)

For all ℓ, explicit bijections $PP^{[\ell]}\left(\Lambda_{OG(6,12)}\right) \cong PP^{[\ell]}\left(\Phi^+_{H_3}\right)$ are given via the combinatorics of K-theoretic Schubert calculus.

- Comes from some geometry on the exceptional Lie group E_7

Thank you!!