Taking the long way home

Orbits of plane partitions

Oliver Pechenik

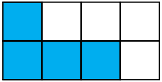
University of Michigan

UM Undergraduate Math Club
October 2017

Mostly based on joint work with
Kevin Dilks and Jessica Striker (NDSU)

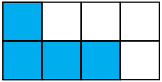
Rowmotion of partitions

- Fix an $a \times b$ rectangle
- Consider ways to stack 1×1 boxes in the lower left corner

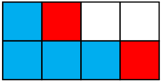
$\lambda =$ 

Rowmotion of partitions

- Fix an $a \times b$ rectangle
- Consider ways to stack 1×1 boxes in the lower left corner

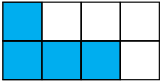
$\lambda =$ 

- Look at all places where you could add a single box

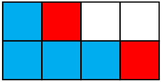


Rowmotion of partitions

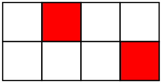
- Fix an $a \times b$ rectangle
- Consider ways to stack 1×1 boxes in the lower left corner

$\lambda =$ 

- Look at all places where you could add a single box

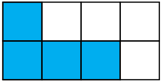


- Remove old boxes

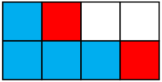


Rowmotion of partitions

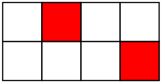
- Fix an $a \times b$ rectangle
- Consider ways to stack 1×1 boxes in the lower left corner

$\lambda =$ 

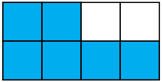
- Look at all places where you could add a single box



- Remove old boxes

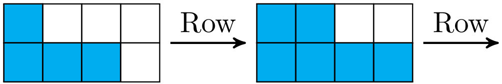


- Add just enough boxes to support the remaining boxes

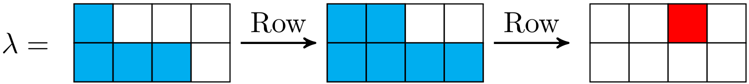
$\mathrm{Row}(\lambda) =$ 

Partition orbit example

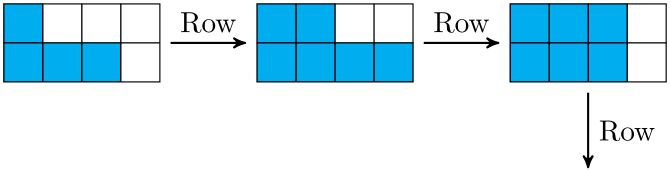
$\lambda =$



Partition orbit example

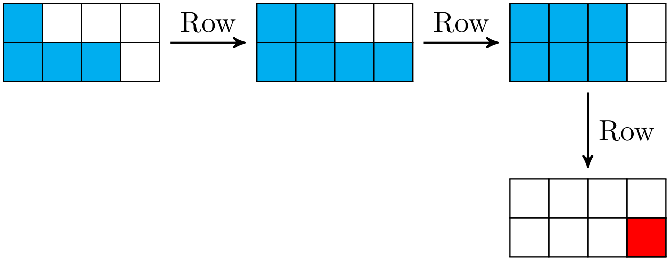


Partition orbit example

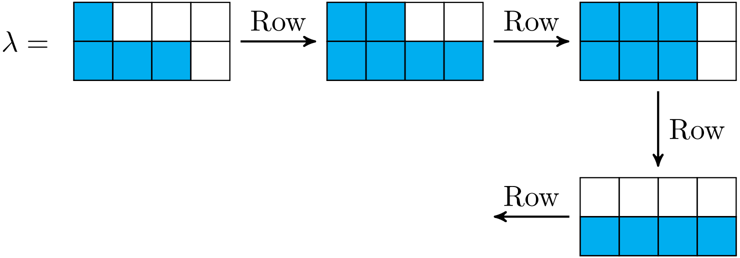
$\lambda =$ 

Partition orbit example

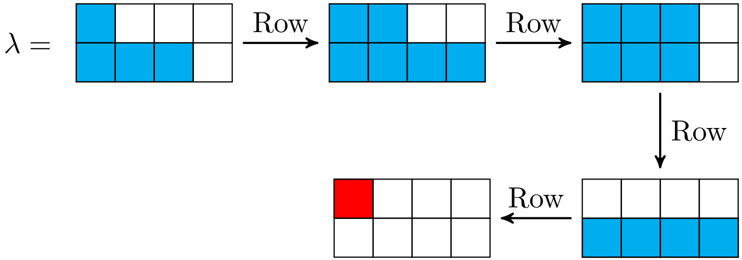
$\lambda =$



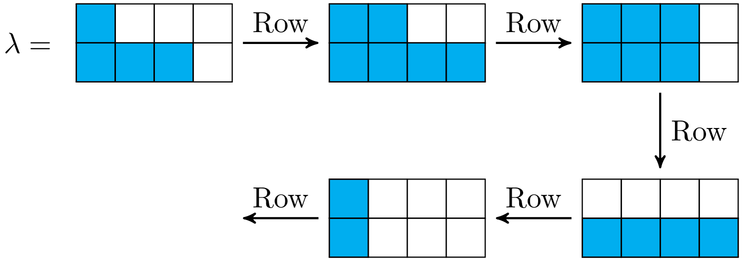
Partition orbit example



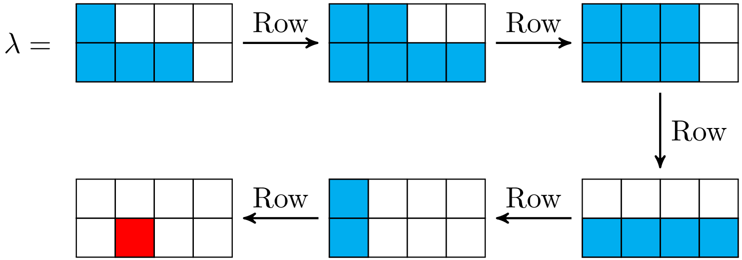
Partition orbit example



Partition orbit example

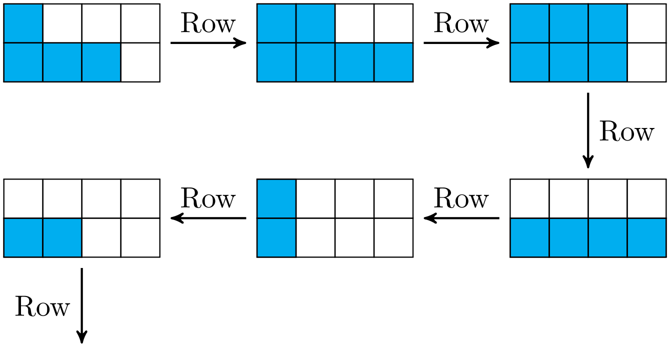


Partition orbit example

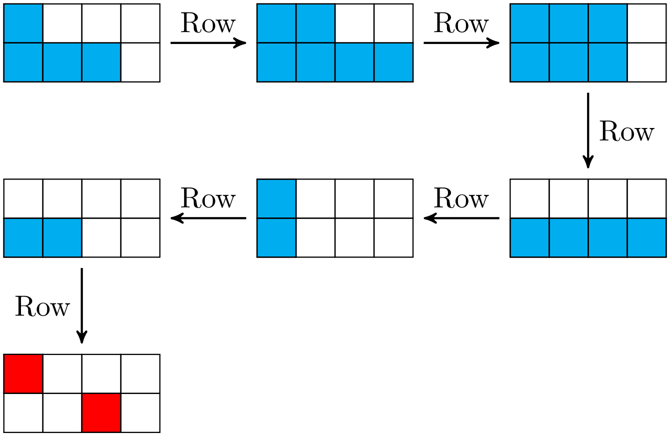


Partition orbit example

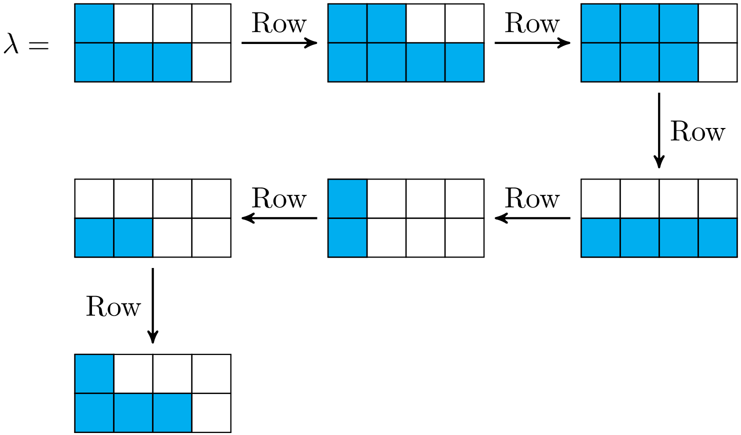
$\lambda =$



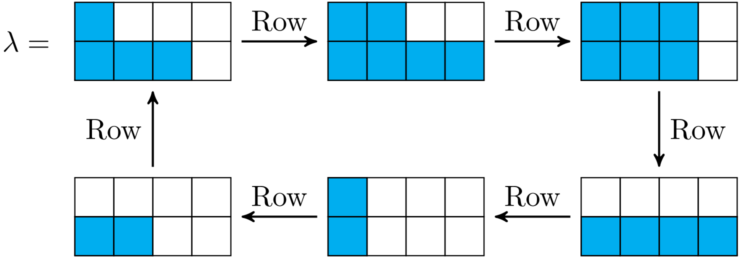
Partition orbit example

$\lambda =$ 

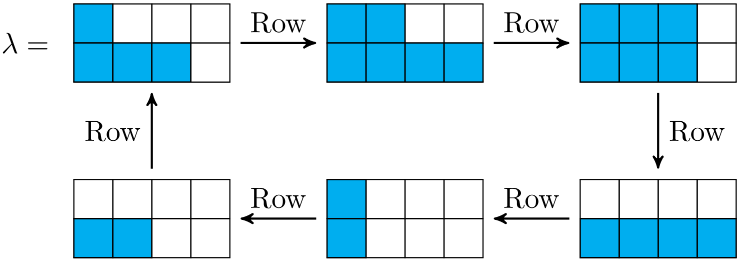
Partition orbit example



Partition orbit example



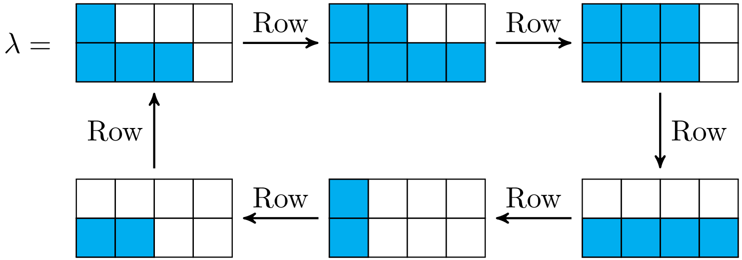
Partition orbit example



Theorem (A. Brouwer–A. Schrijver 1974)

For $\lambda \in J(\mathbf{a} \times \mathbf{b})$, $\mathrm{Row}^{a+b}(\lambda) = \lambda$.

Partition orbit example



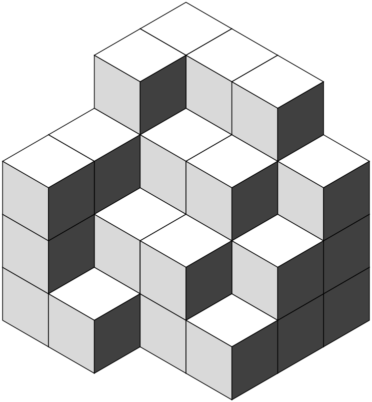
Theorem (A. Brouwer–A. Schrijver 1974)

For $\lambda \in J(\mathbf{a} \times \mathbf{b})$, $\mathrm{Row}^{a+b}(\lambda) = \lambda$.

Theorem (J. Propp–T. Roby 2015)

For each Row*-orbit in* $J(\mathbf{a} \times \mathbf{b})$, *the average number of boxes is* $\frac{ab}{2}$.

P. Cameron–D. Fon-der-Flaass (1995) considered the 3D analogue (*plane partitions* / *stepped surfaces*)



Rowmotion on plane partitions

Theorem (P. Cameron–D. Fon-der-Flaass 1995)

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.

Rowmotion on plane partitions

Theorem (P. Cameron–D. Fon-der-Flaass 1995)

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.

Naive Guess

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$, $\mathrm{Row}^{a+b+c-1}(\lambda) = \lambda$.

Rowmotion on plane partitions

Theorem (P. Cameron–D. Fon-der-Flaass 1995)

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.

Naive Guess

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$, $\mathrm{Row}^{a+b+c-1}(\lambda) = \lambda$.

But for $J(\mathbf{4} \times \mathbf{4} \times \mathbf{4})$, there are orbits of size 33.

Rowmotion on plane partitions

Theorem (P. Cameron–D. Fon-der-Flaass 1995)

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.

Naive Guess

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$, $\mathrm{Row}^{a+b+c-1}(\lambda) = \lambda$.

But for $J(\mathbf{4} \times \mathbf{4} \times \mathbf{4})$, there are orbits of size 33.

Conjecture (P. Cameron–D. Fon-der-Flaass 1995)

If $p = a + b + c - 1$ *is prime, then the length of every* Row*-orbit on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* p.

Theorem (P. Cameron–D. Fon-der-Flaass 1995)

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.

Naive Guess

For $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$, $\mathrm{Row}^{a+b+c-1}(\lambda) = \lambda$.

But for $J(\mathbf{4} \times \mathbf{4} \times \mathbf{4})$, there are orbits of size 33.

Conjecture (P. Cameron–D. Fon-der-Flaass 1995)

If $p = a + b + c - 1$ *is prime, then the length of every* Row*-orbit on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* p.

Theorem (P. Cameron–D. Fon-der-Flaass 1995)

The conjecture holds when $c > ab - a - b + 1$.

Promotion of standard tableaux

$<$

	1	2	5
$\wedge$	3	4	7
	6	8	9

Promotion of standard tableaux

$<$

$\wedge$

1	2	5
3	4	7
6	8	9

Promotion of standard tableaux

	2	5
3	4	7
6	8	9

Promotion of standard tableaux

2		5
3	4	7
6	8	9

Promotion of standard tableaux

2	4	5
3		7
6	8	9

Promotion of standard tableaux

2	4	5
3	7	
6	8	9

Promotion of standard tableaux

2	4	5
3	7	9
6	8	

Promotion of standard tableaux

2	4	5
3	7	9
6	8	10

Promotion of standard tableaux

1	3	4
2	6	8
5	7	9

Promotion of standard tableaux

1	3	4
2	6	8
5	7	9

Theorem (M.-P. Schützenberger, M. Haiman)

For $T \in \mathrm{SYT}(a \times b)$, $\mathrm{Pro}^{ab}(T) = T$.

K-jeu de taquin for increasing tableaux

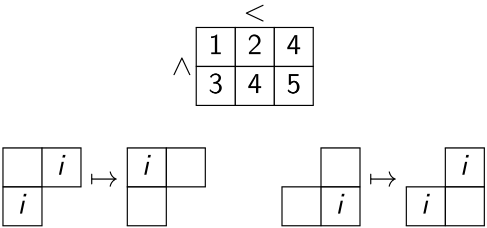
- This sliding algorithm (*jeu de taquin*) computes products in the cohomology rings of Grassmannians $\mathrm{Gr}_k(\mathbb{C}^n)$

K-*jeu de taquin* for increasing tableaux

- This sliding algorithm (*jeu de taquin*) computes products in the cohomology rings of Grassmannians $\mathrm{Gr}_k(\mathbb{C}^n)$
- Natural to extend to the richer K-theory ring of algebraic vector bundles over $\mathrm{Gr}_k(\mathbb{C}^n)$

K-jeu de taquin for increasing tableaux

- This sliding algorithm (*jeu de taquin*) computes products in the cohomology rings of Grassmannians $\mathrm{Gr}_k(\mathbb{C}^n)$
- Natural to extend to the richer K-theory ring of algebraic vector bundles over $\mathrm{Gr}_k(\mathbb{C}^n)$
- H. Thomas–A. Yong (2009) developed a *K-jeu de taquin* for increasing tableaux in this context



Cyclic sieving of increasing tableaux

Theorem (P 2014)

For $T \in \mathrm{Inc}^M(2 \times n)$, $\mathrm{Pro}^M(T) = T$.

- The q-integer $[i]_q := 1 + q + q^2 + \cdots + q^{i-1}$.
- The q-factorial $[i]_q! := [i]_q[i-1]_q \ldots [1]_q$.
- The q-binomial coefficient $\begin{bmatrix} k \\ n \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$.

Theorem (P 2014)

Let ζ be a primitive $(2n-k)^{th}$ root of 1 and $f(q) = \frac{\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-k \\ n-k-1 \end{bmatrix}_q}{[n-k]_q}$.
Then for all m, $f(\zeta^m)$ is a nonnegative integer.
Moreover, $f(\zeta^m)$ counts $T \in \mathrm{Inc}^{2n-k}(2 \times n)$ fixed by Pro^m.

Theorem (M.-P. Schützenberger, M. Haiman)

For $T \in \mathrm{SYT}(m \times n)$, $\mathrm{Pro}^{mn}(T) = T$.

Theorem (P 2014)

For $T \in \mathrm{Inc}^{M}(2 \times n)$, $\mathrm{Pro}^{M}(T) = T$.

K-promotion of increasing tableaux

Theorem (M.-P. Schützenberger, M. Haiman)

For $T \in \mathrm{SYT}(m \times n)$, $\mathrm{Pro}^{mn}(T) = T$.

Theorem (P 2014)

For $T \in \mathrm{Inc}^M(2 \times n)$, $\mathrm{Pro}^M(T) = T$.

Naive Guess

For $T \in \mathrm{Inc}^M(m \times n)$, $\mathrm{Pro}^M(T) = T$.

K-promotion of increasing tableaux

Theorem (M.-P. Schützenberger, M. Haiman)

For $T \in \mathrm{SYT}(m \times n)$, $\mathrm{Pro}^{mn}(T) = T$.

Theorem (P 2014)

For $T \in \mathrm{Inc}^M(2 \times n)$, $\mathrm{Pro}^M(T) = T$.

Naive Guess

For $T \in \mathrm{Inc}^M(m \times n)$, $\mathrm{Pro}^M(T) = T$.

Example

If $T =$

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

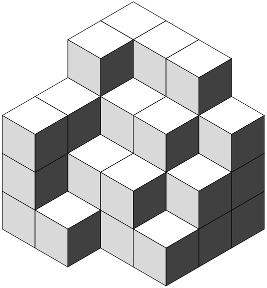
, then $\mathrm{Pro}^{11}(T) =$

1	2	4	7
3	4	6	8
5	6	8	10
7	9	10	11

.

The least k such that $\mathrm{Pro}^k(T) = T$ is $k = 33$.

The magic bijection

$P =$ 

Project to bottom face →

4	4	4	3
4	3	3	2
3	2	2	1
3	1	0	0

Rotate 180°

0	0	1	3
1	2	2	3
2	3	3	4
3	4	4	4

Add 1+rank →

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

$= \Psi_3(P)$

Theorem (K. Dilks–P–J. Striker 2017)

The following diagram commutes:

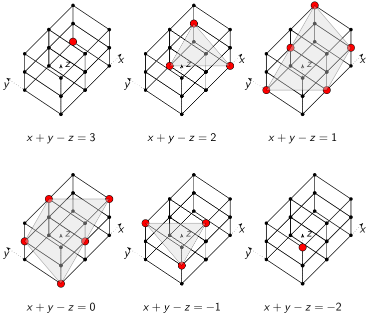
$$\begin{array}{ccc} J(\mathbf{a}\times\mathbf{b}\times\mathbf{c}) & \xrightarrow{\Psi_3} & \mathrm{Inc}^{a+b+c-1}(a\times b) \\ \downarrow \mathrm{Pro}_{\mathrm{id},(1,1,-1)} & & \downarrow \mathrm{Pro} \\ J(\mathbf{a}\times\mathbf{b}\times\mathbf{c}) & \xrightarrow{\Psi_3} & \mathrm{Inc}^{a+b+c-1}(a\times b) \end{array}$$

That is, Ψ_3 *takes* $\mathrm{Pro}_{\mathrm{id},(1,1,-1)}$ *to* Pro.

Affine hyperplane toggles

Theorem (K. Dilks–P–J. Striker 2017)

For every $v \in \{\pm 1\}^n$, $\mathrm{Pro}_{\mathrm{id},v}$ *has the same orbit structure.*



Corollary (K. Dilks–P–J. Striker 2017)

$J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *under* $\mathrm{Row} = \mathrm{Pro}_{\mathrm{id},(1,1,1)}$ *is in equivariant bijection with* $\mathrm{Inc}^{a+b+c-1}(a \times b)$ *under* Pro.

Some corollaries

Corollary (K. Dilks–P–J. Striker 2017)

The following are equivalent:

- *(BS '74) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$, $\mathrm{Row}^{a+b}(\lambda) = \lambda$.
- *(Easy) For* $T \in \mathrm{Inc}^q(1 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n}(m \times n)$, $\mathsf{Pro}^{m+n}(T) = T$.

Some corollaries

Corollary (K. Dilks–P–J. Striker 2017)

The following are equivalent:

- *(BS '74) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$, $\mathrm{Row}^{a+b}(\lambda) = \lambda$.
- *(Easy) For* $T \in \mathrm{Inc}^q(1 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n}(m \times n)$, $\mathsf{Pro}^{m+n}(T) = T$.

As are:

- *(CFdF '95) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.
- *(P '14) For* $T \in \mathrm{Inc}^q(2 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n+1}(m \times n)$, $\mathsf{Pro}^{m+n+1}(T) = T$.

Some corollaries

Corollary (K. Dilks–P–J. Striker 2017)

The following are equivalent:

- *(BS '74) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$, $\mathrm{Row}^{a+b}(\lambda) = \lambda$.
- *(Easy) For* $T \in \mathrm{Inc}^q(1 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n}(m \times n)$, $\mathsf{Pro}^{m+n}(T) = T$.

As are:

- *(CFdF '95) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.
- *(P '14) For* $T \in \mathrm{Inc}^q(2 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n+1}(m \times n)$, $\mathsf{Pro}^{m+n+1}(T) = T$.

Corollary (P. Cameron–D. Fon-der-Flaass 1995)

If $p = a + b + c - 1$ *is prime and* $c > ab - a - b + 1$*, then the length of every* Row*-orbit on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* p.

Some corollaries

Corollary (K. Dilks–P–J. Striker 2017)

The following are equivalent:

- *(BS '74) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$, $\mathrm{Row}^{a+b}(\lambda) = \lambda$.
- *(Easy) For* $T \in \mathrm{Inc}^q(1 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n}(m \times n)$, $\mathsf{Pro}^{m+n}(T) = T$.

As are:

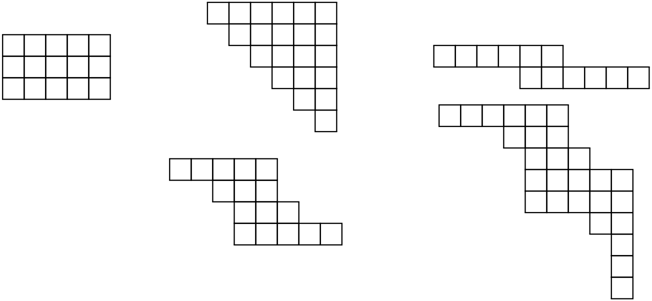
- *(CFdF '95) For* $\lambda \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $\mathrm{Row}^{a+b+1}(\lambda) = \lambda$.
- *(P '14) For* $T \in \mathrm{Inc}^q(2 \times n)$, $\mathsf{Pro}^q(T) = T$.
- *(DPS '17) For* $T \in \mathrm{Inc}^{m+n+1}(m \times n)$, $\mathsf{Pro}^{m+n+1}(T) = T$.

Corollary (K. Dilks–P–J. Striker 2017)

If $p = a + b + c - 1$ *is prime and* $c > \frac{2}{3}ab - a - b + \frac{4}{3}$, *then the length of every* Row*-orbit on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ *is a multiple of* p.

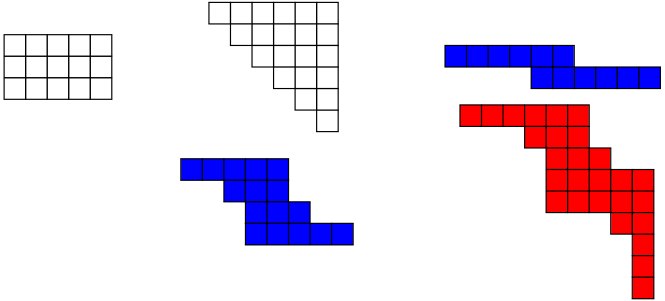
Other floor plans

- One can also consider plane partitions over bases other than rectangles.
- Especially interesting are the **minuscule** cases:



Other floor plans

- One can also consider plane partitions over bases other than rectangles.
- Especially interesting are the **minuscule** cases:



- In forthcoming work with Holly Mandel (UC Berkeley), we analyze three of these cases, proving/disproving conjectures of Rush and Shi (2013).

Thanks!

Thank you!!