Rhombic tilings and Bott-Samelson varieties

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Based on joint work with
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Alexander Yong (UIUC)
arXiv:1605.05613

The flag variety

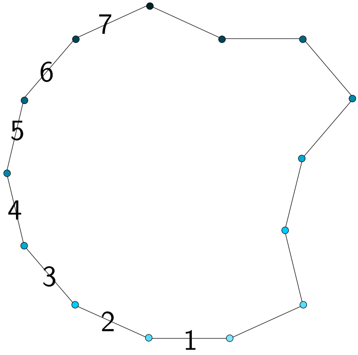
- Let $X = \mathrm{Flags}(\mathbb{C}^n)$ be the variety of complete flags
$$\mathbb{C}^0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n.$$
- The group $\mathrm{GL_n}(\mathbb{C})$ acts on the variety X by change of basis, as does its subgroup B of invertible upper triangular matrices and its maximal torus T of invertible diagonal matrices.
- The T-fixed points are in bijection with permutations w in the symmetric group $\mathfrak{S}_n$: they are the flags $F_\bullet^{(w)}$ defined by $F_k^{(w)} = \langle \vec{e}_{w(1)}, \vec{e}_{w(2)}, \ldots, \vec{e}_{w(k)} \rangle$ where $\vec{e}_i$ is the i-th standard basis vector.
- The **Schubert variety** X_w is the B-orbit closure of $F_\bullet^{(w)}$.

Singularities and resolutions

- Schubert varieties are generally singular.
- H.C. Hansen (1973) and M. Demazure (1974) introduced **Bott-Samelson varieties** $BS^{(i_1,i_2,\ldots,i_{\ell(w)})}$, which are resolutions of singularities for X_w, one for each reduced word $s_{i_1}s_{i_2}\cdots s_{i_{\ell(w)}}$ of w.
- We show how Bott-Samelsons are encoded by rhombic tilings.

The Elnitsky polygon

Given a permutation $w \in S_n$, the **Elnitsky** $2n$**-gon** $\mathrm{E}(w)$ has sides labeled $1, 2, \ldots, n, w(n), w(n-1), \ldots, w(1)$, in which the first n labels form half of a regular $2n$-gon, and sides with the same label are parallel.



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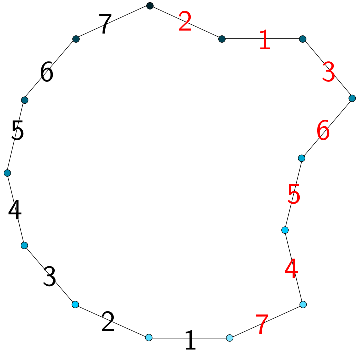

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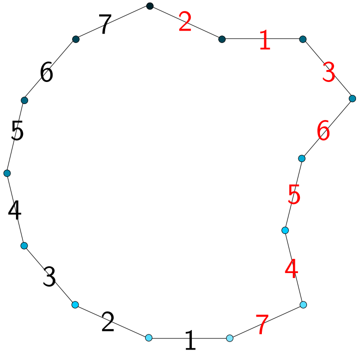
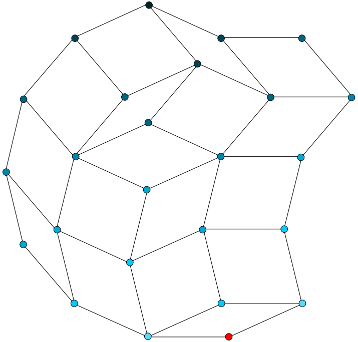


$w = 7456312$

Elnitsky tilings

- Let $\mathcal{T}(w)$ be the set of **rhombic tilings** of $E(w)$ in which the rhombi have sides of length one and edges parallel to edges of $E(w)$.



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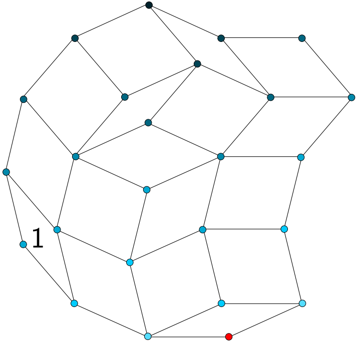

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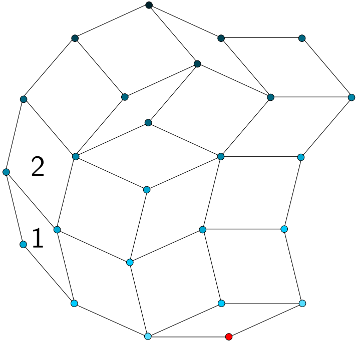

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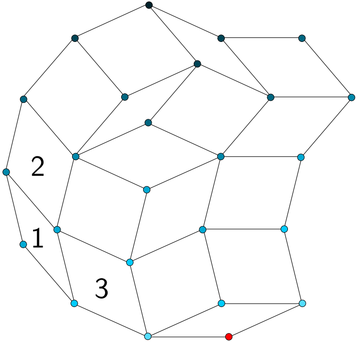

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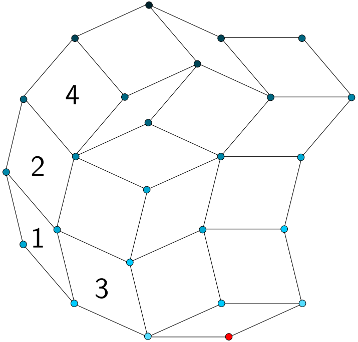

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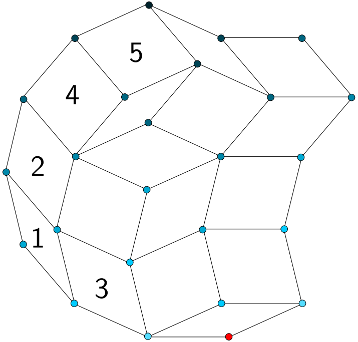

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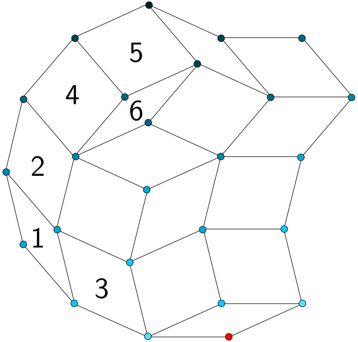

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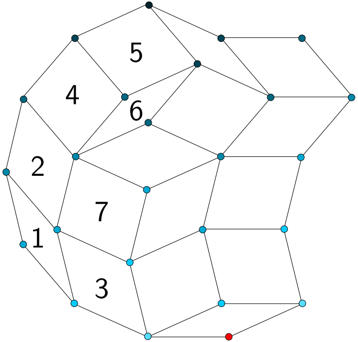

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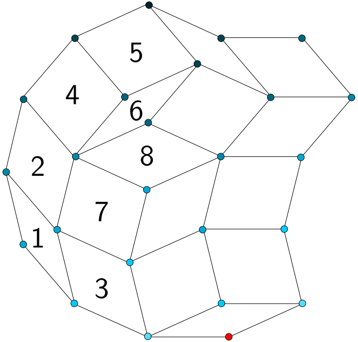

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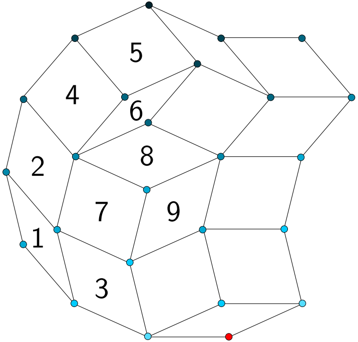

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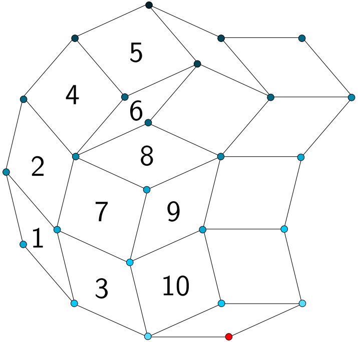

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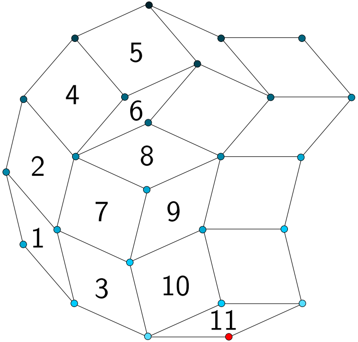

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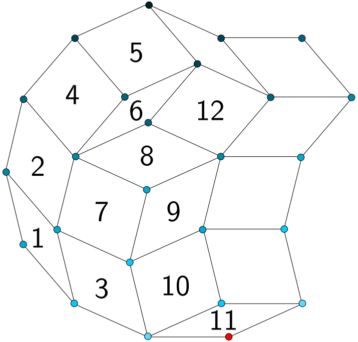

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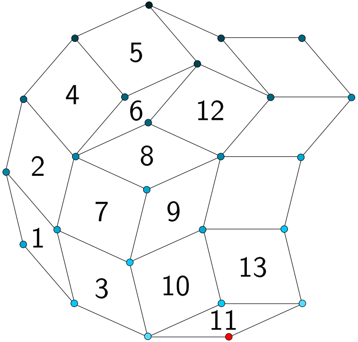

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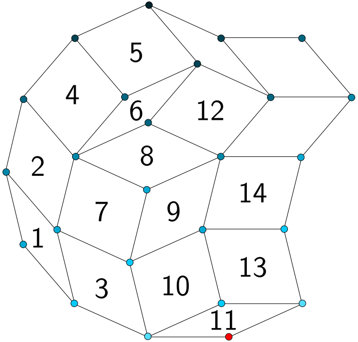

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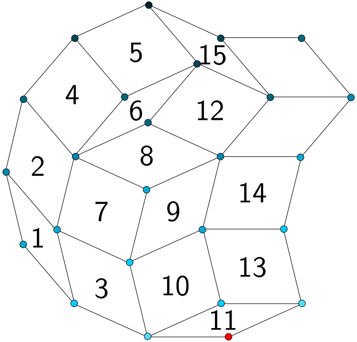

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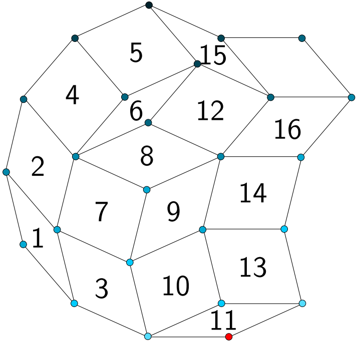

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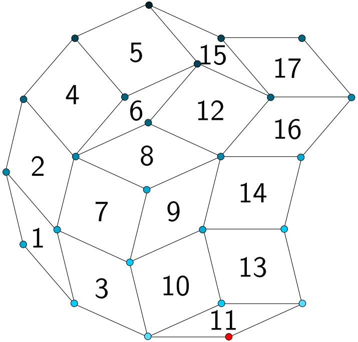

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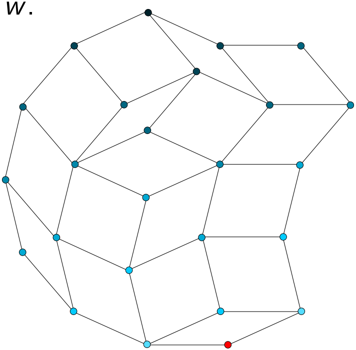
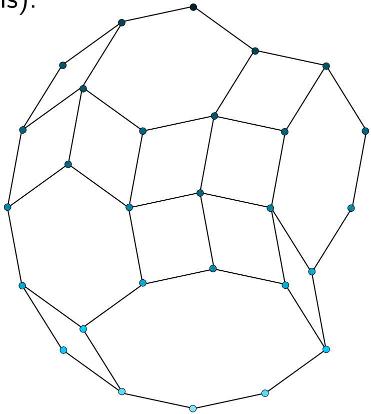
Reading a Bott-Samelson

Key fact: Bott-Samelsons only depend on a commutation class of reduced words for w.



- Attach a vector space V_x to each vertex x with dimension = distance from x to $\bullet$. (Standard flag along left border.)
- For each edge $x - y$, impose the relation $V_x \subset V_y$.
- The space of all such assignments is a smooth subvariety of $\prod_{x \in \texttt{Vert(T)}} \mathrm{Gr}_{\dim V_x}(\mathbb{C}^n)$ and is the Bott-Samelson for that commutation class.

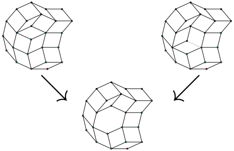
Can consider more general tilings by zonotopes (centrally symmetric polygons):



Obtain other resolutions of singularities, **generalized Bott-Samelson varieties**

Posets of resolutions and tilings

We have a **well-understood** poset of zonotopal tilings by refinement:



This corresponds to a poset of resolutions of singularities:

$$\begin{array}{ccc} \mathcal{Z}_{T_1} & & \mathcal{Z}_{T_2} \\ & \searrow \quad \swarrow & \\ & \mathcal{Z}_Z & \end{array}$$

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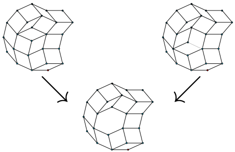

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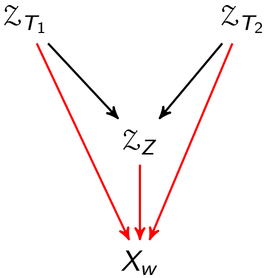

Thanks!

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