PROBABILITY

STAT 220/230 NOTES (2011-12 Edition)

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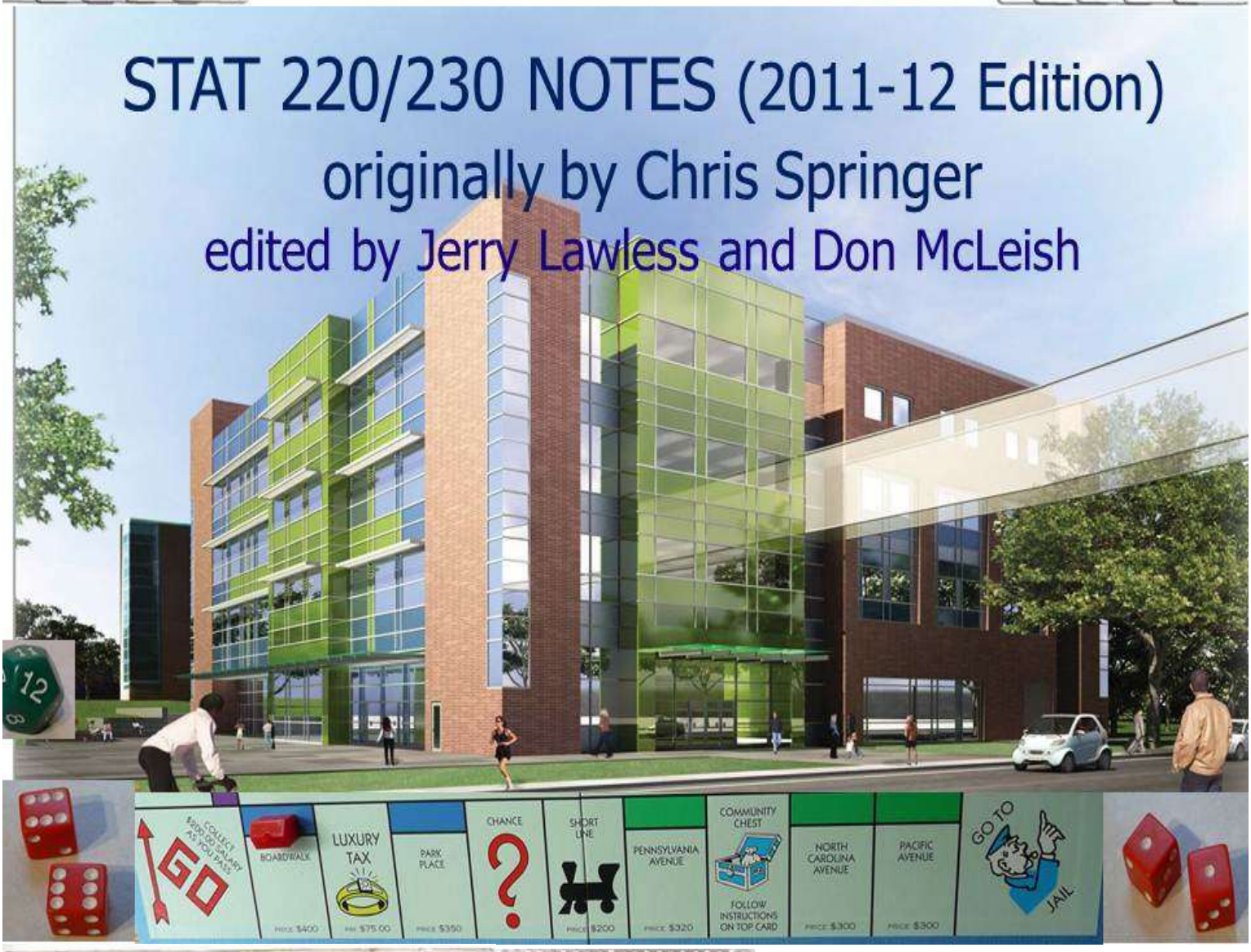

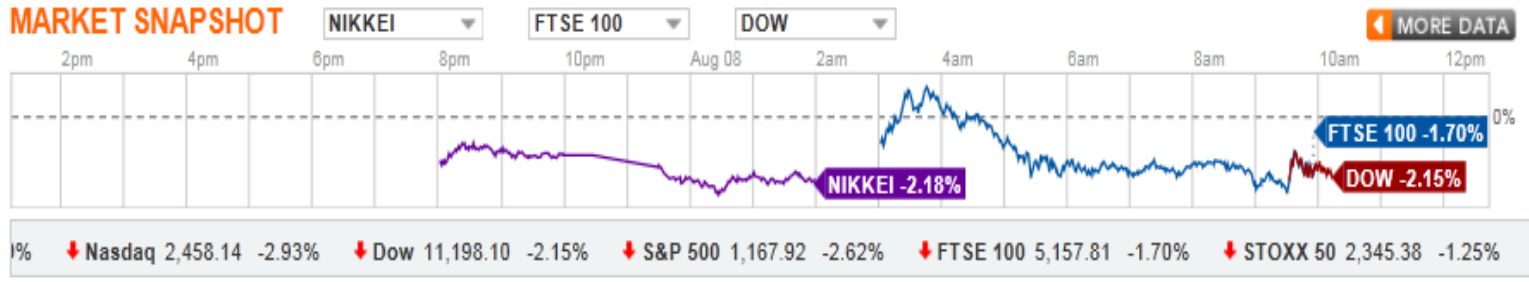

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[1] ◊ =optional for Stat 220

★ =optional for Stat 230

1. Introduction to Probability

1.1 Definitions of Probability

You are the product of a random universe. From the Big Bang to your own conception and birth, random events have determined who we are as a species, who you are as a person, and much of your experience to date. Ironic therefore that we are not well-tuned to understanding the randomness around us, perhaps because millions of years of evolution have cultivated our ability to see regularity, certainty and deterministic cause-and-effect in the events and environment about us. We are good at finding patterns in numbers and symbols, or relating the eating of certain plants with illness and others with a healthy meal. In many areas, such as mathematics or logic, we assume we know the results of certain processes with certainty (e.g., 2+3=5), though even these are often subject to assumed axioms. Most of the real world, however, from the biological sciences to quantum physics[2], involves variability and uncertainty. For example, it is uncertain whether it will rain tomorrow; the price of a given stock a week from today is uncertain; the number of claims that a car insurance policy holder will make over a one-year period is uncertain. Uncertainty or "randomness" (i.e. variability of results) is usually due to some mixture of at least two factors including: (1) *variability in populations* consisting of animate or inanimate objects (e.g., people vary in size, weight, blood type etc.), and (2) *variability in processes* or phenomena (e.g., the random selection of 6 numbers from 49 in a lottery draw can lead to a very large number of different outcomes). Which of these would you use to describe the fluctuations in stock prices or currency exchange rates?

Variability and uncertainty in a system make it more difficult to plan or to make decisions without suitable tools. We cannot eliminate uncertainty but it is usually possible to describe, quantify and deal with variability and uncertainty using the theory of probability. This course develops both the mathematical theory and some of the applications of probability. The applications of this methodology are far-reaching, from finance to the life-sciences, from the analysis of computer algorithms to simulation of queues and networks or the spread of epidemics. Of course we do not have the time in this course

[2] "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" Albert Einstein, 1921.

to develop these applications in detail, but some of the end-of-chapter problems will give a hint of the extraordinary range of application of the mathematical theory of probability and statistics.

It seems logical to begin by defining probability. People have attempted to do this by giving definitions that reflect the uncertainty whether some specified outcome or “event" will occur in a given setting. The setting is often termed an “experiment" or “process" for the sake of discussion. We often consider simple “toy" examples: it is uncertain whether the number 2 will turn up when a 6-sided die is rolled. It is similarly uncertain whether the Canadian dollar will be higher tomorrow, relative to the U.S. dollar, than it is today. So one step in defining probability requires envisioning a random experiment with a number of possible outcomes. We refer to the set of all possible distinct outcomes to a random experiment as the **sample space** (usually denoted by S). Groups or sets of outcomes of possible interest, subsets of the sample space, we will call events. Then we might define probability in three different ways:

1. The **classical** definition: The probability of some event is

$$\frac{\text{number of ways the event can occur}}{\text{number of outcomes in S}},$$

   provided all points in the sample space S are equally likely. For example, when a die is rolled the probability of getting a 2 is $\frac{1}{6}$ because one of the six faces is a 2.

2. The **relative frequency** definition: The probability of an event is the (limiting) proportion (or fraction) of times the event occurs in a very long series of repetitions of an experiment or process. For example, this definition could be used to argue that the probability of getting a 2 from a rolled die is $\frac{1}{6}$.

3. The **subjective probability** definition: The probability of an event is a measure of how sure the person making the statement is that the event will happen. For example, after considering all available data, a weather forecaster might say that the probability of rain today is 30% or 0.3.

Unfortunately, all three of these definitions have serious limitations.

Classical Definition: What does “equally likely” mean? This appears to use the concept of probability while trying to define it! We could remove the phrase “provided all outcomes are equally likely”, but then the definition would clearly be unusable in many settings where the outcomes in S did not tend to occur equally often.

Relative Frequency Definition: Since we can never repeat an experiment or process indefinitely, we can never know the probability of any event from the relative frequency definition. In many cases we

can’t even obtain a long series of repetitions due to time, cost, or other limitations. For example, the probability of rain today can’t really be obtained by the relative frequency definition since today can’t be repeated again under identical conditions. Intuitively, however, if a probability is correct, we expect it to be close to relative frequency, when the experiment is repeated many times.

Subjective Probability: This definition gives no rational basis for people to agree on a right answer, and thus would disqualify probability as an objective science. Are everyone’s opinions equally valid or should we only consult "experts". There is some controversy about when, if ever, to use subjective probability except for personal decision-making but it does play a part in a branch of Statistics that is often called "Bayesian Statistics". This will not be discussed in Stat 230, but it is a common and useful method for updating subjective probabilities with objective experimental results.

The difficulties in producing a satisfactory definition can be overcome by treating probability as a mathematical system defined by a set of axioms. We do not worry about the numerical values of probabilities until we consider a specific application. This is consistent with the way that other branches of mathematics are defined and then used in specific applications (e.g., the way calculus and real-valued functions are used to model and describe the physics of gravity and motion).

The mathematical approach that we will develop and use in the remaining chapters is based on the following description of a **probability model**:

- a sample space of all possible outcomes of a random experiment is defined
- a set of events, subsets of the sample space to which we can assign probabilities, is defined
- a mechanism for assigning probabilities (numbers between 0 and 1) to events is specified.

Of course in a given run of the random experiment, a particular event may or may not occur.

In order to understand the material in these notes, you may need to review your understanding of basic counting arguments, elementary set theory as well as some of the important series that you have encountered in Calculus that provide a basis for some of the distributions discussed in these notes. In the next chapter, we begin a more mathematical description of probability theory.

1.2 Problems on Chapter 1

1.1 Try to think of examples of probabilities you have encountered which might have been obtained by each of the three “definitions".

1.2 Which definitions do you think could be used for obtaining the following probabilities?

(a) You have a claim on your car insurance in the next year.

(b) There is a meltdown at a nuclear power plant during the next 5 years.

(c) A person’s birthday is in April.

1.3 Give examples of how probability applies to each of the following areas.

(a) Lottery draws

(b) Auditing of expense items in a financial statement

(c) Disease transmission (e.g. measles, tuberculosis, STD’s)

(d) Public opinion polls

1.4 Which of the following can be accurately described by a "deterministic" model, i.e. a model which does not require any concept of probability?

(a) The position of a small particle in space

(b) The velocity of an object dropped from the leaning tower of Pisa

(c) The value of a stock which you purchased for $20 one month ago

(d) The purchasing power of $20 CAN according to the Consumer Price Index in one month.

2. Mathematical Probability Models

2.3 Sample Spaces and Probability

Consider some phenomenon or process which is repeatable, at least in theory, and suppose that certain events or outcomes $A_1, A_2, A_3, \ldots$ are defined. We will often term the phenomenon or process an **"experiment"** and refer to a single repetition of the experiment as a **"trial"**. The probability of an event A, denoted $P(A)$, is a number between 0 and 1. For probability to be a useful mathematical concept, it should possess some other properties. For example, if our "experiment" consists of tossing a coin with two sides, Head and Tail, then we might wish to consider the two events A_1 = "Head turns up" and A_2 = "Tail turns up". It does not make much sense to allow $P(A_1) = 0.6$ and $P(A_2) = 0.6$, so that $P(A_1) + P(A_2) > 1$. (Why is this so? Is there a fundamental reason or have we simply adopted 1 as a convenient scale?) To avoid this sort of thing we begin with the following definition.

Definition 1 *A **sample space** S is a set of distinct outcomes for an experiment or process, with the property that in a single trial, one and only one of these outcomes occurs.*

The outcomes that make up the sample space may sometimes be called "sample points" or just "points" on occasion. A sample space is defined as part of the probability model in a given setting but it is not necessarily uniquely defined, as the following example shows.

Example: Roll a 6-sided die, and define the events

$$a_i = \text{top face is } i, \text{ for } i = 1, 2, 3, 4, 5, 6.$$

Then we could take the sample space as $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. (Note we use the curly brackets "{...}" to indicate the elements of a set). Instead of using this definition of the sample space we could instead define events

E is the event that an even number turns up

O is the event that an odd number turns up

and take $S = \{E, O\}$. Both sample spaces satisfy the definition. Which one we use would depends on what we wanted to use the probability model for. If we expect **never** to have to consider events like "

a number less than 3 turns up" then the space $S = \{E, O\}$ will suffice, but in most cases, if possible, we choose sample points that are the smallest possible or "indivisible". Thus the first sample space is likely preferred in this example.

Sample spaces may be either **discrete** or **non-discrete**; S is discrete if it consists of a finite or countably infinite set of simple events. Recall that a countably infinite sequence is one that can be put in one-one correspondence with the positive integers, so for example $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...\}$ is countably infinite as is the set of all rational numbers. The two sample spaces in the preceding example are discrete. A sample space $S = \{1, 2, 3, \dots\}$ consisting of all the positive integers is discrete, but a sample space $S = \{x : x > 0\}$ consisting of all positive real numbers is not. For the next few chapters we consider only discrete sample spaces. For discrete sample spaces it is much easier to specify the class of events to which we may wish to assign probabilities; we will allow all possible subsets of the sample space. For example if $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is the sample space then $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{a_6\}$ and S itself are all examples of events.

Definition 2 *An event in a discrete sample space is a subset $A \subset S$. If the event is indivisible so it contains only one point, e.g. $A_1 = \{a_1\}$ we call it a **simple event**. An event A made up of two or more simple events such as $A = \{a_1, a_2\}$ is called a **compound event.***

Our notation will often not distinguish between the point a_i and the simple event $A_i = \{a_i\}$ which has this point as its only element, although they differ as mathematical objects. When we mean the probability of the event $A_1 = \{a_1\}$, we should write $P(A_1)$ or $P(\{a_1\})$ but the latter is often shortened to $P(a_i)$. In the case of a discrete sample space it is easy to specify probabilities of events since they are determined by the probabilities of simple events.

Definition 3 *Let $S = \{a_1, a_2, a_3, \dots\}$ be a discrete sample space. Then **probabilities** $P(a_i)$ are numbers attached to the a_i's $(i = 1, 2, 3, \dots)$ such that the following two conditions hold:*

(1) $0 \leq P(a_i) \leq 1$

(2) $\sum_i P(a_i) = 1$

The above function $P(*)$ on S which describes the set of probabilities $\{P(a_i),\, i = 1, 2, \dots\}$ is called a **probability distribution on** S. The condition $\sum_i P(a_i) = 1$ above reflects the idea that when the process or experiment happens, one or other of the simple events $\{a_i\}$ in S must occur (recall that the sample space includes all possible outcomes). The probability of a more general event A (not necessarily a simple event) is then defined as follows:

Definition 4 *The probability $P(A)$ of an event A is the sum of the probabilities for all the simple events that make up A or $P(A) = \sum_{a \in A} P(a)$.*

For example, the probability of the compound event $A = \{a_1, a_2, a_3\}$ is $P(a_1) + P(a_2) + P(a_3)$. Probability theory does not say what numbers to assign to the simple events for a given application, only those properties guaranteeing mathematical consistency. In an actual application of a probability model, we try to specify numerical values of the probabilities that are more or less consistent with the frequencies of events when the experiment is repeated. In other words we try to specify probabilities that are consistent with the real world. There is nothing mathematically wrong with a probability model for a toss of a coin that specifies that the probability of heads is zero, except that it likely won't agree with the frequencies we obtain when the experiment is repeated.

Example: Suppose a 6-sided die is rolled, and let the sample space be $S = \{1, 2, 3, 4, 5, 6\}$, where 1 means the top face is 1, and so on. If the die is an ordinary one, (a *fair* die) we would likely define probabilities as

$$P(i) = 1/6 \text{ for } i = 1, 2, 3, 4, 5, 6, \tag{2.1}$$

because if the die were tossed repeatedly by a fair roller (as in some games or gambling situations) then each number would occur close to $1/6$ of the time. However, if the die were weighted in some way, or if the roller were able to manipulate the die so that 1 is more likely, these numerical values would not be so useful. To have a useful mathematical model, some degree of compromise or approximation is usually required. Is it likely that the die or the roller are perfectly "fair"? Given (2.1), if we wish to consider some compound event, the probability is easily obtained. For example, if A = "even number obtains" then because $A = \{2, 4, 6\}$ we get $P(A) = P(2) + P(4) + P(6) = 1/2$.

We now consider some additional examples, starting with some simple "toy" problems involving cards, coins and dice. Once again, to calculate probability for discrete sample spaces, we usually approach a given problem using three steps:

(1) Specify a sample space S.

(2) Assign numerical probabilities to the simple events in S.

(3) For any compound event A, find $P(A)$ by adding the probabilities of all the simple events that make up A.

Later we will discover that having a detailed specification or list of the elements of the sample space may be difficult. Indeed in many cases the sample space is so large that at best we can describe

it in words. For the present we will solve problems that are stated as "Find the probability that ..." by carrying out step (2) above, assigning probabilities that we expect should reflect the long run relative frequencies of the simple events in repeated trials, and then summing these probabilities to obtain $P(A)$.

Some Examples

When S has only a few points, one of the easiest methods for finding the probability of an event is to list all outcomes. In many problems a sample space S with equally probable simple events can be used, and the first few examples are of this type.

Example: *Draw 1 card from a standard well-shuffled deck (13 cards of each of 4 suits - spades, hearts, diamonds, clubs). Find the probability the card is a club.*
Solution 1: Let S = { spade, heart, diamond, club}. Then S has 4 points, with 1 of them being "club", so P(club)$= \frac{1}{4}$.

Solution 2: Let $S =$ { 2♦,3♦,4♦, $...A$♦, 2♡, $...A$♣}. Then each of the 52 cards in S has probability $\frac{1}{52}$. The event A of interest is

$$A = \{2\clubsuit, 3\clubsuit, ...A\clubsuit\}$$

and this event has 13 simple outcomes in it all with the same probability $\frac{1}{52.}$. Therefore

$$P(A) = \frac{1}{52} + \frac{1}{52} + ...\frac{1}{52} = \frac{13}{52} = \frac{1}{4}.$$

Note 1: A sample space is not necessarily unique, as mentioned earlier. The two solutions illustrate this. Note that in the first solution the event A = "the card is a club" is a simple event because of the way the sample space was defined, but in the second it is a compound event.

Note 2: In solving the problem we have assumed that each simple event in S is equally probable. For example in Solution 1 each simple event has probability $1/4$. This seems to be the only sensible choice of numerical value in this setting, but you will encounter problems later on where it is not obvious whether outcomes all are equiprobable.

The term "odds" is sometimes used in describing probabilities. In this card example the odds in favour of clubs are 1:3; we could also say the odds against clubs are 3:1. In general,

Figure 2.1: 9 tosses of two coins each

Definition 5 *The odds in favour of an event A is the probability the event occurs divided by the probability it does not occur or $\frac{P(A)}{1-P(A)}$. The odds against the event is the reciprocal of this, $\frac{1-P(A)}{P(A)}$.*

If the odds against a given horse winning a race are 20 to 1 (or 20:1), what is the corresponding probability that the horse will win the race? According to the definition above $\frac{1-P(A)}{P(A)} = 20$, which gives $P(A) = \frac{1}{21}$. Note that these odds are derived from bettor's collective opinion and therefore subjective.

Example: *Toss a coin twice. Find the probability of getting one head. (In this course, "one head" is taken to mean **exactly** one head. If we meant "at least one head" we would say so.)*

Solution 1: Let $S = \{HH, HT, TH, TT\}$ and assume the simple events each have probability $\frac{1}{4}$. (Here, the notation HT means head on the 1^{st} toss and tails on the 2^{nd}.) Since one head occurs for simple events HT and TH, the event of interest is $A = \{HT, TH\}$ and we get $P(A) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$.

Solution 2: Let $S =$ { 0 heads, 1 head, 2 heads } and assume the simple events each have probability $\frac{1}{3}$. Then P(1 head) = $\frac{1}{3}$.

Which solution is right? Both are mathematically "correct" in the sense that they are both consequences of probability models. However, we want a solution that reflects the relative frequency of occurrence in repeated trials in the real world, not just one that agrees with some mathematical model. In that respect, the points in solution 2 are **not** equally likely. The event $\{1$ head$\}$ occurs more often than either $\{0$ head$\}$ or $\{2$ heads$\}$ in actual repeated trials. You can experiment to verify this (for example of the nine replications of the experiment in Figure 2.1, 2 heads occurred 2 of the nine times, 1 head occurred 6 of the 9 times. For more certainty you should replicate this experiment many times. You can do this without benefit of coin at http://shazam.econ.ubc.ca/flip/index.html). So we say solution 2 is incorrect for ordinary fair coins because it is based on an incorrect model. If we were determined to use the

sample space in solution 2, we could do it by assigning appropriate probabilities to each of the three simple events but then 0 heads would need to have a probability of $\frac{1}{4}$, 1 head \a probability of $\frac{1}{2}$ and 2 heads $\frac{1}{4}$. We do not usually do this because there seems little point in using a sample space whose points are not equally probable when one with equally probable points is readily available.

Example: *Roll a red die and a green die. Find the probability the total is 5.*
Solution: Let (x, y) represent getting x on the red die and y on the green die.
Then, with these as simple events, the sample space is

$$\begin{array}{lccccc} S = \{ & (1,1) & (1,2) & (1,3) & \cdots & (1,6) \\ & (2,1) & (2,2) & (2,3) & \cdots & (2,6) \\ & (3,1) & (3,2) & (3,3) & \cdots & (3,6) \\ & & -- & -- & -- & \\ & (6,1) & (6,2) & (6,3) & \cdots & (6,6)\} \end{array}$$

Each simple event, for example $\{(1,1)\}$ is assigned probability $\frac{1}{36}$. Then the event of interest is the event that the total is 5, $A = \{(1,4)(2,3)(3,2),(4,1)\}$. Therefore $P(A) = \frac{4}{36}$

Example: *Suppose the 2 dice were now identical red dice or equivalently that the observer is color-blind. Find the probability the total is 5.*

Solution 1: Since we can no longer distinguish between (x, y) and (y, x), the only distinguishable points in S are :

$$\begin{array}{lccccc} S = \{ & (1,1) & (1,2) & (1,3) & \cdots & (1,6) \\ & & (2,2) & (2,3) & \cdots & (2,6) \\ & & & (3,3) & \cdots & (3,6) \\ & & & & .. & .. \\ & & & & & (6,6)\} \end{array}$$

Using this sample space, we get a total of 5 from points $(1,4)$ and $(2,3)$ only. If we assign equal probability $\frac{1}{21}$ to each point (simple event) then we get P(total is 5) = $\frac{2}{21}$.
At this point you should be suspicious since $\frac{2}{21} \neq \frac{4}{36}$. The colour of the dice shouldn't have any effect on what total we get. The universe does not change the frequency of real physical events depending on whether the observer is colour-blind or not, so one answer must be wrong! The problem is that the 21 points in S here are not equally likely. There was nothing theoretically wrong with the probability model except that if this experiment is repeated in the real world, the point (1, 2) occurs about twice as often in the long run as the point (1,1). So the only sensible way to use this sample space consistent with the real world is to assign probability weights $\frac{1}{36}$ to the points of the form (x, x) and $\frac{2}{36}$ to the points (x, y) for $x \neq y$. We can compare these probabilities with experimental evidence. On the website

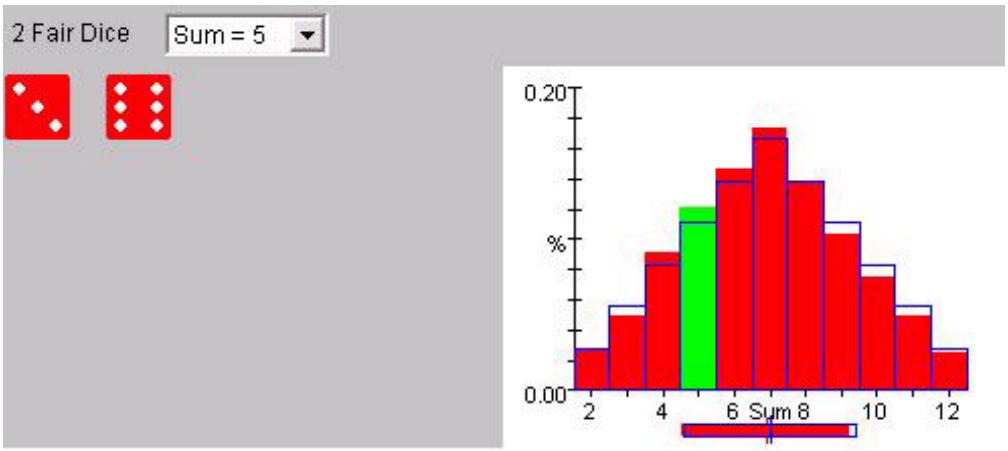


Figure 2.2: Results of 1000 throws of 2 dice

http://www.math.duke.edu/education/postcalc/probability/dice/index.html you may throw virtual dice up to 10,000 times and record the results. For example on 1000 throws of two dice (see Figure 2.2), there were 121 occasions when the sum of the values on the dice was 5, indicating the probability is around 121/1000 or 0.121 This compares with the true probability $4/36 = 0.111$.

Solution 2: For a more straightforward solution to the above problem, pretend the dice can be distinguished even though they can't. (Imagine, for example, that we put tiny mark on one die, or label one of them differently.) We then get the same 36 sample points as in the example with the red die and the green die. The fact that one die has a tiny mark cannot change the probabilities so that

$$P(\text{total is } 5) = \frac{4}{36}$$

The laws determining the probabilities associated with these two dice do not, of course, know whether your eyesight is so keen that you can or cannot distinguish the dice. These probabilities must be the same in either case. In many problems when objects are indistinguishable and we are interested in calculating a probability, you will discover that the calculation is made easier by pretending the objects can be distinguished.

This illustrates a common pitfall. When treating objects in an experiment as distinguishable leads to a different answer from treating them as identical, the points in the sample space for identical objects are usually not "equally likely" in terms of their long run relative frequencies. It is generally safer to pretend objects can be distinguished even when they can't be, in order to get equally likely sample points.

While the method of finding probability by listing all the points in S can be useful, it isn't practical when there are a lot of points to write out (e.g., if 3 dice were tossed there would be 216 points in S). We need to have more efficient ways of figuring out the number of outcomes in S or in a compound event without having to list them all. Chapter 3 considers ways to do this, and then Chapter 4 develops other ways to manipulate and calculate probabilities.
Although we often use "toy" problems involving things such as coins, dice and simple games for examples, probability is used to deal with a huge variety of practical problems from finance to clinical trials. In some settings such as in question 2.6 and 2.7 below, we need to rely on previous repetitions of an experiment, or on related scientific data, to assign numerical probabilities to events.

2.4 Problems on Chapter 2

2.1 Students in a particular program have the same 4 math profs. Two students in the program each independently ask one of their math professors[3] for a letter of reference. Assume each is equally likely to ask any of the math profs.

a) List a sample space for this "experiment".

b) Use this sample space to find the probability both students ask the same prof.

2.2 a) List a sample space for tossing a fair coin 3 times.

b) What is the probability of 2 consecutive tails (but not 3)?

2.3 You wish to choose 2 different numbers without replacement (so the same number can not be chosen twice) from $\{1, 2, 3, 4, 5\}$. List all possible pairs you could obtain, assume all pairs are equally probable, and find the probability the numbers chosen differ by 1 (i.e. the two numbers are consecutive).

2.4 Four letters addressed to individuals W, X, Y and Z are randomly placed in four addressed envelopes, one letter in each envelope.

(a) List a 24-point sample space for this experiment. Be sure to explain your notation.

(b) List the sample points belonging to each of the following events:
A: "W's letter goes into the correct envelope";
B: "no letters go into the correct envelopes";

[3] "America believes in education: the average professor earns more money in a year than a professional athlete earns in a whole week." Evan Esar (1899 - 1995)

C: "exactly two letters go into the correct envelopes";
D: "exactly three letters go into the correct envelopes".

(c) Assuming that the 24 sample points are equally probable, find the probabilities of the four events in (b).

2.5 (a) Three balls are placed at random in three boxes, with no restriction on the number of balls per box; list the 27 possible outcomes of this experiment. Be sure to explain your notation. Assuming that the outcomes are all equally probable, find the probability of each of the following events:
A: "the first box is empty";
B: "the first two boxes are empty";
C: "no box contains more than one ball".

(b) Find the probabilities of events A, B and C when three balls are placed at random in n boxes $(n \geq 3)$.

(c) Find the probabilities of events A, B and C when r balls are placed in n boxes $(n \geq r)$.

2.6 **Diagnostic Tests**. Suppose that in a large population some persons have a specific disease at a given point in time. A person can be tested for the disease, but inexpensive tests are often imperfect, and may give either a "false positive" result (the person does not have the disease but the test says they do) or a "false negative" result (the person has the disease but the test says they do not).

In a random sample of 1000 people, individuals with the disease were identified according to a completely accurate but expensive test, and also according to a less accurate but inexpensive test. The results for the less accurate test were that

- 920 persons without the disease tested negative
- 60 persons without the disease tested positive
- 18 persons with the disease tested positive
- 2 persons with the disease tested negative.

(a) Estimate the fraction of the population that has the disease and tests positive using the inexpensive test.

(b) Estimate the fraction of the population that has the disease.

(c) Suppose that someone randomly selected from the same population as those tested above was administered the inexpensive test and it indicated positive. Based on the above information, how would you estimate the probability that they actually have the disease.

2.7 **Machine Recognition of Handwritten Digits**. Suppose that you have an optical scanner and associated software for determining which of the digits $0, 1, ..., 9$ an individual has written in a square box. The system may of course be wrong sometimes, depending on the legibility of the handwritten number.

(a) Describe a sample space S that includes points (x, y), where x stands for the number actually written, and y stands for the number that the machine identifies.

(b) Suppose that the machine is asked to identify very large numbers of digits, of which $0, 1, ..., 9$ occur equally often, and suppose that the following probabilities apply to the points in your sample space:

$p(0, 6) = p(6, 0) = .004$; $p(0, 0) = p(6, 6) = .096$
$p(5, 9) = p(9, 5) = .005$; $p(5, 5) = p(9, 9) = .095$
$p(4, 7) = p(7, 4) = .002$; $p(4, 4) = p(7, 7) = .098$
$p(y, y) = .100$ for $y = 1, 2, 3, 8$

Give a table with probabilities for each point (x, y) in S. What fraction of numbers is correctly identified?

2.8 [1]Anonymous professor X has an integer ($1 \leq m \leq 9$) in mind and asks two students, $Allan$ and $Beth$ to pick numbers between 1 and 9. Whichever is closer to m gets 90% and the other 80% in Stat 230. If they are equally close, they both get 85%. If the professor's number and that of Allen are chosen purely at random and Allen announces his number out loud, describe a sample space and a strategy which leads Beth to the highest possible mark.

2.9 [1]In questions 2.4-2.7, what can you say about how appropriate you think the probability model is for the experiment being modelled?

[1]Solution not in appendix

3. Probability – Counting Techniques

Some probability problems can be attacked by specifying a sample space $S = \{a_1, a_2, \ldots, a_n\}$ in which each simple event has probability $\frac{1}{n}$ (i.e. is “equally likely"). This is referred to a a uniform distribution over the set $\{a_1, a_2, \ldots, a_n\}$. If a compound event A contains r points, then $P(A) = \frac{r}{n}$. In other words, we need to be able to count the number of events in S which are in A. We review first some basic ways to count outcomes from “experiments".

3.1 Counting Arguments

There are two helpful rules for counting, phrased in terms of “jobs" which are to be done.

1. The **Addition Rule:** *Suppose we can do job 1 in p ways and job 2 in q ways. Then we can do either job 1 **OR** job 2 (but not both), in $p + q$ ways.*

For example, suppose a class has 30 men and 25 women. There are $30 + 25 = 55$ ways the prof. can pick one student to answer a question. If there are 5 vowels and 20 consonants on a list and I must pick one letter, this can be done in 5+20 ways.

2. The **Multiplication Rule:** *Suppose we can do job 1 in p ways and, **for each of these ways**, we can do job 2 in q ways. Then we can do both job 1 **AND** job 2 in $p \times q$ ways.*

For example, if there are 5 vowels and 20 consonants and I must choose one consonant followed by one vowel for a two-letter word, this can be done in 20×5 ways (there are 100 such words). To ride a bike, you must have the chain on both a front sprocket and a rear sprocket. For a 21 speed bike there are 3 ways to select the front sprocket and 7 ways to select the rear sprocket, i.e. $3 \times 7 = 21$ such combinations.

This interpretation of "OR" as addition and "AND" as multiplication evident in the addition and multiplication rules above will occur throughout probability, so it is helpful to make this association in your mind. Of course questions do not always have an AND or an OR in them and you may have to play around with re-wording the question to discover implied AND’s or OR’s.

Example: Suppose we pick 2 numbers from digits 1, 2, 3, 4, 5 with replacement. (Note: "with replacement" means that after the first number is picked it is "replaced" in the set of numbers, so it could be picked again as the second number.) Assume a uniform distribution on the sample space, i.e. that every pair of numbers has the same probability. Let us find the probability that one number is even. This can be reworded as: "The first number is even AND the second is odd (this can be done in 2×3 ways) OR the first is odd AND the second is even (done in 3×2 ways)." Since these are connected with the word OR, we combine them using the addition rule to calculate that there are $(2 \times 3) + (3 \times 2) = 12$ ways for this event to occur. Since the first number can be chosen in 5 ways AND the second in 5 ways, S contains $5 \times 5 = 25$ points and since each point has the same probability, they all have probability $\frac{1}{25}$.

$$\text{Therefore } P(\text{one number is even}) = \frac{12}{25}$$

When objects are selected and replaced after each draw, the addition and multiplication rules are generally sufficient to find probabilities. When objects are drawn **without** being replaced, some special rules may simplify the solution.

Note: The phrases *at random*, or *uniformly* are often used to mean that all of the points in the sample space are equally likely so that in the above problem, every possible pair of numbers chosen from this set has the same probability $\frac{1}{25}$.

Problems:

3.1.1 (a) A course has 4 sections with no limit on how many can enrol in each section. Three students each pick a section at random.

(i) Specify the sample space S.

(ii) Find the probability they all end up in the same section

(iii) Find the probability they all end up in different sections

(iv) Find the probability nobody picks section 1.

(b) Repeat (a) in the case when there are n sections and s students $(n \geq s)$.

3.1.2 Canadian postal codes consist of 3 letters (of 26 possible letters) alternated with 3 digits (of the 10 possible), starting with a letter (e.g. N2L 3G1). Assume no other restrictions on the construction of postal codes. For a postal code chosen at random, what is the probability:

(a) all 3 letters are the same?

(b) the digits are all even or all odd? Treat 0 as being neither even nor odd.

3.1.3 Suppose a password has to contain between six and eight digits, with each digit either a letter or a number from 1 to 9. There must be at least one number present.

(a) What is the total number of possible passwords?

(b) If you started to try passwords in random order, what is the probability you would find the correct password for a given situation within the first 1,000 passwords you tried?

We have already discussed a special class of discrete probability models, the uniform model, in which all of the outcomes have the same probability. In such a model, we can calculate the probability of any event A by counting the number of outcomes in the event A,

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Total Number of outcomes in } S}$$

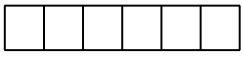
Here we look at some formal counting methods to help calculate probabilities in uniform models.

Counting Arrangements: In many problems, the sample space is a set of arrangements or sequences. These are classically called *permutations*. A key step in the argument is to be sure to understand what it is you are counting. It is helpful to invent a notation for the outcomes in the sample space and the events of interest (these are the objects you are counting).

Example. Suppose the letters are arranged at random to form a six-letter word (an arrangement) – we must use each letter once only. The sample space

$$S = \{abcdef, abcdfe, ..., fedcba\}$$

has a large number of outcomes and, because we formed the word "at random", we assign the same probability to each. To count the number of words in S, count the number of ways that we can construct such a word – each way corresponds to a unique word. Consider filling the boxes corresponding to the six positions in the arrangement



We can fill the first box in 6 ways with any one of the letters. For each of these choices, we can fill the second box in 5 ways with any one of the remaining letters. Thus there are $6 \times 5 = 30$ ways to fill the first two boxes. (If you are not convinced by this argument, list all the possible ways that the first two boxes can be filled.)

For each of these 30 choices, we can fill the third box in 4 ways using any one of the remaining letters so there are $6 \times 5 \times 4 = 120$ ways to fill the first three boxes. Applying the same reasoning, we

see that there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways to fill the 6 boxes and hence 720 equally probable words in S.

Now consider events such as A: the second letter is e or f so *A={ afbcde,aebcdf,...,efdcba}*. We can count the number of outcomes in A using a similar argument if we start with the second box.

We can fill the second box in 2 ways i.e., with an e or f. For each of these choices, we can then fill the first box in 5 ways, so now we can fill the first two boxes in $2 \times 5 = 10$ ways. For each of these choices, we can fill the remaining four boxes in $4 \times 3 \times 2 \times 1 = 24$ ways so the number of outcomes in A is $10 \times 24 = 240$. Since we have a uniform probability model, we have

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } S} = \frac{240}{720} = \frac{1}{3}.$$

In determining the number of outcomes in A, it is important that we start with the second box. Suppose, instead, we start by saying there are 6 ways to fill the first box. Now the number of ways of filling the second box depends on what happened in the first. If we used e or f in the first box, there is only one way to fill the second. If we used a, b, c or d for the first box, there are 2 ways of filling the second. We avoid this complication by starting with the second box.

We can generalize the above problem in several ways. In each case we count the number of arrangements by counting the number of ways we can fill the positions in the arrangement. Suppose we start with n symbols. Then we can make

• $n \times (n-1) \times ... \times 1$ arrangements of length n using each symbol once and only once. This product is denoted by $n!$ (read "n factorial"). Note that $n! = n \times (n-1)!$.

• $n \times (n-1) \times ... \times (n-r+1)$ arrangements of length r using each symbol at most once. This product is denoted by $n^{(r)}$ (read "n to r factors"). Note that $n^{(r)} = \frac{n!}{(n-r)!}$.

• $n \times n \times ... \times n = n^r$ arrangements of length r using each symbol as often as we wish.

The terms above, especially the factorial $n!$ grow at an extraordinary rate as a function of n. For example (we will discuss $0!$ shortly),

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5040	40320	362880	3628800

There is an approximation to $n!$ called Stirling's formula which is often used for large n. First what would it mean for two sequences of numbers which are growing very quickly to be asymptotically equal? Suppose we wish to approximate one sequence a_n with another sequence b_n and want the percentage error of the approximation to approach zero as n grows. This is equivalent to saying $a_n/b_n \to 1$ as $n \to \infty$ and under these circumstances we will call the two sequences *asymptotically equivalent*. Stirling's approximation says that $n!$ is *asymptotically equivalent* to $n^n e^{-n}\sqrt{2\pi n}$. The error in Stirling's approximation is less than 1% if $n \geq 8$ and becomes very small quite quickly as n increases.

For many problems involving sampling from a deck of cards or a reasonably large population, counting the number of cases by simple conventional means is virtually impossible, and we need the counting arguments dealt with here. The extraordinarily large size of populations, in part due to the large size of quantities like n^n and $n!$, is part of the reason that statistics, sampling, counting methods and probability calculations play such an important part in modern science and business.

Example. A pin number of length 4 is formed by randomly selecting (with replacement) 4 digits from the set $\{0, 1, 2, ..., 9\}$. Find the probability of the events:

A: the pin number is even

B: the pin number has only even digits

C: all of the digits are unique

D: the pin number contains at least one 1.

Since we pick the digits with replacement, the outcomes in the sample space can have repeated digits.

The sample space is $S = \{0000, 0001, ..., 9999\}$ with 10^4 equally probable outcomes. For the event $A = \{0000, 0002, ..., 9998\}$, we can select the last digit to be any one of $0, 2, 6, 4, 8$ in 5 ways. Then for each of these choices, we can select the first digit in 10 ways and so on. There are 5×10^3 outcomes in A and

$$P(A) = \frac{5 \times 10^3}{10^4} = \frac{1}{2}.$$

The event $B = \{0000, 0002, ..., 8888\}$. We can select the first digit in 5 ways, and for each of these choices, the second in 5 ways, and so on. There are 5^4 outcomes in B and

$$P(B) = \frac{5^4}{10^4} = \frac{1}{16}.$$

The event $C = \{0123, 0124, ..., 9876\}$. We can select the first digit in 10 ways and for each of these choices, the second in 9 ways and so on. There are $10 \times 9 \times 8 \times 7$ outcomes in C and so

$$P(C) = \frac{10 \times 9 \times 8 \times 7}{10^4} = \frac{63}{125}.$$

The event $D = \{0001, 0011, 0111, 1111, ...\}$. To count the number of outcomes, consider the complement of D, or the set of all outcomes in S but not in D. We denote this event $\overline{D} = \{0000, 0002, ..., 9999\}$. There are 9^4 outcomes in $\overline{D}$ and so there are $10^4 - 9^4$ outcomes in D and

$$P(D) = \frac{10^4 - 9^4}{10^4} = \frac{3439}{10000}.$$

For a general event A, the *complement* of A denoted $\overline{A}$ is the set of all outcomes in S which are not in A. It is often easier to count outcomes in the complement rather than in the event itself.

Example. A pin number of length 4 is formed by randomly selecting (without replacement) 4 digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the probability of the events:

A: the pin number is even.

B: the pin number has only even digits.

C: the pin number begins or ends with a 1.

D: the pin number contains 1.

The sample space is

$$S = \{0123, 0132, ..., 6789\}$$

with $10^{(4)}$ equally probable outcomes. For the event $A = \{1230, 0134, ..., 9876\}$, we can select the last digit to be any one of $0, 2, 6, 4, 8$ in 5 ways. Then for each of these choices, we can select the first digit in 9 ways, the third in 8 ways and so on. There are $5 \times 9 \times 8 \times 7$ outcomes in A and

$$P(A) = \frac{5 \times 9 \times 8 \times 7}{10^{(4)}} = \frac{1}{2}.$$

The event $B = \{0246, 0248, ..., 8642\}$. The pin numbers in B are all $5^{(4)}$ arrangements of length 4 using only the even digits $\{0, 2, 4, 6, 8\}$ and so

$$P(B) = \frac{5^{(4)}}{10^{(4)}} = \frac{5 \times 4 \times 3 \times 2}{10 \times 9 \times 8 \times 7} = \frac{1}{42}.$$

The event $C = \{1023, 0231, ..., 9871\}$. There are 2 positions for the 1. For each of these choices, we can fill the remaining three positions in $9^{(3)}$ ways and so

$$P(C) = \frac{2 \times 9^{(3)}}{10^{(4)}} = \frac{1}{5}.$$

The event $D = \{1234, 2134, ..., 9871\}$. We can use the complement and count the number of pin numbers that do not contain a 1. There are $9^{(4)}$ pin numbers that do not contain 1 and so there are $10^{(4)} - 9^{(4)}$ that do contain a 1. Therefore

$$P(D) = \frac{10^{(4)} - 9^{(4)}}{10^{(4)}} = 1 - \frac{9^{(4)}}{10^{(4)}} = \frac{2}{5}.$$

Note that this is $1 - P(\overline{D})$ where $\overline{D}$ is the complement of D.

Counting Subsets. In some problems, the outcomes in the sample space are subsets of a fixed size. Here we look at counting such subsets. Again, it is useful to write a short list of the subsets you are counting.

Example. Suppose we randomly select a subset of 3 digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ so that the sample space is

$$S = \{\{1, 2, 3\}, \{0, 1, 3\}, \{0, 1, 4\}, ...\{7, 8, 9\}\}.$$

All the digits in each outcome are unique i.e. we do not consider $\{1, 1, 2\}$ to be a subset of S. Also, the order of the elements in a subset is not relevant. This is true in general for sets; the subsets $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same. To count the number of outcomes in S, we use what we have learned about counting arrangements. Suppose there are m such subsets. Using the elements of any subset of size 3, we can form 3! arrangements of length 3. For example, the subset $\{1, 2, 3\}$ generates the $3! = 6$ arrangements

$$123, 132, 213, 231, 312, 321$$

and any other subset generates a different 3! arrangements so that the total number of arrangements of 3 digits taken without replacement from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is $m \times 3!$. But we know the total number of arrangements is $10^{(3)}$ so $m \times 3! = 10^{(3)}$. Solving we get

$$m = \frac{10^{(3)}}{3!} = 120.$$

Number of subsets of size r. We use the combinatorial symbol $\binom{n}{r}$(read n *choose* r) to denote the number of subsets of size r that can be selected from a set of n objects. By an argument similar to that above, if m denotes the number of subsets of size r that can be selected from n things, then $m \times r! = n^{(r)}$ and so we have m is equal

$$\binom{n}{r} = \frac{n^{(r)}}{r!}.$$

In the example, since we selected the subset at random, each of the 120 subsets has the same probability $1/120$. Now find the probability of the following events.

A: the digit 1 is included in the selected subset

B: all the digits in the selected subset are even

C: at least one of the digits in the selected subset is less than 5

The event A**:** To count the outcomes, we must have 1 in the subset and we can select the other two elements from the remaining 9 digits in $\binom{9}{2}$ ways. And so

$$P(A) = \frac{\binom{9}{2}}{\binom{10}{3}} = \frac{9^{(2)}/2!}{10^{(3)}/3!} = \frac{3}{10}.$$

The event $B = \{\{0, 2, 4\}, \{0, 2, 6\}, ...\}$. We can form the outcomes in B by selecting 3 digits from the five even digits $\{0, 2, 4, 6, 8\}$ in $\binom{5}{3}$ ways. And so

$$P(B) = \frac{\binom{5}{3}}{\binom{10}{3}}.$$

The event $C = \{\{0,1,2\},\{0,1,6\},\{0,6,7\},...\}$. Here it is convenient to consider the complement $\overline{C}$ in which the outcomes are $\{\{6,7,8\},\{6,7,9\},...\}$ i.e. subsets with all elements greater than 5. We can form the subsets in $\overline{C}$ by selecting a subset of size 3 from the set $\{6,7,8,9\}$ in $\binom{4}{3}$ ways. Therefore the number of points in C is $\binom{10}{3} - \binom{4}{3}$ and its probability is

$$P(C) = \frac{\binom{10}{3} - \binom{4}{3}}{\binom{10}{3}} = 1 - \frac{\binom{4}{3}}{\binom{10}{3}} = 1 - P(\overline{C}).$$

Example. Suppose a box contains 10 balls of which 3 are red, 4 are white and 3 are green. A sample of 4 balls is selected at random without replacement. Find the probability of the events

E: the sample contains 2 red balls

F: the sample contains 2 red, 1 white and 1 green ball

G: the sample contains 2 or more red balls

Imagine that we label the balls from 1 to 10 with labels $1,2,3$ being red, $4,5,6,7$ being white and $8,9,10$ being green. Construct a uniform probability model in which all subsets of size 4 are equally probable. The sample space is

$$S = \{\{1,2,3,4\},\{1,2,3,5\},...,\{7,8,9,10\}\}$$

and each outcome has probability $1/\binom{10}{4}$.

The event E: To count the number of outcomes in E, we can construct a subset with two red balls by first choosing the two red balls from the three in $\binom{3}{2}$ ways. For each of these choices we can select the other two balls from the seven non-red balls in $\binom{7}{2}$ ways so there $\binom{3}{2} \times \binom{7}{2}$ are outcomes in E and

$$P(E) = \frac{\binom{3}{2} \times \binom{7}{2}}{\binom{10}{4}} = \frac{3}{10}.$$

The event $F = \{\{1,2,4,8\},\{1,2,4,9\},...\}$: To count the number of outcomes in F, we can select the two red balls in $\binom{3}{2}$ ways, then the white ball in $\binom{4}{1}$ ways and the green ball in $\binom{3}{1}$ ways. So we have

$$P(F) = \frac{\binom{3}{2}\binom{4}{1}\binom{3}{1}}{\binom{10}{4}} = \frac{6}{35}.$$

The event $G = \{\{1,2,3,4\},\{1,2,4,5\},...\}$ has outcomes with both 2 and 3 red balls. We need to count these separately (see below). There are $\binom{3}{2}\binom{7}{2}$ outcomes with exactly two red balls and $\binom{3}{3}\binom{7}{1}$ outcomes with three red balls. Hence we have

$$P(G) = \frac{\binom{3}{2}\binom{7}{2} + \binom{3}{3}\binom{7}{1}}{\binom{10}{4}} = \frac{1}{3}.$$

A **common mistake** is to count the outcomes in G as follows. There are $\binom{3}{2}$ ways to select two red balls and then for each of these choices we can select the remaining two balls from the remaining eight in $\binom{8}{2}$ ways. So the number of outcomes in G is $\binom{3}{2} \times \binom{8}{2}$. You can easily check that this is greater than $\binom{3}{2}\binom{7}{2} + \binom{3}{3}\binom{7}{1}$. The reason for the error is that some of the outcomes in G have been counted more than once. For example, you might pick red balls 1,2 and then other balls 3,4 to get the subset $\{1, 2, 3, 4\}$. Or you may pick red balls 1,3 and then other balls 2,4 to get the subset $\{1, 3, 2, 4\}$. These are counted as two separate outcomes but they are in fact the same subset. To avoid this counting error, whenever you are asked about events defined in terms such as "at most…", "more than …", " fewer than..." etc., break the events into pieces where each piece has outcomes with specific values e.g. two red balls, three red balls.

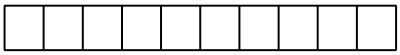
Properties of $\binom{n}{r}$. You should be able to prove the following:

1. $n^{(r)} = \frac{n!}{(n-r)!} = n(n-1)^{(r-1)}$ for $r \geq 1$.
2. $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n^{(r)}}{r!}$
3. $\binom{n}{r} = \binom{n}{n-r}$ for all $r = 0, 1, ..., n$.
4. If we define $0! = 1$, then the formulas above make sense for $\binom{n}{0} = \binom{n}{n} = 1$
5. $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + ... + \binom{n}{r}x^n$ (this is the binomial theorem)

In many problems, we can combine counting arguments for arrangements and subsets as in the following example.

Example. A binary sequence is an arrangement of zeros and ones. Suppose we have a uniform probability model on the sample space of all binary sequences of length 10. What is the probability that the sequence has exactly 5 zeros?

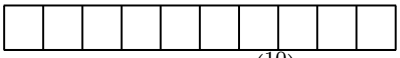
The sample space is $S = \{0000000000, 0000000001, ..., 1111111111\}$. We can fill each of the 10 positions in the sequence in 2 ways and hence S has 2^{10} outcomes each with probability $\frac{1}{2^{10}}$. The event E with exactly 5 zeros and 5 ones is $E = \{0000011111, 1000001111, ..., 1111100000\}$. To count the outcomes in E, think of constructing the sequence by filling boxes below



We can choose the 5 boxes for the zeros in $\binom{10}{5}$ ways and then the ones go in the remaining boxes in 1 way. Hence we have

$$P(E) = \frac{\binom{10}{5}}{2^{10}}.$$

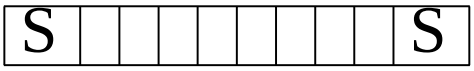
Example. Suppose the letters of the word STATISTICS are arranged at random. Find the probability of the event G that the arrangement begins and ends with S. The sample space is $S = \{SSSTTTIIAC, SSSTTTIICA, ...\}$. Here we need to count arrangements when some of the elements are the same. We use the same idea as in the last example. We construct the arrangements by filling 10 boxes corresponding to the positions in the arrangement.



We can choose the three positions for the three S's in $\binom{10}{3}$ ways. For each of these choices, we can choose the positions for the three T's in $\binom{7}{3}$ ways. Then we can place the two Is in $\binom{4}{2}$ ways, then the C in $\binom{2}{1}$ ways and finally the A in $\binom{1}{1}$ ways. The number of equally probable outcomes in S is

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{10!}{3!7!}\frac{7!}{3!4!}\frac{4!}{2!2!}\frac{2!}{1!1!}\frac{1!}{1!0!} = \frac{10!}{3!3!2!1!1!}$$

The event $G = \{SSTTTIIAC, SSTTTIICAS, ...\}$: To count the outcomes in G we must have S in the first and last position



Now we can use the same technique to arrange the remaining 8 letters. Having placed two of the S's, there remain 8 free boxes, in which we are to place three Ts in $\binom{8}{3}$ ways, two Is in $\binom{5}{2}$ ways, one C in $\binom{3}{1}$ ways, one A in $\binom{2}{1}$ ways and finally the remaining S in the last empty box in $\binom{1}{1}$ way. There are

$$\binom{8}{3}\binom{5}{2}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{8!}{3!2!1!1!1!} = 3360$$

elements in G and

$$P(G) = \frac{\frac{8!}{3!2!1!1!1!}}{\frac{10!}{3!3!2!1!1!}} = \frac{3360}{50400} = \frac{1}{15}.$$

Number of Arrangements when some symbols are alike: In general, if we have n_i symbols of type $i, i = 1, 2, ..., k$ with $n_1 + n_2 + ... + n_k = n$, then the number of arrangements using all of the symbols is

$$\binom{n}{n_1} \times \binom{n - n_1}{n_2} \times \binom{n - n_1 - n_2}{n_3} \times ... \times \binom{n_k}{n_k} = \frac{n!}{n_1!n_2!...n_k!}$$

Example. Suppose we make a random arrangement of length 3 using letters from the set $\{a, b, c, d, e, f, g, h, i, j\}$. What is the probability of the event B that the letters are in alphabetic order if

a) letters are selected without replacement

b) letters are selected with replacement

For (a), the sample space is $\{abc, bac, ..., hij\}$ with $10^{(3)}$ equally probable outcomes.

The event $B = \{abc, abd, ..., hij\}$. To count the outcomes in B, we first select the three (different) letters to form the arrangement in $\binom{10}{3}$ ways. There is then 1 way to make an arrangement with the selected letters in alphabetic order. So we have

$$P(B) = \frac{\binom{10}{3}}{10^{(3)}} = \frac{1}{6}.$$

For (b), the sample space is $\{aaa, aab, abc, ...\}$ with 10^3 equally probable outcomes. To count the elements in B, consider the following cases

Case 1: all three letters are the same. There are 10 such arrangements $\{aaa, bbb, ccc, ...\}$ all in alphabetic order.

Case 2: there are two different letters e.g. $\{aab, aba, baa, abb, bab, bba\}$. We can choose the two letters in $\binom{10}{2}$ ways. For each of these choices, we can then make 2 arrangements with the letters in alphabetic order e.g. $\{aab, abb\}$ There are $\binom{10}{2} \times 2$ arrangements in this case.

Case 3: all three letters are different. We can select the three letters in $\binom{10}{3}$ ways and then make 1 arrangement that is in alphabetic order (as in part (a)).

Combining the three cases, we have

$$P(B) = \frac{10 + \binom{10}{2} \times 2 + \binom{10}{3}}{10^3} = \frac{11}{50}$$

Example: We form a 4 digit number by randomly selecting and arranging 4 digits from 1, 2, 3,... 7 without replacement. Find the probability the number formed is (a) even (b) over 3000 (c) an even number over 3000.

Solution: Let S be the set of all possible 4 digit numbers using digits $1, 2, \ldots, 7$ sampled without replacement. Then S has $7^{(4)}$ outcomes.

(a) For a number to be even, the last digit must be even. We can fill this last position with a 2, 4, or 6; i.e. in 3 ways. The first 3 positions can be filled by choosing and arranging 3 of the 6 digits not used in the final position. i.e. in $6^{(3)}$ ways. Then there are $3 \times 6^{(3)}$ ways to fill the final position AND the first 3 positions to produce an even number. Therefore the probability the number is even is $\frac{3 \times 6^{(3)}}{7^{(4)}} = \frac{3}{7}$. Alternatively, the four digit number is even if and only if the last digit is even. The last digit is equally likely to be any one of the numbers 1, ..., 7 so the probability it is even is the probability it is either 2,4, or 6 or $\frac{3}{7}$.

(b) To get a number over 3000, we require the first digit to be 3, 4, 5, 6, or 7; i.e. it can be chosen in 5 ways. The remaining 3 positions can be filled in $6^{(3)}$ ways. Therefore the probability the

number is greater than 3000 is $\frac{5\times 6^{(3)}}{7^{(4)}} = \frac{5}{7}$. Alternatively, note that the four digit number is over 3000 if and only if the first digit is one of 3, 4, 5, 6 or 7. Since each of 1, ..., 7 is equally likely to be the first digit, we get the probability the number is greater than 3000 is $\frac{5}{7}$.

In both (a) and (b) we dealt with positions which had restrictions first, before considering positions with no restrictions. This is generally the best approach to follow in applying counting techniques.

(c) This part has restrictions on both the first and last positions. To illustrate the complication this introduces, suppose we decide to fill positions in the order 1 then 4 then the middle two. We can fill position 1 in 5 ways. How many ways can we then fill position 4? The answer is either 2 or 3 ways, depending on whether the first position was filled with an even or odd digit. Whenever we encounter a situation such as this, we have to break the solution into separate cases. One case is where the first digit is even. The positions can be filled in 2 ways for the first (i.e. with a 4 or 6), 2 ways for the last, and then $5^{(2)}$ ways to arrange 2 of the remaining 5 digits in the middle positions. This first case then occurs in $2 \times 2 \times 5^{(2)}$ ways. The second case has an odd digit in position one. There are 3 ways to fill position one (3, 5, or 7), 3 ways to fill position four (2, 4, or 6), and $5^{(2)}$ ways to fill the remaining positions. Case 2 then occurs in $3 \times 3 \times 5^{(2)}$ ways. We need case 1 OR case 2. Therefore the probability we obtain an even number greater than 3000 is

$$\frac{2 \times 2 \times 5^{(2)} + 3 \times 3 \times 5^{(2)}}{7^{(4)}} = \frac{13 \times 5^{(2)}}{7 \times 6 \times 5^{(2)}} = \frac{13}{42}.$$

Another way to do this is to realize that we need only to consider the first and last digit, and to find P(first digit is ≥ 3 **and** last digit is even). There are $7 \times 6 = 42$ different choices for (first digit, last digit) and it is easy to see there are 13 choices for which first digit ≥ 3, last digit is even (5×3 minus the impossible outcomes (4, 4) and (6, 6)). Thus the desired probability is $\frac{13}{42}$.

Exercise: Try to solve part (c) by filling positions in the order 4, 1, middle. You should get the same answer.

Exercise: Can you spot the flaw in the following argument? There are $3 \times 6^{(3)}$ ways to get an even number (part (a)). There are $5 \times 6^{(3)}$ ways to get a number ≥ 3000 (part (b)). Therefore by the multiplication rule there are $[3 \times 6^{(3)}] \times [5 \times 6^{(3)}]$ ways to get a number which is even and > 3000.

Example: 5 men and 3 women are placed in random seats in a row. Find the probability that

(a) the same gender is at each end

(b) the women all sit together.

What are you assuming in your solution? Is it likely in real life that individuals are randomly seated?

Solution: If we treat the people as being 8 objects, 5 of one type and 3 of another, i.e. $5M$ and $3W$, our sample space will have $\frac{8!}{5!3!} = 56$ points.

(a) To get the same gender at each end we need either

M							M

OR

W							W

The number of distinct arrangements with a man at each end is $\frac{6!}{3!3!} = 20$, since we are arranging $3M$'s and $3W$'s in the middle 6 positions. The number with a woman at each end is $\frac{6!}{5!1!} = 6$. Thus

$$P(\text{same gender at each end}) = \frac{20+6}{56} = \frac{13}{28}$$

assuming each arrangement is equally likely.

(b) Treating WWW as a single unit, we are arranging 6 objects, $5M$'s and 1 object we might call "WWW''. There are $\frac{6!}{5!1!} = 6$ arrangements. Thus,

$$P(\text{women sit together}) = \frac{6}{56} = \frac{3}{28}.$$

Our solution is based on the assumption that all points in S are equally likely. This would mean the people sit in a purely random order. Random seating is unlikely in real life, since friends are more likely to sit together.

Problems:

3.1.4 Digits 1, 2, 3, . . . , 7 are arranged at random to form a 7 digit number. Find the probability that

(a) the even digits occur together, in any order

(b) the digits at the 2 ends are both even or both odd.

3.1.5 The letters of the word EXCELLENT are arranged in a random order. Find the probability that

(a) the same letter occurs at each end.

(b) $X, C,$ and N occur together, in any order.

(c) the letters occur in alphabetical order.

Example: In the Lotto 6/49 lottery, six numbers are drawn at random, without replacement, from the numbers 1 to 49. Find the probability that

(a) the numbers drawn are $\{1, 2, 3, 4, 5, 6\}$.

(b) no even number is drawn.

Solution:

(a) Let the sample space S consist of all subsets of 6 numbers from 1, ..., 49; there are $\binom{49}{6}$ of them. Since 1, 2, 3, 4, 5, 6 consist of one of these subsets, the probability of this particular set is $1/\binom{49}{6}$, which is about 1 in 13.9 million.

(b) There are 25 odd and 24 even numbers, so there are $\binom{25}{6}$ choices in which all the numbers are odd. Therefore the probability no even number is drawn is the probability they are all odd, or

$$\frac{\binom{25}{6}}{\binom{49}{6}} \simeq 0.0127.$$

Example: Find the probability a bridge hand (13 cards picked at random from a standard deck[4] without replacement) has

(a) 3 aces

(b) at least 1 ace

(c) 6 spades, 4 hearts, 2 diamonds, 1 club

(d) a 6-4-2-1 split between the 4 suits

(e) a 5-4-2-2 split.

Solution: Since order of selection does not matter, we take S to have $\binom{52}{13}$ outcomes, each with the same probability.

(a) We can choose 3 aces in $\binom{4}{3}$ ways. We also have to choose 10 other cards from the 48 non-aces. This can be done in $\binom{48}{10}$ ways. Hence the probability of exactly three aces is $\frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}}$

[4] A standard deck has 13 cards in each of four suits, hearts, diamonds, clubs and spades for a total of 52 cards. There are four aces in the deck (one of each suit).

(b) **Solution 1:** At least 1 ace means 1 ace or 2 aces or 3 aces or 4 aces. Calculate each part as in (a) and use the addition rule to get that the probability of at least one ace is

$$\frac{\binom{4}{1}\binom{48}{12}+\binom{4}{2}\binom{48}{11}+\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}}{\binom{52}{13}}.$$

Solution 2: If we subtract all cases with 0 aces from the $\binom{52}{13}$ points in S we are left with all points having at least 1 ace. There are $\binom{4}{0}\binom{48}{13}=\binom{48}{13}$ possible hands with 0 aces since all cards must be drawn from the non-aces. (The term $\binom{4}{0}$ can be omitted since $\binom{4}{0}=1$, but was included here to show that we were choosing 0 of the 4 aces) This gives that the probability of at least one ace is

$$\frac{\binom{52}{13}-\binom{48}{13}}{\binom{52}{13}}=1-\frac{\binom{48}{13}}{\binom{52}{13}}$$

This solution is incorrect, but illustrates a common error. Choose 1 of the 4 aces then any 12 of the remaining 51 cards. This guarantees we have at least 1 ace, so the probability of at least one ace is $\frac{\binom{4}{1}\binom{51}{12}}{\binom{52}{13}}$. The flaw in this solution is that it counts some points more than once by partially keeping track of order. For example, we could get the ace of spades on the first choice and happen to get the ace of clubs in the last 12 draws. We also could get the ace of clubs on the first draw and then get the ace of spades in the last 12 draws. Though in both cases we have the same outcome, they would be counted as 2 different outcomes. The strategies in solution 1 and 2 above are safer. We often need to inspect a solution carefully to avoid double or multiple counting.

(c) Choose the 6 spades in $\binom{13}{6}$ ways and the hearts in $\binom{13}{4}$ ways and the diamonds in $\binom{13}{2}$ ways and the clubs in $\binom{13}{1}$ ways. Therefore the probability of 6 spades, 4 hearts, 2 diamonds and one clubs is

$$\frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}\simeq 0.00196$$

(d) The split in (c) is only 1 of several possible 6-4-2-1 splits. In fact, filling in the numbers 6, 4, 2 and 1 in the spaces below

Spades	Hearts	Diamonds	Clubs

defines a 6-4-2-1 split. There are $4!$ ways to do this, and having done this, there are $\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}$ ways to pick the cards from these suits. Therefore the probability of a a 6-4-2-1 split between the 4 suits is

$$\frac{4!\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}} \simeq 0.047$$

(e) This is the same question as (d) except the numbers 5-4-2-2 are not all different. There are $\frac{4!}{2!}$ different arrangements of 5-4-2-2 in the spaces below.

Spades	Hearts	Diamonds	Clubs

Therefore, the probability of a a 5-4-2-2 split is

$$\frac{\frac{4!}{2!}\binom{13}{5}\binom{13}{4}\binom{13}{2}\binom{13}{2}}{\binom{52}{13}} \simeq 0.1058$$

Notes. While $n^{(r)}$ only has a physical interpretation when n and r are positive integers with $n \geq r$, it still has meaning when n is not a positive integer, as long as r is a non-negative integer. In general we can define $n^{(r)} = n(n-1)....(n-r+1)$. For example:

$$(-2)^{(3)} = (-2)(-2-1)(-2-2) = (-2)(-3)(-4) = -24 \text{ and}$$
$$1.3^{(2)} = (1.3)(1.3-1) = 0.39$$

Note that in order for $\binom{n}{0} = \binom{n}{n} = 1$ we must define

$$n^{(0)} = \frac{n!}{(n-0)!} = 1 \text{ and } 0! = 1.$$

Also $\binom{n}{r}$ loses its physical meaning when n is not a non-negative integer $\geq r$ but we can use

$$\binom{n}{r} = \frac{n^{(r)}}{r!}$$

to define it when n is not a positive integer but r is. For example,

$$\binom{\frac{1}{2}}{3} = \frac{(\frac{1}{2})^{(3)}}{3!} = \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!} = \frac{1}{16}$$

Also, when n and r are non-negative integers and $r > n$ notice that $\binom{n}{r} = \frac{n^{(r)}}{r!} = \frac{n(n-1)...(0)...}{r!} = 0$.

Problems:

3.2.1 A factory parking lot has 160 cars in it, of which 35 have faulty emission controls. An air quality inspector does spot checks on 8 cars on the lot.

(a) Give an expression for the probability that at least 3 of these 8 cars will have faulty emission controls.

(b) What assumption does your answer to (a) require? How likely is it that this assumption holds if the inspector hopes to catch as many cars with faulty controls as possible?

3.2.2 In a race, the 15 runners are randomly assigned the numbers $1, 2, \cdots, 15$. Find the probability that

(a) 4 of the first 6 finishers have single digit numbers.

(b) the fifth runner to finish is the 3rd finisher with a single digit number.

(c) number 13 is the highest number among the first 7 finishers.

3.2 Review of Useful Series and Sums

We will be making use the following series and sums.

1. **Geometric Series:**

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a\left(1 - r^n\right)}{1 - r} = \frac{a\left(r^n - 1\right)}{r - 1} \text{ for } r \neq 1$$

If $|r| < 1$, then

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r}$$

2. **Binomial Theorem:** There are various forms of this theorem. We will use the form

$$(1 + a)^n = 1 + \binom{n}{1}a^1 + \binom{n}{2}a^2 + ... + \binom{n}{n}a^n = \sum_{x=0}^{n}\binom{n}{x}a^x.$$

Justification: One way of verifying this formula uses the counting arguments of this chapter. Imagine a product of the individual terms:

$$(1 + a)\,(1 + a)\,(1 + a)\,...\,(1 + a)$$

To evaluate this product we must add together all of the possibilities obtained by taking one of the two possible terms from the first bracketed expression, i.e. one of $\{1, a)$, multiplying by one $\{1, a)$ taken from the second bracketed expression. etc. In how many ways do we obtain the term a^x where $x = 0, 1, 2, ..., n$? We might choose a from each of the first x terms above and then 1 from the remaining terms, or indeed we could choose a from any x of the terms in $\binom{n}{x}$ ways and then 1 from the remaining terms.

3. **Binomial Theorem:** There is a more general version of the binomial theorem that results in an infinite series and that holds when n is not a positive integer:

$$(1+a)^n = \sum_{x=0}^{\infty} \binom{n}{x} a^x \text{ if } |a| < 1.$$

Proof: Recall from Calculus the Maclaurin's series which says that a sufficiently smooth function $f(x)$ can be written as an infinite series using an expansion around $x = 0$,

$$f(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2!}x^2 + ...$$

provided that this series is convergent. In this case, with $f(a) = (1+a)^n$, $f(0) = 1, f'(0) = n, f''(0) = n(n-1)$ and $f^{(r)}(0) = n^{(r)}$. Substituting,

$$f(a) = 1 + \frac{n}{1}a + \frac{n(n-1)}{2!}a^2 + ... + \frac{n^{(r)}}{r!}a^r + ... = \sum_{x=0}^{\infty} \binom{n}{x} a^x$$

It is not hard to show that this converges whenever $|a| < 1$.

4. **Multinomial Theorem:** A generalization of the binomial theorem is

$$(a_1 + a_2 + \cdots + a_k)^n = \sum \frac{n!}{x_1!x_2!\cdots x_k!} a_1^{x_1} a_2^{x_2} \cdots a_k^{x_k}.$$

with the summation over all $x_1, x_2, \cdots, x_k$ with $\sum x_i = n$.

Justification: Again we could verify this formula uses the counting arguments. Imagine a product of the individual terms:

$$(a_1 + a_2 + \cdots + a_k)(a_1 + a_2 + \cdots + a_k)...(a_1 + a_2 + \cdots + a_k)$$

To evaluate this product we must add together all of the possibilities obtained by taking one of the terms from the first bracketed expression, i.e. one of $\{a_1, a_2, \cdots, a_k\}$, multiplying by one $\{a_1, a_2, \cdots, a_k\}$ taken from the second bracketed expression. etc. In how many ways do we obtain the term $a_1^{x_1} a_2^{x_2} \cdots a_k^{x_k}$ where $\sum x_i = n$? We can choose a_1 a total of $x_1$times from any of the n terms in $\binom{n}{x_1}$ ways, and then a_2 from any of the remaining $n - x_1$ terms in $\binom{n-x_1}{x_2}$ ways, and so on so there are

$$\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3}...\binom{x_k}{x_k} = \frac{n!}{x_1!x_2!\cdots x_k!}$$

ways or obtaining this term in the product. The case $k = 2$ gives the binomial theorem in the form

$$(a_1 + a_2)^n = \sum_{x_1=0}^{n} \binom{n}{x_1} a_1^{x_1} a_2^{n-x_1}$$

5. **Hypergeometric Identity:**

$$\sum_{x=0}^{\infty} \binom{a}{x}\binom{b}{n-x} = \binom{a+b}{n}.$$

There will not be an infinite number of terms if a and b are positive integers since the terms become 0 eventually. For example

$$\binom{4}{5} = \frac{4^{(5)}}{5!} = \frac{(4)(3)(2)(1)(0)}{5!} = 0$$

Proof: We prove this in the case that a and b are non-negative integers. Obviously

$$(1+y)^{a+b} = (1+y)^a \times (1+y)^b.$$

If we expand each term using the binomial theorem we obtain

$$\sum_{k=0}^{a+b} \binom{a+b}{k} y^k = \sum_{i=0}^{a} \binom{a+b}{i} y^i \times \sum_{j=0}^{b} \binom{b}{j} y^j.$$

Note that the coefficient of y^k on the right side is $\sum_{i=0}^{a} \binom{a}{i}\binom{b}{k-i}$ and so this must equal $\binom{a+b}{k}$, the coefficient of y^k on the left side.

6. **Exponential series:** This is another example of a Maclaurin series expansion, if we let $f(x) = e^x$, then $f^{(r)}(0) = 1$ and so

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We will also use the limit definition of the exponential function: for all real x,

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

7. **Special series involving integers:**

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example: Find

$$\sum_{x=0}^{\infty} x\,(x-1)\binom{a}{x}\binom{b}{n-x}$$

Solution: For $x = 0$ or 1 the term becomes 0, so we can start summing at $x = 2$. For $x \geq 2$, we can expand $x!$ as $x(x-1)(x-2)!$

$$\sum_{x=0}^{\infty} x(x-1)\binom{a}{x}\binom{b}{n-x} = \sum_{x=2}^{\infty} x(x-1)\frac{a!}{x(x-1)(x-2)!(a-x)!}\binom{b}{n-x}.$$

Cancel the $x(x-1)$ terms and try to re-group the factorial terms as "something choose something".

$$\frac{a!}{(x-2)!(a-x)!} = \frac{a(a-1)(a-2)!}{(x-2)!\,[(a-2)-(x-2)]!} = a(a-1)\binom{a-2}{x-2}.$$

Then

$$\sum_{x=0}^{\infty} x(x-1)\binom{a}{x}\binom{b}{n-x} = \sum_{x=2}^{\infty} a(a-1)\binom{a-2}{x-2}\binom{b}{n-x}.$$

Factor out $a(a-1)$ and let $y = x - 2$ to get

$$a(a-1)\sum_{y=0}^{\infty}\binom{a-2}{y}\binom{b}{n-(y+2)} = a(a-1)\binom{a+b-2}{n-2}$$

by the hypergeometric identity.

3.3 Problems on Chapter 3

3.1 Six digits from 2, 3, 4, ..., 8 are chosen and arranged in a row without replacement. Find the probability that

(a) the number is divisible by 2

(b) the digits 2 and 3 appear consecutively in the proper order (i.e. 23)

(c) digits 2 and 3 appear in the proper order but not consecutively.

3.2 Suppose r passengers get on an elevator at the basement floor. There are n floors above (numbered 1, 2, 3, ..., n) where passengers may get off.

(a) Find the probability

(i) no passenger gets off at floor 1

(ii) passengers all get off at different floors $(n \geq r)$.

(b) What assumption(s) underlies your answer to (a)? Comment briefly on how likely it is that the assumption(s) is valid.

3.3 There are 6 stops left on a subway line and 4 passengers on a train. Assume they are each equally likely to get off at any stop. What is the probability

(a) they all get off at different stops?

(b) 2 get off at one stop and 2 at another stop?

3.4 Give an expression for the probability a bridge hand of 13 cards contains 2 aces, 4 face cards (Jack, Queen or King) and 7 others. You might investigate the various permutations and combinations relating to card hands using the Java applet at $http://www.wcrl.ars.usda.gov/cec/java/comb.htm$

3.5 The letters of the word STATISTICS are arranged in a random order. Find the probability

(a) they spell statistics

(b) the same letter occurs at each end.

3.6 Three digits are chosen in order from 0, 1, 2, ..., 9. Find the probability the digits are drawn in increasing order; (i.e., the first $<$ the second $<$ the third) if

(a) draws are made without replacement

(b) draws are made with replacement.

3.7 **The Birthday Problem.** [5] Suppose there are r persons in a room. Ignoring February 29 and assuming that every person is equally likely to have been born on any of the 365 other days in a year, find the probability that no two persons in the room have the same birthday. Find the numerical value of this probability for $r = 20, 40$ and 60. There is a graphic Java applet for illustrating the frequency of common birthdays at http://www-stat.stanford.edu/%7Esusan/surprise/Birthday.html

3.8 You have n identical looking keys on a chain, and one opens your office door. If you try the keys in random order then

(a) what is the probability the k'th key opens the door?

(b) what is the probability one of the first two keys opens the door (assume $n \geq 3$)?

(c) Determine numerical values for the answer in part (b) for the cases $n = 3, 5, 7$.

3.9 From a set of $2n + 1$ consecutively numbered tickets, three are selected at random without replacement. Find the probability that the numbers of the tickets form an arithmetic progression. [The *order* in which the tickets are selected does *not* matter.]

3.10 The 10,000 tickets for a lottery are numbered 0000 to 9999. A four-digit winning number is drawn and a prize is paid on each ticket whose four-digit number is any *arrangement* of the number drawn. For instance, if winning number 0011 is drawn, prizes are paid on tickets numbered 0011, 0101, 0110, 1001, 1010, and 1100. A ticket costs \$1 and each prize is \$500.

(a) What is the probability of winning a prize (i) with ticket number 7337? (ii) with ticket number 7235? What advice would you give to someone buying a ticket for this lottery?

(b) Assuming that all tickets are sold, what is the probability that the operator will lose money on the lottery?

[5] " My birthday was a natural disaster, a shower of paper full of flattery under which one almost drowned" Albert Einstein, 1954 on his seventy-fifth birthday.

3.11 (a) There are 25 deer in a certain forested area, and 6 have been caught temporarily and tagged. Some time later, 5 deer are caught. Find the probability that 2 of them are tagged. (What assumption did you make to do this?)

(b) Suppose that the total number of deer in the area was unknown to you. Describe how you could estimate the number of deer based on the information that 6 deer were tagged earlier, and later when 5 deer are caught, 2 are found to be tagged. What estimate do you get?

3.12 **Lotto 6/49**. In Lotto 6/49 you purchase a lottery ticket with 6 different numbers, selected from the set $\{1, 2, ..., 49\}$. In the draw, six (different) numbers are randomly selected. Find the probability that

(a) Your ticket has the 6 numbers which are drawn. (This means you win the main Jackpot.)

(b) Your ticket matches exactly 5 of the 6 numbers drawn.

(c) Your ticket matches exactly 4 of the 6 numbers drawn.

(d) Your ticket matches exactly 3 of the 6 numbers drawn.

3.13 (**Texas Hold-em**) Texas Hold-em is a poker game in which players are each dealt two cards face down (called your hole or pocket cards), from a standard deck of 52 cards, followed by a round of betting, and then five cards are dealt face up on the table with various breaks to permit players to bet the farm. These are communal cards that anyone can use in combination with their two pocket cards to form a poker hand. Players can use any five of the face-up cards and their two cards to form a five card poker hand. Probability calculations for this game are not only required at the end, but also at intermediate steps and are quite complicated so that usually simulation is used to determine the odds that you will win given your current information, so consider a simple example. Suppose we were dealt 2 Jacks in the first round.

(a) What is the probability that the next three cards (face up) include at least one Jack?

(b) Given that there was no Jack among these next three cards, what is the probability that there is at least one among the last two cards dealt face-up?

(c) What is the probability that the 5 face-up cards show two Jacks, given that I have two in my pocket cards?

3.14 Show that

$$\sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = np$$

(use the binomial theorem).

3.15 I have a quarter which turns up heads with probability 0.6, and a fair dime. The quarter is flipped until a head occurs. Independently the dime is flipped until a head occurs. Find the probability that the number of flips is the same for both coins.

3.16 Some other summation formulas can be obtained by differentiating the above equations on both sides. Show that $a + 2ar + 3ar^2 + \cdots = \frac{a}{(1-r)^2}$ by starting with the geometric series formula. Assume $|r| < 1$.

3.17 Players A and B decide to play chess until one of them wins. Assume games are independent with $P(A$ wins) = .3, $P(B$ wins) = .25 and P(draw) = .45 on each game. If the game ends in a draw another game will be played. Find the probability A wins before B.

4. Probability Rules and Conditional Probability

4.1 General Methods

Recall that a probability model consists of a sample space S, a set of events or subsets of the sample space to which we can assign probabilities and a mechanism for assigning these probabilities. The probability of an arbitrary event A can be determined by summing the probabilities of simple events in A and so we have the following rules:

Rule 1 $P(S) = 1$
Proof: $P(S) = \sum_{a \in S} P(a) = \sum_{\text{all } a} P(a) = 1$

Rule 2 For any event A, $0 \leq P(A) \leq 1$.
Proof: $P(A) = \sum_{a \in A} P(a) \leq \sum_{a \in S} P(a) = 1$ and so since each $P(a) \geq 0$, we have $0 \leq P(A) \leq 1$.

Rule 3 If A and B are two events with $A \subseteq B$ (that is, all of the points in A are also in B), then $P(A) \leq P(B)$.
Proof: $P(A) = \sum_{a \in A} P(a) \leq \sum_{a \in B} P(a) = P(B)$ so $P(A) \leq P(B)$.

Before continuing with the set-theoretic description of a probability model, let us review some of the basic ideas in set theory. First what do sets have to do with the occurrence of events? Suppose a random experiment having sample space S is run (for example a die with $S = \{1, 2, 3, 4, 5, 6\}$ is thrown). When would we say an event $A \subset S$, or in the case of the die, the event $A = \{2, 4, 6\}$ occurs? In the latter case, the event A means that the number showing is even, i.e. in general that *one of the simple outcomes in A occurred.* We often illustrate the relationship among sets using *Venn diagrams.* In the drawings below, think of S consisting of all of the points in a rectangle of area one[6]. To illustrate

[6] As you may know, however, the number of points in a rectangle is NOT countable, so this is not a discrete sample space. Nevertheless this definition of S is used to illustrate various combinations of sets

the event A we can draw a region within the rectangle with area roughly proportional to the probability of the event A. We might think of the random experiment as throwing a dart at the rectangle in Figure 4.3, and we say the event A occurs if the dart lands within the region A.

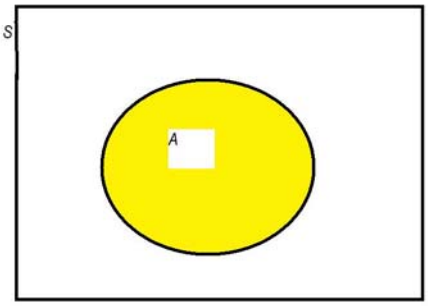


Figure 4.3: Set A in sample space S

What if we combine two events A, B by including all of the points in either A or B or both. This is the union of the two events or $A \cup B$ illustrated in Figure 4.4. The union of the events occurs if one

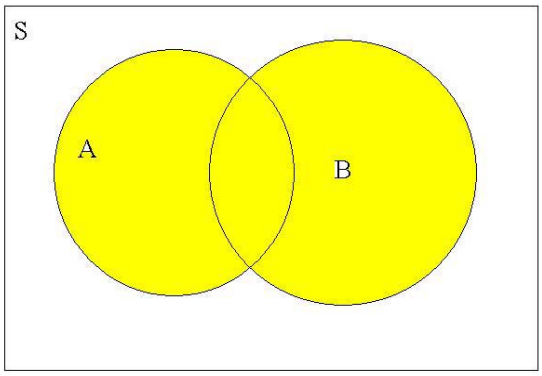


Figure 4.4: The union of two sets $A \cup B$

of the outcomes in either A or B or both occurs. In language we refer to this as the event "A or B" with the understanding that in this course we will use the word "or" inclusively to also permit both. Another way of expressing a union is $A \cup B$ occurs if at least one of A, B occurs. Similarly if we have three events $A, B, C,$ the event $A \cup B \cup C$ means "at least one of A, B, C".

What about the intersection of two events $(A \cap B)$ or the set of all points in S that are in both A **and** B? This is illustrated in Figure 4.5. The event $A \cap B$ occurs if and only if a point in the intersection occurs which means **both A and B occur**. It is common to shorten the notation for the intersection of

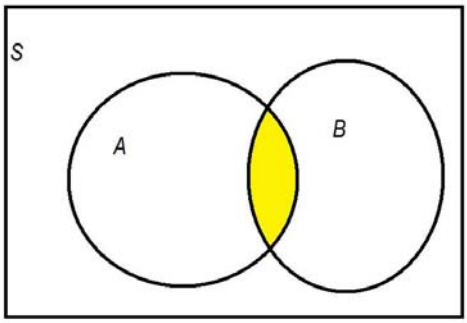


Figure 4.5: The intersection of two events $A \cap B$

two events so that AB means $A \cap B$ and ABC means $A \cap B \cap C$. Finally the complement of the event A is denoted $\bar{A}$ and means the set of all points which are in S but **not in** A as in Figure 4.6.

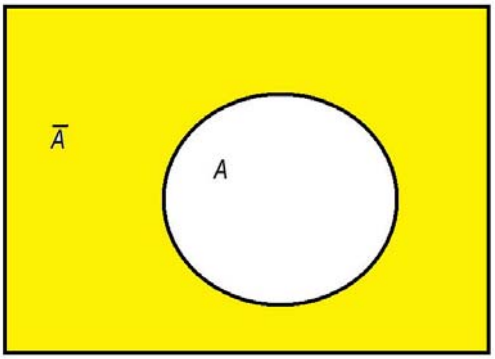


Figure 4.6: $\bar{A}$ =the complement of the event A

There are two special events in a probability model that we will use. One is the whole sample space S. Because $P(S) = 1$, this event is certain to occur. Another is the *empty event*, or the *null set* φ. This is a set with no elements at all and so it must have probability 0. Notice that $\varphi = \overline{S}$.

The illustrations above showing the relationship among sets are examples of Venn diagrams. At the URL http://stat-www.berkeley.edu/users/stark/Java/Venn.htm, there is an applet which allows you to vary the area of the intersection and construct Venn diagrams for a variety of purposes. Since probability theory is built from the relationships among sets, it is often helpful to use Venn diagrams in solving problems. For example there are rules (De Morgan‘s laws) governing taking the complements of unions and intersections that can easily be verified using Venn diagrams.

Exercise: Verify de Morgan‘s laws:

a. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

b. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof of a: One can argue such set theoretic rules using the definitions of the sets. For example when is a point a in the set $\overline{A \cup B}$. This means $a \in S$ but a is not in $A \cup B$, which in turn implies a is not in A **and** it is not in B, or $a \in \overline{A}$ **and** $a \in \overline{B}$,equivalently $a \in \overline{A} \cap \overline{B}$. As and alternative demonstration, we can use a Venn diagram (Figure 4.7) in which $\overline{A}$ is indicated with vertical lines, $\overline{B}$ with horizontal lines and so $\overline{A} \cap \overline{B}$ is the region with cross hatching. This agrees with the shaded region $\overline{A \cup B}$.

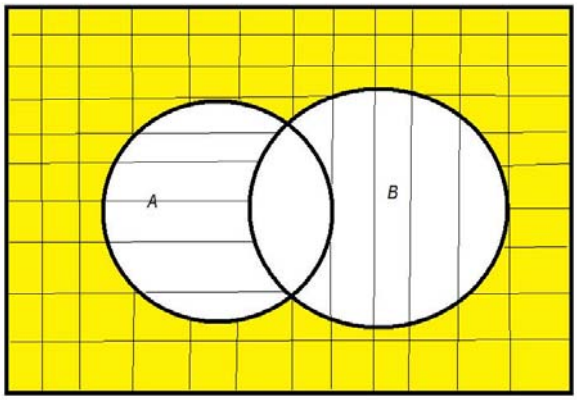


Figure 4.7: Illustration of De Morgan‘s law using a Venn diagram. The region indicated with vertical bars is $\overline{A}$ and with horizonal lines, $\overline{B}$, The shaded region, $\overline{A \cup B}$ is identical to $\overline{A} \cap \overline{B}$

The following example demonstrates solving a problem using a Venn diagram.

Example: Suppose for students finishing second year Math that 22% have a math average greater than 80%, 24% have a STAT 230 mark greater than 80%, 20% have an overall average greater than 80%, 14% have both a math average and STAT 230 greater than 80%, 13% have both an overall average and STAT 230 greater than 80%, 10% have all 3 of these averages greater than 80%, and 67% have none of these 3 averages greater than 80%. Find the probability a randomly chosen math student finishing 2A has math and overall averages both greater than 80% and STAT 230 less than or equal to 80%.

Solution: When using rules of probability it is generally helpful to begin by labeling the events of interest. Imagine a student is chosen at random from all students finishing second year Math. For this student, let

A be the event "math average greater than 80%"
B be the event "overall average greater than 80%"
C be the event "STAT 230 mark greater than 80%"

In terms of these symbols, we are given:

$$\begin{aligned} P(A) &= 0.22, & P(BC) &= 0.13, \\ P(B) &= 0.20, & P(ABC) &= 0.1, \\ P(C) &= 0.24, & P(\bar{A}\bar{B}\bar{C}) &= 0.67 \\ P(AC) &= 0.14, \end{aligned}$$

Let us interpret some of these expressions; for example $\bar{A}\bar{B}\bar{C}$ means $\bar{A} \cap \bar{B} \cap \bar{C}$ or (not A) **and** (not B) **and** (not C), or that none of the marks or averages are greater than 80% for the randomly chosen student. We are asked to find $P(AB\bar{C})$, the shaded region in Figure 4.8. Filling in this information on a Venn diagram, in the order indicated by (1), (2), (3), etc. below (and rather loosely identifying the area of a set with its probability)

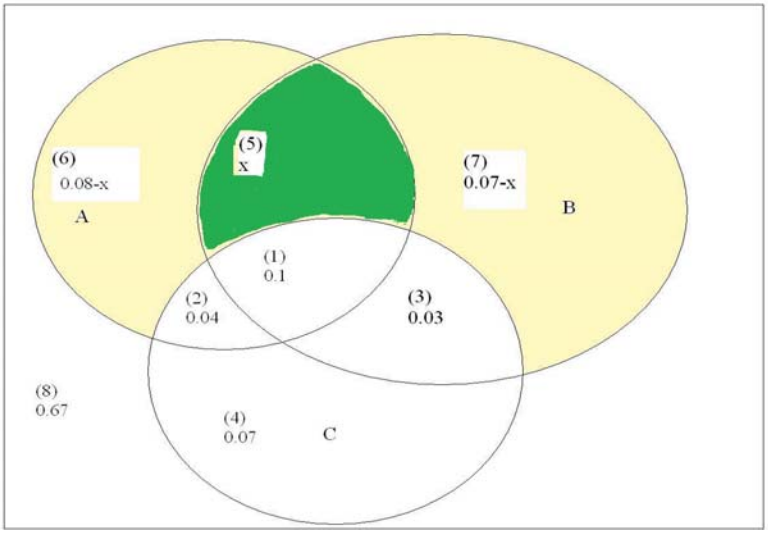


Figure 4.8: Venn Diagram for Math Averages Example

(1) $P(ABC)$ is given$= 0.1$
(2) $P(AC) - P(ABC) = 0.14 - 0.1 = 0.04$
(3) $P(BC) - P(ABC) = 0.13 - 0.1 = 0.03$
(4) $P(C) - P(AC) - 0.03 = 0.24 - 0.14 - 0.03 = 0.07$
(5) $P(AB\overline{C})$ is unknown, so let $P(AB\overline{C}) = x$
(6) $P(A) - P(AC) - P(AB\overline{C}) = 0.22 - 0.14 - x = 0.08 - x$
(7) $P(B) - P(BC) - P(AB\overline{C}) = 0.20 - 0.13 - x = 0.07 - x$
(8) $P(\overline{A \cup B \cup C}) = 0.67$ is given.

Adding all probabilities from (1) to (8) we obtain, since $P(S) = 1$,

$$0.1 + 0.04 + 0.03 + 0.07 + x + 0.08 - x + 0.07 - x + 0.67 = 1$$

giving $1.06 - x = 1$ and solving for x, $P(AB\bar{C}) = x = 0.06$.

Problems:

4.1.1 In a typical year, 20% of the days have a high temperature $> 22^o$C. On 40% of these days there is no rain. In the rest of the year, when the high temperature $\leq 22^o$C, 70% of the days have no rain. What percent of days in the year have rain and a high temperature $\leq 22^o$C?

4.1.2 According to a survey of people on the last Ontario voters list, 55% are female, 55% are politically to the right, and 15% are male and politically to the left. What percent are female and politically to the right? Assume voter attitudes are classified simply as left or right.

4.2 Rules for Unions of Events

In addition to the two rules which govern probabilities listed in Section 4.1, we have the following

Rule 4 a (*probability of unions*) $P(A \cup B) = P(A) + P(B) - P(AB)$

Proof: Suppose we denote set differences by $A/B = A \cap \overline{B}$, the set of points which are in A but not in B. Then

$$\begin{aligned} P(A) + P(B) &= \sum_{a \in A} P(a) + \sum_{a \in B} P(a) \\ &= \left(\sum_{a \in A/B} P(a) + \sum_{a \in AB} P(a) \right) + \left(\sum_{a \in B/A} P(a) + \sum_{a \in AB} P(a) \right) \\ &= \left(\sum_{a \in A/B} P(a) + \sum_{a \in AB} P(a) + \sum_{a \in B/A} P(a) \right) + \sum_{a \in AB} P(a) \\ &= \sum_{a \in A \cup B} P(a) + \sum_{a \in AB} P(a) \\ &= P(A \cup B) + P(AB) \end{aligned}$$

Subtracting $P(AB)$ we obtain $P(A \cup B) = P(A) + P(B) - P(AB)$ as required. This can also be justified by using a Venn diagram. Each point in $A \cup B$ must be counted once. In the expression $P(A) + P(B),$ however, points in AB have their probability counted twice - once in $P(A)$ and once in $P(B)$ - so they need to be subtracted once.

Rule 4 b (the probability of the union of three events) By a similar argument, we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \qquad (4.2)$$

(see Figure 4.9). The proof is similar. In the sum $P(A) + P(B) + P(C)$ those points in the regions labelled D, H, J in Figure 4.9 lie in only one of the events and their probabilities are added only once. However points in the regions labelled G, E, I, for example, lie in two of the events. We can compensate for this double-counting by subtracting these probabilities once, e.g. using $P(A)+P(B)+P(C) - [P(AB) + P(AC) + P(BC)]$. However, now those points in all three sets, i.e. those points in $F = ABC$ have their probabilities added in three times and then subtracted three times so they are not included at all: we must correct the formula to give (4.2).

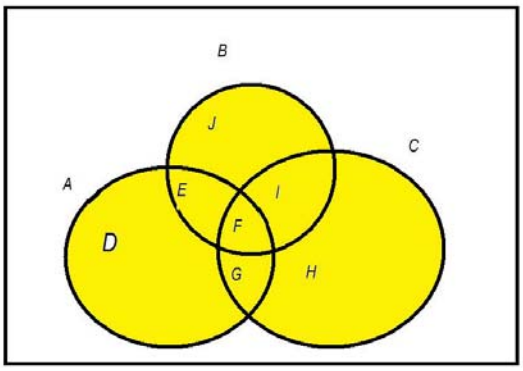


Figure 4.9: The union $A \cup B \cup C$

Rule 4 c There is an obvious generalization of the above formula to n events $A_1, ...A_n$. This is often referred to as the *inclusion-exclusion principle* because of the process discussed above for constructing it:

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \sum_{i<j<k<l} P(A_i A_j A_k A_l) + \cdots \tag{4.3}$$

(where the subscripts are all distinct, for example $i < j < k < l$).

Proof: This is easy to prove using rule 4a and induction. Let $B_n = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ for $n = 1, 2, ...$ Then 4a shows that (4.3) holds for $n = 2$. Suppose the rule is true for n. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \cup A_{n+1}) &= P(B_n \cup A_{n+1}) \\ &= P(B_n) + P(A_{n+1}) - P(B_n A_{n+1}) \\ &= \sum_{i \le n} P(A_i) - \sum_{i<j \le n} P(A_i A_j) + \sum_{i<j<k \le n} P(A_i A_j A_k) + ... + P(A_{n+1}) \\ &- \sum_{i \le n} P(A_i A_{n+1}) + \sum_{i<j \le n} P(A_i A_j A_{n+1}) - \sum_{i<j<k \le n} P(A_i A_j A_k A_{n+1}) + ... \end{aligned}$$

We will use (4.3) rarely in this course[7].

Definition 6 *Events A and B are **mutually exclusive** if $AB = \varphi$ (the empty event).*

[7] i.e. do not memorize

Since mutually exclusive events A and B have no common points, $P(AB) = P(\varphi) = 0$.

In general, events $A_1, A_2 \cdots, A_n$ are mutually exclusive if $A_i A_j = \varphi$ for all $i \neq j$. This means that there is no chance of two or more of these events occurring together, we either have exactly one of the events occur, or none. For example, if a die is rolled twice, the events

A is the event that 2 occurs on the first roll,

B is the event that the total is 10,

are mutually exclusive. Similarly the events $A_2, A_3, ...A_{12}$ where A_j is the event that the total on the two dice is j are all mutually exclusive events. In the case of mutually exclusive events, rule 4 above simplifies to rule 5 below.

Rule 5 a (*unions of mutually exclusive events*). Let A and B be mutually exclusive events. Then $P(A \cup B) = P(A) + P(B)$. This is a consequence or rule 4a and the fact that $P(AB) = P(\varphi) = 0$.

Rule 5 b In general, let $A_1, A_2, \cdots A_n$ be mutually exclusive events. Then $P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i)$.
This is easily proven from rule 5a above using induction or as an immediate consequence of 4c.

Rule 6 (*probability of complements*) $P(A) = 1 - P(\bar{A})$.

Proof: A and $\bar{A}$ are mutually exclusive and $A \cup \bar{A} = S,$ so by Rule 5a,

$$P(A \cup \bar{A}) = P(A) + P(\bar{A}).$$

But since $P(A \cup \bar{A}) = P(S) = 1$,

$$1 = P(A) + P(\bar{A}) \text{ or}$$
$$P(A) = 1 - P(\bar{A}).$$

This result is useful whenever $P(\bar{A})$ is easier to obtain than $P(A)$.

Example: Two ordinary dice are rolled. Find the probability that at least one of them turns up a six.

Solution 1: The sample space is $S = (1,1),(1,2),(1,3),...$Let A be the event that we obtain 6 on the first die , B be the event that we obtain 6 on the second die and note (by rule 4) that

$$\begin{aligned} P(\text{at least one die shows } 6) &= P(A \cup B) \\ &= P(A) + P(B) - P(AB) \\ &= \tfrac{1}{6} + \tfrac{1}{6} - \tfrac{1}{36} \\ &= \tfrac{11}{36} \end{aligned}$$

Solution 2: This is an example where it is perhaps somewhat easier to obtain the complement of the event $A \cup B$ since the complement is the event that there is no six showing on either die, and there are exactly 25 such points, $(1,1),...(1,5),(2,1),...(2,5),...(5,5)$. Therefore

$$\begin{aligned} P(\text{at least one die shows } 6) &= 1 - P(\text{no 6 on either die}) \\ &= 1 - \tfrac{25}{36} \\ &= \tfrac{11}{36} \end{aligned}$$

Example: Roll a die 3 times. Find the probability of getting at least one 6.

Solution 1: Let A be the event "least one die shows 6". Then $\bar{A}$ is the event that no 6 on any die shows. Using counting arguments, there are 6 outcomes on each roll, so $S = \{(1,1,1),(1,1,2)....(6,6,6)\}$ has $6 \times 6 \times 6 = 216$ points. For $\bar{A}$ to occur we can't have a 6 on any roll. Then $\bar{A}$ can occur in $5 \times 5 \times 5 = 125$ ways.

$$\text{Therefore } P(\bar{A}) = \frac{125}{216}. \quad \text{Hence } P(A) = 1 - \frac{125}{216} = \frac{91}{216}$$

Solution 2: Can you spot the flaw in the following argument? Let

A be the event that 6 occurs on the first roll

B be the event that 6 occurs on the second roll

C be the event that 6 occurs on the third roll

Then

$$\begin{aligned} P(\text{one or more six}) &= P(A \cup B \cup C) \\ &= P(A) + P(B) + P(C) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

You should have noticed that $A, B,$ and C are **not** mutually exclusive events, so we should have used

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

Each of $AB, AC,$ and BC occurs 6 times in the 216 point sample space and so $P(AB) = \frac{1}{36} = P(BC) = P(AC)$. Also $P(ABC) = \frac{1}{216}$

$$\text{Therefore } P(A \cup B \cup C) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{36} - \frac{1}{36} - \frac{1}{36} + \frac{1}{216} = \frac{91}{216}.$$

Note: Rules 3, 4, and (indirectly) 5 link the concepts of addition of probabilities with unions of events, and complements. The next segment will consider intersection, multiplication of probabilities, and a concept known as independence. Making these linkages will make problem solving and the construction of probability models easier.

Problems:

4.2.1 Let $A, B,$ and C be events for which

$$P(A) = 0.2, P(B) = 0.5, P(C) = 0.3 \text{ and } P(AB) = 0.1$$

(a) Find the largest possible value for $P(A \cup B \cup C)$

(b) For this largest value to occur, are the events A and C mutually exclusive, not mutually exclusive, or can this not be determined?

4.2.2 Prove that $P(A \cup B) = 1 - P(\overline{A}\,\overline{B})$ for arbitrary events A and B in S.

4.3 Intersections of Events and Independence

Dependent and Independent Events:

Consider the events A : airplane engine fails in flight and B : airplane reaches its destination safely. Do we normally consider these events as related or dependent in some way. Certainly if a Canada Goose is sucked into one jet engine, that effects the probability that the airplane safely reaches its destination, i.e. it effects the probability that should be assigned to the event B. Suppose we toss a fair coin twice. What about the two events $A : H$ is obtained on first toss and $B : H$ is obtained on both tosses. Again there appears to be some dependence. On the other hand if we replace B by $B : H$ is obtained on second toss, we do not think that the occurrence of A affects the chances that B will occur. When we should reassess the probability of one event B given that the event A occurred we call a pair of events *dependent*, and otherwise we call them *independent*. We formalize this concept in the following mathematical definition.

Definition 7 *Events A and B are **independent** if and only if $P(AB) = P(A)P(B)$. If they are not independent, we call the events **dependent**.*

When we used Venn diagrams, we imagined that the probability of events was roughly proportional to their area. This is justified in part because area and probability are to examples of "measures" in mathematics and share much the same properties. Let us continue this tradition, so that in the figure below, the probability of events is represented by the area of the corresponding region. Then if two events are independent, the "size" of their intersection as measured by the probability measure is required to be the product of the individual probabilities. This means, of course, that the intersection must be non-empty, and so the events are not mutually exclusive[8]. For example in the Venn diagram depicted in Figure 4.10, $P(A) = 0.3, P(B) = 0.4$ and $P(AB) = 0.12$ so in this case the two events are independent. If you were to hold the rectangle A in place and move the rectangle B down and to the right, the probability of the intersection as represented by the area would decrease and the events would become dependent.

For another example, suppose we toss a fair coin twice. Let A = {head on 1st toss} and B = {head on 2nd toss}. Clearly A and B are independent since the outcome on each toss is unrelated to other tosses, so $P(A) = \frac{1}{2}, P(B) = \frac{1}{2}, P(AB) = \frac{1}{4} = P(A)P(B)$.

[8]Can you think of a pair of events that are both independent and mutually exclusive? Suppose $P(A) = 0.5$ and B is an event such that $A \cap B = \varphi$ and $P(B) = 0$. Then $P(A)P(B) = 0 = P(A \cap B)$ so this pair of events is independent. Does this make sense to you?

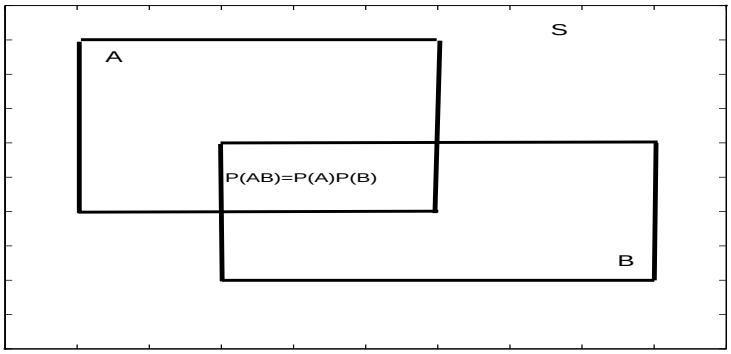


Figure 4.10: Suppose that the probability of a region is equal to its area (so that the aread of S is 1). Then this illustrates *Independent* events A, B

However, if we roll a die once and let A = {the number is even} and B = {number > 3} the events will be dependent since

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{2}, P(AB) = P(4 \text{ or } 6 \text{ occurs}) = \frac{2}{6} \neq P(A)P(B).$$

(Rationale: B only happens half the time. If A occurs we know the number is 2, 4, or 6. So B occurs $\frac{2}{3}$ of the time when A occurs. The occurrence of A does affect the chances of B occurring so A and B are <u>not</u> independent.)

When there are more than 2 events, the above definition generalizes to:

Definition 8 *The events $A_1, A_2, \cdots, A_n$ are independent if and only if*

$$P(A_{i_1}, A_{i_2}, \cdots, A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

for all sets $(i_1, i_2, \cdots, i_k)$ of distinct subscripts chosen from $(1, 2, \cdots, n)$[9]

For example, for $n = 3$, we need

$$\begin{aligned} P(A_1A_2) &= P(A_1)P(A_2), \\ P(A_1A_3) &= P(A_1)P(A_3), \\ P(A_2A_3) &= P(A_2)P(A_3) \end{aligned}$$

[9]We need all subsets so that events are independent of combinations of other events. For example if A_1 is independent of A_2 and A_4 is to be independent of A_1A_2 then, $P(A_1A_2A_4) = P(A_1A_2)P(A_4) = P(A_1)P(A_2)P(A_4)$

and

$$P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3)$$

Technically, we have defined "mutually independent" events, but we will shorten the name to "independent" to reduce confusion with "mutually exclusive."
The definition of independence works two ways. If we can find $P(A), P(B)$, and $P(AB)$ then we can determine whether A and B are independent. Conversely, if we know (or assume) that A and B are independent, then we can use the definition as a rule of probability to calculate $P(AB)$. Examples of each follow.

Example: Toss a die twice. Let A be the event that the first toss is a 3 and B the event that the total is 7. Are A and B independent? (What do you think?) Using the definition to check, we get $P(A) = \frac{1}{6}, P(B) = \frac{6}{36}$ (points (1,6), (2,5), (3,4), (4,3), (5,2) and (6,1) give a total of 7) and $P(AB) = \frac{1}{36}$ (only the point (3,4) makes AB occur).
Therefore, $P(AB) = P(A)P(B)$ and so A and B are independent events.
Now suppose we define C to be the event that the total is 8. This is a minor change from the definition of B.
Then

$$P(A) = \frac{1}{6}, \;\; P(C) = \frac{5}{36} \;\text{ and }\; P(AC) = \frac{1}{36}$$

Therefore $P(AC) \neq P(A)P(C)$

and consequently A and C are dependent events.

This example often puzzles students. Why are they independent if B is a total of 7 but dependent for C: total is 8? The key is that regardless of the first toss, there is always one number on the 2nd toss which makes the total 7. Since the probability of getting a total of 7 started off being $\frac{6}{36} = \frac{1}{6}$, the outcome of the 1st toss doesn't affect the chances. However, for any total other than 7, the outcome of the 1st toss does affect the chances of getting that total (e.g., a first toss of 1 guarantees the total cannot be 8)[10].

Example: A random number generator on the computer can give a sequence of independent random digits chosen from $S = \{0, 1, \ldots, 9\}$. This means that (i) each digit has probability of $\frac{1}{10}$ of being any of $0, 1, \ldots, 9$, and (ii) events determined by the different trials are independent of one another. We call this an "experiment with independent trials". Determine the probability that

[10]This argument is in terms of "conditional probability" closely related to independence and to be treated in the next section.

(a) in a sequence of 5 trials, all the digits generated are odd

(b) the number 9 occurs for the first time on trial 10.

Solution:

(a) Define the events A_i: digit from trial i is odd, $i = 1, \ldots, 5$.
Then

$$\begin{aligned} P(\text{all digits are odd}) &= P(A_1 A_2 A_3 A_4 A_5) \\ &= \prod_{i=1}^{5} P(A_i), \end{aligned}$$

since the A_i's are mutually independent. Since $P(A_i) = \frac{1}{2}$, we get P(all digits are odd)$= \frac{1}{2^5}$.

(b) Define events A_i: 9 occurs on trial i, for $i = 1, 2, \ldots$. Then we want

$$\begin{aligned} P(\bar{A}_1 \bar{A}_2 \ldots \bar{A}_9 A_{10}) &= P(\bar{A}_1) P(\bar{A}_2) \ldots P(\bar{A}_9) P(A_{10}) \\ &= (.9)^9 (.1), \end{aligned}$$

because the A_i's are independent, and $P(A_i) = 1 - P(\bar{A}_i) = 0.1$.

Note: We have used the fact here that if A and B are independent events, then so are $\bar{A}$ and B. To see this note that

$$B = AB \cup \bar{A}B \text{ where } AB \text{ and } \bar{A}B \text{ are mutually exclusive events, so}$$
$$P(B) = P(AB) + P(\bar{A}B).$$

Therefore

$$\begin{aligned} P(\bar{A}B) &= P(B) - P(AB) \\ &= P(B) - P(A)P(B) \text{ (since } A \text{ and } B \text{ are independent)} \\ &= (1 - P(A))P(B) \\ &= P(\bar{A})P(B). \end{aligned}$$

Note: We have implicitly assumed independence of events by using the discrete uniform model some of our earlier probability calculations. For example, suppose a coin is tossed 3 times, and we consider the sample space

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Assuming that the outcomes on the three tosses are independent, and that

$$P(H) = P(T) = \frac{1}{2}$$

on any single toss, we get that

$$P(HHH) = P(H)P(H)P(H) = (\frac{1}{2})^3 = \frac{1}{8}.$$

Similarly, all the other simple events have probability $\frac{1}{8}$. In earlier calculations we implicitly assumed this was true by assigning the same probability $\frac{1}{8}$ to all possible outcomes without thinking directly about independence. However, it is clear that if somehow the 3 tosses were not independent then it might be a bad idea to assume each outcome had probability $\frac{1}{8}$. (For example, instead of heads and tails, suppose H stands for "rain" and T stands for "no rain" on a given day; now consider 3 consecutive days. Would you want to assign a probability of $\frac{1}{8}$ to each of the 8 simple events even if this were in a season when the probability of rain on a day was $\frac{1}{2}$?)

Note: The definition of independent events can thus be used either to check for independence or, if events are known to be independent, to calculate $P(AB)$. Many problems are not obvious, and scientific study is needed to determine if two events are independent. For example, are the events A and B independent if, for a random child living in a country, the events are defined as A: the child lives within 5 Kim. of a nuclear power plant and B: the child has leukemia? Determining whether such events are dependent and if so the extent of the dependence are problems of substantial importance, and can be handled by methods in later statistics courses.

Problems:

4.3.1 A weighted die is such that $P(1) = P(2) = P(3) = 0.1, P(4) = P(5) = 0.2$, and $P(6) = 0.3$. Assume that events determined by different throws of the die are independent.

(a) If the die is thrown twice what is the probability the total is 9?

(b) If a die is thrown twice, and this process repeated 4 times, what is the probability the total will be 9 on exactly 1 of the 4 repetitions?

4.3.2 Suppose among UW students that 15% speaks French and 45% are women. Suppose also that 20% of the women speak French. A committee of 10 students is formed by randomly selecting from UW students. What is the probability there will be at least 1 woman and at least 1 French speaking student on the committee[11]?

4.3.3 Prove that $\overline{A}$ and $\overline{B}$ are independent events if and only if $\overline{A}$ and B are independent.

[11] Although the sampling is conducted without replacement, because the population is very large, whether we replace or

4.4 Conditional Probability

In many situations we may want to determine the probability of some event A, while knowing that some other event B has already occurred. For example, what is the probability a randomly selected person is over 6 feet tall, given that she is female? Let the symbol $P(A|B)$ represent the probability that event A occurs, when we know that B occurs. We call this the conditional probability of A given B. While we will give a definition of $P(A|B)$, let's first consider an example we looked at earlier, to get some sense of why $P(A|B)$ is defined as it is.

Example: Suppose we roll a die once so that sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Let A be the event that the number is even and B the event that the number is greater than 3. If we know that B occurs, that tells us that we have a 4, 5, or 6. Of the times when B occurs, we have an even number $\frac{2}{3}$ of the time. So $P(A|B) = \frac{2}{3}$. More formally, we could obtain this result by calculating $\frac{P(AB)}{P(B)}$, since $P(AB) = P(4\ or\ 6) = \frac{2}{6}$ and $P(B) = \frac{3}{6}$.

Definition 9 *the conditional probability of event A, given event B, is*

$$P(A|B) = \frac{P(AB)}{P(B)}, \textit{ provided } P(B) \neq 0.$$

Note: If A and B are independent,

$$P(AB) = P(A)P(B) \text{ so}$$
$$P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A).$$

This can be taken as an equivalent definition of independence; that is, A and B are independent iff $P(A|B) = P(A)$. We did not use this definition simply because it does not apply in the case that $P(B) = 0$. You should investigate the behaviour of the conditional probabilities as we move the events around on the web-site http://stat-www.berkeley.edu/%7Estark/Java/Venn3.htm.

Example: If a fair coin is tossed 3 times, find the probability that if at least 1 Head occurs, then exactly 1 Head occurs.

Solution: The sample space is $S = \{HHH, HHT, HTH, ...\}$. Define the events $A:$ we obtain1 Head, and $B:$ we obtain at least 1 Head. What we are being asked to find is $P(A|B)$. This equals $P(AB)/P(B)$, and so we find

$$P(B) = 1 - P(0 \text{ heads}) = \frac{7}{8}$$

not will make little difference. Therefore assume in your calculations that sampling is *with replacement* so the 10 draws are independent.

and

$$\begin{aligned} P(AB) &= P(\text{we obtain one head AND we obtain at least one head}) \\ &= P(\text{we obtain one head}) \\ &= P(\{HTT, THT, TTH\}) \\ &= \frac{3}{8} \end{aligned}$$

using either the sample space with equally probably points, or the fact that the 3 tosses are independent. Thus,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{3}{8}}{\frac{7}{8}} = \frac{3}{7}.$$

Example: The probability a randomly selected male is colour-blind is .05, whereas the probability a female is colour-blind is only .0025. If the population is 50% male, what is the fraction that is colour-blind?

Solution: Let$C$ be the event that the person selected is colour-blind, M the event that the person selected is male and $F = \overline{M}$ the event that the person selected is female. We are asked to find $P(C)$. We are told that

$$\begin{aligned} P(C|M) &= 0.05, \\ P(C|F) &= 0.0025, \quad \text{and} \\ P(M) &= 0.5 = P(F). \end{aligned}$$

Note that from the definition of conditional probability

$$P(C|M)P(M) = \frac{P(CM)}{P(M)}P(M) = P(CM) \text{ and similarly } P(C|F)P(F) = P(CF).$$

To get $P(C)$ we can therefore use the fact that

$$\begin{aligned} C &= CM \cup C\overline{M} \text{ and the events } CM \text{ and } C\overline{M} \text{ are mutually exclusive so} \\ P(C) &= P(CM) + P(CF) \\ &= P(C|M)P(M) + P(C|F)P(F) \\ &= (0.05)(0.5) + (0.0025)(0.5) \\ &= 0.02625. \end{aligned}$$

4.5 Multiplication and Partition Rules

The preceding example suggests two more useful probability rules. They are based on breaking events of interest into mutually exclusive pieces.

Rule 6. **Product rules.** *Let $A, B, C, D, \ldots$ be arbitrary events in a sample space. Assume that* $P(A) > 0, P(AB) > 0,$ and $P(ABC) > 0$. *Then*

$$\begin{aligned} P(AB) &= P(A)P(B|A) \\ P(ABC) &= P(A)P(B|A)P(C|AB) \\ P(ABCD) &= P(A)P(B|A)P(C|AB)P(D|ABC) \end{aligned}$$

and so on.

Proof:

The first rule comes directly from the definition $P(B|A)$ since

$$P(A)P(B|A) = P(A)\frac{P(AB)}{P(A)} = P(AB) \text{ assuming } P(A) > 0.$$

The right hand side of the second rule equals (assuming $P(AB) > 0$ and $P(A) > 0$)

$$\begin{aligned} P(A)P(B|A)P(C|AB) &= P(A)\frac{P(AB)}{P(A)}P(C|AB) \\ &= P(AB)P(C|AB) \\ &= P(AB)\frac{P(CAB)}{P(AB)} \\ &= P(ABC), \end{aligned}$$

and so on.

In order to remember these rules you can imagine that the events unfold in some chronological order, even if they do not. For example $P(ABCD) = P(A)P(B|A)P(C|AB)P(D|ABC)$ could be interpreted as the probability that “A occurs” (first) and then "given A occurs, that B occurs" (next), etc.

Partition Rule. *Let $A_1, \ldots, A_k$ be a partition of the sample space S into disjoint (mutually exclusive) events, that is*

$$A_1 \cup A_2 \cup \cdots \cup A_k = S. \text{ and } A_i \cap A_j = \varphi \text{ if } i \neq j$$

Let B be an arbitrary event in S. Then

$$\begin{aligned} P(B) &= P(BA_1) + P(BA_2) + \cdots + P(BA_k) \\ &= \sum_{i=1}^{k} P(B|A_i)P(A_i) \end{aligned}$$

Proof: Note that the events $BA_1, \ldots BA_k$ are all mutually exclusive and their union is B, that is $B = (BA_1) \cup \cdots \cup (BA_k)$. Therefore $P(B) = P(BA_1) + P(BA_2) + \cdots + P(BA_k)$. By the product rule, $P(BA_i) = P(B|A_i)P(A_i)$ so this becomes $P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_k)P(A_k)$.

Example: In an insurance portfolio 10% of the policy holders are in Class A_1 (high risk), 40% are in Class A_2 (medium risk), and 50% are in Class A_3 (low risk). The probability there is a claim on a Class A_1 policy in a given year is .10; similar probabilities for Classes A_2 and A_3 are .05 and .02. Find the probability that if a claim is made, it is made on a Class A_1 policy.

Solution: For a randomly selected policy, let

B = {policy has a claim }

A_i = {policy is of Class A_i}, $i = 1, 2, 3$

We are asked to find $P(A_1|B)$. Note that

$$P(A_1|B) = \frac{P(A_1B)}{P(B)}$$

and that

$$P(B) = P(A_1B) + P(A_2B) + P(A_3B).$$

We are told that

$$P(A_1) = 0.10, P(A_2) = 0.40, P(A_3) = 0.50$$

and that

$$P(B|A_1) = 0.10, P(B|A_2) = 0.05, P(B|A_3) = 0.02.$$

Thus

$$\begin{aligned} P(A_1B) = P(A_1)P(B|A_1) &= .01 \\ P(A_2B) = P(A_2)P(B|A_2) &= .02 \\ P(A_3B) = P(A_3)P(B|A_3) &= .01 \end{aligned}$$

Therefore $P(B) = .04$ and $P(A_1|B) = .01/.04 = .25$.

Tree Diagrams

Tree diagrams can be a useful device for keeping track of conditional probabilities when using multiplication and partition rules. The idea is to draw a tree where each path represents a sequence of events. On any given branch of the tree we write the conditional probability of that event given all the events on branches leading to it. The probability at any node of the tree is obtained by multiplying the

probabilities on the branches leading to the node, and equals the probability of the intersection of the events leading to it.

For example, the immediately preceding example could be represented by the tree in Figure 4.11. Note that the probabilities on the terminal nodes must add up to 1.

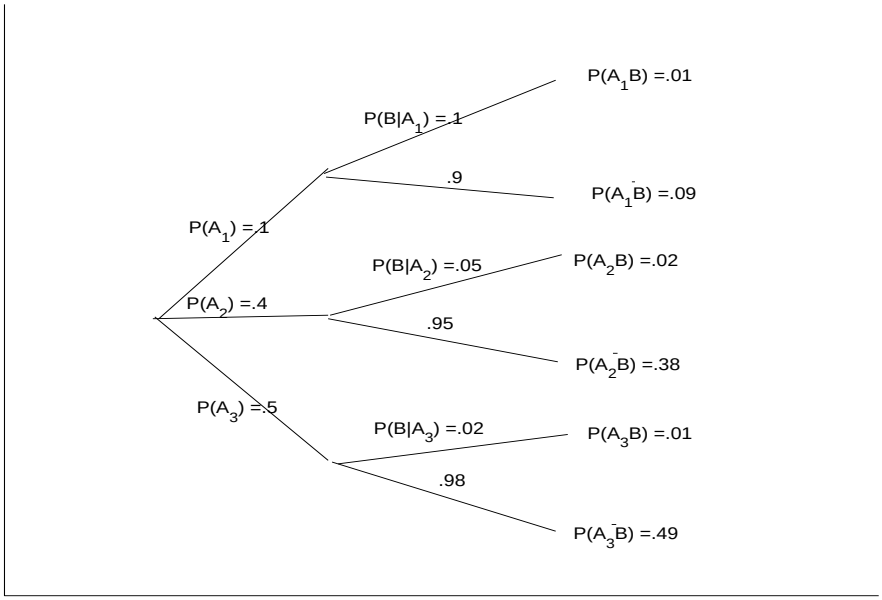


Figure 4.11:

Here is another example involving diagnostic tests for disease. See if you can represent the problem by a tree.

Example. Testing for HIV

Tests used to diagnose medical conditions are often imperfect, and give false positive or false negative results, as described in Problem 2.6 of Chapter 2. A fairly cheap blood test for the Human Immunodeficiency Virus (HIV) that causes AIDS (Acquired Immune Deficiency Syndrome) has the following characteristics: the false negative rate is 2% and the false positive rate is 0.5%. It is assumed that around .04% of Canadian males are infected with HIV.

Find the probability that if a male tests positive for HIV, he actually has HIV.

Solution: Suppose a male is randomly selected from the population, and define the events

A = {selected male has HIV}

B = {blood test is positive}

We are asked to find $P(A|B)$. From the information given we know that

$$P(A) = .0004, \quad P(\bar{A}) = .9996$$
$$P(B|A) = .98, \quad P(B|\bar{A}) = .005$$

Therefore we can find

$$\begin{array}{rrcll} & P(AB) & = & P(A)P(B|A) & = & .000392 \\ & P(\bar{A}B) & = & P(\bar{A})P(B|\bar{A}) & = & .004998 \\ \text{Therefore} & P(B) & = & P(AB) + P(\bar{A}B) & = & .00539 \end{array}$$

and

$$P(A|B) = \frac{P(AB)}{P(B)} = .0727$$

Thus, if a randomly selected male tests positive, there is still only a small probability (.0727) that they actually have HIV!

Exercise: Try to explain in ordinary words why this is the case.

Note: Bayes Theorem. Bayes theorem allows us to write conditional probabilities in terms of similar conditional probabilities but with the order of conditioning reversed:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|\overline{A})P(\overline{A}) + P(B|A)P(A)}$$

The proof of this result is simple since using the product rule,

$$\frac{P(B|A)P(A)}{P(B|\overline{A})P(\overline{A}) + P(B|A)P(A)} = \frac{P(AB)}{P(\overline{A}B) + P(AB)} = \frac{P(AB)}{P(B)} \text{ by the partition rule}$$
$$= P(A|B)$$

This result is called Bayes Theorem, after a mathematician[12] who proved it in the 1700's. It is a simple theorem, but it has inspired approaches to problems in statistics and other areas such as machine learning, classification and pattern recognition. In these areas the term "Bayesian methods" is often used.

Problems:

4.4.1 If you take a bus to work in the morning there is a 20% chance you'll arrive late. When you go by bicycle there is a 10% chance you'll be late. 70% of the time you go by bike, and 30% by bus. Given that you arrive late, what is the probability you took the bus?

[12](Rev) Thomas Bayes (1702-1761) was an English Nonconformist minister, turned Presbyterian. He may have been tutored by De Moivre. His famous paper introducing this rule was published after his death. "Bayesians" are statisticians who opt for a purely probabilistic view of inference. All unknowns obtain from some distribution and ultimately, the distribution says it all.

4.4.2 A box contains 4 coins – 3 fair coins and 1 biased coin for which P(heads) = .8. A coin is picked at random and tossed 6 times. It shows 5 heads. Find the probability this coin is fair.

4.4.3 At a police spot check, 10% of cars stopped have defective headlights and a faulty muffler. 15% have defective headlights and a muffler which is satisfactory. If a car which is stopped has defective headlights, what is the probability that the muffler is also faulty?

4.6 Problems on Chapter 4

4.1 If A and B are mutually exclusive events with $P(A) = 0.25$ and $P(B) = 0.4$, find the probability of each of the following events:

$$\bar{A};\ \ \bar{B};\ \ A \cup B;\ \ A \cap B;\ \ \bar{A} \cup \bar{B};\ \ \bar{A} \cap \bar{B};\ \ \overline{A \cap B}.$$

4.2 Three digits are chosen at random with replacement from $0, 1, ..., 9$; find the probability of each of the following events.

C: "the digits are all nonzero";
A: "all three digits are the same"; D: "the digits all exceed 4";
B: "all three digits are different"; E "digits all have the same parity (all odd or all even)".

Then find the probability of each of the following events, which are combinations of the previous five events:

$$BE;\ \ B \cup D;\ \ B \cup D \cup E;\ \ (A \cup B)D;\ \ A \cup (BD).$$

Show the last two of these events in Venn diagrams.

4.3 Let A and B be events defined on the same sample space, with $P(A) = 0.3$, $P(B) = 0.4$ and $P(A|B) = 0.5$. Given that event B does not occur, what is the probability of event A?

4.4 A die is loaded to give the probabilities:

number	1	2	3	4	5	6
probability	.3	.1	.15	.15	.15	.15

The die is thrown 8 times. Events determined by different throws of the die are assumed independent. Find the probability

(a) 1 does not occur

(b) 2 does not occur

(c) neither 1 nor 2 occurs

(d) both 1 and 2 occur.

4.5 Events A and B are independent with $P(A) = .3$ and $P(B) = .2$. Find $P(A \cup B)$.

4.6 Students A, B and C each independently answer a question on a test. The probability of getting the correct answer is .9 for A, .7 for B and .4 for C. If 2 of them get the correct answer, what is the probability C was the one with the wrong answer?

4.7 Customers at a store independently decide whether to pay by credit card or with cash. Suppose the probability is 70% that a customer pays by credit card. Find the probability

(a) 3 out of 5 customers pay by credit card

(b) the 5th customer is the 3rd one to pay by credit card.

4.8 Let E and F be independent with $E = A \cup B$ and $F = AB$. Prove that either $P(AB) = 0$ or else $P\left(\overline{A}\,\overline{B}\right) = 0$.

4.9 In a large population, people are one of 3 genetic types A, B and C: 30% are type A, 60% type B and 10% type C. The probability a person carries another gene making them susceptible for a disease is .05 for A, .04 for B and .02 for C. If ten unrelated persons are selected, what is the probability at least one is susceptible for the disease?

4.10 Two baseball teams play a best-of-seven series, in which the series ends as soon as one team wins four games. The first two games are to be played on A's field, the next three games on B's field, and the last two on A's field. The probability that A wins a game is 0.7 at home and 0.5 away. Assume that the results of the games are independent. Find the probability that:

(a) A wins the series in 4 games; in 5 games;

(b) the series does not go to 6 games.

4.11 A population consists of F females and M males; the population includes f female smokers and m male smokers. An individual is chosen at random from the population. If A is the event that this individual is female and B is the event he or she is a smoker, find necessary and sufficient conditions on f, m, F and M so that A and B are independent events.

4.12 An experiment has three possible outcomes A, B and C with respective probabilities p, q and r, where $p + q + r = 1$. The experiment is repeated until either outcome A or outcome B occurs. Show that A occurs before B with probability $p/(p + q)$.

4.13 In the game of craps, a player rolls two dice. They win at once if the total is 7 or 11, and lose at once if the total is 2, 3, or 12. Otherwise, they continue rolling the dice until they either win by throwing their initial total again, or lose by rolling 7.
Show that the probability they win is 0.493.
(Hint: You can use the result of Problem 4.12)

4.14 A researcher wishes to estimate the proportion p of university students who have cheated on an examination. The researcher prepares a box containing 100 cards, 20 of which contain Question A and 80 Question B.

Question A: Were you born in July or August?

Question B: Have you ever cheated on an examination?

Each student who is interviewed draws a card at random with replacement from the box and answers the question it contains. Since only the student knows which question he or she is answering, confidentiality is assured and so the researcher hopes that the answers will be truthful[13]. It is known that one-sixth of birthdays fall in July or August.

(a) What is the probability that a student answers 'yes'?

(b) If x of n students answer 'yes', estimate p.

(c) What proportion of the students who answer 'yes' are responding to Question B?

4.15 **Diagnostic tests.** Recall the discussion of diagnostic tests in Problem 2.6 for Chapter 2. For a randomly selected person let $D =$ 'person has the disease' and $R =$ 'the test result is positive'. Give estimates of the following probabilities: $P(R|D)$, $P(R|\bar{D})$, $P(R)$.

4.16 **Slot machines.** Standard slot machines have three wheels, each marked with some number of symbols at equally spaced positions around the wheel. For this problem suppose there are 10 positions on each wheel, with three different types of symbols being used: flower, dog, and house. The three wheels spin independently and each has probability 0.1 of landing at any position. Each of the symbols (flower, dog, house) is used in a total of 10 positions across the three wheels. A payout occurs whenever all three symbols showing are the same.

(a) If wheels 1, 2, 3 have 2, 6, and 2 flowers, respectively, what is the probability all three positions show a flower?

(b) In order to minimize the probability of all three positions showing a flower, what number of flowers should go on wheels 1, 2 and 3? Assume that each wheel must have at least one flower.

4.17 **Spam detection 1**. Many methods of spam detection are based on words or features that appear much more frequently in spam than in regular email. Conditional probability methods are then used to decide whether an email is spam or not. For example, suppose we define the following events associated with a random email message.

Spam	=	"Message is spam"
Not Spam	=	"Message is not spam ("regular")"
A	=	"Message contains the word Viagra"

If we know the values of the probabilities P(Spam), $P(A|$ Spam) and $P(A|$ Not Spam), then we can find the probabilities P(Spam$|A)$ and P(Not Spam$|A)$.

[13] "A foolish faith in authority is the worst enemy of truth" Albert Einsten, 1901.

(a) From a study of email messages coming into a certain system it is estimated that P(Spam) = .5, $P(A|\text{Spam})$ = .2, and $P(A|\text{Not Spam})$ = .001. Find $P(\text{Spam}|A)$ and $P(\text{Not Spam}|A)$.

(b) If you declared that any email containing the word Viagra was Spam, then find what fraction of regular emails would be incorrectly identified as Spam.

4.18 **Spam detection 2.** The method in part (b) of the preceding question would only filter out 20% of Spam messages. (Why?) To increase the probability of detecting spam, we can use a larger set of email "features"; these could be words or other features of a message which tend to occur with much different probabilities in spam and in regular email. (From your experience, what might be some useful features?) Suppose we identify n binary features, and define events

A_i = feature i appears in a message.

We will assume that $A_1, \ldots, A_n$ are independent events, given that a message is spam, and that they are also independent events, given that a message is regular.

Suppose $n = 3$ and that

$P(A_1|\text{Spam})$ = .2 $\quad$ $P(A_1|\text{Not Spam})$ = .005
$P(A_2|\text{Spam})$ = .1 $\quad$ $P(A_2|\text{Not Spam})$ = .004
$P(A_3|\text{Spam})$ = .1 $\quad$ $P(A_3|\text{Not Spam})$ = .005

Assume as in the preceding question that P(Spam) = .5.

(a) Suppose a message has all of features 1, 2, and 3 present. Determine $P(\text{Spam}\,|A_1A_2A_3)$.

(b) Suppose a message has features 1 and 2 present, but feature 3 is not present. Determine $P(\text{Spam}\,|A_1A_2\bar{A}_3)$.

(c) If you declared as spam any message with one or more of features 1, 2 or 3 present, what fraction of spam emails would you detect?

4.19 **Online fraud detection.** Methods like those in problems 4.17 and 4.18 are also used in monitoring events such as credit card transactions for potential fraud. Unlike the case of spam email, however, the fraction of transactions that are fraudulent is usually very small. What we hope to do in this case is to "flag" certain transactions so that they can be checked for potential fraud, and perhaps to block (deny) certain transactions. This is done by identifying features of a transaction so that if F = "transaction is fraudulent", then

$$r = \frac{P(\text{feature present}|F)}{P(\text{feature present}|\bar{F})}$$

is large.

(a) Suppose $P(F)$ =0.0005 and that $P(\text{feature present}|\bar{F}) = .02$. Determine $P(F|$ feature present) as a function of r, and give the values when $r = 10$,30 and 100.

(b) Suppose $r = 100$ and you decide to flag transactions with the feature present. What percentage of transactions would be flagged? Does this seem like a good idea?

4.20* **Challenge problem:** n music lovers have reserved seats in a theatre containing a total of $n + k$ seats (k seats are unassigned). The first person who enters the theatre, however, lost his seat assignment and chooses a seat at random. Subsequently, people enter the theatre one at a time and sit in their assigned seat unless it is already occupied. If it is, they choose a seat at random from the remaining empty seats. What is the probability that person n, the last person to enter the theatre, finds their seat already occupied?

4.21* **Challenge problem: (Monty Hall)** You have been chosen as finalist on a television show. For your prize, the host shows you three doors. Behind one door is a sports car, and behind the other two are goats.. After you choose one door, the host, who knows what is behind each of the three doors, opens one (never the one you chose or the one with the car) and then says:"You are allowed to switch the door you chose if you find that advantageous". Should you switch?

5. Discrete Random Variables and Probability Models

5.1 Random Variables and Probability Functions

Probability models are used to describe outcomes associated with random processes. So far we have used sets $A, B, C, \ldots$ in sample spaces to describe such outcomes. In this chapter we introduce numerical-valued variables $X, Y, \ldots$ to describe outcomes. This allows probability models to be manipulated easily using ideas from algebra, calculus, or geometry.

A random variable (r.v.) is a numerical-valued variable that represents outcomes in an experiment or random process. For example, suppose a coin is tossed 3 times; then

$$X = \text{Number of Heads that occur}$$

would be a random variable. Associated with any random variable is a **range** A, which is the set of possible values for the variable. For example, the random variable X defined above has range $A = \{0, 1, 2, 3\}$.

Random variables are denoted by capital letters like $X, Y, \ldots$ and their possible values are denoted by $x, y, \ldots$. This gives a nice short-hand notation for outcomes: for example, "$X = 2$" in the experiment above stands for "2 heads occurred".

Random variables are always defined for every outcome of the random experiment, i.e. for every outcome $a \in S$. For each possible value x of the random variable X, there is a corresponding set of outcomes a in the sample space S which results in this value of x (i.e. so that "$X = x$" occurs). In rigorous mathematical treatments of probability, a random variable is defined as a function on a sample space, as follows:

Definition 10 *A **random variable** is a function that assigns a real number to each point in a sample space S.*

To understand this definition, consider the experiment in which a coin is tossed 3 times, and suppose

that we used the sample space

$$S = \{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$$

and define a random variable as $X =$Number of heads. In this case the range of the random variable, or the set of possible values of X is the set $\{0, 1, 2, 3\}$. For points in the sample space, for example $a = THH$, the value of the function $X(a)$ is obtained by counting the number of heads, $X(a) = 2$ in this case. Each of the outcomes "$X = x$" (where X = number of heads) represents an event (either simple or compound). For example they are as follows:

Events	Definition of this event
$X = 0$	$\{TTT\}$
$X = 1$	$\{HTT, THT, TTH\}$
$X = 2$	$\{HHT, HTH, THH\}$
$X = 3$	$\{HHH\}$

Table 4.1

and since some value of X in the range A must occur, the events of the form "$X = x$" for $x \in A$ form a partition of the sample space S. For example the events in the second column of Table 4.1 are mutually exclusive (for example $\{TTT\} \cap \{HTT, THT, TTH\} = \varphi$) and their union is the whole sample space: $\{TTT\} \cup \{HTT, THT, TTH\} \cup \{HHT, HTH, THH\} \cup \{HHH\} = S$.

As you may recall, a function is a mapping of each point in a domain into a unique point. e.g. The function $f(x) = x^3$ maps the point $x = 2$ in the domain into the point $f(2) = 8$ in the range. We are familiar with this rule for mapping being defined by a mathematical formula. However, the rule for mapping a point in the sample space (domain) into the real number in the range of a random variable is often given in words rather than by a formula. As mentioned above, we generally denote random variables, in the abstract, by capital letters $(X, Y$, etc.) and denote the actual numbers taken by random variables by small letters $(x, y$, etc.). You should know that there is a difference between a **function** ($f(x)$ or $X(a)$) and the **value of a function** (for example $f(2)$ or $X(a) = 2$).

Since "$X = x$" represents an event of some kind, we will be interested in its probability, which we write as $P(X = x)$. In the above example in which a fair coin is tossed three times, we might wish the probability that X is equal to 2, or $P(X = 2)$. This is $P(\{HHT, HTH, THH\}) = \frac{3}{8}$ in the example.

We classify random variables into two types, according to how big their range of values is:

Discrete random variables take integer values or, more generally, values in a countable set (recall that a set is countable if its elements can be placed in a one-one correspondence with a subset of the positive integers).

Continuous random variables take values in some interval of real numbers like $(0, 1)$ or $(0, \infty)$ or $(-\infty, \infty)$. You should be aware that the cardinality of the real numbers in an interval is NOT countable.

Examples of each might be

Discrete	**Continuous**[14]
number of people in a car	total weight of people in a car
number of cars in a parking lot	distance between cars in a parking lot
number of phone calls to 911	time between calls to 911.

In theory there could also be mixed random variables which are discrete-valued over part of their range and continuous-valued over some other portion of their range. We will ignore this possibility here and concentrate first on discrete random variables. Continuous random variables are considered in Chapter 9.

Our aim is to set up general models which describe how the probability is distributed among the possible values a random variable can take. To do this we define for any discrete random variable X the probability function.

Definition 11 *The **probability function** (p.f.) of a random variable X is the function*

$$f(x) = P(X = x), \text{ defined for all } x \in A.$$

The set of pairs $\{(x, f(x)) : x \in A\}$ is called the **probability distribution** of X. All probability functions must have two properties:

1. $f(x) \geq 0$ for all values of x (i.e. for $x \in A$)
2. $\sum_{\text{all } x \in A} f(x) = 1$

By implication, these properties ensure that $f(x) \leq 1$ for all x. We consider a few "toy" examples before dealing with more complicated problems.

Example: Let X be the number obtained when a die is thrown. We would normally use the probability function $f(x) = 1/6$ for $x = 1, 2, 3, \cdots, 6$. In fact there probably is no absolutely perfect die in existence. For most dice, however, the 6 sides will be close enough to being equally likely that $f(x) = 1/6$ is a satisfactory model for the distribution of probability among the possible outcomes.

Example: Suppose a "fair" coin is tossed 3 times, with the results on the three tosses independent, and let X be the total number of heads occurring. Refer to Table 4.1 and compute the probabilities of the four events listed there; you obtain

Events	Definition of this event	$P(X = x)$
$X = 0$	$\{TTT\}$	$\frac{1}{8}$
$X = 1$	$\{HTT, THT, TTH\}$	$\frac{3}{8}$
$X = 2$	$\{HHT, HTH, THH\}$	$\frac{3}{8}$
$X = 3$	$\{HHH\}$	$\frac{1}{8}$

Table 4.2

Thus the probability function has values $f(0) = \frac{1}{8}, f(1) = \frac{3}{8}, f(2) = \frac{3}{8}, f(3) = \frac{1}{8}$. In this case it is easy to see that the number of points in each of the four events of the form "$X = x$" is $\binom{3}{x}$ using the counting arguments of Chapter 3, so we can give a simple algebraic expression for the probability function,

$$f(x) = \frac{\binom{3}{x}}{8} \quad \text{for } x = 0, 1, 2, 3.$$

Example 3: Find the value of k which makes $f(x)$ below a probability function.

x	0	1	2	3
$f(x)$	k	$2k$	0.3	$4k$

Since the probability of all possible outcomes must add to one, $\sum_{x=0}^{3} f(x) = 1$ giving $7k + 0.3 = 1$. Hence $k = 0.1$.

While the probability function is the most common way of describing a probability model, there are other possibilities. One of them is by using the **cumulative distribution function** (c.d.f.).

Definition 12 *The cumulative distribution function (c.d.f.) of X is the function usually denoted by $F(x)$*

$$F(x) = P(X \leq x)$$

defined for all real numbers x.

In the last example, with $k = 0.1$, the range of values for the random variable is $A = \{0, 1, 2, 3\}$ and we have for $x \in A$

x	$f(x)$	$F(x) = P(X \leq x)$
0	0.1	0.1
1	0.2	0.3
2	0.3	0.6
3	0.4	1

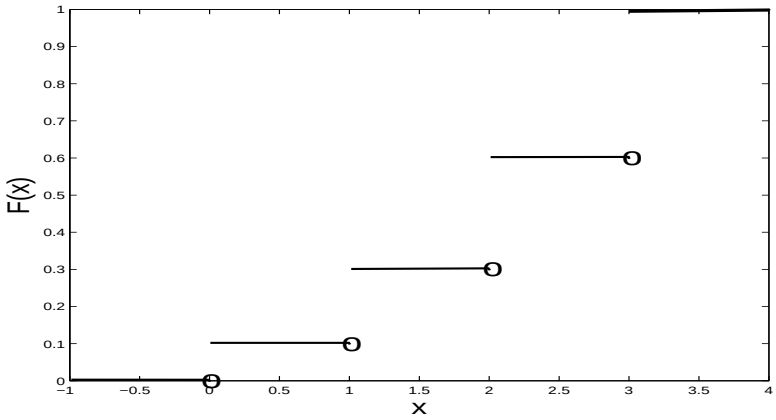


Figure 5.1: A simple cumulative distribution function

Note that the values in the third column are partial sums of the values of the probability function in the second column. For example ,

$$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = f(0) + f(1) = 0.3$$
$$F(2) = P(X \leq 2) = f(0) + f(1) + f(2) = 0.6.$$

Similarly, $F(x)$ is defined for real numbers $x \notin A$ not in the range of the random variable, for example

$$F(2.5) = F(2) = 0.6 \text{ and } F(3.8) = 1.$$

The c.d.f. for this example is plotted in Figure 5.1.

In general, $F(x)$ can be obtained from $f(x)$ by the fact that

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u).$$

A c.d.f. $F(x)$ has certain properties, just as a probability function $f(x)$ does. Obviously, since it represents a probability, $F(x)$ must be between 0 and 1. In addition it must be a non-decreasing function (e.g. $P(X \leq 8)$ cannot be less than $P(X \leq 7)$). Thus we note the following properties of a c.d.f. $F(x)$**:**

1. $F(x)$ is a non-decreasing function of x.
2. $0 \leq F(x) \leq 1$ for all x.
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

We have noted above that $F(x)$ can be obtained from $f(x)$. The opposite is also true; for example the following result holds:

If X takes on integer values then for values x such that $x \in A$ and $x - 1 \in A$,

$$f(x) = F(x) - F(x-1)$$

This says that $f(x)$ is the size of the jump in $F(x)$ at the point x.
To prove this, just note that

$$F(x) - F(x-1) = P(X \leq x) - P(X \leq x-1) = P(X = x).$$

When a random variable has been defined it is sometimes simpler to find its probability function (p.f.) $f(x)$ first, and sometimes it is simpler to find $F(x)$ first. The following example gives two approaches for the same problem.

Example: Suppose that N balls labelled $1, 2, \ldots, N$ are placed in a box, and n balls $(n \leq N)$ are randomly selected without replacement. Define the r.v.

$$X = \text{largest number selected}$$

Find the probability function for X.

Solution 1: If $X = x$ then we must select the number x plus $n-1$ numbers from the set $\{1, 2, \ldots, x-1\}$. (Note that this means we need $x \geq n$.) This gives

$$f(x) = P(X = x) = \frac{\binom{1}{1}\binom{x-1}{n-1}}{\binom{N}{n}} = \frac{\binom{x-1}{n-1}}{\binom{N}{n}} \quad x = n, n+1, \ldots, N$$

Solution 2: First find $F(x) = P(X \leq x)$. Noting that $X \leq x$ if and only if all n balls selected are from the set $\{1, 2, \ldots, x\}$, we get

$$F(x) = \frac{\binom{x}{n}}{\binom{N}{n}} \text{ for } x = n, n+1, \ldots N$$

We can now find

$$\begin{aligned} f(x) &= F(x) - F(x-1) \\ &= \frac{\binom{x}{n} - \binom{x-1}{n}}{\binom{N}{n}} \\ &= \frac{\binom{x-1}{n-1}}{\binom{N}{n}} \end{aligned}$$

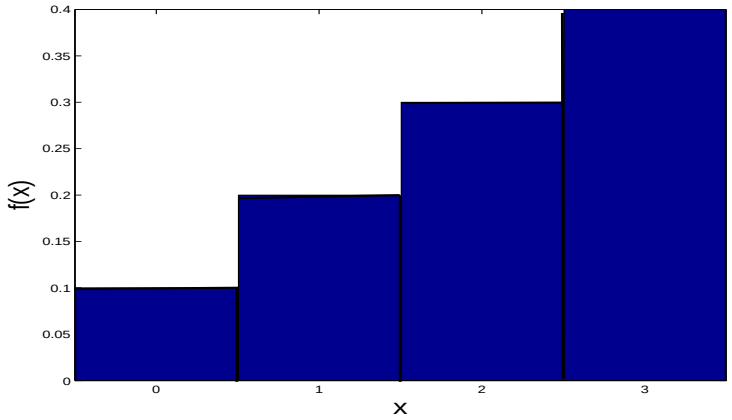


Figure 5.2: Probability histogram for $f(x) = \frac{x+1}{10}, x = 0, 1, 2, 3$

as before.

Remark: When you write down a probability function, don't forget to give its domain (i.e. the possible values of the random variable, or the values $x$for which $f(x)$ is defined). This is an essential part of the function's definition.

We frequently graph the probability function $f(x)$ using a (probability) **histogram**. For now, we'll define this only for random variables whose range is some set of consecutive integers $\{0, 1, 2, \dots\}$. A histogram of $f(x)$ is then a graph consisting of adjacent bars or rectangles. At each x we place a rectangle with base on $(x - .5, x + .5)$ and with height $f(x)$. In the above Example 3, a histogram of $f(x)$ looks like that in Figure 5.2.

Notice that the areas of these rectangles correspond to the probabilities, so for example $P(X = 1)$ is the area of the bar above and centered around the value 1 and $P(1 \leq X \leq 3)$ is the sum of the area of the three rectangles above the points $1, 2,$ and 3 (actually the area of the region above between the points $x = 0.5$ and $x = 3.5$). In general in a probability histogram, probabilities are depicted by areas.

Model Distributions:

Many processes or problems have the same structure. In the remainder of this course we will identify common types of problems and develop probability distributions that represent them. In doing this it is important to be able to strip away the particular wording of a problem and look for its essential features. For example, the following three problems are all essentially the same.

(a) A fair coin is tossed 10 times and the "number of heads obtained" (X) is recorded.

(b) Twenty seeds are planted in separate pots and the "number of seeds germinating" (X) is recorded.

(c) Twelve items are picked at random from a factory's production line and examined for defects. The number of items having no defects (X) is recorded.

What are the common features? In each case the process consists of "trials" which are repeated a stated number of times - 10, 20, and 12. In each repetition there are two types of outcomes - heads/tails, germinate/don't germinate, and no defects/defects. These repetitions are independent (as far as we can determine), with the probability of each type of outcome remaining constant for each repetition. The random variable we record is the number of times one of these two types of outcome occurred.
Six model distributions for discrete random variables will be developed in the rest of this chapter. Students often have trouble deciding which one (if any) to use in a given setting, so be sure you understand the physical setup which leads to each one. Also, as illustrated above you will need to learn to focus on the essential features of the situation as well as the particular content of the problem.

Statistical Computing

A number of major software systems have been developed for probability and statistics. We will use a system called R, which has a wide variety of features and which has Unix and Windows versions. Appendix 6.1 at the end of this chapter gives a brief introduction to R, and how to access it. For this course, R can compute probabilities for all the distributions we consider, can graph functions or data, and can simulate random processes. In the sections below we will indicate how R can be used for some of these tasks.

Problems:

5.1.1 Let X have probability function $\begin{array}{c|ccc} x & 0 & 1 & 2 \\ \hline f(x) & 9c^2 & 9c & c^2 \end{array}$. Find c.

5.1.2 Suppose that 5 people, including you and a friend, line up at random. Let X be the number of people standing between you and your friend. Tabulate the probability function and the cumulative distribution function for X.

5.2 Discrete Uniform Distribution

We define each model in terms of an abstract “physical setup", or setting, and then consider specific examples of the setup.

Physical Setup: Suppose X takes values $a, a+1, a+2, \cdots, b$ with all values being equally likely. Then X has a discrete uniform distribution, on the set $\{a, a+1, a+2, \cdots, b\}$.

Illustrations:

1. If X is the number obtained when a die is rolled, then X has a discrete uniform distribution with $a = 1$ and $b = 6$.
2. Computer random number generators give uniform $[1, N]$ variables, for a specified positive integer N. These are used for many purposes, e.g. generating lottery numbers or providing automated random sampling from a set of N items.

Probability Function: There are $b-a+1$ values X can take so the probability at each of these values must be $\frac{1}{b-a+1}$ in order that $\sum_{x=a}^{b} f(x) = 1$. Therefore

$$f(x) = \begin{cases} \frac{1}{b-a+1}; & x = a, a+1, \cdots, b \\ 0; & \text{otherwise} \end{cases}$$

Example. Suppose a fair die is thrown once and let X be the number on the face. First find the c.d.f., $F(x)$ of X.

This is an example of a discrete uniform distribution on the set $\{1, 2, 3, 4, 5, 6\}$ having $a = 1, b = 6$ and probability function

$$f(x) = \begin{cases} \frac{1}{6}; & x = 1, 2, \cdots, 6 \\ 0; & \text{otherwise} \end{cases}$$

The cumulative distribution function is $F(x) = P(X \leq x)$,

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{[x]}{6} & \text{if } 1 \leq x < 6 \\ 1 & \text{if } x \geq 6 \end{cases}$$

where by $[x]$ we mean the integer part of the real number x or the largest whole number less than or equal to x.

Many distributions are constructed using discrete uniform random variables. For example we might throw two dice and sum the values on their faces.

Example. Suppose two fair dice (suppose for simplicity one is red and the other is green) are thrown. Let X be the sum of the values on their faces. Find the c.d.f., $F(x)$ of X.
In this case we can consider the sample space to be

$$S = \{(1,1),(1,2),(1,3),...,(5,6),(6,6)\}$$

where for example the outcome (i,j) means we obtained i on the red die and j on the green. There are 36 outcomes in this sample space, all with the same probability $\frac{1}{36}$. The probability function of X is easily found. For example $f(5)$ is the probability of the event $X = 5$ or the probability of $\{(1,4),(2,3),(3,2),(4,1)\}$ so $f(5) = \frac{4}{36}$. The probability function and the cumulative distribution function is as listed below:

$x =$	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$F(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{15}{36}$	$\frac{21}{36}$	$\frac{26}{36}$	$\frac{30}{36}$	$\frac{33}{36}$	$\frac{35}{36}$	1

Although it is a bit more difficult to give a formula for the c.d.f. for general argument x in this case, it is clear for example that $F(x) = F([x])$ and $F(x) = 0$ for $x < 2$, $F(x) = 1$ for $x \geq 12$.
Example. Let X be the largest number when a die is rolled 3 times. First find the c.d.f., $F(x)$, and then find the probability function, $f(x)$ of X.

This is another example of a distribution constructed from the discrete uniform. In this case the sample space

$$S = \{(1,1,1),(1,1,2),...,(6,6,6)\}$$

consists of all 6^3 possible outcomes of the 3 dice, with each outcome having probability $\frac{1}{216}$. Suppose that x is an integer between 1 and 6. What is the probability that the largest of these three numbers is less than or equal to x?This requires that all three of the dice show numbers less than or equal to x, and there are exactly x^3 points in S which satisfy this requirement. Therefore the probability that the largest number is less than or equal to x is, for $x = 1,2,3,4,5,$or 6,

$$F(x) = \frac{x^3}{6^3}$$

and more generally if x is not an integer between 1 and 6,

$$F(x) = \begin{cases} \frac{[x]^3}{216} & \text{for } 1 \leq x < 6 \\ 0 & \text{for } x < 1 \\ 1 & \text{for } x \geq 6 \end{cases}$$

To find the probability function we may use the fact that for x in the domain of the probability function (in this case for $x \in \{1,2,3,4,5,6\}$) we have $P(X = x) = P(X \leq x) - P(X < x)$ so that for $x \in \{1,2,3,4,5,6\}$,

$$\begin{aligned} f(x) &= F(x) - F(x-1) \\ &= \frac{x^3 - (x-1)^3}{216} \\ &= \frac{[x-(x-1)][x^2 + x(x-1) + (x-1)^2]}{216} \\ &= \frac{3x^2 - 3x + 1}{216} \end{aligned}$$

5.3 Hypergeometric Distribution♢

[15]**Physical Setup:** We have a collection of N objects which can be classified into two distinct types. Call one type "success"[16] (S) and the other type "failure" (F). There are r successes and $N-r$ failures. Pick n objects at random **without replacement**. Let X be the number of successes obtained. Then X has a hypergeometric distribution.

Illustrations:

1. The number of aces X in a bridge hand has a hypergeometric distribution with $N = 52,\ r = 4$, and $n = 13$.
2. In a fleet of 200 trucks there are 12 which have defective brakes. In a safety check 10 trucks are picked at random for inspection. The number of trucks X with defective brakes chosen for inspection has a hypergeometric distribution with $N = 200, r = 12, n = 10$.

Probability Function: Using counting techniques we note there are $\binom{N}{n}$ points in the sample space S if we don't consider order of selection. There are $\binom{r}{x}$ ways to choose the x success objects from the r available and $\binom{N-r}{n-x}$ ways to choose the remaining $(n-x)$ objects from the $(N-r)$ failures. Hence

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

The range of values for x is somewhat complicated. Of course, $x \geq 0$. However if the number, n, picked exceeds the number, $N-r$, of failures, the difference, $n-(N-r)$ must be successes. So $x \geq \max(0, n-N+r)$. Also, $x \leq r$ since we can't get more successes than the number available. But $x \leq n$, since we can't get more successes than the number of objects chosen. Therefore $x \leq \min(r, n)$.

Example: In Lotto 6/49 a player selects a set of six numbers (with no repeats) from the set $\{1, 2, \ldots, 49\}$. In the lottery draw six numbers are selected at random. Find the probability function for X, the number from your set which are drawn.

Solution: Think of your numbers as the S objects and the remainder as the F objects. Then X has a hypergeometric distribution with $N = 49, r = 6$ and $n = 6$, so

$$P(X = x) = f(x) = \frac{\binom{6}{x}\binom{43}{6-x}}{\binom{49}{6}}, \text{ for } x = 0, 1, \ldots, 6$$

[15]♢ This section optional for stat 220

[16]"If A is a success in life, then A equals x plus y plus z. Work is x; y is play; and z is keeping your mouth shut." Albert Einstein, 1950

For example, you win the jackpot prize if $X = 6$; the probability of this is $\binom{6}{6}/\binom{49}{6}$, or about 1 in 13.9 million.

Remark: When parameter values are large, Hypergeometric probabilities may be tedious to compute using a basic calculator. The R functions $dhyper$ and $phyper$ can be used to evaluate $f(x)$ and the c.d.f $F(x)$. In particular, $dhyper(x, r, N-r, n)$ gives $f(x)$ and $phyper(x, r, N-r, n)$ gives $F(x)$. Using this we find for the Lotto 6/49 problem here, for example, that $f(6)$ is calculated by typing $dhyper(6, 6, 43, 6)$ in R, which returns the answer 7.151124×10^{-8} or $1/13,983,186$.

For all of our model distributions we can also confirm that $\sum\limits_{\text{all } x} f(x) = 1$. To do this here we use a summation result from Chapter 5 called the hypergeometric identity. Letting $a = r, b = N - r$ in that identity we get

$$\sum_{\text{all } x} f(x) = \sum \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \frac{1}{\binom{N}{n}} \sum \binom{r}{x}\binom{N-r}{n-x} = \frac{\binom{r+N-r}{n}}{\binom{N}{n}} = 1$$

Problems:

5.3.1 A box of 12 tins of tuna contains d which are tainted. Suppose 7 tins are opened for inspection and none of these 7 is tainted.

a) Calculate the probability that none of the 7 is tainted for $d = 0, 1, 2, 3$.

b) Do you think it is likely that the box contains as many as 3 tainted tins?

5.3.2 Suppose our sample space distinguishes points with a different orders of selection. For example suppose that $S = \{SSSFFF....,\}$ consists of all words of length n where letters are drawn without replacement from a total of r *S*s and $N - r$ *F*s. Derive a formula for the probability that the word contains exactly X Ss. In other words, determine the hypergeometric probability function using a sample space in which order of selection is considered.

5.4 Binomial Distribution

Physical Setup:
Suppose an "experiment" has two types of distinct outcomes. Call these types "success" (S) and "failure" (F), and let their probabilities be p (for S) and $1-p$ (for F). Repeat the experiment n **independent** times. Let X be the number of successes obtained. Then X has what is called a **binomial distribution**. (We write $X \sim Bi(n,p)$ as a shorthand for "X is distributed according to a binomial distribution with n repetitions and probability p of success".) The n individual experiments in the process just described are often called "trials" or "Bernoulli trials" and the process is called a Bernoulli[17] process or a binomial process.

Illustrations:

1. Toss a fair die 10 times and let X be the number of sixes that occur. Then $X \sim Bi(10, 1/6)$.

2. In a microcircuit manufacturing process, 90% of the chips produced work (10% are defective). Suppose we select 25 chips, independently[18] and let X be the number that work. Then $X \sim Bi(25, .6)$.

Comment: We must think carefully whether the physical process we are considering is closely approximated by a binomial process, for which the key assumptions are that (i) the probability p of success is constant over the n trials, and (ii) the outcome (S or F) on any trial is independent of the outcome on the other trials. For Illustration 1 these assumptions seem appropriate. For Illustration 2 we would need to think about the manufacturing process. Microcircuit chips are produced on "wafers" containing a large number of chips and it is common for defective chips to cluster on wafers. This could mean that if we selected 25 chips from the same wafer, or from only 2 or 3 wafers, that the "trials" (chips) might not be independent, or perhaps that the probability of defectives changes.

[17] After James (Jakob) Bernoulli (1654 – 1705), a Swiss member of a family of eight mathematicians. Nicolaus Bernoulli was an important citizen of Basel, being a member of the town council and a magistrate. Jacob Bernoulli's mother also came from an important Basel family of bankers and local councillors. Jacob Bernoulli was the brother of Johann Bernoulli and the uncle of Daniel Bernoulli. He was compelled to study philosophy and theology by his parents, graduated from the University of Basel with a master's degree in philosophy and a licentiate in theology but against his parents wishes, studied mathematics and astronomy . He was offered an appointment in the Church he turned it down instead taught mechanics at the University in Basel from 1683, giving lectures on the mechanics of solids and liquids. Jakob Bernoulli is responsible for many of the combinatorial results dealing with independent random variables which take values 0 or 1 in these notes. He was also a fierce rival of his younger brother Johann Bernoulli, also a mathematician, who would have liked the chair of mathematics at Basel which Jakob held.

[18] for example we select at random with replacement or without replacement from a *very large number* of chips.

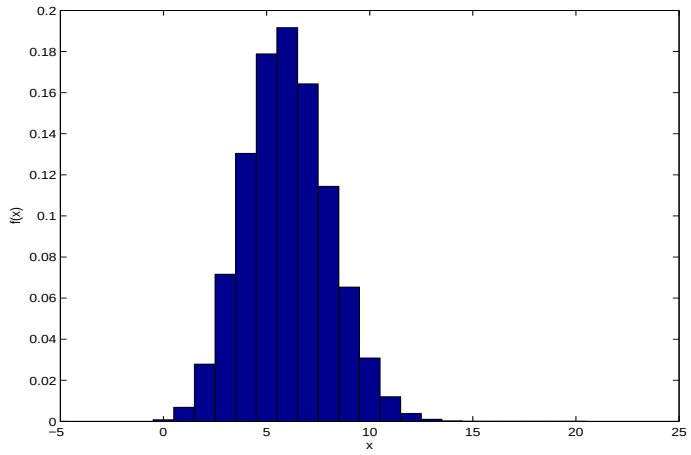


Figure 5.3: The Binomial$(20, 0.3)$ probability histogram.

Probability Function: There are $\frac{n!}{x!(n-x)!} = \binom{n}{x}$ different arrangements of x Ss and $(n-x)$ Fs over the n trials. The probability for each of these arrangements has p multiplied together x times and $(1-p)$ multiplied $(n-x)$ times, in some order, since the trials are independent. So each arrangement has probability $p^x(1-p)^{n-x}$.

$$\text{Therefore } f(x) = \binom{n}{x} p^x (1-p)^{n-x}; \quad x = 0, 1, 2, \cdots, n.$$

Checking that $\sum f(x) = 1$:

$$\begin{aligned} \sum_{x=0}^{n} f(x) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (1-p)^n \sum_{x=0}^{n} \binom{n}{x} \left(\tfrac{p}{1-p}\right)^x \\ &= (1-p)^n \left(1 + \tfrac{p}{1-p}\right)^n \text{ by the binomial theorem} \\ &= (1-p)^n \left(\tfrac{1-p+p}{1-p}\right)^n = 1^n = 1. \end{aligned}$$

We graph in Figure 5.3 the probability function for the Binomial distribution with parameters $n = 20$ and $p = 0.3$. Although the formula for $f(x)$ may seem complicated this shape is increasing to a maximum value near np and then decreasing thereafter.

Computation: Many software packages and some calculators give binomial probabilities. In R we use the function $dbinom(x, n, p)$ to compute $f(x)$ and $pbinom(x, n, p)$ to compute the corresponding c.d.f. $F(x) = P(X \leq x)$.

Example Suppose that in a weekly lottery you have probability .02 of winning a prize with a single ticket. If you buy 1 ticket per week for 52 weeks, what is the probability that (a) you win no prizes, and (b) that you win 3 or more prizes?

Solution: Let X be the number of weeks that you win; then $X \sim Bi(52, .02)$. We find

(a) $P(X = 0) = f(0) = \binom{52}{0}(.02)^0(.98)^{52} = 0.350$

$$\begin{aligned}(\mathrm{b}) \quad P(X \geq 3) &= 1 - P(X \leq 2) \\ &= 1 - f(0) - f(1) - f(2) \\ &= 0.0859\end{aligned}$$

(Note that $P(X \leq 2)$ is given by the R command $pbinom(2, 52, .02)$.)

Comparison of Binomial and Hypergeometric Distributions:
These distributions are similar in that an experiment with 2 types of outcome (S and F) is repeated n times and X is the number of successes. The key difference is that the binomial requires independent repetitions with the same probability of S, whereas the draws in the hypergeometric are made from a fixed collection of objects **without** replacement. The trials (draws) are therefore not independent. For example, if there are $r = 10$ S objects and $N - r = 10$ F objects, then the probability of getting an S on draw 2 depends on what was obtained in draw 1. If these draws had been made **with** replacement, however, they would be independent and we'd use the binomial rather than the hypergeometric model. If N is large and the number, n, being drawn is relatively small in the hypergeometric setup then we are unlikely to get the same object more than once even if we do replace it. So it makes little practical difference whether we draw with or without replacement. This suggests that when we are drawing a fairly small proportion of a large collection of objects the binomial and the hypergeometric models should produce similar probabilities. As the binomial is easier to calculate, it is often used as an approximation to the hypergeometric in such cases.

Example: Suppose we have 15 cans of soup with no labels, but 6 are tomato and 9 are pea soup. We randomly pick 8 cans and open them. Find the probability 3 are tomato.

Solution: The correct solution uses hypergeometric, and is (with X = number of tomato soups picked)

$$f(3) = P(X = 3) = \frac{\binom{6}{3}\binom{9}{5}}{\binom{15}{8}} = 0.396.$$

If we incorrectly used binomial, we'd get

$$f(3) = \binom{8}{3}\left(\frac{6}{15}\right)^3\left(\frac{9}{15}\right)^5 = 0.279$$

As expected, this is a poor approximation since we're picking over half of a fairly small collection of cans.
However, if we had 1500 cans - 600 tomato and 900 pea, we're not likely to get the same can again even if we did replace each of the 8 cans after opening it. (Put another way, the probability we get

a tomato soup on each pick is very close to .4, regardless of what the other picks give.) The exact, hypergeometric, probability is now $\frac{\binom{600}{3}\binom{900}{5}}{\binom{1500}{8}} = .2794$. Here the binomial probability,

$$\binom{8}{3}\left(\frac{600}{1500}\right)^3\left(\frac{900}{1500}\right)^5 = 0.279$$

is a very good approximation.

Problems:

5.4.1 Megan audits 130 clients during a year and finds irregularities for 26 of them.

a) Give an expression for the probability that 2 clients will have irregularities when 6 of her clients are picked at random,

b) Evaluate your answer to (a) using a suitable approximation.

5.4.2 The flash mechanism on camera A fails on 10% of shots, while that of camera B fails on 5% of shots. The two cameras being identical in appearance, a photographer selects one at random and takes 10 indoor shots using the flash.

(a) Give the probability that the flash mechanism fails exactly twice. What assumption(s) are you making?

(b) Given that the flash mechanism failed exactly twice, what is the probability camera A was selected?

5.5 Negative Binomial Distribution$\diamondsuit$

[19]**Physical Setup:**
The setup for this distribution is almost the same as for binomial; i.e. an experiment (trial) has two distinct types of outcome (S and F) and is repeated independently with the same probability, p, of success each time. Continue doing the experiment until a specified number, k, of success have been obtained. Let X be the number of failures obtained before the k^{th} success. Then X has a negative binomial distribution. We often write $X \sim NB(k,p)$ to denote this.
Illustrations:

(1) If a fair coin is tossed until we get our 5^{th} head, the number of tails we obtain has a negative binomial distribution with $k = 5$ and $p = \frac{1}{2}$.

(2) As a rough approximation, the number of half credit failures a student collects before successfully completing 40 half credits for an honours degree has a negative binomial distribution. (Assume all course attempts are independent, with the same probability of being successful, and ignore the fact that getting more than 6 half credit failures prevents a student from continuing toward an honours degree.)

Probability Function: In all there will be $x + k$ trials (x F's and k S's) and the last trial must be a success. In the first $x + k - 1$ trials we therefore need x failures and $(k-1)$ successes, in any order. There are $\frac{(x+k-1)!}{x!(k-1)!} = \binom{x+k-1}{x}$ different orders. Each order will have probability $p^k(1-p)^x$ since there must be x trials which are failures and k which are success. Hence

$$f(x) = \binom{x+k-1}{x} p^k (1-p)^x; \quad x = 0, 1, 2, \cdots$$

Note: An alternate version of the negative binomial distribution defines X to be the total number of trials needed to get the k^{th} success. This is equivalent to our version. For example, asking for the probability of getting 3 tails before the 5^{th} head is exactly the same as asking for a total of 8 tosses in order to get the 5^{th} head. You need to be careful to read how X is defined in a problem rather than mechanically "plugging in" numbers in the above formula for $f(x)$.

Checking that $\sum f(x) = 1$ requires somewhat more work for the negative binomial distribution. We first re-arrange the $\binom{x+k-1}{x}$ term,

$$\binom{x+k-1}{x} = \frac{(x+k-1)^{(x)}}{x!} = \frac{(x+k-1)(x+k-2)\cdots(k+1)(k)}{x!}$$

[19]$\diamondsuit$ This section optional for stat 220

Factor a (-1) out of each of the x terms in the numerator, and re-write these terms in reverse order,

$$\binom{x+k-1}{x} = (-1)^x \frac{(-k)(-k-1)\cdots(-k-x+2)(-k-x+1)}{x!}$$
$$= (-1)^x \frac{(-k)^{(x)}}{x!} = (-1)^x \binom{-k}{x}$$

Then (using the binomial theorem)

$$\sum_{x=0}^{\infty} f(x) = \sum_{x=0}^{\infty} \binom{-k}{x} (-1)^x p^k (1-p)^x$$
$$= p^k \sum_{x=0}^{\infty} \binom{-k}{x} [(-1)(1-p)]^x = p^k [1 + (-1)(1-p)]^{-k}$$
$$= p^k p^{-k} = 1$$

Comparison of Binomial and Negative Binomial Distributions

These should be easily distinguished because they reverse what is specified or known in advance and what is variable.

- **Binomial:** we know the number n if trials in advance but we do not know the number of successes we will obtain until after the experiment.
- **Negative Binomial:** We know the number k of successes in advance but do not know the number of trials that will be needed to obtain this number of successes until after the experiment.

Example: *The fraction of a large population that has a specific blood type T is .08 (8%). For blood donation purposes it is necessary to find 5 people with type T blood. If randomly selected individuals from the population are tested one after another, then (a) What is the probability y persons have to be tested to get 5 type T persons, and (b) What is the probability that over 80 people have to be tested?*

Solution: Think of a type T person as a success (S) and a non-type T as an F. Let Y = number of persons who have to be tested and let X = number of non-type T persons in order to get 5 S's. Then $X \sim NB(k = 5, p = .08)$ and

$$P(X = x) = f(x) = \binom{x+4}{x} (.08)^5 (.92)^x \quad x = 0, 1, 2, \ldots$$

We are actually asked here about $Y = X + 5$. Thus

$$P(Y = y) = P(X = y - 5)$$
$$= f(y-5)$$
$$= \binom{y-1}{y-5} (.08)^5 (.92)^{y-5} \quad \text{for } y = 5, 6, 7, \ldots$$

Thus we have the answer to (a) as given above, and for (b)

$$P(Y > 80) = P(X > 75) = 1 - P(X \leq 75)$$
$$= 1 - \sum_{x=0}^{75} f(x) = 0.2235$$

Note: Calculating such probabilities is easy with R. To get $f(x)$ we use $dnbinom(x, k, p)$ and to get $F(x) = P(X \leq x)$ we use $pnbinom(x, k, p)$.

Problems:

5.5.1 You can get a group rate on tickets to a play if you can find 25 people to go. Assume each person you ask responds independently and has a 20% chance of agreeing to buy a ticket. Let X be the total number of people you have to ask in order to find 25 who agree to buy a ticket. Find the probability function of X.

5.5.2 A shipment of 2500 car headlights contains 200 which are defective. You choose from this shipment without replacement until you have 18 which are not defective. Let X be the number of defective headlights you obtain.

(a) Give the probability function, $f(x)$.

(b) Using a suitable approximation, find $f(2)$.

5.6 Geometric Distribution

Physical Setup: Consider the negative binomial distribution with $k = 1$. In this case we repeat independent Bernoulli trials with two types of outcome (S and F) each time, and the same probability, p, of success each time until we obtain the first success. Let X be the number of failures obtained before the first success.

Illustrations:

(1) The probability you win a lottery prize in any given week is a constant p. The number of weeks **before** you win a prize for the first time has a geometric distribution.

(2) If you take STAT 230 until you pass it and attempts are independent with the same probability of a pass each time[20], then the number of failures would have a geometric distribution. (Thankfully these assumptions are unlikely to be true for most persons! Why is this?)

Probability Function: **T**here is only the one arrangement with x failures followed by 1 success. This arrangement has probability

$$f(x) = (1-p)^x p; \quad x = 0, 1, 2, \cdots$$

Alternatively if we substitute $k = 1$ in the probability function for the negative binomial, we obtain

$$\begin{aligned} f(x) &= \binom{x+1-1}{x} p^1 (1-p)^x; \text{ for } x = 0, 1, 2, \cdots \\ &= p(1-p)^x \text{ for } x = 0, 1, 2, \cdots \end{aligned}$$

which is the same. To checking that $\sum f(x) = 1$, we will need to evaluate a geometric series,

$$\begin{aligned} \sum_{x=0}^{\infty} f(x) &= \sum_{x=0}^{\infty} (1-p)^x p = p + (1-p)p + (1-p)^2 p + \cdots \\ &= \frac{p}{1-(1-p)} = \frac{p}{p} = 1 \end{aligned}$$

Note: The names of the models so far derive from the summation results which show $f(x)$ sums to 1. The geometric distribution involved a geometric series; the hypergeometric distribution used the hypergeometric identity; both the binomial and negative binomial distributions used the binomial theorem.

[20] you burn all notes and purge your memory of the course after each failure

Bernoulli Trials. Once again remember that the binomial, negative binomial and geometric models all involve trials (experiments) which:

(1) are independent

(2) have 2 distinct types of outcome (S and F)

(3) have the same probability p of "success" (S) each time.

Such trials are known as Bernoulli trials.

Problem 5.6.1

Suppose there is a 30% chance of a car from a certain production line having a leaky windshield. The probability an inspector will have to check at least n cars to find the first one with a leaky windshield is .05. Find n.

5.7 Poisson Distribution from Binomial

The **Poisson**[21] **distribution** has probability function (p.f.) of the form

$$f(x) = e^{-\mu}\frac{\mu^x}{x!} \quad x = 0, 1, 2, \ldots$$

where $\mu > 0$ is a parameter whose value depends on the setting for the model. Mathematically, we can see that $f(x)$ has the properties of a p.f., since $f(x) \geq 0$ for $x = 0, 1, 2, \ldots$ and since

$$\begin{aligned} \sum_{x=0}^{\infty} f(x) &= e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!} \\ &= e^{-\mu}(e^{\mu}) = 1 \end{aligned}$$

The Poisson distribution arises in physical settings where the random variable X represents the number of events of some type. In this section we show how it arises from a binomial process, and in the following section we consider another derivation of the model.

We will sometimes write $X \sim$ Poisson (μ) to denote that X has the p.f. above.

Physical Setup: One way the Poisson distribution arises is as a limiting case of the binomial distribution as $n \to \infty$ and $p \to 0$. In particular, we keep the product np fixed at some constant value, μ, while letting $n \to \infty$. This automatically makes $p \to 0$. Let us see what the limit of the binomial p.f. $f(x)$ is in this case.

[21] After Siméon Denis Poisson (1781-1840), a French mathematician who was supposed to become a surgeon but, fortunately for his patients, failed medical school for lack of coordination. He was forced to do theoretical research, being too clumsy for anything in the lab. He wrote a major work on probability and the law, *Recherchés sur la probabilité des jugements en matière criminelle et matière civile (1837),* discovered the Poisson distribution (called law of large numbers) and to him is ascribed one of the more depressing quotes in our discipline "Life is good for only two things: to study mathematics and to teach it"

Probability Function: Since $np = \mu$, Therefore $p = \frac{\mu}{n}$ and for x fixed,

$$\begin{aligned}
f(x) &= \binom{n}{x} p^x (1-p)^{n-x} = \frac{n^{(x)}}{x!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x} \\
&= \frac{\mu^x}{x!} \frac{\overbrace{n(n-1)(n-2)\cdots(n-x+1)}^{x \text{ terms}}}{(n)(n) \quad (n) \quad \cdots \quad (n)} \left(1 - \frac{\mu}{n}\right)^{n-x} \\
&= \frac{\mu^x}{x!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \left(\frac{n-2}{n}\right) \cdots \left(\frac{n-x+1}{n}\right) \left(1 - \frac{\mu}{n}\right)^n \left(1 - \frac{\mu}{n}\right)^{-x} \\
&= \frac{\mu^x}{x!}(1) \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{x-1}{n}\right) \left(1 - \frac{\mu}{n}\right)^n \left(1 - \frac{\mu}{n}\right)^{-x} \\
\lim_{n\to\infty} f(x) &= \frac{\mu^x}{x!} \underbrace{(1)(1)(1)\cdots(1)}_{x \text{ terms}} e^{-\mu}(1)^{-x} \quad \left(\text{since } e^k = \lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n\right) \\
&= \frac{\mu^x e^{-\mu}}{x!}; \quad \text{for } x = 0, 1, 2, \cdots
\end{aligned}$$

(For the binomial the upper limit on x is n, but we are letting $n \to \infty$.) This result allows us to use the Poisson distribution with $\mu = np$ as a close approximation to the binomial distribution $Bi(n, p)$ in processes for which n is large and p is small.

Example: 200 people are at a party. What is the probability that 2 of them were born on Jan. 1?

Solution: Assuming all days of the year are equally likely for a birthday (and ignoring February 29) and that the birthdays are independent (e.g. no twins!) we can use the binomial distribution with $n = 200$ and $p = 1/365$ for X = number born on January 1, giving

$$f(2) = \binom{200}{2} \left(\frac{1}{365}\right)^2 \left(1 - \frac{1}{365}\right)^{198} = .086767$$

Since n is large and p is close to 0, we can use the Poisson distribution to approximate this binomial probability, with $\mu = np = \frac{200}{365}$, giving

$$f(2) = \frac{\left(\frac{200}{365}\right)^2 e^{-\left(\frac{200}{365}\right)}}{2!} = .086791$$

As might be expected, this is a very good approximation.

Notes:

(1) If p is close to 1 we can also use the Poisson distribution to approximate the binomial. By interchanging the labels "success" and "failure", we can get the probability of "success" (formerly labelled "failure") close to 0.

(2) The Poisson distribution used to be very useful for approximating binomial probabilities with n large and p near 0 since the calculations are easier. (This assumes values of e^x to be available.) With the advent of computers, it is just as easy to calculate the exact binomial probabilities as the Poisson probabilities. However, the Poisson approximation is useful when employing a calculator without a built in binomial function.

(3) The R functions $dpois(x, \mu)$ and $ppois(x, \mu)$ give $f(x)$ and $F(x)$.

Problem 5.7.1

An airline knows that 97% of the passengers who buy tickets for a certain flight will show up on time. The plane has 120 seats.

a) They sell 122 tickets. Find the probability that more people will show up than can be carried on the flight. Compare this answer with the answer given by the Poisson approximation.

b) What assumptions does your answer depend on? How well would you expect these assumptions to be met?

5.8 Poisson Distribution from Poisson Process♢

[22]We now derive the Poisson distribution as a model for the number of a certain kind of event or occurrence (e.g. births, insurance claims, web site hits) that occur at points in time or in space. To this end, we use the "order" notation $g(\Delta t) = o(\Delta t)$ as $\Delta t \to 0$ to mean that the function g approaches 0 faster than Δt as Δt approaches zero, or that

$$\frac{g(\Delta t)}{\Delta t} \to 0 \text{ as } \Delta t \to 0.$$

For example $g(\Delta t) = (\Delta t)^2 = o(\Delta t)$ but $(\Delta t)^{1/2}$ is not $o(\Delta t)$.

Physical Setup: Consider a situation in which a certain type of event occurs at random points in time (or space) according to the following conditions:

1. **Independence:** the number of occurrences in non-overlapping intervals are independent.

2. **Individuality:** for sufficiently short time periods of length Δt, the probability of 2 or more events occurring in the interval is close to zero i.e. events occur singly not in clusters. More precisely, as $\Delta t \to 0$, the probability of two or more events in the interval of length Δt must go to zero faster than $\Delta t \to 0$. or that

$$P(\text{2 or more events in } (t, t + \Delta t)) = o(\Delta t) \text{ as } \Delta t \to 0.$$

3. **Homogeneity or Uniformity:** events occur at a uniform or homogeneous rate λ over time so that the probability of one occurrence in an interval $(t, t + \Delta t)$ is approximately $\lambda \Delta t$ for small Δt for any value of t. More precisely,

$$P(\text{one event in } (t, t + \Delta t)) = \lambda \Delta t + o(\Delta t).$$

These three conditions together define a **Poisson Process.**

Let X be the number of event occurrences in a time period of length t. Then it can be shown (see below) that X has a Poisson distribution with $\mu = \lambda t$.

Illustrations:

[22]♢ This section optional for stat 220

(1) The emission of radioactive particles from a substance follows a Poisson process. (This is used in medical imaging and other areas.)

(2) Hits on a web site during a given time period often follow a Poisson process.

(3) Occurrences of certain non-communicable diseases sometimes follow a Poisson process.

Probability Function: We can derive the probability function $f(x) = P(X = x)$ from the conditions above. We are interested in time intervals of arbitrary length t, so as a temporary notation, let $f_t(x)$ be the probability of x occurrences in a time interval of length t. We now relate $f_t(x)$ and $f_{t+\Delta t}(x)$. From that we can determine what $f_t(x)$ is. To find $f_{t+\Delta t}(x)$ we note that for Δt small there are only 2 ways to get a total of x event occurrences by time $t + \Delta t$. Either there are x events by time t and no more from t to $t + \Delta t$ or there are $x - 1$ by time t and 1 more from t to $t + \Delta t$. (since $P(2$ or more events in $(t, t + \Delta t)) = o(\Delta t)$, other possibilities are negligible if Δt is small). This and condition 1 above (independence) imply that

$$f_{t+\Delta t}(x) \doteq f_t(x)(1 - \lambda\Delta t) + f_t(x - 1)(\lambda\Delta t) + o(\Delta t)$$

Re-arranging gives $\frac{f_{t+\Delta t}(x)-f_t(x)}{\Delta t} \doteq \lambda\left[f_t(x - 1) - f_t(x)\right] + o(1)$. Taking the limit as $\Delta t \to 0$ we get

$$\frac{d}{dt}f_t(x) = \lambda\left[f_t(x - 1) - f_t(x)\right]. \tag{5.4}$$

This provides a "differential-difference" equation that needs to be solved for the functions $f_t(x)$ as functions of t for each fixed integer value of x. We know that in interval of length 0, zero events will occur, so that $f_0(0) = 1$ and $f_0(x) = 0$ for $x = 1, 2, 3, \cdots$. At the moment we may not know how to solve such a system but let's approach the problem using the binomial approximation of the last section. Suppose that the interval $(0, t)$ is divided into $n = \frac{t}{\Delta t}$ small subintervals of length Δt. The probability that an event falls in any subinterval (record this as a success) is approximately $p = \lambda\Delta t$ provided the interval length is small. The probability of two or more events falling in any one subinterval is less than $nP(2$ or more events in $(t, t + \Delta t)) = n \times o(\Delta t)$ which goes to 0 as $\Delta t \to 0$ so we can ignore the possibility that one of the subintervals has 2 or more events in it. Also the "successes" are independent on the n different subintervals or "trials", and so the total number of successes recorded, X, is approximately binomial(n, p). Therefore

$$P(X = x) \simeq \binom{n}{x}p^x(1 - p)^{n-x} = \frac{n^{(x)}p^x}{x!}(1 - p)^n\left(\frac{1}{1 - p}\right)^x$$

Notice that for fixed t, x, as $\Delta t \to 0$, $p = \lambda\Delta t \to 0$ and $n = \frac{t}{\Delta t} \to \infty$, and $(1 - p)^n \to e^{-\lambda t}$. Also, for fixed x, $n^{(x)}p^x \to (\lambda t)^x$. This yields the approximation

$$P(X = x) \simeq \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

You can easily confirm that this, i.e.

$$f_t(x) = f(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}; \quad x = 0, 1, 2, \cdots$$

provides a solution to the system (5.4) with the required initial conditions. If we let $\mu = \lambda t$, we can re-write $f(x)$ as $f(x) = \frac{\mu^x e^{-\mu}}{x!}$, which is the Poisson distribution from Section 5.7. That is:

In a Poisson process with rate of occurrence λ, the number of event occurrences X in a time interval of length t has a Poisson distribution with $\mu = \lambda t$.

Interpretation of μ and λ: λ is referred to as the **intensity or rate of occurrence** parameter for the events. It represents the average rate of occurrence of events per unit of time (or area or volume, as discussed below). Then $\lambda t = \mu$ represents the average number of occurrences in t units of time. It is important to note that the value of λ depends on the units used to measure time. For example, if phone calls arrive at a store at an average rate of 20 per hour, then $\lambda = 20$ when time is in hours and the average in 3 hours will be 3×20 or 60. However, if time is measured in minutes then $\lambda = 20/60 = 1/3$; the average in 180 minutes (3 hours) is still $(1/3)(180) = 60$.

Example *Suppose earthquakes recorded in Ontario each year follow a Poisson process with an average of 6 per year. What is the probability that 7 will be recorded in a 2-year period?*

In this case $t = 2$(years) and the intensity of earthquakes is $\lambda = 6$. Therefore X, the number of earthquakes in the two-year period follows a Poisson distribution with parameter $\mu = \lambda t = 12$. The probability that 7 earthquakes will be recorded in a 2 year period is $f(7) = \frac{12^7 e^{-12}}{7!} = .0437$.

Example *At a nuclear power station an average of 8 leaks of heavy water are reported per year. Find the probability of 2 or more leaks in 1 month, if leaks follow a Poisson process.*

Solution: Assume leaks satisfy the conditions for a Poisson process and that a month is $1/12$ of a year. Let X be the number of leaks in one month. Then X has the the Poisson distribution with $\lambda = 8$ and $t = 1/12$, so $\mu = \lambda t = 8/12$. Thus

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) \\ &= 1 - [f(0) + f(1)] \\ &= 1 - \left[\frac{(8/12)^0 e^{-8/12}}{0!} + \frac{\left(\frac{8}{12}\right)^1 e^{-8/12}}{1!} \right] \\ &\simeq 0.1443. \end{aligned}$$

Random Occurrence of Events in Space: The Poisson process also applies when "events" occur randomly in space (either 2 or 3 dimensions). For example, the "events" might be bacteria in a volume of water or blemishes in the finish of a paint job on a metal surface. If X is the number of events in a volume or area in space of size v and if λ is the average number of events per unit volume (or area), then X has a Poisson distribution with $\mu = \lambda v$. For this model to be valid, it is assumed that the Poisson process conditions given previously apply here, with "time" replaced by "volume" or "area". Once again, note that the value of λ depends on the units used to measure volume or area.

Example: Coliform bacteria occur in river water with an average intensity of 1 bacteria per 10 cubic centimeters (cc) of water. Find (a) the probability there are no bacteria in a 20cc sample of water which is tested, and (b) the probability there are 5 or more bacteria in a 50cc sample. (To do this assume that a Poisson process describes the location of bacteria in the water at any given time.)

Solution: Let X = number of bacteria in a sample of volume v cc. Since λ = 0.1 bacteria per cc (1 per 10cc) the p.f. of X is Poisson with $\mu = .1v$,

$$f(x) = e^{-.1v}\frac{(.1v)^x}{x!} \quad x = 0, 1, 2, \ldots$$

Thus we find

(a) With $v = 20, \mu = 2$ so $P(X = 0) = f(0) = e^{-2} = .135$

(b) With $v = 50, \mu = 5$ so $f(x) = e^{-5}5^x/x!$ and $P(X \geq 5) = 1 - P(X \leq 4) = .440$

(Note: we can use the R command $ppois(4, 5)$ to get $P(X \leq 4)$.)

Exercise: In each of the above examples, how well are each of the conditions for a Poisson process likely to be satisfied?

Distinguishing Poisson from Binomial and Other Distributions

Students often have trouble knowing when to use the Poisson distribution and when not to use it. To be certain, the three conditions for a Poisson process need to be checked. However, a quick decision can often be made by asking yourself the following questions:

1. *Can we specify in advance the maximum value which X can take?*
   If we can, then the distribution is <u>not</u> Poisson. If there is no fixed upper limit, the distribution might be Poisson, but is certainly not binomial or hypergeometric, e.g. the number of seeds which germinate out of a package of 25 does not have a Poisson distribution since we know in advance that $X \leq 25$. The number of cardinals sighted at a bird feeding station in a week might

be Poisson since we can't specify a fixed upper limit on X. At any rate, this number would not have a binomial or hypergeometric distribution. Of course if it is binomial with a very large value of n and a small value of p we may still use the Poisson distribution, but in this case it is being used to approximate a binomial.

2. *Does it make sense to ask how often the event did* not *occur?*
   If it does make sense, the distribution is not Poisson. If it does not make sense, the distribution might be Poisson. For example, it does not make sense to ask how often a person did not hiccup during an hour. So the number of hiccups in an hour might have a Poisson distribution. It would certainly not be binomial, negative Binomial, or hypergeometric. If a coin were tossed until the 3^{rd} head occurs it does make sense to ask how often heads did not come up. So the distribution would not be Poisson. (In fact, we'd use negative binomial for the number of non-heads; i.e. tails.)

Problems:

5.8.1 Suppose that emergency calls to 911 follow a Poisson process with an average of 3 calls per minute. Find the probability there will be

a) 6 calls in a period of $2\frac{1}{2}$ minutes.

b) 2 calls in the first minute of a $2\frac{1}{2}$ minute period, given that 6 calls occur in the entire period.

5.8.2 Misprints are distributed randomly and uniformly in a book, at a rate of 2 per 100 lines.

(a) What is the probability a line is free of misprints?

(b) Two pages are selected at random. One page has 80 lines and the other 90 lines. What is the probability that there are exactly 2 misprints on each of the two pages?

5.9 Combining Other Models with the Poisson Process♢

[23]While we've considered the model distributions in this chapter one at a time, we will sometimes need to use two or more distributions to answer a question. To handle this type of problem you'll need to be very clear about the characteristics of each model. Here is a somewhat artificial illustration. Lots of other examples are given in the problems at the end of the chapter.

Example: A very large (essentially infinite) number of ladybugs is released in a large orchard. They scatter randomly so that on average a tree has 6 ladybugs on it. Trees are all the same size.

a) Find the probability a tree has > 3 ladybugs on it.

b) When 10 trees are picked at random, what is the probability 8 of these trees have > 3 ladybugs on them?

c) Trees are checked until 5 with > 3 ladybugs are found. Let X be the total number of trees checked. Find the probability function, $f(x)$.

d) Find the probability a tree with > 3 ladybugs on it has exactly 6.

e) On 2 trees there are a total of t ladybugs. Find the probability that x of these are on the first of these 2 trees.

Solution:

a) If the ladybugs are randomly scattered the most suitable model is the Poisson distribution with $\lambda = 6$ and $v = 1$ (i.e. any tree has a "volume" of 1 unit), so $\mu = 6$ and

$$\begin{aligned} P(X > 3) &= 1 - P(X \leq 3) = 1 - [f(0) + f(1) + f(2) + f(3)] \\ &= 1 - \left[\tfrac{6^0 e^{-6}}{0!} + \tfrac{6^1 e^{-6}}{1!} + \tfrac{6^2 e^{-6}}{2!} + \tfrac{6^3 e^{-6}}{3!}\right] = .8488 \end{aligned}$$

b) Using the binomial distribution where "success" means > 3 ladybugs on a tree, we have $n = 10, p = .8488$ and

$$f(8) = \binom{10}{8}(.8488)^8(1 - .8488)^2 = .2772$$

c) Using the negative binomial distribution, we need the number of successes, k, to be 5, and the number of failures to be $(x - 5)$. Then

$$\begin{aligned} f(x) &= \tbinom{x-5+5-1}{x-5}(.8488)^5(1 - .8488)^{x-5} \\ &= \tbinom{x-1}{x-5}(.8488)^5(1 - .8488)^{x-5} \text{ or } \tbinom{x-1}{4}(.8488)^5(.1512)^{x-5}; \;\; x = 5, 6, 7, \cdots \end{aligned}$$

[23]♢ This section optional for stat 220

d) This is conditional probability. Let A = { 6 ladybugs} and
B = { more than 3 ladybugs }. Then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(6 \text{ ladybugs})}{P(\text{more than 3 ladybugs})} = \frac{\frac{6^6 e^{-6}}{6!}}{0.8488} = 0.1892.$$

e) Again we need to use conditional probability.

$$P(x \text{ on } 1^{\text{st}} \text{ tree}|\text{total of } t) = \frac{P(x \text{ on } 1^{\text{st}} \text{ tree and total of } t)}{P(\text{total of } t)}$$

$$\begin{aligned} &= \frac{P(x \text{ on } 1^{\text{st}} \text{ tree and } t-x \text{ on } 2^{\text{nd}} \text{ tree})}{P(\text{ total of } t)} \\ &= \frac{P(x \text{ on } 1^{\text{st}} \text{ tree}).\ P(t-x \text{ on } 2^{\text{nd}} \text{ tree})}{P(\text{total of } t)} \end{aligned}$$

Use the Poisson distribution to calculate each, with $\mu = 6 \times 2 = 12$ in the denominator since there are 2 trees.

$$\begin{aligned} P(x \text{ on } 1^{\text{st}} \text{ tree}|\text{total of } t) &= \frac{\left(\frac{6^x e^{-6}}{x!}\right)\left(\frac{6^{t-x} e^{-6}}{(t-x)!}\right)}{\frac{12^t e^{-12}}{t!}} \\ &= \frac{t!}{x!(t-x)!}\left(\frac{6}{12}\right)^x \left(\frac{6}{12}\right)^{t-x} \\ &= \binom{t}{x}\left(\frac{1}{2}\right)^x \left(1-\frac{1}{2}\right)^{t-x}, \quad x = 0, 1, \cdots, t. \end{aligned}$$

Caution: Don't forget to give the range of x. If the total is t, there couldn't be more than t ladybugs on the 1^{st} tree.

Exercise: The answer to (e) is a binomial probability function. Can you reach this answer by general reasoning rather than using conditional probability to derive it?

Problems:

5.9.1 In a Poisson process the average number of occurrences is λ per minute. Independent 1 minute intervals are observed until the first minute with no occurrences is found. Let X be the number of 1 minute intervals required, including the last one. Find the probability function, $f(x)$.

5.9.2 Calls arrive at a telephone distress centre during the evening according to the conditions for a Poisson process. On average there are 1.25 calls per hour.

(a) Find the probability there are no calls during a 3 hour shift.

(b) Give an expression for the probability a person who starts working at this centre will have the first shift with no calls on the 15^{th} shift.

(c) A person works one hundred 3 hour evening shifts during the year. Give an expression for the probability there are no calls on at least 4 of these 100 shifts. Calculate a numerical answer using a Poisson approximation.

5.10 Summary of Single Variable Discrete Models

Name	Probability Function
Discrete Uniform	$f(x) = \frac{1}{b-a+1};\ x = a, a+1, a+2, \cdots b$
Hypergeometric	$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}};\ x = \max(0, n-(N-r)), \cdots, \min(n, r)$
Binomial	$f(x) = \binom{n}{x} p^x (1-p)^{n-x};\ x = 0, 1, 2, \cdots n$
Negative Binomial	$f(x) = \binom{x+k-1}{x} p^k (1-p)^x;\ x = 0, 1, 2, \cdots$
Geometric	$f(x) = p(1-p)^x;\ x = 0, 1, 2, \cdots$
Poisson	$f(x) = \frac{e^{-\mu}\mu^x}{x!};\ x = 0, 1, 2, \cdots$

5.11 Problems on Chapter 5

5.1 Suppose that the probability $p(x)$ a person born in 1950 lives at least to certain ages x is as given in the table below.

x:	30	50	70	80	90
Females	.980	.955	.910	.595	.240
Males	.960	.920	.680	.375	.095

(a) If a female lives to age 50, what is the probability she lives to age 80? To age 90? What are the corresponding probabilities for males?

(b) If 51% of persons born in 1950 were male, find the fraction of the total population (males and females) that will live to age 90.

5.2 Let X be a non-negative discrete random variable with cumulative distribution function

$$F(x) = 1 - 2^{-x} \text{ for } x = 0, 1, 2, ...$$

(a) Find the probability function of X.

(b) Find the probability of the event $X = 5$; the event $X \geq 5$.

5.3 Two balls are drawn at random from a box containing ten balls numbered $0, 1, ..., 9$. Let random variable X be the *larger* of the numbers on the two balls and random variable Y be their *total*.

(a) Tabulate the p.f. of X and of Y if the sampling is *without* replacement.

(b) Repeat (a) if the sampling is *with* replacement.

5.4 Let X have a geometric distribution with $f(x) = p(1-p)^x \, ; x = 0, 1, 2, \cdots$. Find the probability function of R, the remainder when X is divided by 4.

5.5 (a) Todd decides to keep buying a lottery ticket each week until he has 4 winners (of some prize). Suppose 30% of the tickets win some prize. Find the probability he will have to buy 10 tickets.

(b) A coffee chain claims that you have a 1 in 9 chance of winning a prize on their “roll up the edge" promotion, where you roll up the edge of your paper cup to see if you win. If so, what is the probability you have no winners in a one week period where you bought 15 cups of coffee?

(c) Over the last week of a month long promotion you and your friends bought 60 cups of coffee, but there was only 1 winner. Find the probability that there would be this few (i.e. 1 or 0) winners. What might you conclude?

5.6 An oil company runs a contest in which there are 500,000 tickets; a motorist receives one ticket with each fill-up of gasoline, and 500 of the tickets are winners.

(a) If a motorist has ten fill-ups during the contest, what is the probability that he or she wins at least one prize?

(b) If a particular gas bar distributes 2,000 tickets during the contest, what is the probability that there is at least one winner among the gas bar's customers?

5.7 **Jury selection**. During jury selection a large number of people are asked to be present, then persons are selected one by one in a random order until the required number of jurors has been chosen. Because the prosecution and defense teams can each reject a certain number of persons, and because some individuals may be exempted by the judge, the total number of persons selected before a full jury is found can be quite large.

(a) Suppose that you are one of 150 persons asked to be present for the selection of a jury. If it is necessary to select 40 persons in order to form the jury, what is the probability you are chosen?

(b) In a recent trial the numbers of men and women present for jury selection were 74 and 76. Let Y be the number of men picked for a jury of 12 persons. Give an expression for $P(Y = y)$, assuming that men and women are equally likely to be picked.

(c) For the trial in part (b), the number of men selected turned out to be two. Find $P(Y \leq 2)$. What might you conclude from this?

5.8 A waste disposal company averages 6.5 spills of toxic waste per month. Assume spills occur randomly at a uniform rate, and independently of each other, with a negligible chance of 2 or more occurring at the same time. Find the probability there are 4 or more spills in a 2 month period.

5.9 Coliform bacteria are distributed randomly and uniformly throughout river water at the average concentration of one per twenty cubic centimeters of water.

(a) What is the probability of finding exactly two coliform bacteria in a 10 cubic centimeters sample of the river water?

(b) What is the probability of finding at least one coliform bacterium in a 1 cubic centimeter sample of the river water?

(c) In testing for the concentration (average number per unit volume) of bacteria it is possible to determine cheaply whether a sample has **any** (i.e. 1 or more) bacteria present or not.

Suppose the average concentration of bacteria in a body of water is λ per cubic centimeter. If 10 independent water samples of 10 c.c. each are tested, let the random variable Y be the number of samples with **no** bacteria. Find $P(Y = y)$.

(d) Suppose that of 10 samples, 3 had no bacteria. Find an estimate for the value of λ.

5.10 In a group of policy holders for house insurance, the average number of claims per 100 policies per year is $\lambda = 8.0$. The number of claims for an individual policy holder is assumed to follow a Poisson distribution.

(a) In a given year, what is the probability an individual policy holder has at least one claim?

(b) In a group of 20 policy holders, what is the probability there are no claims in a given year? What is the probability there are two or more claims?

5.11 Assume power failures occur independently of each other at a uniform rate through the months of the year, with little chance of 2 or more occurring simultaneously. Suppose that 80% of months have no power failures.

a) Seven months are picked at random. What is the probability that 5 of these months have no power failures?

b) Months are picked at random until 5 months without power failures have been found. What is the probability that 7 months will have to be picked?

c) What is the probability a month has more than one power failure?

5.12 a) Let $f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$, and keep $p = \frac{r}{N}$ fixed. (e.g. If N doubles then r also doubles.) Prove that $\lim_{N\to\infty} f(x) = \binom{n}{x}p^x(1-p)^{n-x}$.

b) What part of the chapter is this related to?

5.13 Spruce budworms are distributed through a forest according to a Poisson process so that the average is λ per hectare.

a) Give an expression for the probability that at least 1 of n one hectare plots contains at least k spruce budworms.

b) Discuss briefly which assumption(s) for a Poisson process may not be well satisfied in this situation.

5.14 A person working in telephone sales has a 20% chance of making a sale on each call, with calls being independent. Assume calls are made at a uniform rate, with the numbers made in non-overlapping periods being independent. On average there are 20 calls made per hour.

a) Find the probability there are 2 sales in 5 calls.

b) Find the probability exactly 8 calls are needed to make 2 sales.

c) If 8 calls were needed to make 2 sales, what is the probability there was 1 sale in the first 3 of these calls?

d) Find the probability of 3 calls being made in a 15 minute period.

5.15 A bin at a hardware store contains 35 forty watt lightbulbs and 70 sixty watt bulbs. A customer wants to buy 8 sixty watt bulbs, and withdraws bulbs without replacement until these 8 bulbs have been found. Let X be the number of 40 watt bulbs drawn from the bin. Find the probability function, $f(x)$.

5.16 During rush hour the number of cars passing through a particular intersection[24] has a Poisson distribution with an average of 540 per hour.

a) Find the probability there are 11 cars in a 30 second interval and the probability there are 11 or more cars.

b) Find the probability that when 20 disjoint 30 second intervals are studied, exactly 2 of them had 11 cars.

c) We want to find 12 intervals having 11 cars in 30 seconds.

(i) Give an expression for the probability 1400 30 second intervals have to be observed to find the 12 having the desired traffic flow.

(ii) Use an approximation which involves the Poisson distribution to evaluate this probability and justify why this approximation is suitable.

5.17 (a) Bubbles are distributed in sheets of glass, as a Poisson process, at an intensity of 1.2 bubbles per square metre. Let X be the number of sheets of glass, in a shipment of n sheets, which have no bubbles. Each sheet is 0.8m^2. Give the probability function of X.

(b) The glass manufacturer wants to have at least 50% of the sheets of glass with no bubbles. How small will the intensity λ need to be to achieve this?

5.18 Random variable X takes values 1,2,3,4,5 and has c.d.f.

x	0	1	2	3	4	5
$F(x)$	0	$.1k$	.2	$.5k$	k	$4k^2$

Find k, $f(x)$ and $P(2 < X \leq 4)$. Draw a histogram of $f(x)$.

[24] "Traffic signals in New York are just rough guidelines." David Letterman (1947 -)

5.19 Let random variable Y have a *geometric distribution* $P(Y = y) = p(1-p)^y$ for $y = 0, 1, 2, \ldots$.

(a) Find an expression for $P(Y \geq y)$, and show that $P(Y \geq s+t|Y \geq s) = P(Y \geq t)$ for all non-negative integers s, t.

(b) What is the most probable value of Y?

(c) Find the probability that Y is divisible by 3.

(d) Find the probability function of random variable R, the *remainder* when Y is divided by 3.

5.20 **Polls and Surveys**. Polls or surveys in which people are selected and their opinions or other characteristics are determined are very widely used. For example, in a survey on cigarette use among teenage girls, we might select a random sample of n girls from the population in question, and determine the number X who are regular smokers. If p is the fraction of girls who smoke, then $X \sim Bi(n, p)$. Since p is unknown (that is why we do the survey) we then estimate it as $\hat{p} = X/n$. (In probability and statistics a "hat" is used to denote an estimate of a model parameter based on data.) The binomial distribution can be used to study how "good" such estimates are, as follows

(a) Suppose $p = .3$ and $n = 100$. Find the probability $P(.27 \leq \frac{X}{n} \leq .33)$. Many surveys try to get an estimate X/n which is within 3% (.03) of p with high probability. What do you conclude here?

(b) Repeat the calculation in (a) if $n = 400$ and $n = 1000$. What do you conclude?

(c) If $p = .5$ instead of .3, find $P(.47 \leq \frac{X}{n} \leq .53)$ when $n = 400$ and 1000.

(d) Your employer asks you to design a survey to estimate the fraction p of persons age 25-34 who download music via the internet. The objective is to get an estimate accurate to within 3%, with probability close to .95. What size of sample (n) would you recommend?

5.21 **Telephone surveys**. In some "random digit dialing" surveys, a computer phones randomly selected telephone numbers. However, not all numbers are "active" (belong to a telephone account) and they may belong to businesses as well as to individual or residences.

Suppose that for a given large set of telephone numbers, 57% are active residential or individual numbers. We will call these "personal" numbers.

Suppose that we wish to interview (over the phone) 1000 persons in a survey.

(a) Suppose that the probability a call to a personal number is answered is .8, and that the probability the person answering agrees to be interviewed is .7. Give the probability distribution for X, the number of calls needed to obtain 1000 interviews.

(b) Use R software to find $P(X \leq x)$ for the values $x = 2900,\ 3000,\ 3100,\ 3200$.

(c) Suppose instead that 3200 randomly selected numbers were dialed. Give the probability distribution for Y, the number of interviews obtained, and find $P(Y \geq 1000)$.

(Note: The R functions *pnbinom* and *pbinom* give negative binomial and binomial probabilities, respectively.)

5.22* **Challenge problem**: Suppose that n independent tosses of a coin having probability p of coming up heads are made. Show that the probability of an even number of heads is given by $\frac{1}{2}[1 + (q - p)^n]$ where $q = 1 - p$.

6. Computational Methods and $R\lozenge\bigstar$

[25]One of the giant steps towards democracy in the last century was the increased democratization of knowledge[26], facilitated by the personal computer, *Wikipedia* and the advent of free open-source (GNU) software such as *Linux*. The statistical software package$R$ implements a dialect of the S language that was developed at AT&T Bell Laboratories by Rick Becker, John Chambers and Allan Wilks. Versions of R are available, at no cost, for 32-bit versions of Microsoft Windows for Linux, for Unix and for Macintosh systems. It is available through the Comprehensive R Archive Network (CRAN) (downloadable for unix, windows or MAC platforms at http://cran.r-project.org/). This means that a community of interested statisticians voluntarily maintain and updates the software. Like the licensed software *Matlab* and *Splus*, R permits easy matrix and numerical calculations, as well as a programming environment for high-level computations. The R software also provides a powerful tool for handling probability distributions, generating random variables, and graphical display. Because it is freely available and used by statisticians world-wide, high level programs in R are often available on the web. These notes provide a glimpse of a few of the features of R. Web resources have much more information and more links can be found on the Stat 230 web page. We will provide a brief description of commands on a windows machine here, but the MAC and UNIX commands will generally be similar once R is started.

6.1 Preliminaries

Begin by installing R on your personal computer and then invoke it on Math Unix machines by typing R or on a windows machine by clicking on the R icon. For these notes, we will simply describe typing commands into the R command window following the R prompt "$>$" in interactive mode. This window is displayed below in Figure 6.1

Objects include variables, functions, vectors, arrays, lists and other items. To see online documentation about something, we use the "help" function. For example, to see documentation on the function mean(), type

[25]$\lozenge\bigstar$ This section optional for Stat 220 and Stat 230

[26]"Knowledge is the most deomocratic source of power" Alvin Toffler

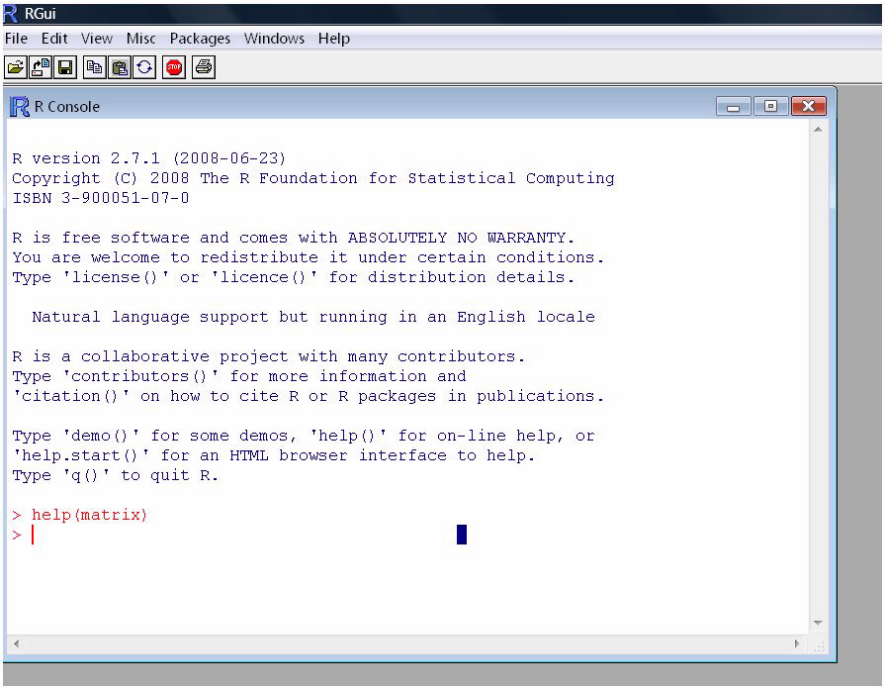


Figure 6.1: An R, version 2.7.1 command window in windows

```
help(mean).
```

In some cases help.search() is helpful. For example

```
help.search("matrix")
```

lists all functions whose help pages have a title or alias in which the text string “matrix” appears.

The <- is a left diamond bracket (<) followed by a minus sign (-). It means “is assigned to”, for example,

```
x<-15
```

assigns the value 15 to variable x. To quit an R session, type

```
q()
```

You need the brackets () because you wish to run the function "q". Typing q on its own, without the parentheses, displays the text of the function on the screen. Try it! Alternatively to quit R, you can click on the "File" menu and then on Exit or on the x in the top right corner of the R window. You are asked whether you want to save the workspace image. Clicking "Yes" (safer) will save all the objects that remain in the workspace both those at the start of the session and those added.

6.2 Vectors

Vectors can consist of numbers or other symbols; we will consider only numbers here. Vectors are defined using c(): for example,

```
x<-c(1,3,5,7,9)
```

defines a vector of length 5 with the elements given. Vectors and other classes of objects possess certain attributes. For example, typing

```
length(x)
```

will give the length of the vector x. Vectors are a convenient way to store values of a function (e.g. a probability function or a c.d.f) or values of a random variable that have been recorded in some experiment or process. We can also read a table of values from a text file that we created earlier called say "mydata.txt" on a disk in drive c:

```
> mydata <- read.table("c:/mydata.txt", header=T)
```

Use of "header=T" causes R to use the first line of the text file to get header information for the columns. If column headings are not included in the file, the argument can be omitted and we obtain a table with just the data. The R object "mydata" is a special form known as a "data frame". Data frames that consist entirely of numeric data have a structure that is similar to that of numeric matrices. The names of the columns can be displayed with the command

```
> names(mydata)
```

6.3 Arithmetic Operations

The following R commands and responses should explain the most basic arithmetic operations.

```
> 7+3
[1] 10
> 7*3
[1] 21
> 7/3
[1] 2.333333
> 2^3
[1] 8
```

In the last example the result is 8. The [1] says basically “first requested element follows” but here there is just one element. The "$>$" indicates that R is ready for another command.

6.4 Some Basic Functions

Functions of many types exist in R. Many operate on vectors in a transparent way, as do arithmetic operations. For example, if x and y are vectors then x+y adds the vectors element-wise; thus x and y must be the same length. Some examples, with comments, follow. Note that anything that follows a # on the command line is taken as comment and ignored by R.

```
> x<- c(1,3,5,7,9)   # Defines a vector x
> x            # displays x
[1] 1 3 5 7 9
> y<- seq(1,2,.25) #seq defines vector whose elements are an arithmetic progression
> y
[1] 1.00 1.25 1.50 1.75 2.00
> y[2]     #displays the second element of vector y
[1] 1.25
> y[c(2,3)]    #displays vector of second and third elements of vector y
[1] 1.25 1.50
> mean(x)      #computes mean of the elements of vector x
[1] 5
> summary(x)    #function which summarizes features of a vector x
 Min.    1st Qu. Median  Mean 3rd Qu. Max.
  1       3        5      5     7     9
> var(x)    # Computes the (sample) variance of the elements of x
[1] 10
> exp(1)    # The exponential function
[1] 2.718282
> exp(y)
[1] 2.718282 3.490343 4.481689 5.754603 7.389056
> round(exp(y),2)   # round(y,n) rounds elements of vector y to n decimals
[1] 2.72 3.49 4.48 5.75 7.39
> x+2*y
[1]  3.0  5.5  8.0 10.5 13.0
```

6.5 R Objects

Type "ls()" to see a list of names of all objects, including functions and data structures, in your workspace.

If you type the name of an object, vector, matrix or function, you are returned its contents. (Try typing "q" or "mean").

Before you quit, you may remove objects that you no longer require with "rm()" and then save the workspace image. The workspace image is automatically loaded when you restart R in that directory.

6.6 Graphs

To open a graphics window in Unix, type x11(). Note that in R, a graphics window opens automatically when a graphical function is used.

There are various plotting and graphical functions. Two useful ones are

```
plot(x,y)  # Gives a scatterplot of x versus y; thus x and y must be
           #vectors of the same length.

hist(x)    # Creates a frequency histogram based on the values in the
           #vector x. To get a relative frequency histogram (areas of
           #rectangles sum to one) use
hist(x,prob=T).
```

Graphs can be tailored with respect to axis labels, titles, numbers of plots to a page etc. Type help(plot), help(hist) or help(par) for some information. Try

```
x<-(0:20)*pi/10
plot(x, sin(x))
```

Is it obvious that these points lie on a sine curve? One can make it more obvious by changing the shape of the graph. Place the cursor over the lower border of the graph sheet, until it becomes a double-sided and then drag the border in towards the top border, to make the graph sheet short and wide.

To save/print a graph in R using UNIX, you generate the graph you would like to save/print in R using a graphing function like plot() and type:

```
dev.print(device,file="filename")
```

where device is the device you would like to save the graph to (i.e. x11) and filename is the name of the file that you would like the graph saved to. To look at a list of the different graphics devices you can save to,

```
type help(Devices).
```

To save/print a graph in R using Windows, you can do one of two things.

a) You can go to the File menu when the graph window is active and save the graph using one of several formats (i.e. postscript, jpeg, etc.) or print it. You may also copy the graph to the clipboard using one of the formats and then paste to an editor, such as MS Word.

b) You can right click on the graph. This gives you a choice of copying the graph and then pasting to an editor, such as MS Word, or saving the graph as a metafile or bitmap or print directly to a printer.

6.7 Distributions

There are functions which compute values of probability or probability density functions, cumulative distribution functions, and quantiles for various distributions. It is also possible to generate (pseudo) random samples from these distributions. Some examples follow for Binomial and Poisson distributions. For other distribution information, type

```
help(rhyper),
help(rnbinom)
```

and so on. Note that R does not have any function specifically designed to generate random samples from a discrete uniform distribution (although there is one for a continuous uniform distribution). To generate n random samples from a discrete UNIF(a,b), use

```
sample(a:b,n,replace=T).
```

```
> y<- rbinom(10,100,0.25)    # Generate 10 random values from the Binomial
               #distribution Bi(100,0.25). The values are  stored in the vector y.
> y    # Display the values
 [1] 24 24 26 18 29 29 33 28 28 28
```

```
> pbinom(3,10,0.5)     # Compute P(Y<=3) for a Bi(10,0.5) random variable.
[1] 0.171875
> qbinom(.95,10,0.5)   # Find the .95 quantile (95th percentile) for
[1] 8                    Bi(10,0.5).

> z<- rpois(10,10)    # Generate 10 random values from the Poisson distribution
                        #Poisson(10). The values are stored in the vector z.
> z   # Display the values
 [1]  6  5 12 10  9  7  9 12  5  9
> ppois(3,10)     # Compute P(Y<=3) for a Poisson(10) random variable.
[1] 0.01033605
> qpois(.95,10)   # Find the .95 quantile (95th percentile) for
[1] 15              Poisson(10).
```

To illustrate how to plot the probability function for a random variable, a Bi(10,0.5) random variable is used.

```
# Assign all possible values of the random variable, X ~ Bi(10,0.5)
x <- seq(0,10,by=1)

# Determine the value of the probability function for possible values of X
x.pf <- dbinom(x,10,0.5)

# Plot the probability function
barplot(x.pf,xlab="X",ylab="Probability Function",
names.arg=c("0","1","2","3","4","5","6","7","8","9","10"))
```

Loops in R are easy to construct but long loops can be slow and should be avoided where possible. For example

```
x=0
for (i in 1:10) x<- c(x,i)
```

can be replaced by

```
x=c(0:10)
```

Commonly used functions.

```
print()      # Prints a single R object
cat()        # Prints multiple objects, one after the other
length()     # Number of elements in a vector or of a list
mean()       # mean of a vector of data
median()     # median of a vector of data
range()      # Range of values of a vector of data
unique()     # Gives the vector of distinct values
diff()       # the vector of first differences so diff(x) has
             # one less element than x
sort()       # Sort elements into order, omitting NAs
order()      # x[order(x)] orders elements of x, with NAs last
cumsum()     #  vector of partial or cumuulative sums
cumprod()    #  vector of partial or cumuulative products
rev()        # reverse the order of vector element
```

6.8 Problems on Chapter 6

6.1 The following ten observations, taken during the years 1970-79, are on October snow cover for Eurasia. (Snow cover is in millions of square kilometers).

Year	Snow.cover
1970	6.5
1971	12
1972	14.9
1973	10
1974	10.7
1975	7.9
1976	21.9
1977	12.5
1978	14.5
1979	9.2

(a) Enter the data into R. To save keystrokes, enter the successive years as 1970:1979

(b) Plot snow.cover versus year.

(c) Use "hist()" to plot a histogram of the snow cover values.

(d) Repeat b and c after taking logarithms of snow cover.

6.2 Input the following data, on damage that had occurred in space shuttle launches prior to the disastrous Challenger space shuttle launch of Jan 28 1986.

Date Temperature Number of damage incidents

Date	**Temperature (F)**	**Number of Damage Incidents**	**Date**	**Temperature (F)**	**Number of Damage Incidents**
4/12/81	66	0	10/5/84	78	0
11/12/81	70	1	11/8/84	67	0
3/22/82	69	0	1/24/85	53	3
6/27/82	80	NA	4/12/85	67	0
1/11/82	68	0	4/29/85	75	0
4/4/83	67	0	6/17/85	70	0
6/18/83	72	0	7/29/85	81	0
8/30/83	73	0	8/27/85	76	0
11/28/83	70	0	10/3/85	79	0
2/3/84	57	1	10/30/85	75	2
4/6/84	63	1	11/26/85	76	0
8/30/84	70	1	1/12/86	58	1

This was then followed by the disasterous CHALLENGER incident on 1/28/86.

(a) Enter the temperature data into a data frame, with (for example) column names temperature, damage.

(b) Plot total incidents against temperature. Do you see any relationship? On the date of the challenger incident the temperature at launch was 31 degrees F. What would you expect for the number of damage incidents?

7. Expected Value and Variance

7.1 Summarizing Data on Random Variables

When we return midterm tests, someone almost always asks what the average was. While we could list out all marks to give a picture of how students performed, this would be tedious. It would also give more detail than could be immediately digested. If we summarize the results by telling a class the average mark, students immediately get a sense of how well the class performed. For this reason, "summary statistics" are often more helpful than giving full details of every outcome.

To illustrate some of the ideas involved, suppose we were to observe cars crossing a toll bridge, and record the number, X, of people in each car. Suppose in a small study[27] data on 25 cars were collected. We could list out all 25 numbers observed, but a more helpful way of presenting the data would be in terms of the **frequency distribution** below, which gives the number of times (the "frequency") each value of X occurred.

X	Frequency Count	Frequency
1	𝍸 |	6
2	𝍸 |||	8
3	𝍸	5
4	|||	3
5	||	2
6	|	1

We could also draw a *frequency* histogram of these frequencies:

Frequency distributions or histograms are good summaries of data because they show the variability in the observed outcomes very clearly. Sometimes, however, we might prefer a single-number summary. The most common such summary is the average, or arithmetic mean of the outcomes. The mean

[27] "Study without desire spoils the memory, and it retains nothing that it takes in." Leonardo da Vinci

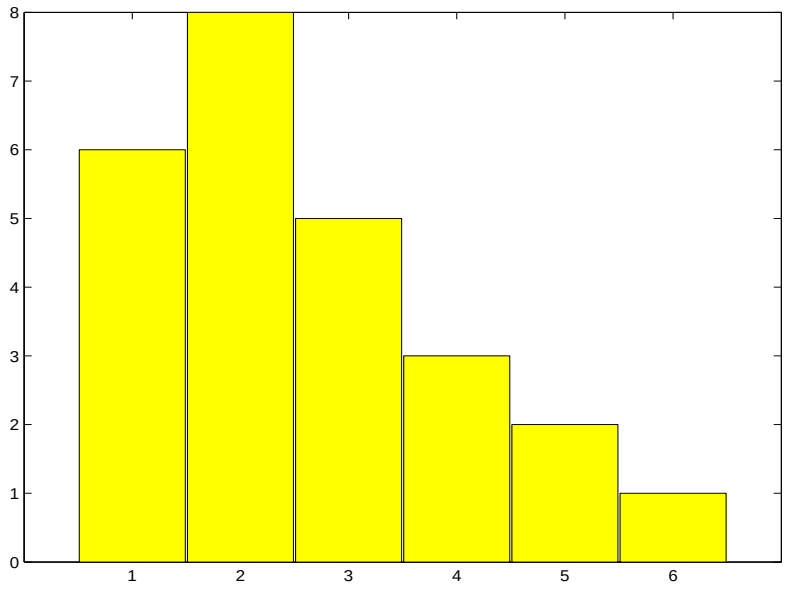


Figure 7.1: Frequency Histogram

of n outcomes $x_1, \ldots, x_n$ for a random variable X is $\sum_{i=1}^{n} x_i/n$, and is denoted by $\bar{x}$. The arithmetic mean for the example above can be calculated as

$$\frac{(6 \times 1) + (8 \times 2) + (5 \times 3) + (3 \times 4) + (2 \times 5) + (1 \times 6)}{25} = \frac{65}{25} = 2.60$$

That is, there was an average of 2.6 persons per car. A set of observed outcomes $x_1, \ldots, x_n$ for a random variable X is termed a **sample** in probability and statistics. To reflect the fact that this is the average for a particular sample, we refer to it as the **sample mean**. Unless somebody deliberately "cooked" the study, we would not expect to get precisely the same sample mean if we repeated it another time. Note also that $\bar{x}$ is not in general an integer, even though X is.

Two other common summary statistics are the median and mode.

Definition 13 *The **median** of a sample is a value such that half the results are below it and half above it, when the results are arranged in numerical order.*

If these 25 results were written in order, the 13^{th} outcome would be a 2. So the median is 2. By convention, we go half way between the middle two values if there are an even number of observations.

Definition 14 *The **mode** of the sample is the value which occurs most often. In this case the mode is 2. There is no guarantee there will be only a single mode.*

Exercise: Give a data set with a total of 11 values for which the median<mode<mean.

7.2 Expectation of a Random Variable

The statistics in the preceding section summarize features of a sample of observed X-values. The same idea can be used to summarize the probability distribution of a random variable X. To illustrate, consider the previous example, where X is the number of persons in a randomly selected car crossing a toll bridge.

Note that we can re-arrange the expression used to calculate $\overline{x}$ for the sample, as

$$\frac{(6\times 1)+(8\times 2)+\cdots+(1\times 6)}{25} = (1)\left(\frac{6}{25}\right)+(2)\left(\frac{8}{25}\right)+(3)\left(\frac{5}{25}\right)+(4)\left(\frac{3}{25}\right)+(5)\left(\frac{2}{25}\right)+(6)\left(\frac{1}{25}\right)$$

$$= \sum_{x=1}^{6} x \times \text{fraction of times } x \text{ occurs}$$

Now suppose we know that the probability function of X is given by

x	1	2	3	4	5	6
$f(x)$	.30	.25	.20	.15	.09	.01

Using the relative frequency "definition" of probability, if we observed a very large number of cars, the fraction (or relative frequency) of times $X = 1$ would be .30, for $X = 2$, this proportion would be .25, etc. So, *in theory*, (according to the probability model) we would expect the mean to be

$$(1)(.30)+(2)(.25)+(3)(.20)+(4)(.15)+(5)(.09)+(6)(.01) = 2.51$$

if we observed an infinite number of cars. This "theoretical" mean is usually denoted by μ or $E(X)$, and requires us to know the distribution of X. With this background we make the following mathematical definition.

Definition 15 *The expected value (also called the mean or the expectation) of a discrete random variable X with probability function $f(x)$ is*

$$E(X) = \sum_{\text{all x}} x f(x).$$

The expected value of X is also often denoted by the Greek letter μ. The expected value [28] of X can be thought of physically as the average of the X-values that would occur in an infinite series of repetitions of the process where X is defined. This value not only describes one aspect of a probability distribution, but is also very important in certain types of applications. For example, if you are playing

[28] Oft expectation fails, and most oft where most it promises; and oft it hits where hope is coldest; and despair most sits. William Shakespeare (1564 - 1616)

a casino game in which X represents the amount you win in a single play, then $E(X)$ represents your average winnings (or losses!) per play.

Sometimes we may not be interested in the average value of X itself, but in some function of X. Consider the toll bridge example once again, and suppose there is a toll which depends on the number of car occupants. For example, a toll of \$1 per car plus 25 cents per occupant would produce an average toll for the 25 cars in the study of Section 7.1 equal to

$$(1.25)\left(\frac{6}{25}\right)+(1.50)\left(\frac{8}{25}\right)+(1.75)\left(\frac{5}{25}\right)+(2.00)\left(\frac{3}{25}\right)+(2.25)\left(\frac{2}{25}\right)+(2.50)\left(\frac{1}{25}\right)=\$1.65$$

If X has the theoretical probability function $f(x)$ given above, then the average value of this \$(.25X + 1) toll would be defined in the same way, as,

$$(1.25)(.30)+(1.50)(.25)+(1.75)(.20)+(2.00)(.15)+(2.25)(.09)+(2.50)(.01)=\$1.6275$$

We call this the expected value of $(0.25X+1)$ and write $E\left(0.25X+1\right)=1.6275$.
As a further illustration, suppose a toll designed to encourage car pooling charged $\$12/x^2$ if there were x people in the car. This scheme would yield an average toll, in theory, of

$$\left(\frac{12}{1}\right)(.30)+\left(\frac{12}{4}\right)(.25)+\left(\frac{12}{9}\right)(.20)+\left(\frac{12}{16}\right)(.15)+\left(\frac{12}{25}\right)(.09)+\left(\frac{12}{36}\right)(.01)=\$4.7757$$

that is,

$$E\left(\frac{12}{X^2}\right)=4.7757$$

is the " expected value " of $\left(\frac{12}{X^2}\right)$.
With this as background, we can now make a formal definition.

Theorem 16 *Suppose the random variable X has probability function $f(x)$. Then the **expected value** of some function $g(X)$ of X is given by*

$$E\left[g(X)\right]=\sum_{\text{all } x} g(x)f(x)$$

Proof. To use definition 15, we need to determine the expected value of the random variable $Y=g(X)$ by first finding the probability function of Y, say $f_Y(y)=P(Y=y)$ and then computing

$$E[g(X)]=E(Y)=\sum_{\text{all } y} y f_Y(y) \tag{7.5}$$

Notice that if we let $D_y = \{x; g(x) = y\}$ be the set of x values with a given value y for $g(x)$, then

$$f_Y(y) = P(g(X) = y) = \sum_{x \in D_y} f(x)$$

Substituting this in (7.5) we obtain

$$\begin{aligned} E[g(X)] &= \sum_{\text{all } y} y f_Y(y) \\ &= \sum_{\text{all } y} y \sum_{x \in D_y} f(x) \\ &= \sum_{\text{all } y} \sum_{x \in D_y} g(x) f(x) \\ &= \sum_{\text{all } x} g(x) f(x) \end{aligned}$$

■

Notes:

(1) You can interpret $E[g(X)]$ as the average value of $g(X)$ in an infinite series of repetitions of the process where X is defined.

(2) $E\left[g(X)\right]$ is also known as the "expected value" of $g(X)$. This name is somewhat misleading since the average value of $g(X)$ may be a value which $g(X)$ never takes - hence unexpected!

(3) The case where $g(x) = x$ reduces to our earlier definition of $E(X)$.

(4) Confusion sometimes arises because we have two notations for the mean of a probability distribution: μ and $E(X)$ mean the same thing. There is a small advantage to using the (lower case) letter μ. It makes it visually clearer that the expected value is NOT a random variable like X but a non-random constant.

(5) When calculating expectations, look at your answer to be sure it makes sense. If X takes values from 1 to 10, you should know you've made an error if you get $E(X) > 10$ or $E(X) < 1$. In physical terms, $E(X)$ is the balance point for the histogram of $f(x)$.

Let us note a couple of mathematical properties of expected value that can help to simplify calculations.

Linearity Properties of Expectation: If your linear algebra is good, it may help if you think of E as being a linear operator, and this may save memorizing these properties.

1. For constants a and b,

$$E\left[ag(X)+b\right] = aE\left[g(X)\right] + b$$

Proof:

$$\begin{aligned} E\left[ag(X)+b\right] &= \sum_{\text{all } x} \left[ag(x)+b\right] f(x) \\ &= \sum_{\text{all } x} \left[ag(x)f(x) + bf(x)\right] \\ &= a\sum_{\text{all } x} g(x)f(x) + b\sum_{\text{all } x} f(x) \\ &= aE\left[g(X)\right] + b \quad \text{since } \sum_{\text{all } x} f(x) = 1 \end{aligned}$$

2. Similarly for constants a and b and two functions g_1 and g_2, it is also easy to show

$$E\left[ag_1(X) + bg_2(X)\right] = aE\left[g_1(X)\right] + bE\left[g_2(X)\right]$$

Don't let expected value intimidate you. Much of it is common sense. For example, using property 1, with we let $a = 0$ and $b = 13$ we obtain $E(13) = 13$. The expected value of a constant b is, of course, equal to b. The property also implies $E\left(2X\right) = 2E(X)$ if we use $a = 2$, $b = 0$, and $g(X) = X$. This is obvious also. Note, however, that for $g(x)$ a *nonlinear* function, it is NOT generally true that $E[g(X)] = g(E(X))$; this is a common mistake. (Check this for the example above when $g(X) = 12/X^2$.)

7.3 Some Applications of Expectation

Because expected value is an average value, it is frequently used in problems where costs or profits are connected with the outcomes of a random variable X. It is also used as a summary statistic; for example, one often hears about the expected life (expectation of lifetime) for a person or the expected return on an investment. Be cautious however. The expected value does NOT tell the whole story about a distribution. One investment could have a higher expected value than another but much much larger probability of large losses.
The following are examples.

Example: Expected Winnings in a Lottery. *A small lottery*[29] *sells 1000 tickets numbered* $000, 001, \ldots, 999$*; the tickets cost $10 each. When all the tickets have been sold the draw takes place: this consists of a single ticket from 000 to 999 being chosen at random. For ticket holders the prize structure is as follows:*

- *Your ticket is drawn - win $5000.*
- *Your ticket has the same first two number as the winning ticket, but the third is different - win $100.*
- *Your ticket has the same first number as the winning ticket, but the second number is different - win $10.*
- *All other cases - win nothing.*

Let the random variable X represent the winnings from a given ticket. Find $E(X)$.

Solution: The possible values for X are 0, 10, 100, 5000 (dollars). First, we need to find the probability function for X. We find (make sure you can do this) that $f(x) = P(X = x)$ has values

$$f(0) = 0.900, \qquad f(10) = 0.090, \quad f(100) = .009, \quad f(5000) = .001$$

The expected winnings are thus the expected value of X, or

$$E(X) = \sum_{\text{all x}} x f(x) = \$6.80$$

Thus, the gross expected winnings per ticket are $6.80. However, since a ticket costs $10 your expected net winnings are negative, -$3.20 (i.e. an expected loss of $3.20).

[29] "Here's something to think about: How come you never see a headline like 'Psychic Wins Lottery'? " Jay Leno (1950 -)

Remark: For any lottery or game of chance the expected net winnings per play is a key value. A fair game is one for which this value is 0. Needless to say, casino games and lotteries are never fair: the expected net winnings for a player are always negative.

Remark: The random variable associated with a given problem may be defined in different ways but the expected winnings will remain the same. For example, instead of defining X as the amount won we could have defined $X = 0, 1, 2, 3$ as follows:

$X = 3$	all 3 digits of number match winning ticket
$X = 2$	1st 2 digits (only) match
$X = 1$	1st digit (but not the 2nd) match
$X = 0$	1st digit does not match

Now, we would define the function $g(x)$ as the winnings when the outcome $X = x$ occurs. Thus,

$$g(0) = 0, \qquad g(1) = 10, \qquad g(2) = 100, \qquad g(3) = 5000$$

The expected winnings are then

$$E(g(X)) = \sum_{x=0}^{3} g(x)f(x) = \$6.80,$$

the same as before.

Example: Diagnostic Medical Tests: Often there are cheaper, less accurate tests for diagnosing the presence of some conditions in a person, along with more expensive, accurate tests. Suppose we have two cheap tests and one expensive test, with the following characteristics. All three tests are positive if a person has the condition (there are no "false negatives"), but the cheap tests give "false positives". Let a person be chosen at random, and let D = {person has the condition}. The three tests are

Test 1:	P (positive test $\|\overline{D}$) = .05; test costs \$5.00
Test 2:	P (positive test $\|\overline{D}$) = .03; test costs \$8.00
Test 3:	P (positive test $\|\overline{D}$) = 0; test costs \$40.00

We want to check a large number of people for the condition, and have to choose among three testing strategies:

(i) Use Test 1, followed by Test 3 if Test 1 is positive[30].

(ii) Use Test 2, followed by Test 3 if Test 2 is positive.

[30] Assume that given D or $\overline{D}$, tests are independent of one another.

(iii) Use Test 3.

Determine the expected cost per person under each of strategies (i), (ii) and (iii). We will then choose the strategy with the lowest expected cost. It is known that about .001 of the population have the condition (i.e. $P(D) = .001, P(\overline{D}) = .999$).

Solution: Define the random variable X as follows (for a random person who is tested):

$X = 1$ if the initial test is negative
$X = 2$ if the initial test is positive

Also let $g(x)$ be the total cost of testing the person. The expected cost per person is then

$$E[g(X)] = \sum_{x=1}^{2} g(x)f(x)$$

The probability function $f(x)$ for X and function $g(x)$ differ for strategies (i), (ii) and (iii). Consider for example strategy (i). Then

$$\begin{aligned} P(X = 2) &= P\,(\text{initial test positive}) \\ &= P(D) + P(\text{positive}|\overline{D})P(\overline{D}) \\ &= .001 + (.05)(.999) = 0.0510 \end{aligned}$$

The rest of the probabilities, associated values of $g(X)$ and $E[g(X)]$ are obtained below.

(i) $f(1) = P(X = 1) = 1 - f(2) = 1 - 0.0510 = 0.949$ (see $f(2)$ below)
$f(2) = 0.0510$ (obtained above)
$g(1) = 5 \qquad g(2) = 45$
$E[g(X)] = 5(.949) + 45(.0510) = \$7.04$

(ii) $f(2) = .001 + (.03)(.999) = .03097$
$f(1) = 1 - f(2) = .96903$
$g(1) = 8 \qquad g(2) = 48$
$E[g(X)] = 8(.96903) + 48(.03097) = \$9.2388$

(iii) $f(2) = .001, f(1) = .999$
$g(0) = g(1) = 40$
$E[g(X)] = \$40.00$

Thus, it is cheapest to use strategy (i).

Problem:

7.3.1 A lottery[31] has tickets numbered 000 to 999 which are sold for $1 each. One ticket is selected at random and a prize of $200 is given to any person whose ticket number is a permutation of the selected ticket number. All 1000 tickets are sold. What is the expected profit or loss to the organization running the lottery?

[31] "I've done the calculation and your chances of winning the lottery are identical whether you play or not." Fran Lebowitz (1950 -)

7.4 Means and Variances of Distributions

Its useful to know the means, $\mu = E(X)$ of probability models derived in Chapter 6.

Example: (Expected Value of the binomial distribution) Let $X \sim Bi(n, p)$. Find $E(X)$.

Solution:

$$\begin{aligned} \mu &= E(X) = \sum_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n} x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} \end{aligned}$$

When $x = 0$ the value of the expression is 0. We can therefore begin our sum at $x = 1$. Provided $x \neq 0$, we can expand $x!$ as $x(x-1)!$ (so it is important to eliminate the term when $x = 0$).

$$\text{Therefore} \mu = \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!} p p^{x-1}(1-p)^{(n-1)-(x-1)}$$

$$= np(1-p)^{n-1} \sum_{x=1}^{n} \binom{n-1}{x-1} \left(\frac{p}{1-p}\right)^{x-1}$$

Let $y = x - 1$ in the sum, to get

$$\begin{aligned} \mu &= np(1-p)^{n-1} \sum_{y=0}^{n-1} \binom{n-1}{y} \left(\tfrac{p}{1-p}\right)^{y} \\ &= np\,(1-p)^{n-1} \left(1 + \tfrac{p}{1-p}\right)^{n-1} \quad (\text{binomial theorem}) \\ &= np\,(1-p)^{n-1} \tfrac{(1-p+p)^{n-1}}{(1-p)^{n-1}} = np \end{aligned}$$

Exercise: Does this result make sense? If you try something 100 times and there is a 20% chance of success each time, how many successes do you expect to get, on average?

Example: (Expected value of the Poisson distribution) Let X have a Poisson distribution where λ is the average rate of occurrence and the time interval is of length t. Find $\mu = E(X)$.

Solution: The probability function of X is $f(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$. Then $\mu = E(X) = \sum_{x=0}^{\infty} x\frac{(\lambda t)^x e^{-\lambda t}}{x!}$. As in the binomial example, we can eliminate the term when $x = 0$ and expand $x!$ as $x(x-1)!$ for

$x = 1, 2, \cdots, \infty$.

$$\begin{aligned}
\mu &= \sum_{x=1}^{\infty} x \frac{(\lambda t)^x \, e^{-\lambda t}}{x!} = \sum_{x=1}^{\infty} x \frac{(\lambda t)^x \, e^{-\lambda t}}{x(x-1)!} \\
&= \sum_{x=1}^{\infty} (\lambda t) e^{-\lambda t} \frac{(\lambda t)^{x-1}}{(x-1)!} = (\lambda t) \, e^{-\lambda t} \sum_{x=1}^{\infty} \frac{(\lambda t)^{x-1}}{(x-1)!} \\
&= (\lambda t) e^{-\lambda t} \sum_{y=0}^{\infty} \frac{(\lambda t)^y}{y!} \text{ letting } y = x - 1 \text{ in the sum} \\
&= (\lambda t) e^{-\lambda t} e^{\lambda t} \text{ since } e^x = \sum_{y=0}^{\infty} \frac{x^y}{y!} \\
&= \lambda t.
\end{aligned}$$

Note that we used the symbol $\mu = \lambda t$ earlier in connection with the Poisson model; this was because we knew (but couldn't show until now) that $E(X) = \mu$.

Exercise: These techniques can also be used to work out the mean for the hypergeometric or negative binomial distributions. Looking back at how we proved that $\sum f(x) = 1$ shows the same method of summation used to find μ. However, in Chapter 8 we will give a simpler method of finding the means of these distributions, which are $E(X) = nr/N$ (hypergeometric) and $E(X) = k(1-p)/p$ (negative binomial).

Variability: While an average or expected value is a useful summary of a set of observations, or a probability distribution, it omits another important piece of information, namely the amount of variability. For example, it would be possible for car doors to be the right width, on average, and still have no doors fit properly. In the case of fitting car doors, we would also want the door widths to all be close to this correct average. We give a way of measuring the amount of variability next. You might think we could use the average difference between X and μ to indicate the amount of variation. In terms of expectation, this would be $E(X - \mu)$. However, $E(X - \mu) = E(X) - \mu$ (since μ is a constant) = 0. We soon realize that for a measure of variability, we can use the expected value of a function that has the same sign for $X > \mu$ and for $X < \mu$. One might try the expected value of the distance between X and its mean, e.g. $E(|X - \mu|)$. An alternative, more mathematically tractable version squares the distance (much as Euclidean distance in $\Re^n$ involves a sum of squared distances) is the variance.

Definition 17 *The **variance** of a r.v X is $E\left[(X-\mu)^2\right]$, and is denoted by σ^2 or by Var (X).*

In words, the variance is the average square of the distance from the mean. This turns out to be a very useful measure of the variability of X.

Example: *Let X be the number of heads when a fair coin is tossed 4 times. Then $X \sim Bi\left(4, \frac{1}{2}\right)$ so $\mu = np = (4)\left(\frac{1}{2}\right) = 2$. Find $Var(X)$.*

Without doing any calculations we know $Var(X) = \sigma^2 \leq 4$. This is because X is always between 0 and 4 and so the maximum possible value for $(X-\mu)^2$ is $(4-2)^2$ or $(0-2)^2$ which is 4. An expected value of a function, say $E(g(x)]$ is always somewhere between the minimum and the maximum value of the function $g(x)$ so in this case $0 \leq Var(X) \leq 4$. The values of $f(x)$ are

x	0	1	2	3	4
$f(x)$	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

since $f(x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x} = \binom{4}{x}\left(\frac{1}{2}\right)^4$

The value of $Var(X)$ (i.e. σ^2) is easily found here:

$$\begin{aligned}\sigma^2 &= E\left[(X-\mu)^2\right] = \sum_{x=0}^{4}(x-\mu)^2 f(x)\\ &= (0-2)^2\left(\tfrac{1}{16}\right) + (1-2)^2\left(\tfrac{4}{16}\right) + (2-2)^2\left(\tfrac{6}{16}\right) + (3-2)^2\left(\tfrac{4}{16}\right) + (4-2)^2\left(\tfrac{1}{16}\right)\\ &= 1\end{aligned}$$

If we keep track of units of measurement the variance will be in peculiar units; e.g. if X is the number of heads in 4 tosses of a coin, σ^2 is in units of heads^2! We can regain the original units by taking (positive) $\sqrt{\text{variance}}$. This is called the standard deviation of X, and is denoted by σ, or as $SD(X)$.

Definition 18 *The standard deviation of a random variable X is $\sigma = \sqrt{E\left[(X-\mu)^2\right]}$*

Both variance and standard deviation are commonly used to measure variability.

The basic definition of variance is often awkward to use for mathematical calculation of σ^2, whereas the following two results are often useful:

$$\begin{aligned}(1)\quad \sigma^2 &= E\left(X^2\right) - \mu^2\\ (2)\quad \sigma^2 &= E\left[X(X-1)\right] + \mu - \mu^2\end{aligned}$$

Proof:

(1) Using properties of expected value ,

$$\begin{aligned}\sigma^2 &= E\left[(X-\mu)^2\right] = E\left[X^2 - 2\mu X + \mu^2\right]\\ &= E\left(X^2\right) - 2\mu E(X) + \mu^2 \quad (\text{since } \mu \text{ is constant})\\ &= E\left(X^2\right) - 2\mu^2 + \mu^2 \quad (\text{ Therefore } E(X) = \mu)\\ &= E\left(X^2\right) - \mu^2\end{aligned}$$

(2) since $X^2 = X(X-1) + X$

$$\begin{aligned} \text{Therefore} \quad E\left(X^2\right) - \mu^2 &= E\left[X\left(X-1\right) + X\right] - \mu^2 \\ &= E\left[X(X-1)\right] + E(X) - \mu^2 \\ &= E\left[X(X-1)\right] + \mu - \mu^2 \end{aligned}$$

Formula (2) is most often used when there is an $x!$ term in the denominator of $f(x)$. Otherwise, formula (1) is generally easier to use.

Example: (Variance of binomial distribution)
Let $X \sim Bi(n,p)$. Find Var (X).

Solution: The probability function for the binomial is

$$f(x) = \frac{n!}{x!(n-x)!}\, p^x\,(1-p)^{n-x}$$

so we'll use formula (2) above,

$$E\left[X(X-1)\right] = \sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!}\;\; p^x(1-p)^{n-x}$$

If $x = 0$ or $x = 1$ the value of the term is 0, so we can begin summing at $x = 2$. For $x \neq 0$ or 1, we can expand the $x!$ as $x(x-1)(x-2)!$

$$\text{Therefore } E\left[X\left(X-1\right)\right] = \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x(1-p)^{n-x}$$

Now re-group to fit the binomial theorem, since that was the summation technique used to show $\sum f(x) = 1$ and to derive $\mu = np$.

$$\begin{aligned} E\left[X(X-1)\right] &= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\left[(n-2)-(x-2)\right]!} p^2 p^{x-2}(1-p)^{(n-2)-(x-2)} \\ &= n(n-1)p^2(1-p)^{n-2} \sum_{x=2}^{n} \binom{n-2}{x-2} \left(\frac{p}{1-p}\right)^{x-2} \end{aligned}$$

Let $y = x - 2$ in the sum, giving

$$\begin{aligned} E\left[X(X-1)\right] &= n(n-1)p^2(1-p)^{n-2} \sum_{y=0}^{n-2} \binom{n-2}{y} \left(\frac{p}{1-p}\right)^{y} \\ &= n(n-1)p^2(1-p)^{n-2}\left(1+\tfrac{p}{1-p}\right)^{n-2} \\ &= n(n-1)p^2(1-p)^{n-2}\tfrac{(1-p+p)^{n-2}}{(1-p)^{n-2}} = n(n-1)p^2 \end{aligned}$$

Then

$$\begin{aligned}\sigma^2 &= E\left[X\left(X-1\right)\right]+\mu-\mu^2\\ &= n(n-1)p^2+np-(np)^2\\ &= n^2p^2-np^2+np-n^2p^2=np(1-p)\end{aligned}$$

Remember that the variance of a binomial distribution is $np(1-p)$, since we'll be using it later in the course.

Example: (Variance of Poisson distribution) Find the variance of the Poisson distribution.
Solution: The probability function of the Poisson is

$$f(x)=\frac{\mu^x e^{-\mu}}{x!}$$

from which we obtain

$$\begin{aligned}E\left[X(X-1)\right] &= \sum_{x=0}^{\infty} x(x-1)\frac{\mu^x e^{-\mu}}{x!}\\ &= \sum_{x=2}^{\infty} x(x-1)\frac{\mu^x e^{-\mu}}{x(x-1)(x-2)!}, \text{ setting the lower limit to 2 and expanding } x!\\ &= \mu^2 e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^{x-2}}{(x-2)!}\end{aligned}$$

Let $y=x-2$ in the sum, giving

$$E\left[X(X-1)\right]=\mu^2e^{-\mu}\sum_{y=0}^{\infty}\frac{\mu^y}{y!}=\mu^2e^{-\mu}e^{\mu}=\mu^2 \text{ so}$$

$$\begin{aligned}\sigma^2 &= E\left[X(X-1)\right]+\mu-\mu^2\\ &= \mu^2+\mu-\mu^2=\mu\end{aligned}$$

(*For the Poisson distribution, the variance equals the mean.*)

Properties of Mean and Variance
If a and b are constants and $Y=aX+b$, then

$$\mu_Y=a\mu_X+b \text{ and } \sigma_Y^2=a^2\sigma_X^2$$

(where μ_X and σ_X^2 are the mean and variance of X and μ_Y and σ_Y^2 are the mean and variance of Y).
Proof:
We already showed that $E(aX+b)=aE(X)+b$.

i.e. $\mu_Y = a\mu_X + b$, and then

$$\begin{aligned}\sigma_Y^2 &= E\left[(Y-\mu_Y)^2\right] = E\left\{[(aX+b)-(a\mu_X+b)]^2\right\} \\ &= E\left[(aX-a\mu_X)^2\right] = E\left[a^2(X-\mu_X)^2\right] \\ &= a^2 E\left[(X-\mu_X)^2\right] = a^2\sigma_X^2\end{aligned}$$

This result is to be expected. Adding a constant, b, to all values of X has no effect on the amount of variability. So it makes sense that Var $(aX+b)$ doesn't depend on the value of b. Also since variance is in squared units, multiplication by a constant results in multiplying the variance by the constant squared. A simple way to relate to this result is to consider a random variable X which represents a temperature in degrees Celsius (even though this is a continuous random variable which we don't study until Chapter 9). Now let Y be the corresponding temperature in degrees Fahrenheit. We know that

$$Y = \frac{9}{5}X + 32$$

and it is clear if we think about it that $\mu_Y = (\frac{9}{5})\mu_X + 32$ and that $\sigma_Y^2 = (\frac{9}{5})^2\sigma_X^2$.

Problems:

7.4.1 An airline knows that there is a 97% chance a passenger for a certain flight will show up, and assumes passengers arrive independently of each other. Tickets cost $100, but if a passenger shows up and can't be carried on the flight the airline has to refund the $100 and pay a penalty of $400 to each such passenger. If the passenger does not show up, the airline must fully refund the price of the ticket. How many tickets should they sell for a plane with 120 seats to maximize their expected ticket revenues after paying any penalty charges? Assume ticket holders who don't show up get a full refund for their unused ticket.

7.4.2 A typist typing at a constant speed of 60 words per minute makes a mistake in any particular word with probability .04, independently from word to word. Each incorrect word must be corrected; a task which takes 15 seconds per word.

(a) Find the mean and variance of the time (in seconds) taken to finish a 450 word passage.

(b) Would it be less time consuming, on average, to type at 45 words per minute if this reduces the probability of an error to .02?

7.5 Moment Generating Functions♢

[32]We have now seen two functions which characterize a distribution, the probability function and the cumulative distribution function. There is a third type of function, the *moment generating function*, which uniquely determines a distribution. The moment generating function is closely related to other transforms used in mathematics, the Laplace and Fourier transforms.

Definition 19 *Consider a discrete random variable X with probability function $f(x)$. The moment generating function (m.g.f.) of X is defined as*

$$M(t) = E(e^{tX}) = \sum_x e^{tx} f(x).$$

We will assume that the moment generating function is defined and finite for values of t in an interval around 0 (i.e. for some $a > 0$, $\sum_x e^{tx} f(x) < \infty$ for all $t \in [-a, a]$).

The *moments* of a random variable X are the expectations of the functions X^r for $r = 1, 2, \ldots$. The expected value $E(X^r)$ is called r^{th} moment of X. The mean $\mu = E(X)$ is therefore the first moment, $E(X^2)$ the second and so on. It is often easy to find the moments of a probability distribution mathematically by using the moment generating function. This often gives easier derivations of means and variances than the direct summation methods in the preceding section. The following theorem gives a useful property of m.g.f.'s.

Theorem 20 *Let the random variable X have m.g.f. $M(t)$. Then*

$$E(X^r) = M^{(r)}(0) \quad r = 1, 2, \ldots$$

where $M^{(r)}(0)$ stands for $d^r M(t)/dt^r$ evaluated at $t = 0$.

Proof:
$M(t) = \sum_x e^{tx} f(x)$ and if the sum converges, then

$$\begin{aligned} M^{(r)}(t) &= \tfrac{d}{dt^r} \sum_x e^{tx} f(x) \\ &= \sum_x \tfrac{d}{dt^r} (e^{tx}) f(x) \\ &= \sum_x x^r e^{tx} f(x) \end{aligned}$$

Therefore $M^{(r)}(0) = \sum_x x^r f(x) = E(X^r)$, as stated.

[32]♢ This section optional for stat 220

This sometimes gives a simple way to find the moments for a distribution.

Example 1. Suppose X has a Binomial(n, p)distribution. Then its moment generating function is

$$\begin{aligned} M(t) &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= (pe^t + 1 - p)^n \end{aligned}$$

Therefore

$$\begin{aligned} M'(t) &= npe^t(pe^t + 1 - p)^{n-1} \\ M''(t) &= npe^t(pe^t + 1 - p)^{n-1} + n(n-1)p^2e^{2t}(pe^t + 1 - p)^{n-2} \end{aligned}$$

and so

$$\begin{aligned} E(X) &= M'(0) = np, \\ E(X^2) &= M"(0) = np + n(n-1)p^2 \\ \text{Var}(X) &= E(X^2) - E(X)^2 = np(1-p) \end{aligned}$$

Exercise. Poisson distribution
Show that the Poisson distribution with probability function

$$f(x) = e^{-\mu}\mu^x/x! \quad x = 0, 1, 2, \ldots$$

has m.g.f. $M(t) = e^{-\mu+\mu e^t}$. Then show that $E(X) = \mu$ and $Var(X) = \mu$.

The m.g.f. also uniquely identifies a distribution in the sense that two different distributions cannot have the same m.g.f. This result is often used to find the distribution of a random variable. For example if I can show somehow that the moment generating function of a random variable X is

$$e^{2(e^t-1)}$$

then I know, from the above exercise that the random variable must have a Poisson(2) distribution. Moment generating functions are often used to identify a given distribution. If two random variables have the same moment generating function, they have the same distribution (so the same probability function, cumulative distribution function, moments, etc.). Of course the moment generating functions must match for all values of t, in other words they agree as *functions*, not just at a few points. Moment generating functions can also be used to determine that a sequence of distributions gets closer and

closer to some limiting distribution. To show this (albeit a bit loosely), suppose that a sequence of probability functions $f_n(x)$ have corresponding moment generating functions

$$M_n(t) = \sum_x e^{tx} f_n(x)$$

Suppose moreover that the probability functions $f_n(x)$ converge to another probability function $f(x)$ pointwise in x as $n \to \infty$. This is what we mean by convergence of discrete distributions. Then since

$$f_n(x) \to f(x) \text{ as } n \to \infty \text{ for each } x, \tag{7.6}$$

$$\sum_x e^{tx} f_n(x) \to \sum_x e^{tx} f(x) \text{ as } n \to \infty \text{ for each } t \tag{7.7}$$

which says that $M_n(t)$ converges to $M(t)$ the moment generating function of the limiting distribution. It shouldn't be too surprising that a very useful converse to this result also holds. (This is strictly an aside and may be of interest only to those with a thing for infinite series, but is it always true that because the individual terms in a series converge as in (7.6) does this guarantee that the sum of the series also converges (7.7)?)

Suppose conversely that X_n has moment generating function $M_n(t)$ and $M_n(t) \to M(t)$ for each t such that $M(t) < \infty$. For example we saw in Chapter 6 that a Binomial(n, p) distribution with very large n and very small p is close to a Poisson distribution with parameter $\mu = np$. Consider the moment generating function of such a binomial random variable

$$\begin{aligned} M(t) &= (pe^t + 1 - p)^n \\ &= \{1 + p(e^t - 1)\}^n \\ &= \{1 + \frac{\mu}{n}(e^t - 1)\}^n \end{aligned} \tag{7.8}$$

Now take the limit of this expression as $n \to \infty$. Since in general

$$(1 + \frac{c}{n})^n \to e^c$$

the limit of (7.8) as $n \to \infty$ is

$$e^{\mu(e^t - 1)} = e^{-\mu + \mu e^t}$$

and this is the moment generating function of a Poisson distribution with parameter μ. This shows a little more formally than we did earlier that the binomial(n, p) distribution with (small) $p = \mu/n$ approaches the Poisson(μ) distribution as $n \to \infty$.

7.6 Problems on Chapter 7

7.1 Let X have probability function $f(x) = \begin{cases} \frac{1}{2x} \text{ for } x = 2, 3, 4, 5, \text{ or } 6 \\ 11/40 \text{ for } x = 1 \end{cases}$ Find the mean and variance for X.

7.2 A game is played where a fair coin is tossed until the first tail occurs. The probability x tosses will be needed is $f(x) = .5^x;\ x = 1, 2, 3, \cdots$. You win $\$2^x$ if x tosses are needed for $x = 1, 2, 3, 4, 5$ but lose \$256 if $x > 5$. Determine your expected winnings.

7.3 Diagnostic tests. Consider diagnostic tests like those discussed above in the example of Section 7.3 and in Problem 15 for Chapter 4. Assume that for a randomly selected person, $P(D) = .02$, $P(R|D) = 1$, $P(R|\overline{D}) = .05$, so that the inexpensive test only gives false positive, and not false negative, results.
Suppose that this inexpensive test costs \$10. If a person tests positive then they are also given a more expensive test, costing \$100, which correctly identifies all persons with the disease. What is the expected cost per person if a population is tested for the disease using the inexpensive test followed, if necessary, by the expensive test?

7.4 Diagnostic tests II. Two percent of the population has a certain condition for which there are two diagnostic tests. Test A, which costs \$1 per person, gives positive results for 80% of persons with the condition and for 5% of persons without the condition. Test B, which costs \$100 per person, gives positive results for all persons with the condition and negative results for all persons without it.

(a) Suppose that test B is given to 150 persons, at a cost of \$15,000. How many cases of the condition would one expect to detect?

(b) Suppose that 2000 persons are given test A, and then only those who test positive are given test B. Show that the expected cost is \$15,000 but that the expected number of cases detected is much larger than in part (a).

7.5 The probability that a roulette wheel stops on a red number is 18/37. For each bet on "red" you are returned twice your bet (including your bet) if the wheel stops on a red number, and lose your money if it does not.

(a) If you bet \$1 on each of 10 consecutive plays, what is your expected winnings? What is your expected winnings if you bet \$10 on a single play?

(b) For each of the two cases in part (a), calculate the probability that you made a profit (that is, your "winnings" are positive, not negative).

7.6 Slot machines. Consider the slot machine discussed above in Problem 16 for Chapter 4. Suppose that the number of each type of symbol on wheels 1, 2 and 3 is as given below:

	Wheel		
Symbols	1	2	3
Flower	2	6	2
Dog	4	3	3
House	4	1	5

If all three wheels stop on a flower, you win \$20 for a \$1 bet. If all three wheels stop on a dog, you win \$10, and if all three stop on a house, you win \$5. Otherwise you win nothing.

Find your expected winnings per dollar spent.

7.7 Suppose that n people take a blood test for a disease, where each person has probability p of having the disease, independent of other persons. To save time and money, blood samples from k people are pooled and analyzed together. If none of the k persons has the disease then the test will be negative, but otherwise it will be positive. If the pooled test is positive then each of the k persons is tested separately (so $k+1$ tests are done in that case).

(a) Let X be the number of tests required for a group of k people. Show that

$$E(X) = k + 1 - k(1-p)^k.$$

(b) What is the expected number of tests required for n/k groups of k people each? If $p = .01$, evaluate this for the cases $k = 1, 5, 10$.

(c) Show that if p is small, the expected number of tests in part (b) is approximately $n(kp + k^{-1})$, and is minimized for $k \doteq p^{-1/2}$.

7.8 A manufacturer of car radios ships them to retailers in cartons of n radios. The profit per radio is \$59.50, less shipping cost of \$25 per carton, so the profit is \$ $(59.5n - 25)$ per carton. To promote sales by assuring high quality, the manufacturer promises to pay the retailer $\$200X^2$ if X radios in the carton are defective. (The retailer is then responsible for repairing any defective radios.) Suppose radios are produced independently and that 5% of radios are defective. How many radios should be packed per carton to maximize expected net profit per carton?

7.9 Let X have a geometric distribution with probability function
$f(x) = p(1-p)^x;\ \ x = 0, 1, 2, ...$

(a) Calculate the m.g.f. $M(t) = E\left(e^{tX}\right)$, where t is a parameter.

(b) Find the mean and variance of X.

(c) Use your result in (b) to show that if p is the probability of "success" (S) in a sequence of Bernoulli trials, then the expected number of trials until the first S occurs is $1/p$. Explain why this is "obvious".

7.10 **Analysis of Algorithms: Quicksort.** Suppose we have a set S of distinct numbers and we wish to sort them from smallest to largest. The quicksort algorithm works as follows: When $n = 2$ it just compares the numbers and puts the smallest one first. For $n > 2$ it starts by choosing a random "pivot" number from the n numbers. It then compares each of the other $n - 1$ numbers with the pivot and divides them into groups S_1 (numbers smaller than the pivot) and $\bar{S}_1$ (numbers bigger than the pivot). It then does the same thing with S_1 and $\bar{S}_1$ as it did with S, and repeats this recursively until the numbers are all sorted. (Try this out with, say $n = 10$ numbers to see how it works.) In computer science it is common to analyze such algorithms by finding the expected number of comparisons (or other operations) needed to sort a list. Thus, let

$$C_n = \text{expected number of comparisons for lists of length } n$$

(a) Show that if X is the number of comparisons needed,

$$C_n = \sum_{i=1}^{n} E(X|\text{ initial pivot is } i\text{th smallest number}) \left(\frac{1}{n}\right)$$

(b) Show that

$$E(X|\text{initial pivot is } i\text{th smallest number}) = n - 1 + C_{i-1} + C_{n-i}$$

and thus that C_n satisfies the recursion (note $C_0 = C_1 = 0$)

$$C_n = n - 1 + \frac{2}{n} \sum_{k=1}^{n-1} C_k \quad n = 2, 3, \ldots$$

(c) Show that

$$(n + 1)C_{n+1} = 2n + (n + 2)C_n \quad n = 1, 2, \ldots$$

(d) (Harder) Use the result of part (c) to show that for large n,

$$\frac{C_{n+1}}{n+1} \sim 2 \log (n + 1)$$

(Note: $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$) This proves a result from computer science which says that for Quicksort, $C_n \sim O(n\ log\ n)$.

7.11 Find the distributions that correspond to the following moment-generating functions:
(a) $M(t) = \frac{1}{3e^{-t}-2}$, for $t < \ln(3/2)$
(b) $M(t) = e^{2(e^t-1)}$, for $t < \infty$

7.12 Find the moment generating function of the discrete uniform distribution X on $\{a, a+1, ..., b\}$;

$$P(X = x) = \frac{1}{b-a+1}, \text{for } x = a, a+1, ..., b.$$

What do you get in the special case $a = b$ and in the case $b = a+1$? Use the moment generating function in these two cases to confirm the expected value and the variance of X.

7.13 Let X be a random variable taking values in the set $\{0, 1, 2\}$ with moments $E(X) = 1$, $E(X^2) = 3/2$.

(a) Find the moment generating function of X
(b) Find the first six moments of X
(c) Find $P(X = i), i = 0, 1, 2$.
(d) Show that any probability distribution on $\{0, 1, 2\}$ is completely determined by its first two moments.

7.14 Assume that each week a stock either increases in value by \$1 with probability $\frac{1}{2}$ or decreases by \$1, these moves independent of the past. The current price of the stock is \$50. I wish to purchase a call option which allows me (if I wish to do so) the option of buying the stock 13 weeks from now at a "strike price" of \$55. Of course if the stock price at that time is \$55 or less there is no benefit to the option and it is not exercised. Assume that the return from the option is

$$g(S_{13}) = \max(S_{13} - 55, 0)$$

where S_{13} is the price of the stock in 13 weeks. What is the fair price of the option today assuming no transaction costs and 0% interest; i.e. what is $E[g(S_{13})]$?

7.15* **Challenge problem:** Let X_n be the number of ascents in a random permutation of the integers $\{1, 2, ...n\}$. For example, the number of ascents in the permutation 213546 is three, since $2, 135, 46$ form ascending sequences.

(a) Show that the following recursion for the probabilities $p_n(k) = P[X_n = k]$.

$$p_n(k) = \frac{k+1}{n} p_{n-1}(k) + \frac{n-k}{n} p_{n-1}(k-1)$$

(b) Cards numbered 1,2,....,n are shuffled, drawn and put into a pile as long as the card drawn has a number lower than its predecessor. A new pile is started whenever a higher card is drawn. Show that the distribution of the number of piles that we end with is that of $1 + X_n$ and that the expected number of piles is $\frac{n+1}{2}$.

8. Discrete Multivariate Distributions

8.1 Basic Terminology and Techniques

Many problems involve more than a single random variable. When there are multiple random variables associated with an experiment or process we usually denote them as $X, Y, \ldots$ or as $X_1, X_2, \ldots$. For example, your final mark in a course might involve X_1=your assignment mark, X_2=your midterm test mark, and X_3 =your exam mark. We need to extend the ideas introduced for single variables to deal with multivariate problems. Here we only consider discrete multivariate problems, though continuous multivariate variables are also common in daily life (e.g. consider a person's height X and weight Y, or X_1 =the return from Stock 1, X_2 =return from stock 2). To introduce the ideas in a simple setting, we'll first consider an example in which there are only a few possible values of the variables. Later we'll apply these concepts to more complex examples. The ideas themselves are simple even though some applications can involve fairly messy algebra.

Joint Probability Functions:

First, suppose there are two random variables X and Y, and define the function

$$\begin{aligned} f(x,y) &= P(X = x \text{ and } Y = y) \\ &= P(X = x, Y = y). \end{aligned}$$

We call $f(x,y)$ the joint probability function of (X,Y). In general,

$$f(x_1, x_2, \cdots, x_n) = P(X_1 = x_1 \text{ and } X_2 = x_2 \text{ and } \ldots \text{ and } X_n = x_n)$$

if there are n random variables $X_1, \ldots, X_n$.
The properties of a joint probability function are similar to those for a single variable; for two random variables we have $f(x,y) \geq 0$ for all (x,y) and

$$\sum_{\text{all(x,y)}} f(x,y) = 1.$$

Example: Consider the following numerical example, where we show $f(x, y)$ in a table.

$f(x,y)$	x = 0	1	2
y = 1	.1	.2	.3
y = 2	.2	.1	.1

for example $f(0, 2) = P(X = 0 \text{ and } Y = 2) = 0.2$. We can check that $f(x, y)$ is a proper joint probability function since $f(x, y) \geq 0$ for all 6 combinations of (x, y) and the sum of these 6 probabilities is 1. When there are only a few values for X and Y it is often easier to tabulate $f(x, y)$ than to find a formula for it. We'll use this example below to illustrate other definitions for multivariate distributions, but first we give a short example where we need to find $f(x, y)$.

Example: Suppose a fair coin is tossed 3 times. Define the random variables X = number of Heads and $Y = 1(0)$ if $H(T)$ occurs on the first toss. Find the joint probability function for (X, Y).

Solution: First we should note the range for (X, Y), which is the set of possible values (x, y) which can occur. Clearly X can be 0, 1, 2, or 3 and Y can be 0 or 1, but we'll see that not all 8 combinations (x, y) are possible.
We can find $f(x, y) = P(X = x, Y = y)$ by just writing down the sample space
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ that we have used before for this process. Then simple counting gives $f(x, y)$ as shown in the following table:

$f(x,y)$	x = 0	1	2	3
y = 0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
y = 1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

For example, $(X, Y) = (0, 0)$ if and only if the outcome is TTT; $(X, Y) = (1, 0)$ iff the outcome is either THT or TTH.
Note that the range or joint p.f. for (X, Y) is a little awkward to write down here in formulas, so we just use the table.

Marginal Distributions: We may be given a joint probability function involving more variables than we're interested in using. How can we eliminate any which are not of interest? Look at the first example above. If we're only interested in X, and don't care what value Y takes, we can see that

$$P(X = 0) = P(X = 0, Y = 1) + P(X = 0, Y = 2),$$

so $P(X = 0) = f(0, 1) + f(0, 2) = 0.3$. Similarly

$$P(X = 1) = f(1, 1) + f(1, 2) = .3 \text{ and}$$
$$P(X = 2) = f(2, 1) + f(2, 2) = .4$$

The distribution of X obtained in this way from the joint distribution is called the marginal probability function of X:

x	0	1	2
$f(x)$	.3	.3	.4

In the same way, if we were only interested in Y, we obtain

$$P(Y = 1) = f(0, 1) + f(1, 1) + f(2, 1) = .6$$

since X can be 0, 1, or 2 when $Y = 1$. The marginal probability function of Y would be:

y	1	2
$f(y)$	.6	.4

Our notation for marginal probability functions is still inadequate. What is $f(1)$? As soon as we substitute a number for x or y, we don't know which variable we're referring to. For this reason, we generally put a subscript on the f to indicate whether it is the marginal probability function for the first or second variable. So $f_1(1)$ would be $P(X = 1) = .3$, while $f_2(1)$ would be $P(Y = 1) = 0.6$. An alternative notation that you may see is $f_X(x)$ and $f_Y(y)$.
In general, to find $f_1(x)$ we add over all values of y where $X = x$, and to find $f_2(y)$ we add over all values of x with $Y = y$. Then

$$f_1(x) = \sum_{\text{all } y} f(x, y) \text{ and}$$
$$f_2(y) = \sum_{\text{all } x} f(x, y).$$

This reasoning can be extended beyond two variables. For example, with three variables (X_1, X_2, X_3),

$$f_1(x_1) = \sum_{\text{all } (x_2, x_3)} f(x_1, x_2, x_3) \text{ and}$$

$$f_{1,3}(x_1, x_3) = \sum_{\text{all } x_2} f(x_1, x_2, x_3) = P(X_1 = x_1, X_3 = x_3)$$

where $f_{1,3}(x_1, x_3)$ is the marginal joint distribution of (X_1, X_3).

Independent Random Variables:

For events A and B, we have defined A and B to be independent if and only if $P(AB) = P(A)\ P(B)$. This definition can be extended to random variables (X, Y): two random variables are independent if their joint probability function is the product of the marginal probability functions.

Definition 21 *X and Y are **independent** random variables iff $f(x, y) = f_1(x) f_2(y)$ for all values (x, y)*

Definition 22 *In general, $X_1, X_2, \cdots, X_n$ are independent random variables iff*

$$f(x_1, x_2, \cdots, x_n) = f_1(x_1) f_2(x_2). \cdots . f_n(x_n) \text{ for all } x_1, x_2, ...x_n$$

In our first example X and Y are not independent since $f_1(x) f_2(y) \neq f(x, y)$ for any of the 6 combinations of (x, y) values; e.g., $f(1, 1) = .2$ but $f_1(1) f_2(1) = (0.3)\ (0.6) \neq 0.2$. Be careful applying this definition. You can only conclude that X and Y are independent after checking all (x, y) combinations. Even a single case where $f_1(x) f_2(y) \neq f(x, y)$ makes X and Y dependent.

Conditional Probability Functions:

Again we can extend a definition from events to random variables. For events A and B, recall that $P(A|B) = \frac{P(AB)}{P(B)}$. Since $P(X = x|Y = y) = P(X = x, Y = y)/P(Y = y)$, we make the following definition.

Definition 23 *The conditional probability function of X given $Y = y$ is $f(x|y) = \frac{f(x,y)}{f_2(y)}$.*
Similarly, $f(y|x) = \frac{f(x,y)}{f_1(x)}$ (provided, of course, the denominator is not zero).

In our first example let us find $f(x|Y=1)$.

$$f(x|Y=1) = \frac{f(x,1)}{f_2(1)}.$$

This gives:

x	0	1	2
$f(x\|Y=1)$	$\frac{.1}{.6} = \frac{1}{6}$	$\frac{.2}{.6} = \frac{1}{3}$	$\frac{.3}{.6} = \frac{1}{2}$

As you would expect, marginal and conditional probability functions are probability functions in that they are always ≥ 0 and their sum is 1.

Functions of Variables:

In an example earlier, your final mark in a course might be a function of the 3 variables X_1, X_2, X_3 - assignment, midterm, and exam marks[33]. Indeed, we often encounter problems where we need to find the probability distribution of a function of two or more random variables. The most general method for finding the probability function for some function of random variables X and Y involves looking at every combination (x, y) to see what value the function takes. For example, if we let $U = 2(Y - X)$ in our example, the possible values of U are seen by looking at the value of $U = 2(y - x)$ for each (x, y) in the range of (X, Y).

		x		
	u	0	1	2
	1	2	0	-2
y				
	2	4	2	0

Then
$$\begin{aligned}
P(U=-2) &= P(X=2 \text{ and } Y=1) = f(2,1) = .3\\
P(U=0) &= P(X=1 \text{ and } Y=1, \text{ or } X=2 \text{ and } Y=2)\\
&= f(1,1) + f(2,2) = .3\\
P(U=2) &= f(0,1) + f(1,2) = .2\\
P(U=4) &= f(0,2) = .2
\end{aligned}$$

The probability function of U is thus

[33] "Don't worry about your marks. Just make sure that you keep up with the work and that you don't have to repeat a year. It s not necessary to have good marks in everything" Albert Einstein in letter to his son, 1916.

u	-2	0	2	4
$f(u)$	.3	.3	.2	.2

For some functions it is possible to approach the problem more systematically. One of the most common functions of this type is the total. Let $T = X + Y$. This gives:

		x		
t		0	1	2
	1	1	2	3
y				
	2	2	3	4

Then $P(T = 3) = f(1, 2) + f(2, 1) = .4$, for example. Continuing in this way, we get

t	1	2	3	4
$f(t)$	.1	.4	.4	.1

(We are being a little sloppy with our notation by using "f" for both $f(t)$ and $f(x, y)$. No confusion arises here, but better notation would be to write $f_T(t)$ for $P(T = t)$.) In fact, to find $P(T = t)$ we are simply adding the probabilities for all (x, y) combinations with $x + y = t$. This could be written as:

$$f(t) = \sum_{\substack{\text{all } (x,y) \\ \text{with } x+y=t}} f(x, y).$$

However, if $x + y = t$, then $y = t - x$. To systematically pick out the right combinations of (x, y), all we really need to do is sum over values of x and then substitute $t - x$ for y. Then,

$$f(t) = \sum_{\text{all } x} f(x, t - x) = \sum_{\text{all } x} P(X = x, Y = t - x)$$

So $P(T = 3)$ would be

$$P(T = 3) = \sum_{\text{all } x} f(x, 3 - x) = f(0, 3) + f(1, 2) + f(2, 1) = 0.4.$$

(note $f(0, 3) = 0$ since Y can't be 3.)

We can summarize the method of finding the probability function for a function $U = g(X, Y)$ of two random variables X and Y as follows:
Let $f(x, y) = P(X = x, Y = y)$ be the probability function for (X, Y). Then the probability function for U is

$$f_U(u) = P(U = u) = \sum_{\substack{\text{all}(x,y): \\ g(x,y)=u}} f(x, y)$$

This can also be extended to functions of three or more random variables $U = g(X_1, X_2, \ldots, X_n)$:

$$f_U(u) = P(U = u) = \sum_{\substack{(x_1,\ldots,x_n):\\ g(x_1,\ldots,x_n)=u}} f(x_1, \ldots, x_n).$$

(**Note:** Do not get confused between the functions f and g in the above: $f(x, y)$ is the joint probability function of the random variables X, Y whereas $U = g(X, Y)$ defines the "new" random variable that is a function of X and Y, and whose distribution we want to find.)

Example: Let X and Y be independent random variables having Poisson distributions with averages (means) of μ_1 and μ_2 respectively. Let $T = X + Y$. Find its probability function, $f(t)$.

Solution: We first need to find $f(x, y)$. Since X and Y are independent we know

$$f(x, y) = f_1(x) f_2(y)$$

Using the Poisson probability function,

$$f(x, y) = \frac{\mu_1^x e^{-\mu_1}}{x!} \frac{\mu_2^y e^{-\mu_2}}{y!}$$

where x and y can equal 0, 1, 2, ... Now,

$$P(T = t) = P(X + Y = t) = \sum_{\text{all } x} P(X = x, Y = t - x).$$

Then

$$\begin{aligned} f(t) &= \sum_{\text{all } x} f(x, t - x) \\ &= \sum_{x=0}^{t} \frac{\mu_1^x e^{-\mu_1}}{x!} \frac{\mu_2^{t-x} e^{-\mu_2}}{(t - x)!} \end{aligned}$$

To evaluate this sum, factor out constant terms and try to regroup in some form which can be evaluated by one of our summation techniques.

$$f(t) = \mu_2^t e^{-(\mu_1+\mu_2)} \sum_{x=0}^{t} \frac{1}{x!(t - x)!} \left(\frac{\mu_1}{\mu_2}\right)^x$$

If we had a $t!$ on the top inside the $\sum_{x=0}^{t}$, the sum would be of the form $\sum_{x=0}^{t} \binom{t}{x} \left(\frac{\mu_1}{\mu_2}\right)^x$. This is the right hand side of the binomial theorem. Multiply top and bottom by $t!$ to get:

$$\begin{aligned} f(t) &= \frac{\mu_2^t e^{-(\mu_1+\mu_2)}}{t!} \sum_{x=0}^{t} \binom{t}{x} \left(\frac{\mu_1}{\mu_2}\right)^x \\ &= \frac{\mu_2^t e^{-(\mu_1+\mu_2)}}{t!} (1 + \frac{\mu_1}{\mu_2})^t \text{ by the binomial theorem.} \end{aligned}$$

Take a common denominator of μ_2 to get

$$f(t) = \frac{\mu_2^t \, e^{-(\mu_1+\mu_2)}}{t!} \frac{(\mu_1+\mu_2)^t}{\mu_2^t} = \frac{(\mu_1+\mu_2)^t}{t!} e^{-(\mu_1+\mu_2)}, \text{ for } t = 0, 1, 2, \cdots$$

Note that we have just shown that the sum of 2 independent Poisson random variables also has a Poisson distribution.

Example: Three sprinters, A, B and C, compete against each other in 10 independent 100 m. races. The probabilities of winning any single race are .5 for A, .4 for B, and .1 for C. Let X_1, X_2 and X_3 be the number of races A, B and C win.

(a) Find the joint probability function, $f(x_1, x_2, x_3)$

(b) Find the marginal probability function, $f_1(x_1)$

(c) Find the conditional probability function, $f(x_2|x_1)$

(d) Are X_1 and X_2 independent? Why?

(e) Let $T = X_1 + X_2$. Find its probability function, $f(t)$.

Solution: Before starting, note that $x_1 + x_2 + x_3 = 10$ since there are 10 races in all. We really only have two variables since $x_3 = 10 - x_1 - x_2$. However it is convenient to use x_3 to save writing and preserve symmetry.

(a) The reasoning will be similar to the way we found the binomial distribution in Chapter 6 except that there are now 3 types of outcome. There are $\frac{10!}{x_1!x_2!x_3!}$ different outcomes (i.e. results for races 1 to 10) in which there are x_1 wins by A, x_2 by B, and x_3 by C. Each of these arrangements has a probability of (.5) multiplied x_1 times, (.4) x_2 times, and (.1) x_3 times in some order;

i.e., $(.5)^{x_1}(.4)^{x_2}(.1)^{x_3}$

Therefore

$$f(x_1, x_2, x_3) = \frac{10!}{x_1!x_2!x_3!}(.5)^{x_1}(.4)^{x_2}(.1)^{x_3}$$

The range for $f(x_1, x_2, x_3)$ is triples (x_1, x_2, x_3) where each x_i is an integer between 0 and 10, and where $x_1 + x_2 + x_3 = 10$.

(b) It would also be acceptable to drop x_3 as a variable and write down the probability function for X_1, X_2 only; this is

$$f(x_1, x_2) \;=\; \frac{10!}{x_1!x_2!(10 - x_1 - x_2)!}\;\; (.5)^{x_1}\;\; (.4)^{x_2}\;\; (.1)^{10-x_1-x_2},$$

because of the fact that X_3 must equal $10 - X_1 - X_2$. For this probability function $x_1 = 0, 1, \cdots, 10;\; x_2 = 0, 1, \cdots, 10$ and $x_1 + x_2 \leq 10$. This simplifies finding $f_1(x_1)$ a little . We now have $f_1(x_1) = \sum\limits_{x_2} f(x_1, x_2)$. The limits of summation need care: x_2 could be as small as 0, but since $x_1 + x_2 \leq 10$, we also require $x_2 \leq 10 - x_1$. (For example if $x_1 = 7$ then B can win $0, 1, 2$, or 3 races.) Thus,

$$\begin{aligned} f_1(x_1) &= \sum_{x_2=0}^{10-x_1} \frac{10!}{x_1!x_2!(10 - x_1 - x_2)!}(.5)^{x_1}(.4)^{x_2}(.1)^{10-x_1-x_2} \\ &= \frac{10!}{x_1!}(.5)^{x_1}(.1)^{10-x_1} \sum_{x_2=0}^{10-x_1} \frac{1}{x_2!(10 - x_1 - x_2)!}\left(\frac{.4}{.1}\right)^{x_2} \end{aligned}$$

(**Hint:** In $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ the 2 terms in the denominator add to the term in the numerator, if we ignore the ! sign.) Multiply top and bottom by $[x_2 + (10 - x_1 - x_2)]! = (10 - x_1)!$ This gives

$$\begin{aligned} f_1(x_1) &= \frac{10!}{x_1!(10 - x_1)!}(0.5)^{x_1}(0.1)^{10-x_1} \sum_{x_2=0}^{10-x_1} \binom{10 - x_1}{x_2}\left(\frac{0.4}{0.1}\right)^{x_2} \\ &= \binom{10}{x_1}(0.5)^{x_1}(0.1)^{10-x_1}(1 + \frac{.4}{.1})^{10-x_1} \;\text{ (again using the binomial theorem)} \\ &= \binom{10}{x_1}(0.5)^{x_1}(0.1)^{10-x_1}\frac{(0.1 + 0.4)^{10-x_1}}{(0.1)^{10-x_1}} = \binom{10}{x_1}(0.5)^{x_1}(0.5)^{10-x_1} \end{aligned}$$

Here $f_1(x_1)$ is defined for $x_1 = 0, 1, 2, \cdots, 10$.

Note: While this derivation is included as an example of how to find marginal distributions by summing a joint probability function, there is a much simpler method for this problem. Note that each race

is either won by A ("success") or it is not won by A ("failure"). Since the races are independent and X_1 is now just the number of "success" outcomes, X_1 must have a binomial distribution, with $n = 10$ and $p = .5$.

Hence $f_1(x_1) = \binom{10}{x_1}(.5)^{x_1}(.5)^{10-x_1}$; for $x_1 = 0, 1, \ldots, 10$, as above.

(c) Remember that $f(x_2|x_1) = P(X_2 = x_2|X_1 = x_1)$, so that

$$f(x_2|x_1) = \frac{f(x_1,x_2)}{f_1(x_1)} = \frac{\frac{10!}{x_1!x_2!(10-x_1-x_2)!}(.5)^{x_1}(.4)^{x_2}(.1)^{10-x_1-x_2}}{\frac{10!}{x_1!(10-x_1)!}(.5)^{x_1}(.5)^{10-x_1}}$$
$$= \frac{(10-x_1)!}{x_2!\,(10-x_1-x_2)!}\frac{(.4)^{x_2}\,(.1)^{10-x_1-x_2}}{(.5)^{x_2}\,(.5)^{10-x_1-x_2}} = \binom{10-x_1}{x_2}\left(\frac{4}{5}\right)^{x_2}\left(\frac{1}{5}\right)^{10-x_1-x_2}$$

For any given value of x_1, x_2 ranges through $0, 1, \ldots, (10 - x_1)$. (So the range of X_2 depends on the value x_1, which makes sense: if B wins x_1 races then the most A can win is $10 - x_1$.)

Note: As in (b), this result can be obtained more simply by general reasoning. Once we are given that A wins x_1 races, the remaining $(10 - x_1)$ races are all won by either B or C. For these races, B wins $\frac{4}{5}$ of the time and C $\frac{1}{5}$ of the time, because $P(B \text{ wins}) = 0.4$ and $P(C \text{ wins}) = 0.1$; i.e., B wins 4 times as often as C. More formally

$$P(B \text{ wins } | B \text{ or } C \text{ wins}) = 0.8.$$

$$\text{Therefore } f(x_2|x_1) = \binom{10-x_1}{x_2}\left(\frac{4}{5}\right)^{x_2}\left(\frac{1}{5}\right)^{10-x_1-x_2}$$

from the binomial distribution.

(d) X_1 and X_2 are clearly not independent since the more races A wins, the fewer races there are for B to win. More formally,

$$f_1(x_1)f_2(x_2) = \binom{10}{x_1}(.5)^{x_1}(.5)^{10-x_1}\binom{10}{x_2}(.4)^{x_2}(.6)^{10-x_2} \neq f(x_1,x_2)$$

(In general, if the range for X_1 depends on the value of X_2, then X_1 and X_2 cannot be independent.)

(e) If $T = X_1 + X_2$ then

$$\begin{aligned} f(t) &= \sum_{x_1} f\left(x_1, t - x_1\right) \\ &= \sum_{x_1=0}^{t} \frac{10!}{x_1!(t-x_1)!\underbrace{(10 - x_1 - (t-x_1))!}_{(10-t)!}} (.5)^{x_1}(.4)^{t-x_1}(.1)^{10-t} \end{aligned}$$

The upper limit on x_1 is t because, for example, if $t = 7$ then A could not have won more than 7 races. Then

$$f(t) = \frac{10!}{(10-t)!}(.4)^t(.1)^{10-t} \sum_{x_1=0}^{t} \frac{1}{x_1!(t-x_1)!} \left(\frac{.5}{.4}\right)^{x_1}$$

What do we need to multiply by on the top and bottom? Can you spot it before looking below?

$$\begin{aligned} f(t) &= \frac{10!}{t!(10-t)!}(.4)^t(.1)^{10-t} \sum_{x_1=0}^{t} \frac{t!}{x_1!(t-x_1)!} \left(\frac{.5}{.4}\right)^{x_1} \\ &= \binom{10}{t}(.4)^t(.1)^{10-t}\left(1 + \frac{.5}{.4}\right)^t \\ &= \binom{10}{t}(.4)^t(.1)^{10-t}\frac{(.4+.5)^t}{(.4)^t} = \binom{10}{t}(.9)^t(.1)^{10-t} \text{ for } t = 0, 1, \ldots, 10. \end{aligned}$$

Exercise: Explain to yourself how this answer can be obtained from the binomial distribution, as we did in the notes following parts (b) and (c).

The following problem is similar to conditional probability problems that we solved in Chapter 4. Now we are dealing with events defined in terms of random variables. Earlier results give us things like

$$P(Y = y) = \sum_{\text{all } x} P(Y = y|X = x)P(X = x) = \sum_{\text{all } x} f(y|x)f_1(x)$$

Example: In an auto parts company an average of μ defective parts are produced per shift. The number, X, of defective parts produced has a Poisson distribution. An inspector checks all parts prior to shipping them, but there is a 10% chance that a defective part will slip by undetected. Let Y be the number of defective parts the inspector finds on a shift. Find $f(x|y)$. (The company wants to know how many defective parts are produced, but can only know the number which were actually detected.)

Solution: Think of $X = x$ being event A and $Y = y$ being event B; we want to find $P(A|B)$. To do this we'll use

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

We know $f_1(x) = \frac{\mu^x e^{-\mu}}{x!} = P(X = x)$. Also, for a given number x of defective items produced, the number, Y, detected has a binomial distribution with $n = y$ and $p = .9$, assuming each inspection takes place independently. Then

$$f\,(y|x) = \binom{x}{y}\,(.9)^y\,(.1)^{x-y} = \frac{f(x,y)}{f_1(x)}.$$

Therefore

$$f(x,y) = f_1(x)f(y|x) = \frac{\mu^x e^{-\mu}}{x!}\frac{x!}{y!(x-y)!}(.9)^y(.1)^{x-y}$$

To get $f(x|y)$ we'll need $f_2(y)$. We have

$$f_2(y) = \sum_{\text{all } x} f(x,y) = \sum_{x=y}^{\infty} \frac{\mu^x e^{-\mu}}{y!(x-y)!}(.9)^y(.1)^{x-y}$$

($x \geq y$ since the number of defective items produced can't be less than the number detected.)

$$= \frac{(.9)^y e^{-\mu}}{y!} \sum_{x=y}^{\infty} \frac{\mu^x (.1)^{x-y}}{(x-y)!}$$

We could fit this into the summation result $e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$ by writing μ^x as $\mu^{x-y}\mu^y$. Then

$$\begin{aligned}
f_2(y) &= \frac{(.9\mu)^y e^{-\mu}}{y!} \sum_{x=y}^{\infty} \frac{(.1\mu)^{x-y}}{(x-y)!} \\
&= \frac{(.9\mu)^y e^{-\mu}}{y!}\left[\frac{(.1\mu)^0}{0!} + \frac{(.1\mu)^1}{1!} + \frac{(.1\mu)^2}{2!} + \cdots\right] \\
&= \frac{(.9\mu)^y e^{-\mu}}{y!} e^{.1\mu} = \frac{(.9\mu)^y e^{-.9\mu}}{y!} \\
f(x|y) &= \frac{f(x,y)}{f_2(y)} = \frac{\frac{\mu^x e^{-\mu}(.9)^y(.1)^{x-y}}{y!((x-y)!}}{\frac{(.9)^y\mu^y e^{-.9\mu}}{y!}} \\
&= \frac{(.1\mu)^{x-y}e^{-.1\mu}}{(x-y)!} \text{ for } x = y, y+1, y+2, \cdots
\end{aligned}$$

Problems:

8.1.1 The joint probability function of (X, Y) is:

		x		
$f(x,y)$		0	1	2
	0	.09	.06	.15
y	1	.15	.05	.20
	2	.06	.09	.15

a) Are X and Y independent? Why?

b) Tabulate the conditional probability function, $f\,(y|X=0)$.

c) Tabulate the probability function of $D = X - Y$.

8.1.2 In problem 6.14, given that x sales were made in a 1 hour period, find the probability function for Y, the number of calls made in that hour.

8.1.3 X and Y are independent, with $f(x) = \binom{x+k-1}{x} p^k (1-p)^x$ and $f(y) = \binom{y+\ell-1}{y} p^\ell (1-p)^y$. Let $T = X + Y$. Find the probability function, $f(t)$. You may use the result $\binom{a+b-1}{a} = (-1)^a \binom{-b}{a}$.

8.2 Multinomial Distribution

There is only this one multivariate model distribution introduced in this course, though other multivariate distributions exist. The multinomial distribution defined below is very important. It is a generalization of the binomial model to the case where each trial has k possible outcomes.

Physical Setup: This distribution is the same as binomial except there are k types of outcome rather than two. An experiment is repeated independently n times with k distinct types of outcome each time. Let the probabilities of these k types be $p_1, p_2, \cdots, p_k$ each time. Let X_1 be the number of times the 1^{st} type occurs, X_2 the number of times the 2^{nd} occurs, $\cdots$, X_k the number of times the k^{th} type occurs. Then $(X_1, X_2, \cdots, X_k)$ has a multinomial distribution.

Notes:

(1) $p_1 + p_2 + \cdots + p_k = 1$

(2) $X_1 + X_2 + \cdots + X_k = n$,

If we wish we can drop one of the variables (say the last), and just note that X_k equals $n - X_1 - X_2 - \cdots - X_{k-1}$.

Illustrations:

(1) In the example of Section 8.1 with sprinters A,B, and C running 10 races we had a multinomial distribution with $n = 10$ and $k = 3$.

(2) Suppose student marks are given in letter grades as A, B, C, D, or F. In a class of 80 students the number getting A, B, ..., F might have a multinomial distribution with $n = 80$ and $k = 5$.

Joint Probability Function: The joint probability function of $X_1, \ldots, X_k$ is given by extending the argument in the sprinters example from $k = 3$ to general k. There are $\frac{n!}{x_1!x_2!\cdots x_k!}$ different outcomes of the n trials in which x_1 are of the 1^{st} type, x_2 are of the 2^{nd} type, etc. Each of these arrangements has probability $p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}$ since p_1 is multiplied x_1 times in some order, etc.

$$\text{Therefore } f(x_1, x_2, \cdots, x_k) = \frac{n!}{x_1!x_2!\cdots x_k!}p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}$$

The restriction on the x_i's are $x_i = 0, 1, \cdots, n$ and $\sum_{i=1}^{k} x_i = n$.
As a check that $\sum f(x_1, x_2, \cdots, x_k) = 1$ we use the multinomial theorem to get

$$\sum \frac{n!}{x_1!x_2!\cdots x_k!}p_1^{x_1}\cdots p_k^{x_k} = (p_1 + p_2 + \cdots + p_k)^n = 1.$$

We have already seen one example of the multinomial distribution in the sprinter example. Here is another simple example.

Example: Every person is one of four blood types: A, B, AB and O. (This is important in determining, for example, who may give a blood transfusion to a person.) In a large population let the fraction that has type A, B, AB and O, respectively, be p_1, p_2, p_3, p_4. Then, if n persons are randomly selected from the population, the numbers X_1, X_2, X_3, X_4 of types A, B, AB, O have a multinomial distribution with $k = 4$ (In Caucasian people the values of the p_i's are approximately $p_1 = .45, p_2 = .08, p_3 = .03, p_4 = .44$.)

Remark: We sometimes use the notation $(X_1, \ldots, X_k) \sim Mult(n; p_1, \ldots, p_k)$ to indicate that $(X_1, \ldots, X_k)$ have a multinomial distribution.

Remark: For some types of problems its helpful to write formulas in terms of $x_1, \ldots, x_{k-1}$ and $p_1, \ldots, p_{k-1}$ using the fact that

$$x_k = n - x_1 - \cdots - x_{k-1} \quad \text{and} \quad p_k = 1 - p_1 - \cdots - p_{k-1}.$$

In this case we can write the joint p.f. as $f(x_1, \ldots, x_{k-1})$ but we must remember then that $x_1, \ldots, x_{k-1}$ satisfy the condition $0 \le x_1 + \cdots + x_{k-1} \le n$.
The multinomial distribution can also arise in combination with other models, and students often have trouble recognizing it then.

Example: A potter is producing teapots one at a time. Assume that they are produced independently of each other and with probability p the pot produced will be "satisfactory"; the rest are sold at a lower price. The number, X, of rejects before producing a satisfactory teapot is recorded. When 12

satisfactory teapots are produced, what is the probability the 12 values of X will consist of six 0's, three 1's, two 2's and one value which is ≥ 3?

Solution: Each time a "satisfactory" pot is produced the value of X falls in one of the four categories $X = 0, X = 1, X = 2, X \geq 3$. Under the assumptions given in this question, X has a geometric distribution with

$$f(x) = p(1-p)^x; \text{ for } x = 0, 1, 2, \cdots$$

so we can find the probability for each of these categories. We have $P(X = x) = f(x)$ for $0, 1, 2$, and we can obtain $P(X \geq 3)$ in various ways:

a)

$$\begin{aligned} P(X \geq 3) &= f(3) + f(4) + f(5) + \cdots = p(1-p)^3 + p(1-p)^4 + p(1-p)^5 + \cdots \\ &= \frac{p(1-p)^3}{1-(1-p)} = (1-p)^3 \end{aligned}$$

since we have a geometric series.

b)

$$P(X \geq 3) = 1 - P(X < 3) = 1 - f(0) - f(1) - f(2).$$

With some re-arranging, this also gives $(1-p)^3$.

c) The only way to have $X \geq 3$ is to have the first 3 pots produced all being rejects. Therefore $P(X \geq 3) =$ P (3 consecutive rejects) = $(1-p)(1-p)(1-p) = (1-p)^3$

Reiterating that each time a pot is successfully produced, the value of X falls in one of 4 categories $(0, 1, 2, or \ \geq 3)$, we see that the probability asked for is given by a multinomial distribution, Mult$(12; f(0), f(1), f(2), P(X \geq 3))$:

$$\begin{aligned} f(6,3,2,1) &= \tfrac{12!}{6!3!2!1!}\left[f(0)\right]^6\left[f(1)\right]^3[f(2)]^2[P(X \geq 3)]^1 \\ &= \tfrac{12!}{6!3!2!1!}p^6\left[p(1-p)\right]^3\left[p(1-p)^2\right]^2\left[(1-p)^3\right]^1 \\ &= \tfrac{12!}{6!3!2!1!}p^{11}(1-p)^{10} \end{aligned}$$

Problems:

8.2.1 An insurance company classifies policy holders as class A,B,C, or D. The probabilities of a randomly selected policy holder being in these categories are .1, .4, .3 and .2, respectively. Give expressions for the probability that 25 randomly chosen policy holders will include

(a) 3A's, 11B's, 7C's, and 4D's.

(b) 3A's and 11B's.

(c) 3A's and 11B's, given that there are 4D's.

8.2.2 Chocolate chip cookies are made from batter containing an average of 0.6 chips per c.c. Chips are distributed according to the conditions for a Poisson process. Each cookie uses 12 c.c. of batter. Give expressions for the probabilities that in a dozen cookies:

(a) 3 have fewer than 5 chips.

(b) 3 have fewer than 5 chips and 7 have more than 9.

(c) 3 have fewer than 5 chips, given that 7 have more than 9.

8.3 Markov Chains♢

[34]Consider a sequence of (discrete) random variables $X_1, X_2, \ldots$ each of which takes integer values $1, 2, \ldots N$ (called *states*). We assume that for a certain matrix P (called the *transition probability matrix*), the conditional probabilities are given by corresponding elements of the matrix; i.e.

$$P[X_{n+1} = j | X_n = i] = P_{ij}, i = 1, \ldots N, j = 1, \ldots N$$

and furthermore that the chain only uses the last state occupied in determining its future; i.e. that

$$P[X_{n+1} = j | X_n = i, X_{n-1} = i_1, X_{n-2} = i_2 ... X_{n-l} = i_l] = P[X_{n+1} = j | X_n = i] = P_{ij}$$

for all $j, i, i_1, i_2, \ldots i_l$, and $l = 2, 3, ...$ Then the sequence of random variables X_n is called a *Markov*[35] *Chain*. Markov Chain models are the most common simple models for dependent variables, and are used to predict weather as well as movements of security prices. They allow the future of the process to depend on the present state of the process, but the past behaviour can influence the future only through the present state.

[34]♢ This section optional for stat 220

[35]After Andrei Andreyevich Markov (1856-1922), a Russian mathematician, Professor at Saint Petersburg University. Markov studied sequences of mutually dependent variables, hoping to establish the limiting laws of probability in their most general form and discovered Markov chains, launched the theory of stochastic processes. As well, Markov applied the method of continued fractions, pioneered by his teacher Pafnuty Chebyshev, to probability theory, completed Chebyschev's proof of the central limit theorem (see Chapter 9) for independent non-identically distributed random variables. For entertainment, Markov was also interested in poetry and studied poetic style.

Example. Rain-No rain

Suppose that the probability that tomorrow is rainy given that today is not raining is α (and it does not otherwise depend on whether it rained in the past) and the probability that tomorrow is dry given that today is rainy is β. If tomorrow's weather depends on the past only through whether today is wet or dry, we can define random variables

$$X_n = \begin{cases} 1 & \text{if} \quad \text{Day } n \text{ is wet} \\ 0 & \text{if} \quad \text{Day } n \text{ is dry} \end{cases}$$

(beginning at some arbitrary time origin, day $n = 0$). Then the random variables $X_n, n = 0, 1, 2, ...$ form a Markov chain with $N = 2$ possible states and having probability transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Properties of the Transition Matrix P

Note that $P_{ij} \geq 0$ for all i, j and $\sum_{j=1}^{N} P_{ij} = 1$ for all i. This last property holds because given that $X_n = i$, X_{n+1} must occupy one of the states $j = 1, 2, ..., N$.

The distribution of X_n

Suppose that the chain is started by randomly choosing a state for X_0 with distribution $P[X_0 = i] = q_i, i = 1, 2, \ldots N$. Then the distribution of X_1 is given by

$$\begin{aligned} P(X_1 = j) &= \sum_{i=1}^{N} P(X_1 = j, X_0 = i) \\ &= \sum_{i=1}^{N} P(X_1 = j | X_0 = i) P(X_0 = i) \\ &= \sum_{i=1}^{N} P_{ij} q_i \end{aligned}$$

and this is the $j'th$ element of the vector $\underline{q}^T P$ where $\underline{q}$ is the column vector of values q_i. To obtain the distribution at time $n = 1$, premultiply the transition matrix P by a vector representing the distribution at time $n = 0$. Similarly the distribution of X_2 is the vector $\underline{q}^T P^2$ where P^2 is the product of the matrix P with itself and the distribution of X_n is $\underline{q}^T P^n$. Under very general conditions, it can be shown that these probabilities converge because the matrix P^n converges pointwise to a limiting matrix as $n \to \infty$. In fact, in many such cases, the limit does not depend on the initial distribution $\underline{q}$ because the limiting matrix has all of its rows identical and equal to some vector of probabilities $\underline{\pi}$. Identifying this vector $\underline{\pi}$ when convergence holds is reasonably easy.

Definition

A *limiting distribution* of a Markov chain is a vector ($\underline{\pi}$ say) of long run probabilities of the individual states so

$$\pi_i = \lim_{t\to\infty} P[X_t = i].$$

Now let us suppose that convergence to this distribution holds for a particular initial distribution $\underline{q}$ so we assume that

$$\underline{q}^T P^n \to \underline{\pi}^T \text{ as } n \to \infty.$$

Then notice that

$$(\underline{q}^T P^n)P \to \underline{\pi}^T P$$

but also

$$(\underline{q}^T P^n)P = \underline{q}^T P^{n+1} \to \underline{\pi}^T \text{ as } n \to \infty$$

so $\underline{\pi}^T$ must have the property that

$$\underline{\pi}^T P = \underline{\pi}^T$$

Any limiting distribution must have this property and this makes it easy in many examples to identify the limiting behaviour of the chain.

Definition 24 *A* stationary distribution *of a Markov chain is the column vector ($\underline{\pi}$ say) of probabilities of the individual states such that $\underline{\pi}^T P = \underline{\pi}^T$.*

Example: (weather continued)

Let us return to the weather example in which the transition probabilities are given by the matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

What is the long-run proportion of rainy days? To determine this we need to solve the equations

$$\underline{\pi}^T P = \underline{\pi}^T$$

$$\begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix}$$

subject to the conditions that the values π_0, π_1 are both probabilities (non-negative) and add to one. It is easy to see that the solution is

$$\pi_0 = \frac{\beta}{\alpha+\beta}$$
$$\pi_1 = \frac{\alpha}{\alpha+\beta}$$

which is intuitively reasonable in that it says that the long-run probability of the two states is proportional to the probability of a switch to that state from the other. So the long-run probability of a dry day is the limit

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0) = \frac{\beta}{\alpha+\beta}.$$

You might try verifying this by computing the powers of the matrix P^n for $n = 1, 2, ...$ and show that P^n approaches the matrix

$$\begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}$$

as $n \to \infty$. There are various mathematical conditions under which the limiting distribution of a Markov chain is unique and independent of the initial state of the chain but roughly they assert that the chain is such that it forgets the more and more distant past.

Independent Random Variables

Consider a Markov chain with transition probability matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ 1-\alpha & \alpha \end{pmatrix}.$$

Notice that both rows of this matrix are identical so $P(X_{n+1} = 1|X_n = 0) = \alpha = P(X_{n+1} = 1|X_n = 1)$. For this chain the conditional distribution of X_{n+1} given $X_n = i$ evidently does not depend on the value of i. This demonstrates independence. Indeed if X and Y are two discrete random variables and if the conditional probability function $f_{y|x}(y|x)$ of Y given X is identical for all possible values of x then it must be equal to the unconditional (marginal) probability function $f_y(y)$. If $f_{y|x}(y|x) = f_y(y)$ for all values of x and y then X and Y are independent random variables. Therefore if a Markov Chain has transition probability matrix with all rows identical, it corresponds to **independent random variables** $X_1, X_2, ...$ This is the most forgetful of all Markov chains. It pays no attention whatever to the current state in determining the next state.

Is the stationary distribution unique? One might wonder whether it is possible for a Markov chain to have more than one stationary distribution and consequently possibly more than one limiting distribution. We have seen that the 2×2 Markov chain with transition probability matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

has a solution of $\underline{\pi}^T P = \underline{\pi}^T$ and $\pi_0 + \pi_1 = 1$ given by $\pi_0 = \frac{\beta}{\alpha+\beta}$, $\pi_1 = \frac{\alpha}{\alpha+\beta}$. Is there is any other solution possible? Rewriting the equation $v^T P = v^T$ in the form $v^T(P - I) = 0$, note that the dimension of the subspace of solutions v^T is one provided that the rank of the matrix $P - I$ is one (i.e. the solutions v^T are all scalar multiples of the vector $\underline{\pi}^T$), and the dimension is 2 provided that the rank of the matrix $P - I$ is 0. Only if rank$(P - I) = 0$ will there be two linear independent solutions and hence two possible candidates for equilibrium distributions. But if $P - I$ has rank 0, then $P = I$, the transition probability matrix of a very stubborn Markov chain which **always stays in the state currently occupied**. For two-dimensional Markov Chains, only in the case $P = I$ is there more than one stationary distribution and any probability vector $\underline{\pi}^T$ satisfies $\underline{\pi}^T P = \underline{\pi}^T$ and is a stationary distribution. This is at the opposite end of the spectrum from the independent case above which pays no attention to the current state in determining the next state. The chain with $P = I$ never leaves the current state.

Example (Gene Model) A simple form of inheritance of traits occurs when a trait is governed by a pair of genes A and a. An individual may have an AA of an Aa combination (in which case they are indistinguishable in appearance, or "A dominates a"). Let us call an AA individual *dominant*, aa, *recessive* and Aa *hybrid*. When two individuals mate, the offspring inherits one gene of the pair from each parent, and we assume that these genes are selected at random. Now let us suppose that two individuals of opposite sex selected at random mate, and then two of their offspring mate, etc. Here the state is determined by a pair of individuals, so the states of our process can be considered to be objects like (AA, Aa) indicating that one of the pair is AA and the other is Aa (we do not distinguish the order of the pair, or male and female-assuming these genes do not depend on the sex of the individual)

Number	State
1	(AA, AA)
2	(AA, Aa)
3	(AA, aa)
4	(Aa, Aa)
5	$Aa, aa)$
6	(aa, aa)

For example, consider the calculation of $P(X_{t+1} = j|X_t = 2)$. In this case each offspring has probability $1/2$ of being a dominant AA, and probability of $1/2$ of being a hybrid (Aa). If two offspring are selected independently from this distribution the possible pairs are $(AA, AA), (AA, Aa), (Aa, Aa)$

with probabilities $1/4, 1/2, 1/4$ respectively. So the transitions have probabilities below:

	(AA, AA)	(AA, Aa)	(AA, aa)	(Aa, Aa)	(Aa, aa)	(aa, aa)
(AA, AA)	1	0	0	0	0	0
(AA, Aa)	.25	.5	0	.25	0	0
(AA, aa)	0	0	0	1	0	0
(Aa, Aa)	.0625	.25	.125	.25	.25	.0625
(Aa, aa)	0	0	0	.25	.5	.25
(aa, aa)	0	0	0	0	0	1

and transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .25 & .5 & 0 & .25 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ .0625 & .25 & .125 & .25 & .25 & .0625 \\ 0 & 0 & 0 & .25 & .5 & .25 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

What is the long-run behaviour in such a system? For example, the two-generation transition probabilities are given by

$$P^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.3906 & 0.3125 & 0.0313 & 0.1875 & 0.0625 & .01156 \\ 0.0625 & 0.25 & 0.125 & 0.25 & 0.25 & 0.0625 \\ 0.1406 & 0.1875 & 0.0312 & 0.3125 & 0.1875 & 0.14063 \\ 0.01562 & 0.0625 & 0.0313 & 0.1875 & 0.3125 & 0.3906 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which seems to indicate a drift to one or other of the extreme states 1 or 6. To confirm the long-run behaviour calculate:

$$P^{100} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.75 & 0 & 0 & 0 & 0 & 0.25 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 & 0.5 \\ 0.25 & 0 & 0 & 0 & 0 & 0.75 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which shows that eventually the chain is absorbed in either of state 1 or state 6, with the probability of absorption depending on the initial state. This chain, unlike the ones studied before, has more than one possible stationary distribution, for example, $\pi^T = (1, 0, 0, 0, 0, 0)$ and $\pi^T = (0, 0, 0, 0, 0, 1)$, and in these circumstances the chain does not have the same limiting distribution for all initial states.

8.4 Expectation for Multivariate Distributions: Covariance and Correlation

It is easy to extend the definition of expected value to multiple variables. Generalizing $E\left[g\left(X\right)\right] = \sum\limits_{\text{all } x} g(x)f(x)$ leads to the definition of expected value in the multivariate case

Definition 25

$$E\left[g\left(X,Y\right)\right] = \sum_{\text{all } (x,y)} g(x,y)f(x,y)$$

and

$$E\left[g\left(X_1,X_2,\cdots,X_n\right)\right] = \sum_{\text{all } (x_1,x_2,\cdots,x_n)} g\left(x_1,x_2,\cdots x_n\right) f\left(x_1,\cdots,x_n\right)$$

As before, these represent the average value of $g(X,Y)$ and $g(X_1,\ldots,X_n)$. $E\left[g\left(X,Y\right)\right]$ could also be determined by finding the probability function $f_Z(z)$ of $Z = g(X,Y)$ and then using the definition of expected value $E(Z) = \sum_{\text{all } z} z f_Z(z)$.

Example: Let the joint probability function, $f(x,y)$, be given by

		x		
$f(x,y)$		0	1	2
	1	.1	.2	.3
y	2	.2	.1	.1

Find $E(XY)$ and $E(X)$.

Solution:

$$\begin{aligned} E\left(XY\right) &= \sum_{\text{all } (x,y)} xyf(x,y) \\ &= (0\times 1\times .1) + (1\times 1\times .2) + (2\times 1\times .3) + (0\times 2\times .2) + (1\times 2\times .1) + (2\times 2\times .1) \\ &= 1.4 \end{aligned}$$

To find $E(X)$ we have a choice of methods. First, taking $g(x,y) = x$ we get

$$\begin{aligned} E(X) &= \sum_{\text{all } (x,y)} xf(x,y) \\ &= (0\times .1) + (1\times .2) + (2\times .3) + (0\times .2) + (1\times .1) + (2\times .1) \\ &= 1.1 \end{aligned}$$

Alternatively, since $E(X)$ only involves X, we could find $f_1(x)$ and use

$$E(X) = \sum_{x=0}^{2} x f_1(x) = (0 \times .3) + (1 \times .3) + (2 \times .4) = 1.1$$

Example: In the example of Section 8.1 with sprinters A, B, and C we had (using only X_1 and X_2 in our formulas)

$$f(x_1, x_2) = \frac{10!}{x_1! x_2! (10 - x_1 - x_2)!} (.5)^{x_1} (.4)^{x_2} (.1)^{10 - x_1 - x_2}$$

where A wins x_1 times and B wins x_2 times in 10 races. Find $E(X_1 X_2)$.

Solution: This will be similar to the way we derived the mean of the binomial distribution but, since this is a multinomial distribution, we'll be using the multinomial theorem to sum.

$$E(X_1 X_2) = \sum x_1 x_2 f(x_1, x_2) = \sum_{\substack{x_1 \neq 0 \\ x_2 \neq 0}} x_1 x_2 \frac{10!}{x_1 (x_1 - 1)! x_2 (x_2 - 1)! (10 - x_1 - x_2)!} (.5)^{x_1} (.4)^{x_2} (.1)^{10 - x_1 - x_2}$$

$$= \sum_{\substack{x_1 \neq 0 \\ x_2 \neq 0}} \frac{(10)(9)(8!)}{(x_1 - 1)!(x_2 - 1)! [(10 - 2) - (x_1 - 1) - (x_2 - 1)]!} (.5)(.5)^{x_1 - 1} (.4)(.4)^{x_2 - 1} (.1)^{(10-2) - (x_1 - 1) - (x_2 - 1)}$$

$$= (10)(9)(.5)(.4) \sum_{\substack{x_1 \neq 0 \\ x_2 \neq 0}} \frac{8!}{(x_1 - 1)!(x_2 - 1)! [8 - (x_1 - 1) - (x_2 - 1)]!} (.5)^{x_1 - 1} (.4)^{x_2 - 1} (.1)^{8 - (x_1 - 1) - (x_2 - 1)}$$

Let $y_1 = x_1 - 1$ and $y_2 = x_2 - 1$ in the sum and we obtain

$$E(X_1 X_2) = (10)(9)(.5)(.4) \sum_{(y_1, y_2)} \frac{8!}{y_1! y_2! (8 - y_1 - y_2)!} (.5)^{y_1} (.4)^{y_2} (.1)^{8 - y_1 - y_2}$$

$$= 18(.5 + .4 + .1)^8 = 18$$

Property of Multivariate Expectation: It is easily proved (make sure you can do this) that

$$E[a g_1(X, Y) + b g_2(X, Y)] = a E[g_1(X, Y)] + b E[g_2(X, Y)]$$

This can be extended beyond 2 functions g_1 and g_2, and beyond 2 variables X and Y.

Relationships between Variables:

Independence is a "yes/no" way of defining a relationship between variables. We all know that there can be different types of relationships between variables which are dependent. For example, if X

is your height in inches and Y your height in centimeters the relationship is one-to-one and linear. More generally, two random variables may be related (non-independent) in a probabilistic sense. For example, a person's weight Y is not an exact linear function of their height X, but Y and X are nevertheless related. We'll look at two ways of measuring the strength of the relationship between two random variables. The first is called covariance.

Definition 26 *The **covariance** of X and Y, denoted $\mathrm{Cov}(X,Y)$ or σ_{XY}, is*

$$\mathrm{Cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

For calculation purposes this definition is usually harder to use than the formula which follows, which is proved noting that

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= E\left[(X-\mu_X)(Y-\mu_Y)\right] = E\left(XY-\mu_X Y - X\mu_Y + \mu_X\mu_Y\right) \\
&= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y \\
&= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y) \\
\text{Therefore } \mathrm{Cov}(X,Y) &= E(XY) - E(X)E(Y)
\end{aligned}$$

Example:
In the example with joint probability function

		x		
$f(x,y)$		0	1	2
	1	.1	.2	.3
y				
	2	.2	.1	.1

find Cov (X,Y).

Solution: We previously calculated $E(XY) = 1.4$ and $E(X) = 1.1$. Similarly, $E(Y) = (1\times.6) + (2\times.4) = 1.4$

$$\text{Therefore Cov}(X,Y) = 1.4 - (1.1)(1.4) = -.14$$

Exercise: Calculate the covariance of X_1 and X_2 for the sprinter example. We have already found that $E\left(X_1X_2\right)$ = 18. The marginal distributions of X_1 and of X_2 are models for which we've already derived the mean. If your solution takes more than a few lines you're missing an easier solution.

Interpretation of Covariance:

(1) Suppose large values of X tend to occur with large values of Y and small values of X with small values of Y. Then $(X - \mu_X)$ and $(Y - \mu_Y)$ will tend to be of the same sign, whether positive or negative. Thus $(X - \mu_X)(Y - \mu_Y)$ will be positive. Hence Cov $(X, Y) > 0$. For example in Figure 8.2 we see several hundred points plotted. Notice that the majority of the points are in the two quadrants (lower left and upper right) labelled with "+" so that for these $(X - \mu_X)(Y - \mu_Y) > 0$. A minority of points are in the other two quadrants labelled "-" and for these $(X - \mu_X)(Y - \mu_Y) < 0$. Moreover the points in the latter two quadrants appear closer to the mean (μ_X, μ_Y) indicating that on average, over all points generated $average((X - \mu_X)(Y - \mu_Y)) > 0$. Presumably this implies that over the joint distribution of (X, Y), $E[(X - \mu_X)(Y - \mu_Y)] > 0$ or $Cov(X, Y) > 0$.

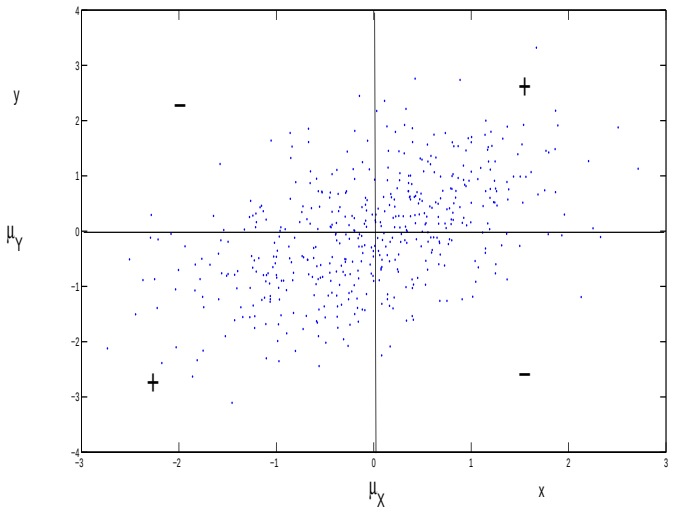


Figure 8.2: Random points (X, Y) with covariance 0.5, variances 1.

For example of X =person's height and Y =person's weight, then these two random variables will have positive covariance.

(2) Suppose large values of X tend to occur with small values of Y and small values of X with large values of Y. Then $(X - \mu_X)$ and $(Y - \mu_Y)$ will tend to be of opposite signs. Thus $(X - \mu_X)(Y - \mu_Y)$ tends to be negative. Hence Cov $(X, Y) < 0$. For example see Figure 8.3

For example if X =thickness of attic insulation in a house and Y =heating cost for the house, then $Cov(X, Y) < 0$.

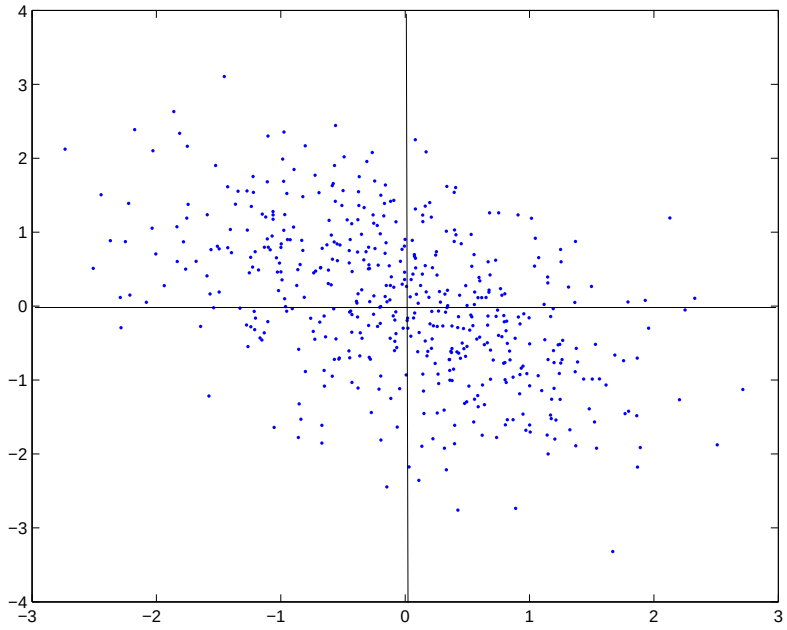


Figure 8.3: Covariance=-0.5, variances=1

Theorem 27 *If X and Y are independent then Cov $(X, Y) = 0$.*

Proof: Recall $E(X - \mu_X) = E(X) - \mu_X = 0$. Let X and Y be independent.
Then $f(x, y) = f_1(x)f_2(y)$.

$$\begin{aligned}
\text{Cov}\,(X,Y) &= E\left[(X-\mu_X)(Y-\mu_Y)\right] = \sum_{\text{all } y}\left[\sum_{\text{all } x}(x-\mu_X)(y-\mu_Y)\,f_1(x)f_2(y)\right] \\
&= \sum_{\text{all } y}\left[(y-\mu_Y)\,f_2(y)\sum_{\text{all } x}(x-\mu_X)\,f_1(x)\right] \\
&= \sum_{\text{all } y}\left[(y-\mu_Y)\,f_2(y)E\,(X-\mu_X)\right] \\
&= \sum_{\text{all } y} 0 = 0
\end{aligned}$$

The following theorem gives a direct proof the result above, and is useful in many other situations.

Theorem 28 *Suppose random variables X and Y are independent. Then, if $g_1(X)$ and $g_2(Y)$ are any two functions,*

$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)].$$

Proof: Since X and Y are independent, $f(x,y) = f_1(x)f_2(y)$. Thus

$$\begin{aligned} E[g_1(X)g_2(Y)] &= \sum_{\text{all}(x,y)} g_1(x)g_2(y)f(x,y) \\ &= \sum_{\text{all } x}\sum_{\text{all } y} g_1(x)f_1(x)g_2(y)f_2(y) \\ &= [\sum_{\text{all } x} g_1(x)f_1(x)][\sum_{\text{all } y} g_2(y)f_2(y)] \\ &= E[g_1(X)]E[g_2(Y)] \end{aligned}$$

□

To prove result (3) above, we just note that if X and Y are independent then

$$\begin{aligned} \text{Cov}(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E(X-\mu_X)E(Y-\mu_Y) = 0 \end{aligned}$$

Caution: This result is <u>not</u> reversible. If Cov $(X,Y) = 0$ we can not conclude that X and Y are independent. For example suppose that the random variable Z is uniformly distributed on the values $\{-1, -0.9, ...0.9, 1\}$ and define $X = \sin(2\pi Z)$ and $Y = \cos(2\pi Z)$. It is easy to see that Cov$(X,Y) = 0$ but the two random variables X, Y are clearly related because the points (X,Y) are always on a circle.

Example: Let (X,Y) have the joint probability function $f(0,0) = 0.2,\ \ f(1,1) = 0.6,\ \ f(2,0) = 0.2$; i.e. (X,Y) only takes 3 values.

x	0	1	2
$f_1(x)$	.2	.6	.2

and

y	0	1
$f_2(y)$	.4	.6

are marginal probability functions. Since $f_1(x)f_2(y) \neq f(x,y)$, therefore, X and Y are not independent. However,

$$\begin{aligned} E\,(XY) &= (0\times 0\times .2) + (1\times 1\times .6) + (2\times 0\times .2) = .6 \\ E(X) &= (0\times .2) + (1\times .6) + (2\times .2) = 1 \text{ and } E(Y) = (0\times .4) + (1\times .6) = .6 \end{aligned}$$

Therefore Cov $(X,Y) = E(XY) - E(X)E(Y) = .6 - (1)(.6) = 0$

So X and Y have covariance 0 but are not independent. If Cov $(X,Y) = 0$ we say that X and Y are uncorrelated, because of the definition of correlation [36] given below.

[36] " The finest things in life include having a clear grasp of correlations. " Albert Einstein, 1919.

(4) The actual numerical value of Cov (X, Y) has no interpretation, so covariance is of limited use in measuring relationships.

Exercise:

(a) Look back at the example in which $f(x, y)$ was tabulated and Cov $(X, Y) = -.14$. Considering how covariance is interpreted, does it make sense that Cov (X, Y) would be negative?

(b) Without looking at the actual covariance for the sprinter exercise, would you expect Cov (X_1, X_2) to be positive or negative? (If A wins more of the 10 races, will B win more races or fewer races?)

We now consider a second, related way to measure the strength of relationship between X and Y.

Definition 29 *The **correlation coefficient** of X and Y is* $\rho = \frac{\text{Cov }(X,Y)}{\sigma_X \sigma_Y}$

The correlation coefficient measures the strength of the linear relationship between X and Y and is simply a rescaled version of the covariance, scaled to lie in the interval $[-1, 1]$. You can attempt to guess the correlation between two variables based on a scatter diagram of values of these variables at the web page

`http://statweb.calpoly.edu/chance/applets/guesscorrelation/GuessCorrelation.html`

For example in Figure 8.4 I guessed a correlation of -0.9 whereas the true correlation coefficient generating these data was $\rho = -0.92$.

Properties of ρ:

1) Since σ_X and σ_Y, the standard deviations of X and Y, are both positive, ρ will have the same sign as Cov (X, Y). Hence the interpretation of the sign of ρ is the same as for Cov (X, Y), and $\rho = 0$ if X and Y are independent. When $\rho = 0$ we say that X and Y are uncorrelated.

2) $-1 \leq \rho \leq 1$ and as $\rho \to \pm 1$ the relation between X and Y becomes one-to-one and linear.

Proof: Define a new random variable $S = X + tY$, where t is some real number. We'll show that the fact that Var$(S) \geq 0$ leads to 2) above. We have

$$\begin{aligned}
\text{Var }(S) &= E\{(S - \mu_S)^2\} \\
&= E\{[(X + tY) - (\mu_X + t\mu_Y)]^2\} \\
&= E\{[(X - \mu_X) + t(Y - \mu_Y)]^2\} \\
&= E\{(X - \mu_X)^2 + 2t(X - \mu_X)(Y - \mu_Y) + t^2(Y - \mu_Y)^2\} \\
&= \sigma_X^2 + 2t\text{Cov}(X, Y) + t^2\sigma_Y^2
\end{aligned}$$

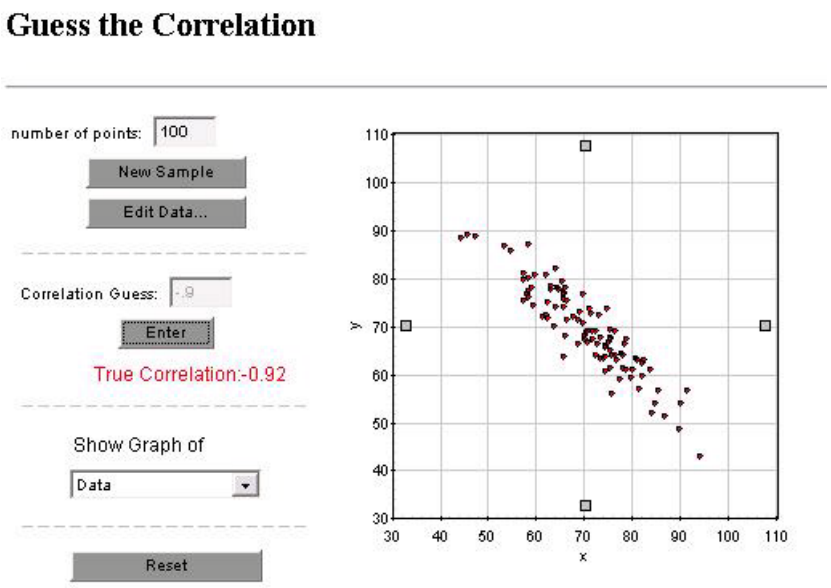


Figure 8.4: Guessing the correlation based on a scatter diagram of points

Since $Var(S) \geq 0$ for any real number t, this quadratic equation must have at most one real root (value of t for which it is zero). Therefore

$$(2\text{Cov}(X,Y))^2 - 4\sigma_X^2\sigma_Y^2 \leq 0$$

leading to the inequality

$$\left|\frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y}\right| \leq 1$$

To see that $\rho = \pm 1$ corresponds to a one-to-one linear relationship between X and Y, note that $\rho = \pm 1$ corresponds to a zero discriminant in the quadratic equation. This means that there exists one real number t^* for which

$$\text{Var}\,(S) = \text{Var}\,(X + t^*Y) = 0$$

But for Var$(X + t^*Y)$ to be zero, $X + t^*Y$ must equal a constant c. Thus X and Y satisfy a linear relationship.

Exercise: Calculate ρ for the sprinter example. Does your answer make sense? (You should already have found Cov (X_1, X_2) in a previous exercise, so little additional work is needed.)

Problems:

8.4.1 The joint probability function of (X, Y) is:

		x	
$f(x,y)$	0	1	2
0	.06	.15	.09
y			
1	.14	.35	.21

Calculate the correlation coefficient, ρ. What does it indicate about the relationship between X and Y?

8.4.2 Suppose that X and Y are random variables with joint probability function:

		x	
$f(x,y)$	2	4	6
-1	1/8	1/4	p
y			
1	1/4	1/8	$\frac{1}{4} - p$

(a) For what value of p are X and Y uncorrelated?

(b) Show that there is no value of p for which X and Y are independent.

8.5 Mean and Variance of a Linear Combination of Random Variables

Many problems require us to consider linear combinations of random variables; examples will be given below and in Chapter 9. Although writing down the formulas is somewhat tedious, we give here some important results about their means and variances.

Results for Means:

1. $E(aX + bY) = aE(X) + bE(Y) = a\mu_X + b\mu_Y$, when a and b are constants. (This follows from the definition of expected value .) In particular, $E(X + Y) = \mu_X + \mu_Y$ and $E(X - Y) = \mu_X - \mu_Y$.

2. Let a_i be constants (real numbers) and $E(X_i) = \mu_i$. Then $E(\sum a_i X_i) = \sum a_i \mu_i$. In particular, $E(\sum X_i) = \sum E(X_i)$.

3. Let $X_1, X_2, \cdots, X_n$ be random variables which have mean μ. (You can imagine these being some sample results from an experiment such as recording the number of occupants in cars travelling over a toll bridge.) The sample mean is $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. Then $E\left(\overline{X}\right) = \mu$.

Proof: From (2), $E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E\left(X_i\right) = \sum_{i=1}^{n} \mu = n\mu$. Thus

$$E\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n}E\left(\sum X_i\right) = \frac{1}{n}n\mu = \mu$$

Results for Covariance:

1. Cov $(X, X) = E\left[\left(X - \mu_X\right)\left(X - \mu_X\right)\right] = E\left[\left(X - \mu\right)^2\right] = Var\left(X\right)$
2. Cov $(aX + bY, cU + dV) = ac\,Cov\ (X, U) + ad\,Cov\ (X, V) + bc\,Cov\ (Y, U) + bd\,Cov\,(Y, V)$ where $a, b, c,$ and d are constants.

Proof:

$$\begin{aligned} Cov\,(aX + bY, cU + dV) &= E\left[\left(aX + bY - a\mu_X - b\mu_Y\right)\left(cU + dV - c\mu_U - d\mu_V\right)\right] \\ &= E\left\{\left[a\left(X - \mu_X\right) + b\left(Y - \mu_Y\right)\right]\left[c\left(U - \mu_U\right) + d\left(V - \mu_V\right)\right]\right\} \\ &= acE\left[\left(X - \mu_X\right)\left(U - \mu_U\right)\right] + adE\left[\left(X - \mu_X\right)\left(V - \mu_V\right)\right] \\ &+ bcE\left[\left(Y - \mu_Y\right)\left(U - \mu_U\right)\right] + bdE\left[\left(Y - \mu_Y\right)\left(V - \mu_V\right)\right] \\ &= ac\ \text{Cov}\ (X, U) + ad\ \text{Cov}\ (X, V) + bc\ \text{Cov}\ (Y, U) + bd\ \text{Cov}\ (Y, V) \end{aligned}$$

This type of result can be generalized, but gets messy to write out.

Results for Variance:

1. **Variance of a linear combination:**

$$\text{Var}\ (aX + bY) = a^2\ \text{Var}\ (X) + b^2\ \text{Var}(Y) + 2ab\ \text{Cov}\ (X, Y)$$

Proof:

$$\begin{aligned} \text{Var}\ (aX + bY) &= E\left[\left(aX + bY - a\mu_X - b\mu_Y\right)^2\right] \\ &= E\left\{\left[a\left(X - \mu_X\right) + b\left(Y - \mu_Y\right)\right]^2\right\} \\ &= E\left[a^2\left(X - \mu_X\right)^2 + b^2\left(Y - \mu_Y\right)^2 + 2ab\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] \\ &= a^2E\left[\left(X - \mu_X\right)^2\right] + b^2E\left[\left(Y - \mu_Y\right)^2\right] + 2abE\left[\left(X - \mu_X\right)\left(Y - \mu_Y\right)\right] \\ &= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\ \text{Cov}\ (X, Y) \end{aligned}$$

Exercise: Try to prove this result by writing $Var\ (aX + bY)$ as Cov $(aX + bY, aX + bY)$ and using properties of covariance.

2. **Variance of a sum of independent random variables:** Let X and Y be independent. Since Cov $(X, Y) = 0$, result 1. gives

$$\text{Var } (X + Y) = \sigma_X^2 + \sigma_Y^2;$$

i.e., for independent variables, the *variance of a sum is the sum of the variances.* Also note

$$\text{Var } (X - Y) = \sigma_X^2 + (-1)^2\sigma_Y^2 = \sigma_X^2 + \sigma_Y^2;$$

i.e., for independent variables, the variance of a difference is the sum of the variances.

3. **Variance of a general linear combination:** Let a_i be constants and Var $(X_i) = \sigma_i^2$. Then

$$\text{Var } \left(\sum a_i X_i\right) = \sum a_i^2\sigma_i^2 + 2\sum_{i<j} a_i a_j \text{ Cov } (X_i, X_j)\,.$$

This is a generalization of result 1. and can be proved using either of the methods used for 1.

4. **Variance of a linear combination of independent:** Special cases of result 3. are:

a) If $X_1, X_2, \cdots, X_n$ are independent then Cov $(X_i, X_j) = 0$, so that

$$\text{Var } \left(\sum a_i X_i\right) = \sum a_i^2\sigma_i^2.$$

b) If $X_1, X_2, \cdots, X_n$ are independent and all have the same variance σ^2, then

$$\text{Var } \left(\overline{X}\right) = \sigma^2/n$$

Proof of 4 (b): $\overline{X} = \frac{1}{n}\sum X_i$. From 4(a), Var $(\sum X_i) = \sum_{i=1}^{n} Var\ (X_i) = n\sigma^2$. Using Var $(aX + b) = a^2\ Var\ (X)$, we get:

$$\text{Var } \left(\overline{X}\right) = \text{Var } \left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\text{Var}\left(\sum X_i\right) = \frac{n\sigma^2}{n^2} = \sigma^2/n.$$

Remark: This result is a very important one in probability and statistics. To recap, it says that if $X_1, \ldots, X_n$ are independent random variables with the same mean μ and some variance σ^2, then the sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ has

$$E(\bar{X}) = \mu$$
$$\text{Var}\,(\bar{X}) = \sigma^2/n$$

This shows that the average $\bar{X}$ of n random variables with the same distribution is less variable than any single observation X_i, and that the larger n is the less variability there is. This explains mathematically why, for example, that if we want to estimate the unknown mean height μ in a population of people, we are better to take the average height for a random sample of $n = 10$ persons than to just take the height of one randomly selected person. A sample of $n = 20$ persons would be better still. There are interesting applets at the url *http://users.ece.gatech.edu/users/gtz/java/samplemean/notes.html* and *http://www.ds.unifi.it/VL/VL_EN/applets/BinomialCoinExperiment.html* which allows one to sample and explore the rate at which the sample mean approaches the expected value. In Chapter 9 we will see how to decide how large a sample we should take for a certain degree of precision. Also note that as $n \to \infty, Var(\bar{X}) \to 0$, which means that $\bar{X}$ becomes arbitrarily close to μ. This is sometimes called the "law of averages[37]". There is a formal theorem which supports the claim that for large sample sizes, sample means approach the expected value, called the "law of large numbers".

Indicator Variables

The results for linear combinations of random variables provide a way of breaking up more complicated problems, involving mean and variance, into simpler pieces using indicator variables; an indicator variable is just a binary variable (0 or 1) that indicates whether or not some event occurs. We'll illustrate this important method with 3 examples.

Example: Mean and Variance of a Binomial R.V.

Let $X \sim Bi(n, p)$ in a binomial process. Define new variables X_i by:

X_i = 0 if the i^{th} trial was a failure

X_i = 1 if the i^{th} trial was a success.

i.e. X_i indicates whether the outcome "success" occurred on the i^{th} trial. The trick we use is that the total number of successes, X, is the sum of the X_i's:

$$X = \sum_{i=1}^{n} X_i.$$

[37] "I feel like a fugitive from the law of averages."
William H. Mauldin (1921 - 2003)

We can find the mean and variance of X_i and then use our results for the mean and variance of a sum to get the mean and variance of X. First,

$$E(X_i) = \sum_{x_i=0}^{1} x_i f(x_i) = 0f(0) + 1f(1) = f(1)$$

But $f(1) = p$ since the probability of success is p on each trial. Therefore $E(X_i) = p$. Since $X_i = 0$ or 1, $X_i = X_i^2$, and therefore

$$E\left(X_i^2\right) = E(X_i) = p.$$

Thus

$$\mathrm{Var}(X_i) = E\left(X_i^2\right) - [E(X_i)]^2 = p - p^2 = p(1-p).$$

In the binomial distribution the trials are independent so the X_i's are also independent. Thus

$$\begin{aligned} E(X) &= E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np \\ \mathrm{Var}(X) &= \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p) \end{aligned}$$

These, of course, are the same as we derived previously for the mean and variance of the binomial distribution. Note how simple the derivation here is!

Remark: If X_i is a binary random variable with $P(X_i = 1) = p = 1 - P(X_i = 0)$ then $E(X_i) = p$ and $\mathrm{Var}(X_i) = p(1-p)$, as shown above. (Note that $X_i \sim Bi(1, p)$ is actually a binomial r.v.) In some problems the X_i's are not independent, and then we also need covariances.

Example: Let X have a hypergeometric distribution. Find the mean and variance of X.

Solution: As above, let us think of the setting, which involves drawing n items at random from a total of N, of which r are "S" and $N - r$ are "F" items. Define

$$X_i = \begin{cases} 0 & \text{if } i^{\text{th}} \text{ draw is a failure } (F) \text{ item} \\ 1 & \text{if } i^{\text{th}} \text{ draw is a success } (S) \text{ item.} \end{cases}$$

Then $X = \sum_{i=1}^{n} X_i$ as for the binomial example, but now the X_i's are dependent. (For example, what we get on the first draw affects the probabilities of S and F for the second draw, and so on.) Therefore we need to find $\mathrm{Cov}(X_i, X_j)$ for $i \neq j$ as well as $E(X_i)$ and $\mathrm{Var}(X_i)$ in order to use our formula for the variance of a sum.

We see first that $P(X_i = 1) = r/N$ for each of $i = 1, \ldots, n$. (If the draws are random then the probability an S occurs in draw i is just equal to the probability position i is an S when we arrange r S's and $N - r$ F's in a row.) This immediately gives

$$E(X_i) = r/N$$
$$\text{Var}(X_i) = \frac{r}{N}(1 - \frac{r}{N})$$

since

$$\text{Var}(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i) - E(X_i)^2.$$

The covariance of X_i and $X_j (i \neq j)$ is equal to $E(X_i X_j) - E(X_i)E(X_j)$, so we need

$$\begin{aligned} E(X_i X_j) &= \sum_{x_i=0}^{1} \sum_{x_j=0}^{1} x_i x_j f(x_i, x_j) \\ &= f(1,1) \\ &= P(X_i = 1, X_j = 1) \end{aligned}$$

The probability of an S on both draws i and j is just

$$r(r-1)/[N(N-1)] = P(X_i = 1)P(X_j = 1 | X_i = 1)$$

Thus,

$$\begin{aligned} \text{Cov}\ (X_i, X_j) &= E\left(X_i X_j\right) - E\left(X_i\right) E\left(X_j\right) \\ &= \tfrac{r(r-1)}{N(N-1)} - \left(\tfrac{r}{N}\right)\left(\tfrac{r}{N}\right) = \left(\tfrac{r}{N}\right)\left(\tfrac{r-1}{N-1} - \tfrac{r}{N}\right) \\ &= -\tfrac{r(N-r)}{N^2(N-1)} \end{aligned}$$

(Does it make sense that Cov (X_i, X_j) is negative? If you draw a success in draw i, are you more or less likely to have a success on draw j?) Now we find $E(X)$ and Var(X). First,

$$E(X) = E\left(\sum X_i\right) = \sum_{i=1}^{n} E\left(X_i\right) = \sum_{i=1}^{n} \left(\frac{r}{N}\right) = n\left(\frac{r}{N}\right)$$

Before finding Var (X), how many combinations X_i, X_j are there for which $i < j$? Each i and j takes values from $1, 2, \cdots, n$ so there are $\binom{n}{2}$ different combinations of (i, j) values. Each of these can only be written in 1 way to make $i < j$. Therefore There are $\binom{n}{2}$ combinations with $i < j$ (e.g. if $i = 1, 2, 3$ and $j = 1, 2, 3$, the combinations with $i < j$ are (1,2) (1,3) and (2,3). So there are $\binom{3}{2} = 3$ different combinations.)

Now we can find

$$\begin{aligned} \text{Var}(X) &= \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}\left(X_i\right) + 2\sum_{i<j} \text{Cov}\left(X_i, X_j\right) \\ &= n\tfrac{r(N-r)}{N^2} + 2\tbinom{n}{2}\left[-\tfrac{r(N-r)}{N^2(N-1)}\right] \\ &= n\left(\tfrac{r}{N}\right)\left(\tfrac{N-r}{N}\right)\left[1 - \tfrac{(n-1)}{(N-1)}\right] \quad \left(\text{since } 2\tbinom{n}{2} = \tfrac{2n(n-1)}{2} = n(n-1)\right) \\ &= n\left(\tfrac{r}{N}\right)\left(1 - \tfrac{r}{N}\right)\left(\tfrac{N-n}{N-1}\right) \end{aligned}$$

In the last two examples, we know $f(x)$, and could have found $E(X)$ and Var(X) without using indicator variables. In the next example $f(x)$ is not known and is hard to find, but we can still use indicator variables for obtaining μ and σ^2. The following example is a famous problem in probability.

Example: We have N letters to N different people, and N envelopes addressed to those N people. One letter is put in each envelope at random. Find the mean and variance of the number of letters placed in the right envelope.

Solution:

$$\text{Let } X_i = \begin{cases} 0; & \text{if letter } i \text{ is not in envelope } \ i \\ 1; & \text{if letter } i \text{ is in envelope } \ i. \end{cases}$$

Then $\sum_{i=1}^{N} X_i$ is the number of correctly placed letters. Once again, the X_i's are dependent (Why?).

First $E(X_i) = \sum_{x_i=0}^{1} x_i f(x_i) = f(1) = \frac{1}{N} = E\left(X_i^2\right)$ (since there is 1 chance in N that letter i will be put in envelope i) and then,

$$\text{Var}(X_i) = E(X_i) - [E(X_i)]^2 = \frac{1}{N} - \frac{1}{N^2} = \frac{1}{N}\left(1 - \frac{1}{N}\right)$$

Exercise: Before calculating cov (X_i, X_j), what sign do you expect it to have? (If letter i is correctly placed does that make it more or less likely that letter j will be placed correctly?)

Next, $E(X_i X_j) = f(1,1)$ (As in the last example, this is the only non-zero term in the sum.) Now, $f(1,1) = \frac{1}{N}\frac{1}{N-1}$ since once letter i is correctly placed there is 1 chance in $N-1$ of letter j going in envelope j.

$$\text{Therefore } E(X_i X_j) = \frac{1}{N(N-1)}$$

For the covariance,

$$\begin{aligned}
\text{Cov } (X_i, X_j) &= E(X_i X_j) - E(X_i)\,E(X_j) = \frac{1}{N(N-1)} - \left(\frac{1}{N}\right)\left(\frac{1}{N}\right) \\
&= \frac{1}{N}\left(\frac{1}{N-1} - \frac{1}{N}\right) = \frac{1}{N^2(N-1)} \\
E\left(\sum_{i=1}^{N} X_i\right) &= \sum_{i=1}^{N} E(X_i) = \sum_{i=1}^{N} \frac{1}{N} = \left(\frac{1}{N}\right) N = 1 \\
\text{Var}\left(\sum_{i=1}^{N} X_i\right) &= \sum_{i=1}^{N} \text{Var}\,(X_i) + 2\sum_{i<j} \text{Cov}\,(X_i, X_j) \\
&= \sum_{i=1}^{N} \frac{1}{N}\left(1 - \frac{1}{N}\right) + 2\binom{N}{2}\frac{1}{N^2(N-1)} \\
&= N\frac{1}{N}\left(1 - \frac{1}{N}\right) + 2\binom{N}{2}\frac{1}{N^2(N-1)} \\
&= 1 - \frac{1}{N} + 2\frac{N(N-1)}{2}\frac{1}{N^2(N-1)} = 1
\end{aligned}$$

(Common sense often helps in this course, but we have found no way of being able to say this result is obvious. On average 1 letter will be correctly placed and the variance will be 1, regardless of how many letters there are.)

Problems:

8.5.1 The joint probability function of (X, Y) is given by:

		x	
$f(x,y)$	0	1	2
0	.15	.1	.05
y			
1	.35	.2	.15

Calculate $E(X)$, Var (X), Cov (X, Y) and Var $(3X-2Y)$. You may use the fact that $E(Y) = .7$ and Var (Y) = .21 without verifying these figures.

8.5.2 In a row of 25 switches, each is considered to be "on" or "off". The probability of being on is .6 for each switch, independently of other switch. Find the mean and variance of the number of unlike pairs among the 24 pairs of adjacent switches.

8.5.3 Suppose Var $(X) = 1.69$, Var $(Y) = 4$, $\rho = 0.5$; and let $U = 2X - Y$. Find the standard deviation of U.

8.5.4 Let $Y_0, Y_1, \cdots, Y_n$ be uncorrelated random variables with mean 0 and variance σ^2. Let $X_1 = Y_0 + Y_1$, $X_2 = Y_1 + Y_2, \cdots$, $X_n = Y_{n-1} + Y_n$. Find Cov (X_{i-1}, X_i) for $i = 2, 3, \cdots, n$ and Var $\left(\sum_{i=1}^{n} X_i\right)$.

8.5.5 A plastic fabricating company produces items in strips of 24, with the items connected by a thin piece of plastic:

Item 1 – Item 2 – ... – Item 24

A cutting machine then cuts the connecting pieces to separate the items, with the 23 cuts made independently. There is a 10% chance the machine will fail to cut a connecting piece. Find the mean and standard deviation of the number of the 24 items which are completely separate after the cuts have been made. (Hint: Let $X_i = 0$ if item i is not completely separate, and $X_i = 1$ if item i is completely separate.)

8.6 Multivariate Moment Generating Functions♢★

[38]Suppose we have two possibly dependent random variables (X, Y) and we wish to characterize their joint distribution using a moment generating function. Just as the probability function and the cumulative distribution function are, in tis case, functions of two arguments, so is the moment generating function.

Definition 30 *The joint moment generating function of* (X, Y) *is*

$$M(s,t) = E\{e^{sX+tY}\}$$

Recall that if X, Y happen to be independent, $g_1(X)$ and $g_2(Y)$ are any two functions,

$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)]. \tag{8.9}$$

and so with $g_1(X) = e^{sX}$ and $g_2(Y) = e^{tY}$ we obtain, for independent random variables X, Y

$$M(s,t) = M_X(s)M_Y(t)$$

the product of the moment generating functions of X and Y respectively.

There is another labour-saving property of moment generating functions for independent random variables. Suppose X, Y are independent random variables with moment generating functions $M_X(t)$

[38]♢★ This section optional for Stat 220 and Stat 230

and $M_Y(t)$. Suppose you wish the moment generating function of the sum $Z = X + Y$. One could attack this problem by first determining the probability function of Z,

$$\begin{aligned} f_Z(z) = P(Z = z) &= \sum_{\text{all } x} P(X = x, Y = z - x) \\ &= \sum_{\text{all } x} P(X = x)P(Y = z - x) \\ &= \sum_{\text{all } x} f_X(x) f_Y(z - x) \end{aligned}$$

and then calculating

$$E(e^{tZ}) = \sum_{\text{all } z} e^{tZ} f_Z(z).$$

Evidently lots of work! On the other hand recycling (8.9) with

$$\begin{aligned} g_1(X) &= e^{tX} \\ g_2(Y) &= e^{tY} \end{aligned}$$

gives

$$M_Z(t) = Ee^{t(X+Y)} = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t).$$

Theorem 31 *The moment generating function of the sum of independent random variables is the product of the individual moment generating functions.*

For example if both X and Y are independent with the same (Bernoulli) distribution

$x =$	0	1
$f(x) =$	$1 - p$	p

then both have moment generating function

$$M_X(t) = M_Y(t) = (1 - p + pe^t)$$

and so the moment generating function of the sum Z is $M_X(t)M_Y(t) = (1-p+pe^t)^2$. Similarly if we add another independent Bernoulli the moment generating function is $(1-p+pe^t)^3$ and in general the sum of n independent Bernoulli random variables is $(1 - p + pe^t)^n$, the moment generating function of a Binomial(n, p) distribution. This confirms that the sum of independent Bernoulli random variables has a Binomial(n, p) distribution.

8.7 Problems on Chapter 8

8.1 The joint probability function of (X, Y) is given by:

		x	
$f(x,y)$	0	1	2
0	.15	.1	.05
y			
1	.35	.2	.15

a) Are X and Y independent? Why?

b) Find $P\,(X > Y)$ and $P\,(X = 1|Y = 0)$

8.2 For a person whose car insurance and house insurance are with the same company, let X and Y represent the number of claims on the car and house policies, respectively, in a given year. Suppose that for a certain group of individuals, $X \sim$ Poisson (mean $= .10$) and $Y \sim$ Poisson (mean $= .05$).

(a) If X and Y are independent, find $P(X+Y > 1)$ and find the mean and variance of $X+Y$.

(b) Suppose it was learned that $P(X = 0, Y = 0)$ was very close to $.94$. Show why X and Y cannot be independent in this case. What might explain the non-independence?

8.3 Consider Problem 2.7 for Chapter 2, which concerned machine recognition of handwritten digits. Recall that $p(x, y)$ was the probability that the number actually written was x, and the number identified by the machine was y.

(a) Are the random variables X and Y independent? Why?

(b) What is $P(X = Y)$, that is, the probability that a random number is correctly identified?

(c) What is the probability that the number 5 is incorrectly identified?

8.4 Blood donors arrive at a clinic and are classified as type A, type O, or other types. Donors' blood types are independent with P (type A) = p, P (type O) = q, and P (other type) = $1 - p - q$. Consider the number, X, of type A and the number, Y, of type O donors arriving before the 10^{th} other type.

a) Find the joint probability function, $f(x, y)$

b) Find the conditional probability function, $f(y|x)$.

8.5 Slot machine payouts. Suppose that in a slot machine there are $n+1$ possible outcomes $A_1, \ldots, A_{n+1}$ for a single play. A single play costs \$1. If outcome A_i occurs, you win $\$a_i$, for $i = 1, \ldots, n$. If outcome A_{n+1} occurs, you win nothing. In other words, if outcome A_i $(i = 1, \ldots, n)$ occurs your net profit is $a_i - 1$; if A_{n+1} occurs your net profit is - 1.

(a) Give a formula for your expected profit from a single play, if the probabilities of the $n + 1$ outcomes are $p_i = P(A_i),\ i = 1, \ldots, n + 1$.

(b) The owner of the slot machine wants the player's expected profit to be negative. Suppose $n = 4$, with $p_1 = .1,\ p_2 = p_3 = p_4 = .04$ and $p_5 = .78$. If the slot machine is set to pay \$3 when outcome A_1 occurs, and \$5 when either of outcomes A_2, A_3, A_4 occur, determine the player's expected profit per play.

(c) The slot machine owner wishes to pay da_i dollars when outcome A_i occurs, where $a_i = \frac{1}{p_i}$ and d is a number between 0 and 1. The owner also wishes his or her expected profit to be \$.05 per play. (The player's expected profit is -.05 per play.) Find d as a function of n and p_{n+1}. What is the value of d if $n = 10$ and $p_{n+1} = .7$?

8.6 Bacteria are distributed through river water according to a Poisson process with an average of 5 per 100 c.c. of water. What is the probability five 50 c.c. samples of water have 1 with no bacteria, 2 with one bacterium, and 2 with two or more?

8.7 A box contains 5 yellow and 3 red balls, from which 4 balls are drawn one at a time, at random, without replacement. Let X be the number of yellow balls on the first two draws and Y the number of yellow balls on all 4 draws.

a) Find the joint probability function, $f(x, y)$.

b) Are X and Y independent? Justify your answer.

8.8 In a quality control inspection items are classified as having a minor defect, a major defect, or as being acceptable. A carton of 10 items contains 2 with minor defects, 1 with a major defect, and 7 acceptable. Three items are chosen at random without replacement. Let X be the number selected with minor defects and Y be the number with major defects.

a) Find the joint probability function of X and Y.

b) Find the marginal probability functions of X and of Y.

c) Evaluate numerically $P\,(X = Y)$ and $P\,(X = 1|Y = 0)$.

8.9 Let X and Y be discrete random variables with joint probability function $f(x, y) = k\frac{2^{x+y}}{x!y!}$ for $x = 0, 1, 2, \cdots$ and $y = 0, 1, 2, \cdots$, where k is a positive constant.

a) Derive the marginal probability function of X.

b) Evaluate k.

c) Are X and Y independent? Explain.

d) Derive the probability function of $T = X + Y$.

8.10 **"Thinning" a Poisson process.** Suppose that events are produced according to a Poisson process with an average of λ events per minute. Each event has a probability p of being a "Type A" event, independent of other events.

(a) Let the random variable Y represent the number of Type A events that occur in a one-minute period. Prove that Y has a Poisson distribution with mean λp. (Hint: let X be the total number of events in a 1 minute period and consider the formula just before the last example in Section 8.1).

(b) Lighting strikes in a large forest region occur over the summer according to a Poisson process with $\lambda = 3$ strikes per day. Each strike has probability .05 of starting a fire. Find the probability that there are at least 5 fires over a 30 day period.

8.11 In a breeding experiment involving horses the offspring are of four genetic types with probabilities:

Type	1	2	3	4
Probability	3/16	5/16	5/16	3/16

A group of 40 independent offspring are observed. Give expressions for the following probabilities:

(a) There are 10 of each type.

(b) The total number of types 1 and 2 is 16.

(c) There are exactly 10 of type 1, given that the total number of types 1 and 2 is 16.

8.12 In a particular city, let the random variable X represent the number of children in a randomly selected household, and let Y represent the number of female children. Assume that the probability a child is female is 0.5, regardless of what size household they live in, and that the marginal distribution of X is as follows:

$$f(0) = .20, f(1) = .25, f(2) = .35, f(3) = .10, f(4) = .05,$$
$$f(5) = .02, f(6) = .01, f(7) = .01, f(8) = .01$$

(a) Determine $E(X)$.

(b) Find the probability function for the number of girls Y in a randomly chosen family. What is $E(Y)$?

8.13 In a particular city, the probability a call to a fire department concerns various situations is as given below:

1. fire in a detached home - $p_1 = .10$
2. fire in a semi detached home - $p_2 = .05$
3. fire in an apartment or multiple unit residence - $p_3 = .05$
4. fire in a non-residential building - $p_4 = .15$
5. non-fire-related emergency - $p_5 = .15$
6. false alarm - $p_6 = .50$

In a set of 10 calls, let $X_1, ..., X_6$ represent the numbers of calls of each of types $1, ..., 6$.

(a) Give the joint probability function for $X_1, ..., X_6$.

(b) What is the probability there is at least one apartment fire, given that there are 4 fire-related calls?

(c) If the average costs of calls of types $1, ..., 6$ are (in \$100 units) 5, 5, 7, 20, 4, 2 respectively, what is the expected total cost of the 10 calls?

8.14 Suppose $X_1, \ldots, X_n$ have joint p.f. $f(x_1, \ldots, x_n)$. If $g(x_1, \ldots, x_n)$ is a function such that $a \leq g(x_1, \ldots, x_n) \leq b$ for all $(x_1, \ldots, x_n)$ in the range of f,

then show that $a \leq E[g(X_1, \ldots, X_n)] \leq b$.

8.15 Let X and Y be random variables with Var $(X) = 13$, Var$(Y) = 34$ and $\rho = -0.7$. Find Var$(X - 2Y)$.

8.16 Let X and Y have a trinomial distribution with joint probability function

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p^x q^y (1-p-q)^{n-x-y}; \quad \begin{matrix} x = 0, 1, \cdots, n \\ y = 0, 1, \cdots, n \end{matrix}$$

and $x + y \leq n$. Let $T = X + Y$.

a) What distribution does T have? Either explain why or derive this result.

b) For the distribution in (a), what is $E(T)$ and Var(T)?

c) Using (b) find Cov(X, Y), and explain why you expect it to have the sign it does.

8.17 Jane and Jack each toss a fair coin twice. Let X be the number of heads Jane obtains and Y the number of heads Jack obtains. Define $U = X + Y$ and $V = X - Y$.

a) Find the means and variances of U and V.

b) Find Cov (U, V)

c) Are U and V independent? Why?

8.18 A multiple choice exam has 100 questions, each with 5 possible answers. One mark is awarded for a correct answer and 1/4 mark is deducted for an incorrect answer. A particular student has probability p_i of knowing the correct answer to the i^{th} question, independently of other questions.

a) Suppose that on a question where the student does not know the answer, he or she guesses randomly. Show that his or her total mark has mean $\sum p_i$ and variance $\sum p_i (1 - p_i) + \frac{(100 - \sum p_i)}{4}$.

b) Show that the total mark for a student who refrains from guessing also has mean $\sum p_i$, but with variance $\sum p_i (1 - p_i)$. Compare the variances when all p_i's equal (i) .9, (ii) .5.

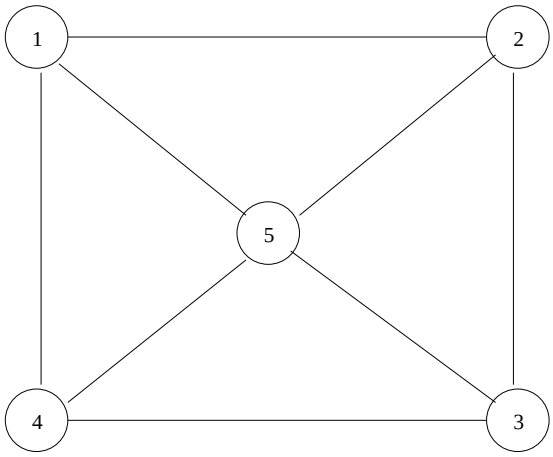
8.19 Let X and Y be independent random variables with $E(X) = E(Y) = 0$, Var$(X) = 1$ and Var $(Y) = 2$. Find Cov$(X + Y, X - Y)$.

8.20 An automobile driveshaft is assembled by placing parts A, B and C end to end in a straight line. The standard deviation in the lengths of parts A, B and C are 0.6, 0.8, and 0.7 respectively.

(a) Find the standard deviation of the length of the assembled driveshaft.

(b) What percent reduction would there be in the standard deviation of the assembled driveshaft if the standard deviation of the length of part B were cut in half?

8.21 The inhabitants of the beautiful and ancient canal city of Pentapolis live on 5 islands separated from each other by water. Bridges cross from one island to another as shown.



On any day, a bridge can be closed, with probability p, for restoration work. Assuming that the 8 bridges are closed independently, find the mean and variance of the number of islands which are completely cut off because of restoration work.

8.22 A Markov chain has a *doubly stochastic* transition matrix if both the row sums and the column sums of the transition matrix P are all 1. Show that for such a Markov chain, the uniform distribution on $\{1, 2, \ldots, N\}$ is a stationary distribution.

8.23 A salesman sells in three cities A,B, and C. He never sells in the same city on successive weeks. If he sells in city A, then the next week he always sells in B. However if he sells in either B or C, then the next week he is twice as likely to sell in city A as in the other city. What is the long-run proportion of time he spends in each of the three cities?

8.24 Find

$$\lim_{n \to \theta} P^n$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

8.25 Suppose X and Y are independent having Poisson distributions with parameters λ_1 and λ_2 respectively. Use moment generating functions to identify the distribution of the sum $X + Y$.

8.26 Waterloo in January is blessed by many things, but not by good weather. There are never two nice days in a row. If there is a nice day, we are just as likely to have snow as rain the next day. If we have snow or rain, there is an even chance of having the same the next day. If there is change

from snow or rain, only half of the time is this a change to a nice day. Taking as states the kinds of weather R, N, and S. the transition probabilities P are as follows

$$P = \begin{pmatrix} & \text{R} & \text{N} & \text{S} \\ \text{R} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \text{N} & \frac{1}{2} & 0 & \frac{1}{2} \\ \text{S} & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

If today is raining, find the probability of Rain, Nice, Snow three days from now. Find the probabilities of the three states in five days, given (1) today is raining (ii) today is nice (iii) today is snowing.

8.27 (**One-card Poker**) A card game, which, for the purposes of this question we will call Metzler Poker, is played as follows. Each of 2 players bets an initial \$1 and is dealt a card from a deck of 13 cards numbered 1-13. Upon looking at their card, each player then decides (unaware of the other's decision) whether or not to increase their bet by \$5 (to a total stake of \$6). If both increase the stake ("raise"), then the player with the higher card wins both stakes-i.e. they get their money back as well as the other player's \$6. If one person increases and the other does not, then the player who increases automatically wins the pot (i.e. money back+\$1). If neither person increases the stake, then it is considered a draw-each player receives their own \$1 back. Suppose that Player A and B have similar strategies, based on threshold numbers {*a,b*} they have chosen between 1 and 13. A chooses to raise whenever their card is greater than or equal to *a* and B whenever B's card is greater than or equal to *b*.

(a) Suppose B always raises (so that *b=1*). What is the expected value of A's win or loss for the different possible values of *a=1,2,...,13.*

(b) Suppose *a* and *b* are arbitrary. Given that both players raise, what is the probability that A wins? What is the expected value of A's win or loss?

(c) Suppose you know that *b=11.* Find your expected win or loss for various values of *a* and determine the optimal value. How much do you expect to make or lose per game under this optimal strategy?

8.28 (**Searching a database**) Suppose that we are given *3* records, R_1, R_2, R_3 *initially* stored in that order. The cost of accessing the *j'th* record in the list is *j* so we would like the more frequently accessed records near the front of the list. Whenever a request for record *j* is processed, the "move-to-front" heuristic stores R_j at the front of the list and the others in the original order. For example if the first request is for record 2, then the records will be re-stored in the order R_2, R_1, R_3. Assume that on each request, record j is requested with probability p_j, for $j = 1, 2, 3$.

(a) Show that if X_j is the permutation that obtains after j requests for records (e.g. $X_2 = (2,1,3)$), then $X_j, j = 1,2,...$ is a Markov chain.

(b) Find the stationary distribution of this Markov chain. (Hint: what is the probability that X_j takes the form $(2,*,*)$ for large j?)

(c) Find the expected long-run cost per record accessed in the case $p_1, p_2, p_3 = 0.1, 0.3, 0.6$ respectively.

(d) How does this expected long-run cost compare with keeping the records in random order, and with keeping them in order of decreasing values of p_j(only possible if we know p_j).

8.29 (**Secretary Problem**) Suppose you are to interview N candidates for a job, one at a time. You must decide immediately after each interview whether to hire the current candidate or not and you wish to maximize your chances of choosing the best person for the job (there is no benefit from choosing the second or third best). For simplicity, assume candidate i has numerical value X_i chosen without replacement from $\{1,2,...,N\}$ where $1 =$worst, $N =$best. Our strategy is to interview k candidates first, and then pick the first of the remaining $N-k$ that has value greater than max($X_1, X_2, ..., X_k$). What is the best choice of k? (Hint: you may use the approximation $\sum_{j=1}^{n-1} \frac{1}{j} \simeq \ln(n)$). For this choice, what is the approximate probability that you do choose the maximum?

8.30 Three stocks are assumed to have returns over the next year X_1, X_2, X_3 which have the same expected value $E(X_i) = 0.08, i = 1,2,3$ and variances $Var(X_1) = (0.2)^2, Var(X_2) = (0.3)^2$, $Var(X_4) = (0.4)^2$. Assuming that the returns are independent, find portfolio weights w_1, w_2, w_3 so that the linear combination

$$w_1X_1 + w_2X_2 + w_3X_3$$

has the smallest variance among all such linear combinations subject to $w_1 + w_2 + w_3 = 1$.

8.31* **Challenge problem:** A drunken probabilist stands n steps from a cliffs edge. He takes random steps, either towards or away from the cliff, each step independent of the past. At any point, the probability of taking a step away is 2/3, or a step toward, 1/3. What are his chances of escaping the cliff?

9. Continuous Probability Distributions

9.1 General Terminology and Notation

Continuous random variables have a range (set of possible values) an interval (or a collection of intervals) on the real number line. They have to be treated a little differently than discrete random variables because $P(X = x)$ is zero for each x. To illustrate a random variable with a *continuous distribution*, consider the simple spinning pointer in Figure 9.1. and suppose that all numbers in the

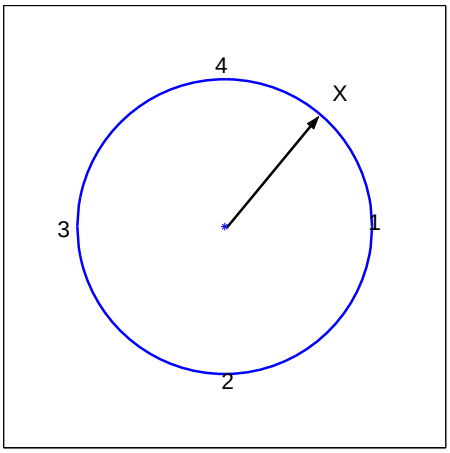


Figure 9.1: Spinner: a device for generating a continuous random variable (in a zero-gravity, virtually frictionless environment)

interval (0,4] are equally likely. The probability of the pointer stopping precisely at any given number x must be zero, because if each number has the same probability $p > 0$, then the probability of $R = \{x : 0 < x \leq 4\}$ is the sum $\sum_{x\epsilon(0,4]} p = \infty$, since the set R is uncountably infinite. For a continuous random variable the probability of each point is 0 and probability functions cannot be used to describe a distribution. On the other hand, intervals of the same length h entirely contained in (0,4], for example the interval $(0, \frac{1}{4}]$ and $(1\frac{3}{4}, 2]$ all have the same probability (1/16 in this case). For continuous random variables we specify the probability of intervals, rather than individual points.

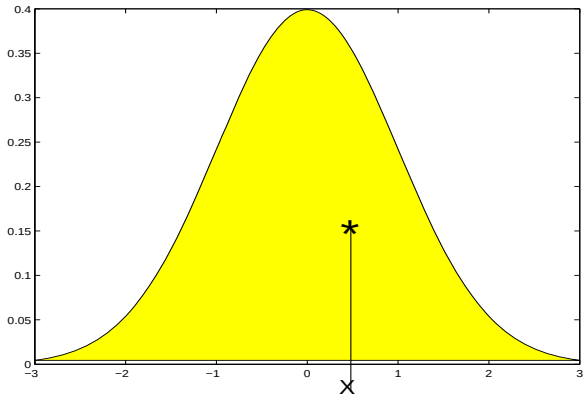


Figure 9.2:

Consider another example produced by choosing a “random point” in a region. Suppose we plot a graph a function $f(x)$ as in Figure 9.2 (assume the function is positive and has finite integral) and then generate a point at random by closing our eyes and firing a dart from a distance until at least one lands in the shaded region under the graph. We assume such a point, here denoted "*" is “uniformly” distributed under the graph. This means that the point is equally likely to fall in any one of many possible regions of a given area located in the shaded region so we only need to know the area of a region to determine the probability that a point falls in it. Consider the x-coordinate X of the point "*" as our random variable (in Figure 9.2 it appears to be around 0.4). Notice that the probability that X falls in a particular interval (a, b) is the measured by the area of the region above this interval, i.e. $\int_a^b f(x)dx$ and so the probability of any particular point $P(X = a)$ is the area of the region immediately above this single point $\int_a^a f(x)dx = 0$. This is another example of a random variable X which has a continuous distribution. For continuous X, there are two commonly used functions which describe its distribution. The first is the cumulative distribution function, used before for discrete distributions, and the second is the probability density function, the derivative of the c.d.f.

Cumulative Distribution Function:

For discrete random variables we defined the c.d.f., $F(x) = P(X \leq x)$ for continuous random variables as well as for discrete. For the spinner, the probability the pointer stops between 0 and 1 is 1/4 if all values x are equally “likely"; between 0 and 2 the probability is 1/2, between 0 and 3 it is 3/4; and so on. In general, $F(x) = x/4$ for $0 < x \leq 4$. Also, $F(x) = 0$ for $x \leq 0$ since there is no chance of the pointer stopping at a number ≤ 0, and $F(x) = 1$ for $x > 4$ since the pointer is certain to stop at number below x if $x > 4$. In our second example in which we generated a point at random under the graph of a function $f(x)$, if we assume that the total area under the graph is one, the cumulative distribution function $F(x)$ is the area under the graph but to the left of the point x as in Figure 9.3.

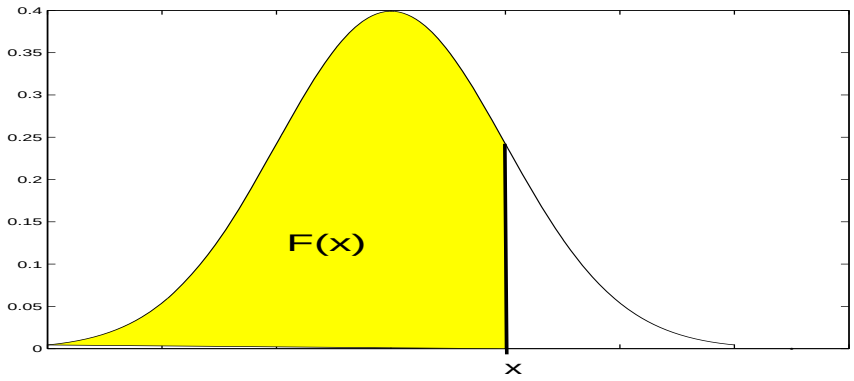


Figure 9.3:

Most properties of a c.d.f. are the same for continuous variables as for discrete variables. These are:

1. $F(-\infty) = 0$;and $F(\infty) = 1$
2. $F(x)$ is a non-decreasing function of x
3. $P\left(a < X \leq b\right) = F(b) - F(a)$.

Note that, as indicated before, for a continuous distribution, we have $0 = P(X = a) = \lim_{\varepsilon \to 0} P(a - \varepsilon < X \leq a) = \lim_{\varepsilon \to 0} F(a) - F(a - \varepsilon)$. This means that $\lim_{\varepsilon \to 0} F(a - \varepsilon) = F(a)$ or that the continuous distribution function is a continuous function (in the sense of continuity in calculus). Also, since the probability is 0 at each point:

$$P(a < X < b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = F(b) - F(a)$$

(For a discrete random variable, each of these 4 probabilities could be different.). For the continuous distributions in this chapter, we do not worry about whether intervals are open, closed, or half-open since the probability of these intervals is the same.

Probability Density Function (p.d.f.): While the c.d.f. can be used to find probabilities, it does not give an intuitive picture of which values of x are more likely, and which are less likely. To develop such a picture suppose that we take a short interval of X-values, $[x, x + \Delta x]$. The probability X lies in the interval is

$$P(x \leq X \leq x + \Delta x) = F(x + \Delta x) - F(x).$$

To compare the probabilities for two intervals, each of length Δx, is easy. Now suppose we consider what happens as Δx becomes small, and we divide the probability by Δx. This leads to the following definition.

Definition 32 *The **probability density function** (p.d.f.) $f(x)$ for a continuous random variable X is*

the derivative

$$f(x) = \frac{dF(x)}{dx}$$

where $F(x)$ is the c.d.f. for X.

Notice that if the function $f(x)$ graphed in Figure 9.3 has total integral one, the c.d.f. or the area to the left of a point x is given by $F(x) = \int_{-\infty}^{x} f(z)dz$ and so the derivative of the c.d.f. is $F'(x) = f(x)$. It is clear from the way in which X was generated that $f(x)$ represents the relative likelihood of (small intervals around) different x-values. To do this we first note some properties of a p.d.f. It is assumed that $f(x)$ is a continuous function of x at all points for which $0 < F(x) < 1$.

Properties of a probability density function

1. $P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)dx$. (This follows from the definition of $f(x)$)
2. $f(x) \geq 0$. (since $F(x)$ is non-decreasing, its derivative is non-negative)
3. $\int_{-\infty}^{\infty} f(x)dx = \int_{\text{all}x} f(x)dx = 1$. (This is because $P(-\infty \leq X \leq \infty) = 1$)
4. $F(x) = \int_{-\infty}^{x} f(u)du$. (This is just property 1 with $a = -\infty$)

To see that $f(x)$ represents the relative likelihood of different outcomes, we note that for Δx small,

$$P(x - \frac{\Delta x}{2} \leq X \leq x + \frac{\Delta x}{2}) = F(x + \frac{\Delta x}{2}) - F(x - \frac{\Delta x}{2}) \doteq f(x)\Delta x.$$

Thus, $f(x) \neq P(X = x)$ **but** $f(x)\Delta x$ is the *approximate probability* that X is inside an interval of length Δx centered about the value x when Δx is small. A plot of the function $f(x)$ shows such values clearly and for this reason it is very common to plot the probability density functions of continuous random variables.

Example: Consider the spinner example, where

$$F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{x}{4} & \text{for } 0 < x \leq 4 \\ 1 & \text{for } x > 4 \end{cases}$$

Thus, the p.d.f. is $f(x) = F'(x)$, or

$$f(x) = \frac{1}{4} \text{ for } 0 < x < 4.$$

and outside this interval the p.d.f. is 0.Figure 9.4 shows the probability density function $f(x)$; for obvious reasons this is called a "uniform" distribution.

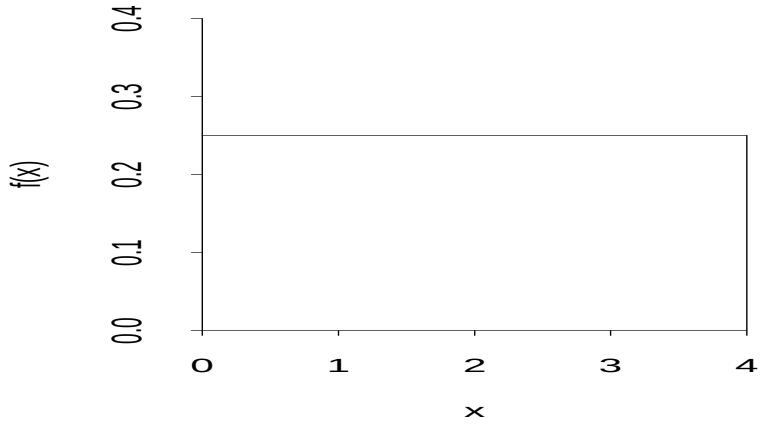


Figure 9.4: Uniform p.d.f.

Remark: Continuous probability distributions are, like discrete distributions, mathematical[39] **models**. Thus, the uniform distribution assumed for the spinner above is a model, though it seems likely it would be a good model for many real spinners.

Remark: It may seem paradoxical that $P(X = x) = 0$ for a continuous r.v. and yet we record the outcomes $X = x$ in real "experiments" with continuous variables. The catch is that all measurements have finite precision; they are in effect discrete. For example, the height $60 + \pi$ inches is within the range of the height X of people in a population but we could never observe the outcome $X = 60 + \pi$ if we selected a person at random and measured their height.
To summarize, in measurements we are actually observing something like

$$P(x - 0.5\Delta \leq X \leq x + 0.5\Delta)$$

where Δ may be very small, but not zero. The probability of this outcome is **not** zero: it is (approximately) $f(x)\Delta$.
We now consider a more complicated mathematical example of a continuous random variable Then we'll consider real problems that involve continuous variables. Remember that it is always a good idea to sketch or plot the p.d.f. $f(x)$ for a random variable.

Example:

[39] "How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?" Albert Einstein.

Let $f(x) = \begin{cases} kx^2; & 0 < x \leq 1 \\ k(2-x); & 1 < x < 2 \\ 0; & \text{otherwise} \end{cases}$ be a p.d.f.

Find

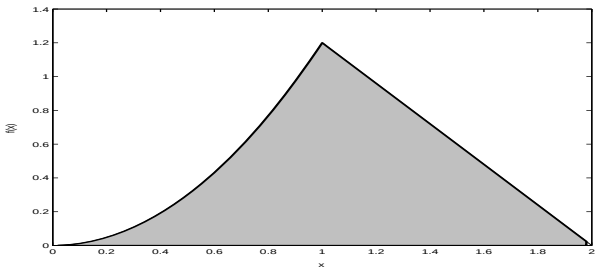
a) k

b) $F(x)$

c) $P\left(1/2 < X < 1\frac{1}{2}\right)$

Solution:

a) Set $\int_{-\infty}^{\infty} f(x)dx = 1$ to solve for k. When finding the area of a region bounded by different functions we split the integral into pieces.



(We normally wouldn't even write down the parts with $\int 0dx$)

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)dx \\ &= \int_{-\infty}^{0} 0dx + \int_{0}^{1} kx^2dx + \int_{1}^{2} k(2-x)dx + \int_{2}^{\infty} 0dx \\ &= 0 + k\int_{0}^{1} x^2dx + k\int_{1}^{2} (2-x)dx + 0 \\ &= k\frac{x^3}{3}\,|_0^1 + k\left(2x - \frac{x^2}{2}\,|_1^2\right) \\ &= \frac{5k}{6} \end{aligned}$$

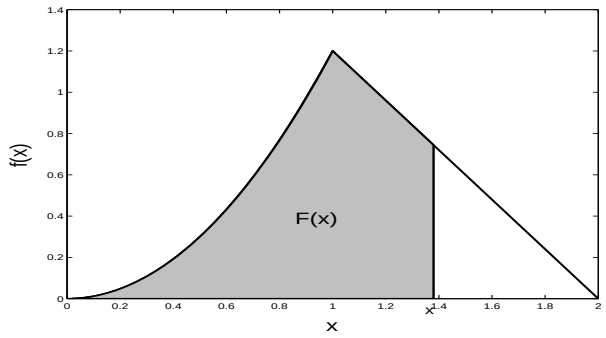
Therefore $k = \frac{6}{5}$.

b) Doing the easy pieces, which are often left out, first:

$$\begin{aligned} F(x) &= 0 \text{ if } x \le 0 \\ \text{and } F(x) &= 1 \text{ if } x \ge 2 \text{ (since all probability is below } x \text{ if } x \text{ is a number above 2.)} \end{aligned}$$

$$\begin{aligned} &\text{For } 0 < x < 1 \quad P(X \le x) = \int_0^x \tfrac{6}{5} z^2 dz = \tfrac{6}{5} \times \tfrac{x^3}{3} \mid_0^x = \tfrac{2x^3}{5} \\ &\text{For } 1 < x < 2, \quad P(X \le x) = \int_0^1 \tfrac{6}{5} z^2 dz + \int_1^x \tfrac{6}{5}(2 - z)\, dz \end{aligned}$$

(see the shaded area below)



$$\begin{aligned} &= \tfrac{6}{5} \tfrac{x^3}{3} \mid_0^1 + \tfrac{6}{5} \left(2x - \tfrac{x^2}{2} \mid_1^x \right. \\ &= \tfrac{12x - 3x^2 - 7}{5} \end{aligned}$$

i.e.

$$F(x) = \begin{cases} 0; & x \le 0 \\ 2x^3/5; & 0 < x \le 1 \\ \frac{12x - 3x^2 - 7}{5}; & 1 < x < 2 \\ 1; & x \ge 2 \end{cases}$$

As a rough check, since for a continuous distribution there is no probability at any point, $F(x)$ should have the same value as we approach each boundary point from above and from below.
e.g.

$$\begin{aligned} &\text{As } x \to 0^+, \quad \tfrac{2x^3}{5} \to 0 \\ &\text{As } x \to 1^-, \quad \tfrac{2x^3}{5} \to \tfrac{2}{5} \\ &\text{As } x \to 1^+, \quad \tfrac{12x - 3x^2 - 7}{5} \to \tfrac{2}{5} \\ &\text{As } x \to 2^-, \quad \tfrac{12x - 3x^2 - 7}{5} \to 1 \end{aligned}$$

This quick check won't prove your answer is right, but will detect many careless errors.

c)

$$\begin{aligned} P\left(\tfrac{1}{2} < X < 1\tfrac{1}{2}\right) &= \int_{1/2}^{1\frac{1}{2}} f(x)dx \\ &\text{or} \quad F\left(1\tfrac{1}{2}\right) - F\left(\tfrac{1}{2}\right) \quad \text{(easier)} \\ &= \frac{12\left(\frac{3}{2}\right) - 3\left(\frac{3}{2}\right)^2 - 7}{5} - \frac{2\left(\frac{1}{2}\right)^3}{5} = 4/5 \end{aligned}$$

Defined Variables or Change of Variable:

When we know the p.d.f. or c.d.f. for a continuous random variable X we sometimes want to find the p.d.f. or c.d.f. for some other random variable Y which is a function of X. The procedure for doing this is summarized below. It is based on the fact that the c.d.f. $F_Y(y)$ for Y equals $P(Y \leq y)$, and this can be rewritten in terms of X since Y is a function of X. Thus:

1) Write the c.d.f. of Y as a function of X.

2) Use $F_X(x)$ to find $F_Y(y)$. Then if you want the p.d.f. $f_Y(y)$, you can differentiate the expression for $F_Y(y)$.

3) Find the range of values of y.

Example: In the earlier spinner example,

$$\begin{aligned} f(x) &= \tfrac{1}{4}; \quad 0 < x \leq 4 \\ \text{and } F(x) &= \tfrac{x}{4}; \quad 0 < x \leq 4 \end{aligned}$$

Let $Y = 1/X$. Find $f(y)$.

Solution:

$$\begin{aligned} F_Y(y) &= P\left(Y \leq y\right) = P\left(\tfrac{1}{X} \leq y\right) = P\left(X \geq \tfrac{1}{y}\right) \\ &= 1 - P\left(X < 1/y\right) \\ &= 1 - F_X\left(1/y\right) \quad \text{(this completes step (1))} \end{aligned}$$

For step (2), we can do either:

$$\begin{aligned} F_Y(y) &= 1 - \frac{\left(\frac{1}{y}\right)}{4} \quad \text{(substituting } \tfrac{1}{y} \text{ for } x \text{ in } F_X(x)\text{)} \\ &= 1 - \tfrac{1}{4y} \\ \text{Therefore } f_Y(y) &= \tfrac{d}{dy}F_Y(y) = \tfrac{1}{4y^2}; \quad \tfrac{1}{4} \leq y < \infty \end{aligned}$$

(As x goes from 0 to 4, $y = \frac{1}{x}$ goes between ∞ and $\frac{1}{4}$.)

$$\begin{array}{rrcl} \text{or}: & f_Y(y) & = & \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(1 - F_X\left(1/y\right)\right) \\ & & = & -\frac{d}{dy}F_X(1/y) = -\frac{d}{dx}F_X(1/y)\frac{dx}{dy}|_{x=1/y} \quad (\text{chain rule}) \\ & & = & -f_X\left(1/y\right)\left(-\frac{1}{y^2}\right) = -\frac{1}{4}\left(-\frac{1}{y^2}\right) = \frac{1}{4y^2};\ \ \frac{1}{4} \leq y < \infty \end{array}$$

Generally if $F_X(x)$ is known it is easier to substitute first, then differentiate. If $F_X(x)$ is in the form of an integral that can't be solved, it is usually easier to differentiate first, then substitute $f_X(x)$.

Extension of Expectation, Mean, and Variance to Continuous Distributions

Definition 33 *When X is continuous, we still define*

$$E(g(X)) = \int_{all\ x} g(x)f(x)dx.$$

With this definition, all of the earlier properties of expected value and variance still hold; for example with $\mu = E(X)$,

$$\sigma^2 = \text{Var}(X) = E[(X-\mu)^2] = E(X^2) - \mu^2.$$

(This definition can be justified by writing $\int_{\text{all } x} g(x)f(x)dx$ as a limit of a Riemann sum and recognizing the Riemann sum as being in the form of an expected value for discrete random variables.)

Example: In the spinner example with $f(x) = \frac{1}{4};\ \ 0 < x \leq 4$

$$\begin{array}{rcl} \mu & = & \int_0^4 x\frac{1}{4}dx = \frac{1}{4}\left(\frac{x^2}{2}\right)|_0^4 = 2 \\ E\left(X^2\right) & = & \int_0^4 x^2\frac{1}{4}dx = \frac{1}{4}\left(\frac{x^3}{3}\right)|_0^4 = \frac{16}{3} \\ \sigma^2 & = & E\left(X^2\right) - \mu^2 = \frac{16}{3} - 4 = 4/3 \end{array}$$

Example: Let X have p.d.f.

$$f(x) = \begin{cases} \frac{6x^2}{5}; & 0 < x \leq 1 \\ \frac{6}{5}(2-x); & 1 < x < 2 \\ 0; & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned}
\mu &= \int_{\text{all } x} x f(x)dx = \int_0^1 x\frac{6}{5}x^2dx + \int_1^2 x\frac{6}{5}(2-x)dx \quad \text{(splitting the integral)} \\
&= \frac{6}{5}\left[\frac{x^4}{4}\,|_0^1 + \left(x^2 - \frac{x^3}{3}\right)|_1^2\right] = 11/10 \text{ or } 1.1 \\
E(X^2) &= \int_0^1 x^2\frac{6}{5}x^2dx + \int_1^2 x^2\frac{6}{5}(2-x)dx \\
&= \frac{6}{5}\left(\frac{x^5}{5}\,|_0^1 + 2\left(\frac{x^3}{3}\right)|_1^2 - \frac{x^4}{4}\,|_1^2\right) = \frac{67}{50} \\
\sigma^2 &= E\left(X^2\right) - \mu^2 = \frac{67}{50} - \left(\frac{11}{10}\right)^2 = \frac{13}{100} \text{ or } 0.13
\end{aligned}$$

Problems:

9.1.1 Let X have p.d.f. $f(x) = \begin{cases} kx^2; & -1 < x < 1. \\ 0; & \text{otherwise} \end{cases}$ Find

a) k

b) the c.d.f., $F(x)$

c) $P(-.1 < X < .2)$

d) the mean and variance of X.

e) let $Y = X^2$. Derive the p.d.f. of Y.

9.1.2 A continuous distribution has c.d.f. $F(x) = \frac{kx^n}{1+x^n}$ for $x > 0$, where n is a positive constant.

(a) Evaluate k.

(b) Find the p.d.f., $f(x)$.

(c) What is the median of this distribution? (The median is the value of x such that half the time we get a value below it and half the time above it.)

9.2 Continuous Uniform Distribution

Just as we did for discrete random variables, we now consider some special types of continuous probability distributions. These distributions arise in certain settings, described below. This section considers what we call uniform distributions.

Physical Setup:
Suppose X takes values in some interval [a,b] (it doesn't actually matter whether interval is open or closed) with all subintervals of a fixed length being equally likely. Then X has a **continuous uniform distribution**. We write $X \sim U[a, b]$.

Illustrations:

(1) In the spinner example $X \sim U(0, 4]$.

(2) Computers can generate a random number X which appears as though it is drawn from the distribution $U(0, 1)$. This is the starting point for many computer simulations of random processes; an example is given below.

The probability density function and the cumulative distribution function:
Since all points are equally likely (more precisely, intervals contained in $[a, b]$ of a given length, say 0.01, all have the same probability), the probability density function must be a constant $f(x) = k;\ \ a \leq x \leq b$ for some constant k. To make $\int_a^b f(x)dx = 1$, we require $k = \frac{1}{b-a}$.

$$\text{Therefore } f(x) = \frac{1}{b-a} \text{ for } \ \ a \leq x \leq b$$

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \int_a^x \frac{1}{b-a}dx = \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

Mean and Variance:

$$\begin{aligned} \mu &= \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}\left(\frac{x^2}{2}\,|_a^b\right) = \frac{b^2-a^2}{2(b-a)} \\ &= \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2} \\ E(X^2) &= \int_a^b x^2\frac{1}{b-a}dx = \frac{1}{(b-a)}\left(\frac{x^3}{3}\,|_a^b\right) \\ &= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)\left(b^2+ab+a^2\right)}{3(b-a)} = \frac{b^2+ab+a^2}{3} \\ \sigma^2 &= E\left(X^2\right) - \mu^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12} \\ &= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12} \end{aligned}$$

Example: Suppose X has the continuous p.d.f.

$$f(x) = .1e^{-.1x} \qquad\qquad x > 0$$

(This is called an exponential distribution and is discussed in the next section. It is used in areas such as queueing theory and reliability.) We'll show that the new random variable

$$Y = e^{-.1X}$$

has a uniform distribution, $U(0, 1)$. To see this, we follow the steps in Section 9.1:

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(e^{-.1X} \le y) \\ &= P(X \ge -10\ ln\ y) \\ &= 1 - P(X < -10\ ln\ y) \\ &= 1 - F_X(-10\ ln\ y) \end{aligned}$$

Since $F_X(x) = \int_0^x .1e^{-.1u}du = 1 - e^{-.1x}$ we get

$$\begin{aligned} F_Y(y) &= 1 - (1 - e^{-.1(-10ln\ y)}) \\ &= y \text{ for } 0 < y < 1 \end{aligned}$$

(The range of Y is (0,1) since $X > 0$.) Thus $f_Y(y) = F_Y'(y) = 1(0 < y < 1)$ and so $Y \sim U(0,1)$.

Many computer software systems have “random number generator" functions that will simulate observations Y from a $U(0,1)$ distribution. (These are more properly called **pseudo-random number generators** because they are based on deterministic algorithms. In addition they give observations Y that have finite precision so they cannot be **exactly** like continuous $U(0,1)$ random variables. However, good generators give Y's that appear indistinguishable in most ways from $U(0,1)$ random variables.) Given such a generator, we can also simulate random variables X with the exponential distribution above by the following algorithm:

1. Generate $Y \sim U(0,1)$ using the computer random number generator.
2. Compute $X = -10\ ln\ Y$.

Then X has the desired distribution. This is a particular case of a method described in Section 9.4 for generating random variables from a general distribution. In R software the command $runif(n)$ produces a vector consisting of n independent $U(0,1)$ values.

Problem:

9.2.1 If X has c.d.f. $F(x)$, then $Y = F(X)$ has a uniform distribution on [0,1]. (Show this.) Suppose you want to simulate observations from a distribution with $f(x) = \frac{3}{2}x^2;\ \ -1 < x < 1$, by using the random number generator on a computer to generate $U[0,1)$ numbers. What value would X take when you generated the random number .27125?

9.3 Exponential Distribution

The continuous random variable X is said to have an **exponential distribution** if its p.d.f. is of the form

$$f(x) = \lambda e^{-\lambda x} \qquad\qquad x > 0$$

where $\lambda > 0$ is a real parameter value. This distribution arises in various problems involving the time until some event occurs. The following gives one such setting.

Physical Setup: In a Poisson process for events in time let X be the length of time we wait for the first event occurrence. We'll show that X has an exponential distribution. (Recall that the number of occurrences in a fixed time has a Poisson distribution. The difference between the Poisson and exponential distributions lies in what is being measured.)

Illustrations:

(1) The length of time X we wait with a Geiger counter until the emission of a radioactive particle is recorded follows an exponential distribution.

(2) The length of time between phone calls to a fire station (assuming calls follow a Poisson process) follows an exponential distribution.

Derivation of the probability density function and the c.d.f.

$$\begin{aligned} F(x) = P(X \le x) &= P(\text{time to } 1^{\text{st}} \text{ occurrence} \le x) \\ &= 1 - P(\text{time to } 1^{\text{st}} \text{ occurrence} > x) \\ &= 1 - P(\text{no occurrences in the interval } (0, x)) \end{aligned}$$

Check that you understand this last step. If the time to the first occurrence $> x$, there must be no occurrences in $(0, x)$, and vice versa. We have now expressed $F(x)$ in terms of the number of occurrences in a Poisson process by time x. But the number of occurrences has a Poisson distribution with mean $\mu = \lambda x$, where λ is the average rate of occurrence.

$$\text{Therefore } F(x) = 1 - \frac{\mu^0 e^{-\mu}}{0!} = 1 - e^{-\mu}.$$

Since $\mu = \lambda x, F(x) = 1 - e^{-\lambda x};$ for $x > 0$. Thus

$$f(x) = \frac{d}{dx} F(x) = \lambda e^{-\lambda x}; \text{for } x > 0$$

which is the formula we gave above.

Alternate Form: It is common to use the parameter $\theta = 1/\lambda$ in the exponential distribution. (We'll see below that $\theta = E(X)$.) This makes

$$\begin{aligned} F(x) &= 1 - e^{-x/\theta} \\ \text{and } f(x) &= \tfrac{1}{\theta} e^{-x/\theta} \end{aligned}$$

Exercise:
Suppose trees in a forest are distributed according to a Poisson process. Let X be the distance from an arbitrary starting point to the nearest tree. The average number of trees per square metre is λ. Derive $f(x)$ the same way we derived the exponential p.d.f. You're now using the Poisson distribution in 2 dimensions (area) rather than 1 dimension (time).

Mean and Variance:
Finding μ and σ^2 directly involves integration by parts. An easier solution uses properties of **gamma functions**, which extends the notion of factorials beyond the integers to the positive real numbers.

Definition 34 ***The Gamma Function:*** $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ *is called the gamma function of* α*, where* $\alpha > 0$*.*

Note that α is 1 more than the power of x in the integrand. e.g. $\int_0^\infty x^4 e^{-x} dx = \Gamma(5)$. There are 3 properties of gamma functions which we'll use.

1. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for $\alpha > 1$
   Proof: Using integration by parts,
   $$\int_0^\infty x^{\alpha-1} e^{-x} dx = -x^{\alpha-1} e^{-x}|_0^\infty + (\alpha - 1) \int_0^\infty x^{\alpha-2} e^{-x} dx$$
   and provided that $\alpha > 1$, $x^{\alpha-1} e^{-x}|_0^\infty = 0$. Therefore
   $$\int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1) \int_0^\infty x^{\alpha-2} e^{-x} dx$$
2. $\Gamma(\alpha) = (\alpha - 1)!$ if α is a positive integer.
   Proof: It is easy to show that $\Gamma(1) = 1$. Using property 1 repeatedly, we obtain
   $$\begin{aligned} \Gamma(2) &= 1\Gamma(1) = 1, \\ \Gamma(3) &= 2\Gamma(2) = 2!, \\ \Gamma(4) &= 3\Gamma(3) = 3!, \text{ etc.} \end{aligned}$$
   In general, $\Gamma(n + 1) = n!$ for integer n.

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
   (This can be proved using double integration.)

Returning to the exponential distribution:

$$\mu = \int_0^\infty x \frac{1}{\theta} e^{-x/\theta} dx$$

Let $y = \frac{x}{\theta}$. Then $dx = \theta dy$ and

$$\begin{aligned} \mu &= \int_0^\infty y e^{-y} \theta dy = \theta \int_0^\infty y^1 e^{-y} dy = \theta \Gamma(2) \\ &= \theta \end{aligned}$$

Note: Read questions carefully. If you're given the average **rate** of occurrence in a Poisson process, that is λ. If you're given the average **time** you wait for an occurrence, that is θ.
To get $\sigma^2 = \mathrm{Var}(X)$, we first find

$$\begin{aligned} E\left(X^2\right) &= \int_0^\infty x^2 \tfrac{1}{\theta} e^{-x/\theta} dx \\ &= \int_0^\infty \theta^2 y^2 \tfrac{1}{\theta} e^{-y} \theta dy = \theta^2 \int_0^\infty y^2 e^{-y} dy \\ &= \theta^2 \Gamma(3) = 2!\theta^2 = 2\theta^2 \\ \text{Therefore } \sigma^2 &= E\left(X^2\right) - \mu^2 = 2\theta^2 - \theta^2 = \theta^2 \end{aligned}$$

Example:
Suppose #7 buses arrive at a bus stop according to a Poisson process with an average of 5 buses per hour. (i.e. $\lambda = 5$/hr. So $\theta = \frac{1}{5}$ hr. or 12 min.) Find the probability (a) you have to wait longer than 15 minutes for a bus (b) you have to wait more than 15 minutes longer, having already been waiting for 6 minutes.

Solution:

a) $P(X > 15) = 1 - P(X \leq 15) = 1 - F(15)$
   $= 1 - \left(1 - e^{-15/12}\right) = e^{-1.25} = .2865$

b) If X is the total waiting time, the question asks for the probability

$$P(X > 21 | X > 6) = \frac{P(X > 21 \text{ and } X > 6)}{P(X > 6)} = \frac{P(X > 21)}{P(X > 6)}$$

$$= \frac{1 - \left(1 - e^{-21/12}\right)}{1 - \left(1 - e^{-6/12}\right)} = \frac{e^{-21/12}}{e^{-6/12}} = e^{-15/12} = e^{-1.25} = .2865$$

Does this surprise you? The fact that you're already waited 6 minutes doesn't seem to matter. This illustrates the "memoryless property" of the exponential distribution:

$$P(X > a + b | X > b) = P(X > a)$$

Fortunately, buses don't follow a Poisson process so this example needn't cause you to stop using the bus.

Problems:

9.3.1 In a bank with on-line terminals, the time the system runs between disruptions has an exponential distribution with mean θ hours. One quarter of the time the system shuts down within 8 hours of the previous disruption. Find θ.

9.3.2 Flaws in painted sheets of metal occur over the surface according to the conditions for a Poisson process, at an intensity of λ per m^2. Let X be the distance from an arbitrary starting point to the second closest flaw. (Assume sheets are of infinite size!)

(a) Find the p.d.f., $f(x)$.

(b) What is the average distance to the second closest flaw?

9.4 A Method for Computer Generation of Random Variables$\diamondsuit$

[40]Most computer software has a built-in "pseudo-random number[41] generator" that will simulate observations U from a $U(0,1)$ distribution, or at least a reasonable approximation to this uniform distribution. If we wish a random variable with a non-uniform distribution, the standard approach is to take a suitable function of U. By far the simplest and most common method for generating non-uniform variates is based on the inverse cumulative distribution function. For arbitrary c.d.f. $F(x)$, define $F^{-1}(y) =$min $\{x; F(x) \geq y\}$. This is a real inverse (i.e. $F(F^{-1}(y)) = F^{-1}(F(y)) = y$) in the case that the c.d.f. is continuous and strictly increasing, so for example for a continuous distribution. However, in the more general case of a possibly discontinuous non-decreasing c.d.f. (such as the c.d.f. of a discrete distribution) the function continues to enjoy at least some of the properties of an inverse. F^{-1} is useful for generating a random variables having c.d.f. $F(x)$ from U, a uniform random variable on the interval $[0,1]$.

[40]$\diamondsuit$ This section optional for stat 220

[41]"The generation of random numbers is too important to be left to chance." Robert R. Coveyou, Oak Ridge National Laboratory

Theorem 35 *If F is an arbitrary c.d.f. and U is uniform on $[0,1]$ then the random variable defined by $X = F^{-1}(U)$ has c.d.f. $F(x)$.*

Proof:

The proof is a consequence of the fact that

$$[U < F(x)] \subset [X \leq x] \subset [U \leq F(x)] \text{ for all } x.$$

You can check this graphically be checking, for example, that if $[U < F(x)]$ then $[F^{-1}(U) \leq x]$ (this confirms the left hand "$\subset$"). Taking probabilities on all sides of this, and using the fact that $P[U \leq F(x)] = P[U < F(x)] = F(x)$, we discover that $P[X \leq x] = F(x)$.

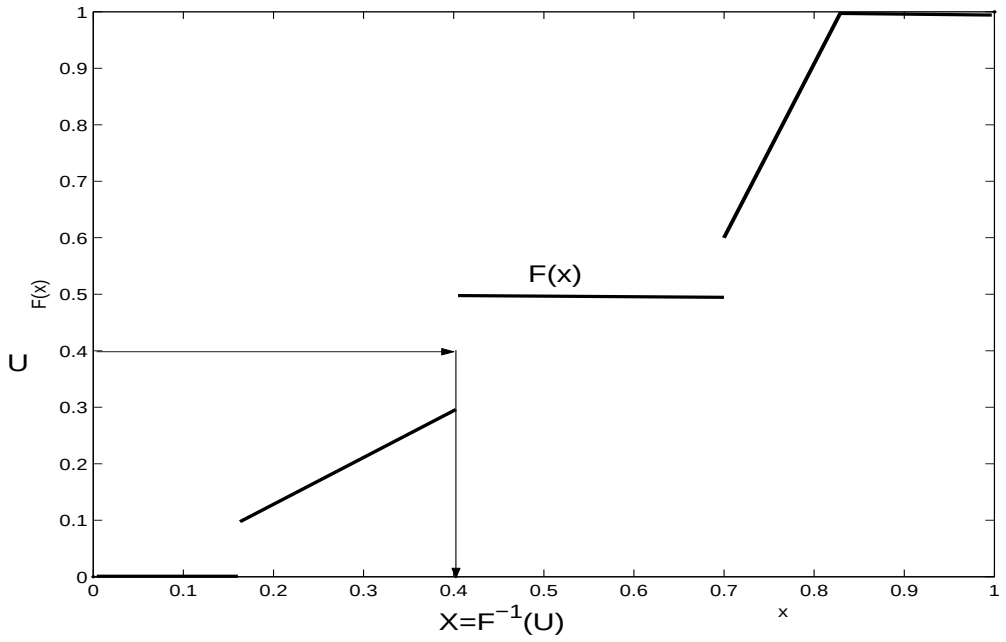


Figure 9.5: Inverting a Cumulative Distribution Function

The relation $X = F^{-1}(U)$ implies that $F(X) \geq U$ and for any point $z < X$, $F(z) < U$. For example, for the rather unusual looking piecewise linear cumulative distribution function in Figure 9.5, we find the solution $X = F^{-1}(U)$ by drawing a horizontal line at U until it strikes the graph of the c.d.f. (or where the graph would have been if we had joined the ends at the jumps) and then X is the $x-coordinate$ of this point. This is true in general, X is the coordinate of the point where a horizontal line first strikes the graph of the c.d.f. We provide one simple example of generating random variables by this method, for the geometric distribution.

Example: A geometric random number generator

For the Geometric distribution, the cumulative distribution function is given by

$$F(x) = 1 - (1-p)^{x+1}, \quad \text{for } x = 0, 1, 2, ...$$

Then if U is a uniform random number in the interval $[0, 1]$, we seek an integer X such that

$$F(X-1) < U \leq F(X)$$

(you should confirm that this is the value of X at which the above horizontal line strikes the graph of the c.d.f) and solving these inequalities gives

$$\begin{aligned} 1 - (1-p)^X < U &\leq 1 - (1-p)^{X+1} \\ (1-p)^X > 1 - U &\geq (1-p)^{X+1} \\ X \ln(1-p) > \ln(1-U) &\geq (X+1)\ln(1-p) \\ X < \frac{\ln(1-U)}{\ln(1-p)} &\leq X + 1 \end{aligned}$$

so we compute the value of

$$\frac{\ln(1-U)}{\ln(1-p)}$$

and round down to the next lower integer.

Exercise: An exponential random number generator.

Show that the inverse transform method above results in the generator for the exponential distribution

$$X = -\frac{1}{\lambda}\ln(1-U)$$

9.5 Normal Distribution

Physical Setup:
A random variable X defined on $(-\infty, \infty)$ has a normal[42] distribution if it has probability density function of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \qquad -\infty < x < \infty$$

[42] "The only normal people are the ones you don't know very well." Joe Ancis,

where $-\infty < \mu < \infty$ and $\sigma > 0$ are parameters. It turns out (and is shown below) that $E(X) = \mu$ and Var$(X) = \sigma^2$ for this distribution; that is why its p.d.f. is written using the symbols μ and σ. We write

$$X \sim N(\mu, \sigma^2)$$

to denote that X has a normal distribution with mean μ and variance σ^2 (standard deviation σ).
The normal distribution is the most widely used distribution in probability and statistics. Physical processes leading to the normal distribution exist but are a little complicated to describe. (For example, it arises in physics via statistical mechanics and maximum entropy arguments.) It is used for many processes where X represents a physical dimension of some kind, but also in many other settings. We'll see other applications of it below. The shape of the p.d.f. $f(x)$ above is what is often termed a "bell shape" or "bell curve", symmetric about 0 as shown in Figure 9.6.(you should be able to verify the shape without graphing the function)

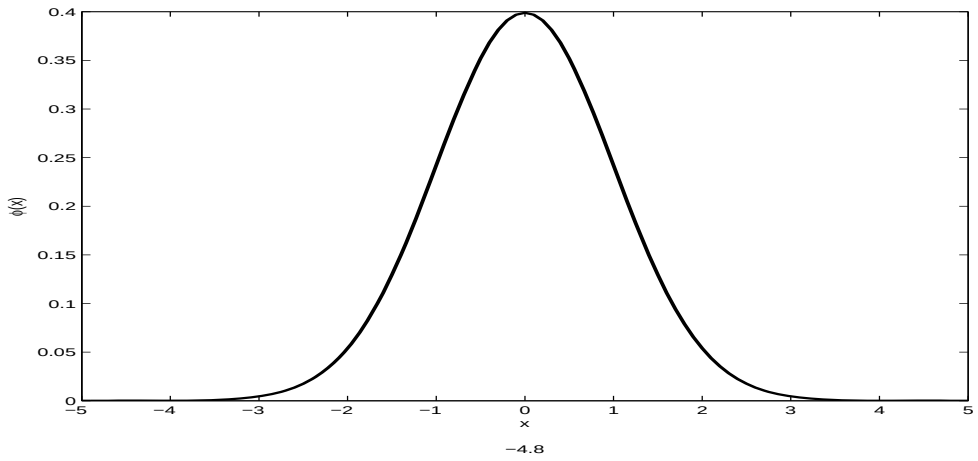


Figure 9.6: The Standard Normal ($N(0, 1)$) probability density function

Illustrations:

(1) Heights or weights of males (or of females) in large populations tend to follow normal distributions.

(2) The logarithms of stock prices are often assumed to be normally distributed.

The cumulative distribution function: The c.d.f. of the normal distribution $N(\mu, \sigma^2)$ is

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2} dy.$$

as shown in Figure 9.7. This integral cannot be given a simple mathematical expression so numerical methods are used to compute its value for given values of x, μ and σ. This function is included in many software packages and some calculators.

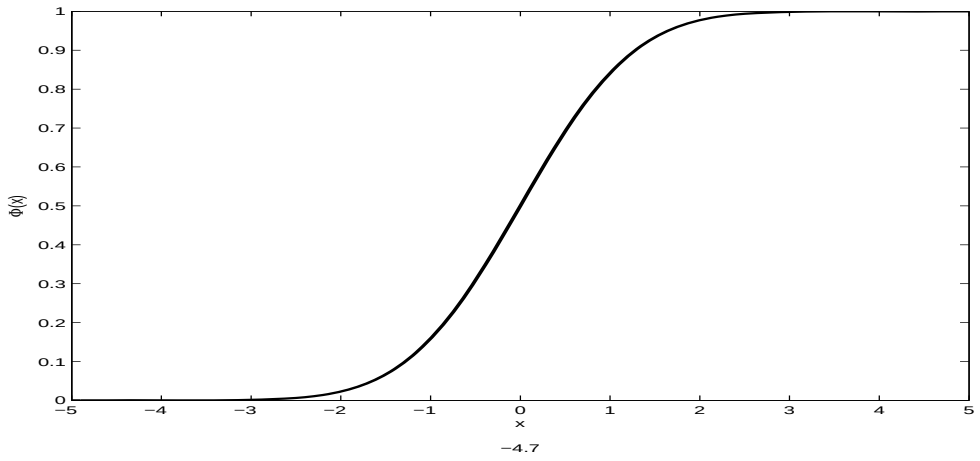


Figure 9.7: The standard normal c.d.f.

In the statistical packages R and S-Plus we get $F(x)$ above using the function $pnorm(x, \mu, \sigma)$. Before computers, people needed to produce tables of probabilities $F(x)$ by numerical integration, using mechanical calculators. Fortunately it is necessary to do this only for a single normal distribution: the one with $\mu = 0$ and $\sigma = 1$. This is called the **"standard" normal distribution** and denoted $N(0, 1)$.

It is easy to see that if $X \sim N(\mu, \sigma^2)$ then the "new" random variable $Z = (X - \mu)/\sigma$ is distributed as $Z \sim N(0, 1)$. (Just use the change of variables methods in Section 9.1.) We'll use this to compute $F(x)$ and probabilities for X below, but first we show that $f(x)$ integrates to 1 and that $E(X) = \mu$ and Var$(X) = \sigma^2$. For the first result, note that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \text{ where we let } z = (x-\mu)/\sigma) \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= 2\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y} \frac{dy}{\sqrt{2}y^{\frac{1}{2}}} \text{ Note: } y = \frac{1}{2}z^2; \text{ and } dz = \frac{dy}{\sqrt{2}y^{\frac{1}{2}}} \\
&= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} y^{-\frac{1}{2}} e^{-y} dy \\
&= \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2}) \qquad (\text{where } \Gamma \text{ is the gamma function}) \\
&= 1
\end{aligned}$$

Mean, Variance, Moment generating function: Recall that an odd function, $f(x)$, has the property that $f(-x) = -f(x)$. If $f(x)$ is an odd function then $\int_{-\infty}^{\infty} f(x)dx = 0$, provided the integral exists.

Consider

$$E\left(X-\mu\right)=\int_{-\infty}^{\infty}\left(x-\mu\right)\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx.$$

Let $y=x-\mu$. Then

$$E\left(X-\mu\right)=\int_{-\infty}^{\infty}y\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}dy,$$

where $y\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{y^2}{2\sigma^2}}$ is an odd function so that $E\left(X-\mu\right)=0$. But since $E\left(X-\mu\right)=E(X)-\mu$, this implies

$$E(X)=\mu,$$

and so μ is the mean. To obtain the variance,

$$\begin{aligned}\text{Var}(X) &= E\left[(X-\mu)^2\right]=\textstyle\int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx\\ &= 2\textstyle\int_{\mu}^{\infty}(x-\mu)^2\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx\ \text{ (since the function is symmetric about } \mu).\end{aligned}$$

We can obtain a gamma function by letting $y=\frac{(x-\mu)^2}{2\sigma^2}$.

$$\begin{aligned}\text{Then } (x-\mu)^2 &= 2\sigma^2y\\ (x-\mu) &= \sigma\sqrt{2y}\ \ (x>\mu,\ \text{so the positive root is taken})\\ dx &= \tfrac{\sigma\sqrt{2}dy}{2\sqrt{y}}=\tfrac{\sigma}{\sqrt{2y}}dy\end{aligned}$$

Then

$$\begin{aligned}\text{Var}(X) &= 2\int_0^{\infty}\left(2\sigma^2y\right)\frac{1}{\sigma\sqrt{2\pi}}e^{-y}\left(\frac{\sigma}{\sqrt{2y}}dy\right)\\ &= \frac{2\sigma^2}{\sqrt{\pi}}\int_0^{\infty}y^{1/2}e^{-y}dy=\frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)=\frac{2\sigma^2}{\sqrt{\pi}}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\frac{2\sigma^2\left(\frac{1}{2}\right)\sqrt{\pi}}{\sqrt{\pi}}\\ &= \sigma^2\end{aligned}$$

and so σ^2 is the variance. We now find the moment generating function of the $N(\mu,\sigma^2)$ distribution.

If X has the $N(\mu, \sigma^2)$ distribution, then

$$\begin{aligned} M_X(t) = E(e^{Xt}) &= \int_{-\infty}^{\infty} e^{xt} f(x) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{xt} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x - 2xt\sigma^2 + \mu^2)} dx \\ &= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x^2 - 2(\mu + t\sigma^2)x + (\mu + t\sigma^2)^2\}} dx \\ &= \frac{e^{\mu t + \sigma^2 t^2/2}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x - (\mu + t\sigma^2)\}^2} dx \\ &= e^{\mu t + \sigma^2 t^2/2} \end{aligned}$$

where the last step follows since

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x - (\mu + t\sigma^2)\}^2} dx$$

is just the integral of a $N(\mu + t\sigma^2, \sigma^2)$ probability density function and is therefore equal to one. This confirms the values we already obtained for the mean and the variance of the normal distribution

$$\begin{aligned} M'_X(0) &= e^{\mu t + \sigma^2 t^2/2}(\mu + t\sigma^2)|_{t=0} = \mu \\ M''_X(0) &= \mu^2 + \sigma^2 = E(X^2) \end{aligned}$$

from which we obtain

$$Var(X) = \sigma^2.$$

Finding Normal Probabilities Via $N(0,1)$ Tables As noted above, $F(x)$ does not have an explicit closed form so numerical computation is needed. The following result shows that if we can compute the c.d.f. for the standard normal distribution $N(0,1)$, then we can compute it for any other normal distribution $N(\mu, \sigma^2)$ as well.

Theorem 36 *Let $X \sim N(\mu, \sigma^2)$ and define $Z = (X - \mu)/\sigma$. Then $Z \sim N(0,1)$ and*

$$\begin{aligned} F_X(x) &= P(X \le x) \\ &= F_Z(\tfrac{x-\mu}{\sigma}). \end{aligned}$$

Proof: The fact that $Z \sim N(0,1)$ has p.d.f.

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

follows immediately by change of variables. Alternatively, we can just note that

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx \\ &= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \qquad (z = \frac{x-\mu}{\sigma}) \\ &= F_Z(\frac{x-\mu}{\sigma}) \qquad \square \end{aligned}$$

A table of probabilities $F_Z(z) = P(Z \leq z)$ is given on the last page of these notes. A space-saving feature is that only the values for $z > 0$ are shown; for negative values we use the fact that $N(0,1)$ p.d.f. is symmetric about 0. The following examples illustrate how to get probabilities for Z using the tables.

Examples: Find the following probabilities, where $Z \sim N(0,1)$.

(a) $P(Z \leq 2.11)$

(b) $P(Z \leq 3.40)$

(c) $P(Z > 1.06)$

(d) $P(Z < -1.06)$

(e) $P(-1.06 < Z < 2.11)$

Solution:

a) Look up 2.11 in the table by going down the left column to 2.1 then across to the heading .01. We find the number .9826. Then $P(Z \leq 2.11) = F(2.11) = .9826$. See Figure 9.8.

b) $P(Z \leq 3.40) = F(3.40) = .99966$

c) $P(Z > 1.06) = 1 - P(Z \leq 1.06) = 1 - F(1.06) = 1 - .8554 = .1446$

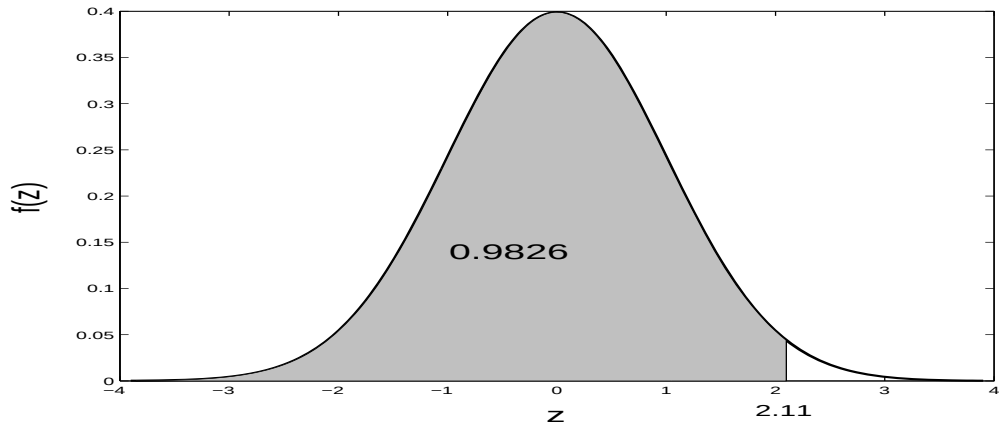


Figure 9.8:

d) Now we have to use symmetry:

$$P\left(Z < -1.06\right) = P\left(Z > 1.06\right) = 1 - P\left(Z \leq 1.06\right) = 1 - F(1.06) = .1446$$

See Figure 9.5.

e) $P\left(-1.06 < Z < 2.11\right) = F(2.11) - F(-1.06)$
$= F(2.11) - P\left(Z \leq -1.06\right) = F(2.11) - [1 - F(1.06)]$
$= .9826 - (1 - .8554) = .8380$

In addition to using the tables to find the probabilities for given numbers, we sometimes are given the probabilities and asked to find the number. With R or S-Plus software , the function qnorm (p, μ, σ) gives the 100 p-th percentile (where $0 < p < 1$). We can also use tables to find desired values.

Examples:

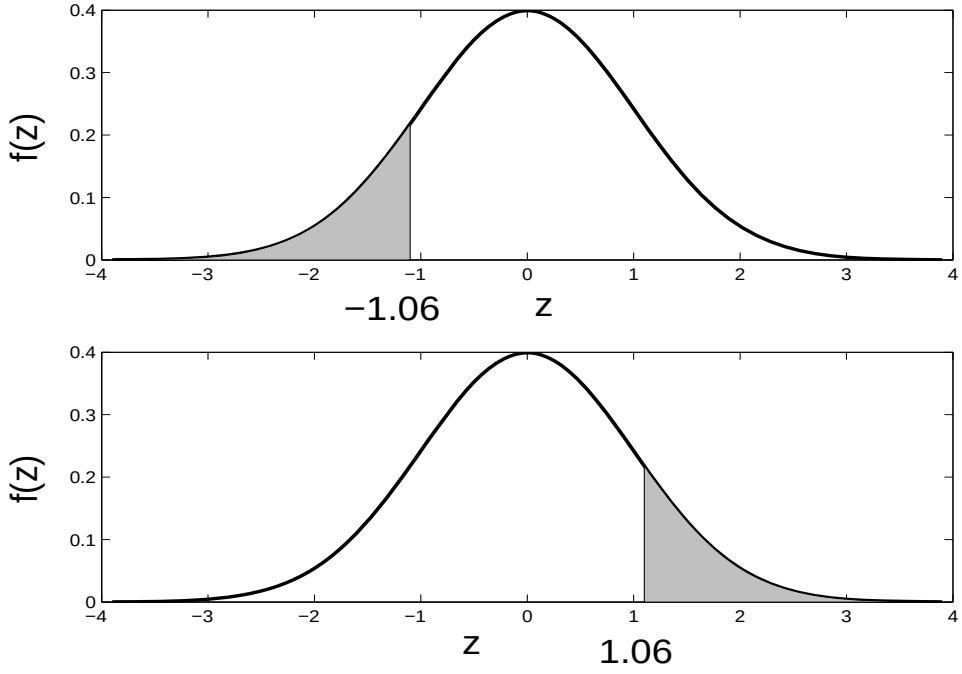
a) Find a number c such that $P\left(Z < c\right) = .85$

b) Find a number d such that $P\left(Z > d\right) = .90$

c) Find a number b such that $P\left(-b < Z < b\right) = .95$

Solutions:

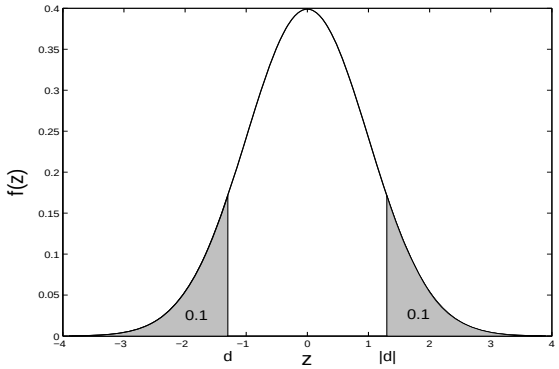
a) We can look in the body of the table to get an entry close to .8500. This occurs for z between 1.03 and 1.04; $z = 1.04$ gives the closest value to .85. For greater accuracy, the table at the



bottom of the last page is designed for finding numbers, given the probability. Looking beside the entry .85 we find $z = 1.0364$.

b) Since $P(Z > d) = .90$ we have $F(d) = P(Z \leq d) = 1 - P(Z > d) = .10$. There is no entry for which $F(z) = .10$ so we again have to use symmetry, since d will be negative.

$P(Z \leq d) = P(Z \geq |d|)$
$= 1 - F(|d|) = .10$
Therefore $F(|d|) = .90$
Therefore $|d| = 1.2816$
Therefore $d = -1.2816$



The key to this solution lies in recognizing that d will be negative. If you can picture the situation it will probably be easier to handle the question than if you rely on algebraic manipulations.

Exercise: Will a be positive or negative if $P(Z > a) = .05$? What if $P(Z < a) = .99$?

c) If $P(-b < Z < b) = .95$ we again use symmetry.

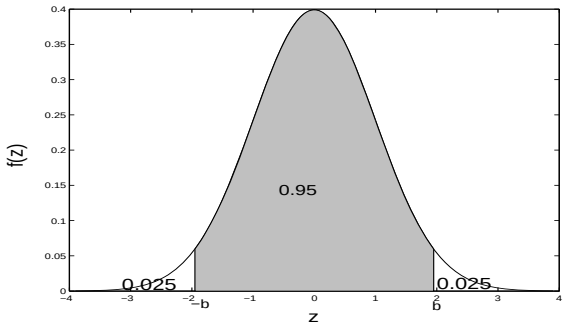


Figure 9.9:

The probability outside the interval $(-b, b)$ must be .05, and this is evenly split between the area above b and the area below $-b$.

$$\begin{aligned} \text{Therefore } P(Z < -b) &= P(Z > b) = .025 \\ \text{and } P(Z \leq b) &= .975 \end{aligned}$$

Looking in the table, $b = 1.96$.

To find $N(\mu, \sigma^2)$ probabilities in general, we use the theorem given earlier, which implies that if $X \sim N(\mu, \sigma^2)$ then

$$\begin{aligned} P(a \leq X \leq b) &= P\left(\tfrac{a-\mu}{\sigma} \leq Z \leq \tfrac{b-\mu}{\sigma}\right) \\ &= F_Z\left(\tfrac{b-\mu}{\sigma}\right) - F_Z\left(\tfrac{a-\mu}{\sigma}\right) \end{aligned}$$

where $Z \sim N(0, 1)$.

Example: Let $X \sim N(3, 25)$.

a) Find $P(X < 2)$

b) Find a number c such that $P(X > c) = .95$.

Solution:

a)

$$P(X<2) = P\left(\frac{X-\mu}{\sigma} < \frac{2-3}{5}\right) = P(Z<-.20) = 1-P(Z<.20)$$
$$= 1-F(.20) = 1-.5793 = .4207$$

b)

$$P(X>c) = P\left(\frac{X-\mu}{\sigma} > \frac{c-3}{5}\right) = P\left(Z > \frac{c-3}{5}\right) = .95$$

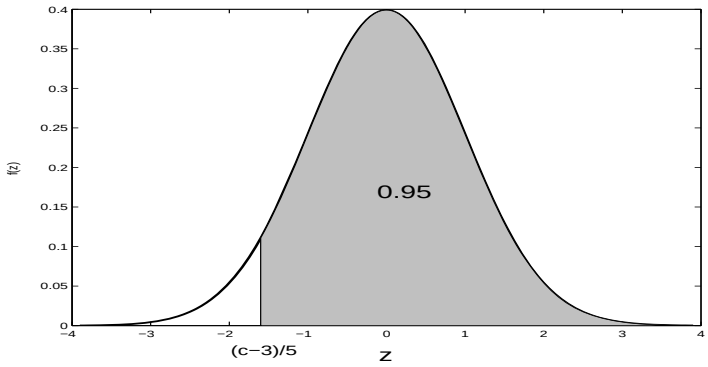


Figure 9.10:

$$\begin{aligned} \text{Therefore } \tfrac{c-3}{5} &= -1.6449 \\ \text{and } c &= -5.2245 \end{aligned}$$

Gaussian Distribution: The normal distribution is also known as the Gaussian[43] distribution. The notation $X \sim G(\mu, \sigma)$ means that X has Gaussian (normal) distribution with mean μ and standard deviation σ. So, for example, if $X \sim N(1,4)$ then we could also write $X \sim G(1,2)$.

Example: The heights of adult males in Canada are close to normally distributed, with a mean of 69.0 inches and a standard deviation of 2.4 inches. Find the 10th and 90th percentiles of the height distribution. (Recall that the a-th percentile is such that a% of the population has height less than this value.)

[43] After Johann Carl Friedrich Gauss (1777-1855), a German mathematician, physicist and astronomer, discoverer of Bode's Law, the Binomial Theorem and a regular 17-gon. He discovered the prime number theorem while an 18 year-old student and used least-squares (what is called statistical regression in most statistics courses) to predict the position of Ceres.

Solution: We are being told that if X is the height of a randomly selected Canadian adult male, then $X \sim G(69.0, 2.4)$, or equivalently $X \sim N(69.0, 5.76)$. To find the 90th percentile c, we use

$$P(X \leq c) = P\left(\frac{X - 69.0}{2.4} \leq \frac{c - 69.0}{2.4}\right)$$
$$= P\left(Z \leq \frac{c - 69.0}{2.4}\right) = .90$$

From the table we see $P(Z \leq 1.2816) = .90$ so we need

$$\frac{c - 69.0}{2.4} = 1.2816,$$

which gives $c = 72.08$ inches. Similarly, to find c such that $P(X \leq c) = .10$ we find that $P(Z \leq -1.2816) = .10$, so we need

$$\frac{c - 69.0}{2.4} - -1.2816,$$

or $c = 65.92$ inches, as the 10th percentile.

Linear Combinations of Independent Normal Random Variables

Linear combinations of normal random variables are important in many applications. Since we have not covered continuous multivariate distributions, we can only quote the second and third of the following results without proof. The first result follows easily from the change of variables method.

1. Let $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, where a and b are constant real numbers. Then $Y \sim N(a\mu + b, a^2\sigma^2)$

2. Let $X \sim N\left(\mu_1, \sigma_1^2\right)$ and $Y \sim N\left(\mu_2, \sigma_2^2\right)$ be independent, and let a and b be constants.
   Then $aX + bY \sim N\left(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2\right)$.
   In general if $X_i \sim N\left(\mu_i, \sigma_i^2\right)$ are independent and a_i are constants,
   then $\sum a_i X_i \sim N\left(\sum a_i\mu_i, \sum a_i^2\sigma_i^2\right)$.

3. Let $X_1, X_2, \cdots, X_n$ be independent $N\left(\mu, \sigma^2\right)$ random variables.
   Then $\sum X_i \sim N\left(n\mu, n\sigma^2\right)$ and $\overline{X} \sim N\left(\mu, \sigma^2/n\right)$.

   Actually, the only new result here is that the distributions are normal. The means and variances of linear combinations of random variables were previously obtained in section 8.3.

Example: Let $X \sim N(3, 5)$ and $Y \sim N(6, 14)$ be independent. Find $P(X > Y)$.

Solution: Whenever we have variables on both sides of the inequality we should collect them on one side, leaving us with a linear combination.

$$P(X > Y) = P(X - Y > 0)$$
$$X - Y \sim N(3 - 6, 5 + 14) \text{ i.e. } N(-3, 19)$$
$$P(X - Y > 0) = P\left(Z > \tfrac{0-(-3)}{\sqrt{19}} = .69\right) = 1 - F(.69) = .2451$$

Example: Three cylindrical parts are joined end to end to make up a shaft in a machine; 2 type A parts and 1 type B. The lengths of the parts vary a little, and have the distributions: $A \sim N\,(6, .4)$ and $B \sim N\,(35.2, .6)$. The overall length of the assembled shaft must lie between 46.8 and 47.5 or else the shaft has to be scrapped. Assume the lengths of different parts are independent. What percent of assembled shafts have to be scrapped?

Exercise: Why would it be wrong to represent the length of the shaft as 2A + B? How would this length differ from the solution given below?

Solution: Let L, the length of the shaft, be $L = A_1 + A_2 + B$.

Then

$$L \sim N\,(6 + 6 + 35.2,\ \ .4 + .4 + .6) = N(47.2, 1.4)$$

and so

$$\begin{aligned} P\,(46.8 < L < 47.5) &= P\left(\tfrac{46.8-47.2}{\sqrt{1.4}} < Z < \tfrac{47.5-47.2}{\sqrt{1.4}}\right) \\ &= P\,(-.34 < Z < .25) = .2318. \end{aligned}$$

i.e. 23.18% are acceptable and 76.82% must be scrapped. Obviously we have to find a way to reduce the variability in the lengths of the parts. This is a common problem in manufacturing.

Exercise: How could we reduce the percent of shafts being scrapped? (What if we reduced the variance of A and B parts each by 50%?)

Example: The heights of adult females in a large population is well represented by a normal distribution with mean 64 in. and variance 6.2 in^2.

(a) Find the proportion of females whose height is between 63 and 65 inches.

(b) Suppose 10 women are randomly selected, and let $\bar{X}$ be their average height (i.e. $\bar{X} = \sum_{i=1}^{10} X_i/10$, where $X_1, \ldots, X_{10}$ are the heights of the 10 women). Find $P(63 \leq \bar{X} \leq 65)$.

(c) How large must n be so that a random sample of n women gives an average height $\bar{X}$ so that $P(|\bar{X} - \mu| \leq 1) \geq .95$?

Solution:

(a) $X \sim N(64, 6.2)$ so for the height X of a random woman,

$$\begin{aligned} P(63 \le X \le 65) &= P\left(\frac{63-64}{\sqrt{6.2}} \le \frac{X-\mu}{\sigma} \le \frac{65-64}{\sqrt{6.2}}\right) \\ &= P(-0.402 \le Z \le 0.402) \\ &= 0.312 \end{aligned}$$

(b) $\bar{X} \sim N\left(64, \frac{6.2}{10}\right)$ so

$$\begin{aligned} P(63 \le \bar{X} \le 65) &= P\left(\tfrac{63-64}{\sqrt{.62}} \le \tfrac{\bar{X}-\mu}{\sigma_{\bar{X}}} \le \tfrac{65-64}{\sqrt{.62}}\right) \\ &= P(-.1.27 \le Z \le 1.27) \\ &= 0.796 \end{aligned}$$

(c) If $\bar{X} \sim N\left(64, \frac{6.2}{n}\right)$ then

$$\begin{aligned} P(|\bar{X}-\mu| \le 1) &= P(|\bar{X}-64| \le 1) \\ &= P(63 \le \bar{X} \le 65) \\ &= P\left(\tfrac{63-64}{\sqrt{6.2/n}} \le \tfrac{\bar{X}-\mu}{\sigma_{\bar{X}}} \le \tfrac{65-64}{\sqrt{6.2/n}}\right) \\ &= P(-0.402\sqrt{n} \le Z \le 0.402\sqrt{n}) = .95 \end{aligned}$$

iff $.402\sqrt{n} = 1.96$. (This is because $P(-1.96 \le Z \le 1.96) = .95$). So $P(|\bar{X}-64| \le 1) \ge .95$ iff $0.402\sqrt{n} \ge 1.96$ which is true if $n \ge (1.96/.402)^2$, or $n \ge 23.77$. Thus we require $n \ge 24$ since n is an integer.

Remark: This shows that if we were to select a random sample of $n = 24$ persons, then their average height $\bar{X}$ would be with 1 inch of the average height μ of the whole population of women. So if we did not know μ then we could estimate it to within ± 1 inch (with probability .95) by taking this small a sample.

Exercise: Find how large n would have to be to make $P(|\bar{X}-\mu| \le .5) \ge .95$.

These ideas form the basis of statistical sampling and estimation of unknown parameter values in populations and processes. If $X \sim N(\mu, \sigma^2)$ and we know roughly what σ is, but don't know μ, then we can use the fact that $\bar{X} \sim N(\mu, \sigma^2/n)$ to find the probability that the mean $\bar{X}$ from a sample of size n will be within a given distance of μ.

Problems:

9.5.1 Let $X \sim N(10, 4)$ and $Y \sim N(3, 100)$ be independent. Find the probability

a) $8.4 < X < 12.2$

b) $2Y > X$

c) $\overline{Y} < 0$ where $\overline{Y}$ is the sample mean of 25 independent observations on Y.

9.5.2 Let X have a normal distribution. What percent of the time does X lie within one standard deviation of the mean? Two standard deviations? Three standard deviations?

9.5.3 Let $X \sim N(5, 4)$. An independent variable Y is also normally distributed with mean 7 and standard deviation 3. Find:

(a) The probability $2X$ differs from Y by more than 4.

(b) The minimum number, n, of independent observations needed on X so that $P\left(|\overline{X} - 5| < 0.1\right) \geq .98$. ($\overline{X} = \sum_{i=1}^{n} X_i/n$ is the sample mean)

9.6 Use of the Normal Distribution in Approximations

The normal distribution can, under certain conditions, be used to approximate probabilities for linear combinations of variables having a non-normal distribution. This remarkable property follows from an amazing result called the central limit theorem. There are actually several versions of the central limit theorem. The version given below is one of the simplest.

Central Limit Theorem (CLT):

The major reason that the normal distribution is so commonly used is that it tends to approximate the distribution of sums of random variables. For example, if we throw n fair dice and S_n is the sum of the outcomes, what is the distribution of S_n? The tables below provide the number of ways in which a given value can be obtained. The corresponding probability is obtained by dividing by 6^n. For example on the throw of $n = 1$ dice the probable outcomes are 1,2,...,6 with probabilities all $1/6$ as indicated in the first panel of the histogram in Figure 9.11.

If we sum the values on two fair dice, the possible outcomes are the values 2,3,...,12 as shown in the following table and the probabilities are the values below:

Values	2	3	4	5	6	7	8	9	10	11	12
Probabilities$\times 36$	1	2	3	4	5	6	5	4	3	2	1

The probability histogram of these values is shown in the second panel. Finally for the sum of the values on three independent dice, the values range from 3 to 18 and have probabilities which, when multiplied by 6^3 result in the values

1 3 6 10 15 21 25 27 27 25 21 15 10 6 3 1

to which we can fit three separate quadratic functions one in the middle region and one in each of the

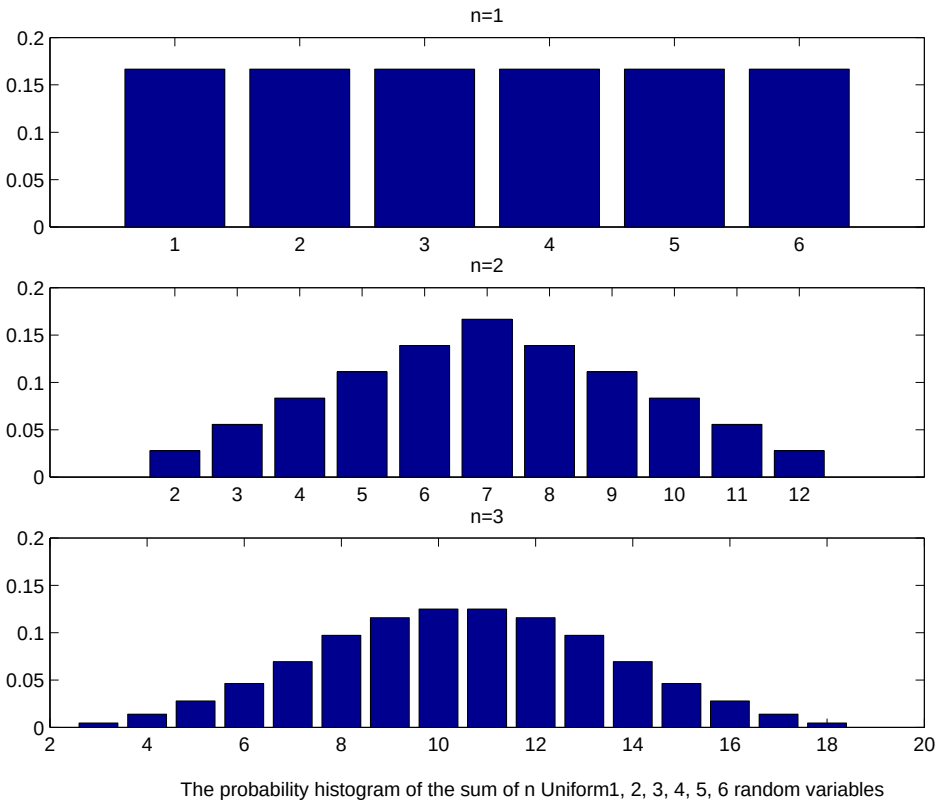


Figure 9.11: The probability histogram of the sum of n=1,2,3 discrete uniform {1,2,3,4,5,6}random variables

two tails. The histogram of these values shown in the third panel of Figure 9.11. and already resembles a normal probability density function.In general, these distributions show a simple pattern. For $n = 1$, the probability function is a constant (polynomial degree 0). For $n = 2,$ two linear functions spliced together. For $n = 3$, the histogram can be constructed from three quadratic pieces (polynomials of degree $n - 1$). These probability histograms rapidly approach the shape of the normal probability density function, as is the case with the sum or the average of independent random variables from most distributions. You can simulate the throws of any number of dice and illustrate the behaviour of the sums on at the url *http://www.math.csusb.edu/faculty/stanton/probstat/clt.html*.

Let $X_1, X_2, \cdots, X_n$ be independent random variables all having the same distribution, with mean μ and variance σ^2. Then as $n \to \infty$,

$$\sum_{i=1}^{n} X_i \sim N\left(n\mu, n\sigma^2\right) \tag{9.10}$$

and

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right). \tag{9.11}$$

This is actually a rough statement of the result since, as $n \to \infty$, both the $N\left(n\mu, n\sigma^2\right)$ and $N\left(\mu, \sigma^2/n\right)$ distributions fail to exist. (The former because both $n\mu$ and $n\sigma^2 \to \infty$, the latter because $\frac{\sigma^2}{n} \to 0$.) A precise version of the results is:

Theorem 37 *If $X_1, X_2, \cdots, X_n$ be independent random variables all having the same distribution, with mean μ and variance σ^2, then as $n \to \infty$, the cumulative distribution function of the random variable*

$$\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}$$

approaches the $N(0,1)$ c.d.f. Similarly, the c.d.f. of

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

approaches the standard normal c.d.f.

Although this is a theorem about limits, we will use it when n is large, but finite, to approximate the distribution of $\sum X_i$ or $\overline{X}$ by a normal distribution, so the rough version of the theorem in (9.10) and (9.11) is adequate for our purposes.

Notes:

(1) This theorem works for essentially all distributions which X_i could have. The only exception occurs when X_i has a distribution whose mean or variance don't exist. There are such distributions, but they are rare.

(2) We will use the Central Limit Theorem to approximate the distribution of sums $\sum_{i=1}^{n} X_i$ or averages $\bar{X}$. The accuracy of the approximation depends on n (bigger is better) and also on the actual distribution the X_i's come from. The approximation works better for small n when X_i's p.d.f. is close to symmetric.

(3) If you look at the section on linear combinations of independent normal random variables you will find two results which are very similar to the central limit theorem. These are:

For $X_1, \cdots, X_n$ independent and $N\left(\mu, \sigma^2\right)$, $\sum X_i \sim N\left(n\mu, n\sigma^2\right)$, and $\overline{X} \sim N\left(\mu, \sigma^2/n\right)$.

Thus, if the X_i's themselves have a normal distribution, then $\sum X_i$ and $\overline{X}$ have exactly normal distributions for all values of n. If the X_i's do not have a normal distribution themselves, then $\sum X_i$ and

$\overline{X}$ have approximately normal distributions when n is large. From this distinction you should be able to guess that if the X_i's distribution is somewhat normal shaped the approximation will be good for smaller values of n than if the X_i's distribution is very non-normal in shape. (This is related to the second remark in (2)).

Example: Hamburger patties are packed 8 to a box, and each box is supposed to have 1 Kg of meat in it. The weights of the patties vary a little because they are mass produced, and the weight X of a single patty is actually a random variable with mean $\mu = 0.128$ kg and standard deviation $\sigma = 0.005$ kg. Find the probability a box has at least 1 kg of meat, assuming that the weights of the 8 patties in any given box are independent.

Solution: Let $X_1, \ldots, X_8$ be the weights of the 8 patties in a box, and $Y = X_1 + \cdots + X_8$ be their total weight. By the Central Limit Theorem, Y is approximately $N(8\mu, 8\sigma^2)$; we'll assume this approximation is reasonable even though $n = 8$ is small. (This is likely OK because X's distribution is likely fairly close to normal itself.) Thus $Y \sim N(1.024, .0002)$ and

$$\begin{aligned} P(Y > 1) &= P\left(Z > \tfrac{1-1.024}{\sqrt{.0002}}\right) \\ &= P(Z > -1.702) \\ &\doteq .9554 \end{aligned}$$

(**Note:** We see that only about 95% of the boxes actually have 1 kg or more of hamburger. What would you recommend be done to increase this probability to 99%?)

Example: Suppose fires reported to a fire station satisfy the conditions for a Poisson process, with a mean of 1 fire every 4 hours. Find the probability the 500^{th} fire of the year is reported on the 84^{th} day of the year.

Solution: Let X_i be the time between the $(i-1)^{\text{st}}$ and i^{th} fires (X_1 is the time to the 1^{st} fire). Then X_i has an exponential distribution with $\theta = 1/\lambda = 4$ hours, or $\theta = 1/6$ day. Since $\sum_{i=1}^{500} X_i$ is the time until the 500th fire, we want to find $P\left(83 < \sum_{i=1}^{500} X_i \leq 84\right)$. While the exponential distribution is not close to normal shaped, we are summing a large number of independent exponential variables. Hence, by the central limit theorem, $\sum X_i$ has approximately a $N\left(500\mu, 500\sigma^2\right)$ distribution, where $\mu = E(X_i)$ and $\sigma^2 = \text{Var}(X_i)$.
For exponential distributions, $\mu = \theta = 1/6$ and $\sigma^2 = \theta^2 = 1/36$ so

$$P\left(83 < \sum X_i \leq 84\right) = P\left(\frac{83 - \frac{500}{6}}{\sqrt{\frac{500}{36}}} < Z \leq \frac{84 - \frac{500}{6}}{\sqrt{\frac{500}{36}}}\right)$$

$$= P\left(-.09 < Z \leq .18\right) = .1073$$

Example: This example is frivolous but shows how the normal distribution can approximate even sums of discrete random variables. In an orchard, suppose the number X of worms in an apple has probability function:

x	0	1	2	3
$f(x)$	.4	.3	.2	.1

Find the probability a basket with 250 apples in it has between 225 and 260 (inclusive) worms in it.

Solution:

$$\begin{aligned} \mu &= E(X) = \sum_{x=0}^{3} x f(x) = 1 \\ E\left(X^2\right) &= \sum_{x=0}^{3} x^2 f(x) = 2 \\ \text{Therefore } \sigma^2 &= E\left(X^2\right) - \mu^2 = 1 \end{aligned}$$

By the central limit theorem, $\sum_{i=1}^{250} X_i$ has approximately a $N\left(250\mu, 250\sigma^2\right)$ distribution, where X_i is the number of worms in the i^{th} apple.
i.e.

$$\sum X_i \sim N\left(250, 250\right)$$

$$\begin{aligned} P\left(225 \leq \sum X_i \leq 260\right) &= P\left(\frac{225-250}{\sqrt{250}} \leq Z \leq \frac{260-250}{\sqrt{250}}\right) \\ &= P\left(-1.58 \leq Z \leq .63\right) = .6786 \end{aligned}$$

While this approximation is adequate, we can improve its accuracy, as follows. When X_i has a discrete distribution, as it does here, $\sum X_i$ will always remain discrete no matter how large n gets. So the distribution of $\sum X_i$, while normal shaped, will never be precisely normal. Consider a probability histogram of the distribution of $\sum X_i$, as shown in Figure 9.12. (Only part of the histogram is shown.) The area of each bar of this histogram is the probability at the x value in the centre of the interval. The smooth curve is the p.d.f. for the approximating normal distribution. Then $\sum_{x=225}^{260} P\left(\sum X_i = x\right)$ is the total area of all bars of the histogram for x from 225 to 260. These bars actually span continuous x values from 224.5 to 260.5. We could then get a more accurate approximation by finding the area under the normal curve from 224.5 to 260.5.
i.e. $P\left(225 \leq \sum X_i \leq 260\right) = P\left(224.5 < \sum X_i < 260.5\right)$

$$\begin{aligned} &= P\left(\tfrac{224.5-250}{\sqrt{250}} < Z < \tfrac{260.5-250}{\sqrt{250}}\right) \\ &= P\left(-1.61 < Z < .66\right) = .6917 \end{aligned}$$

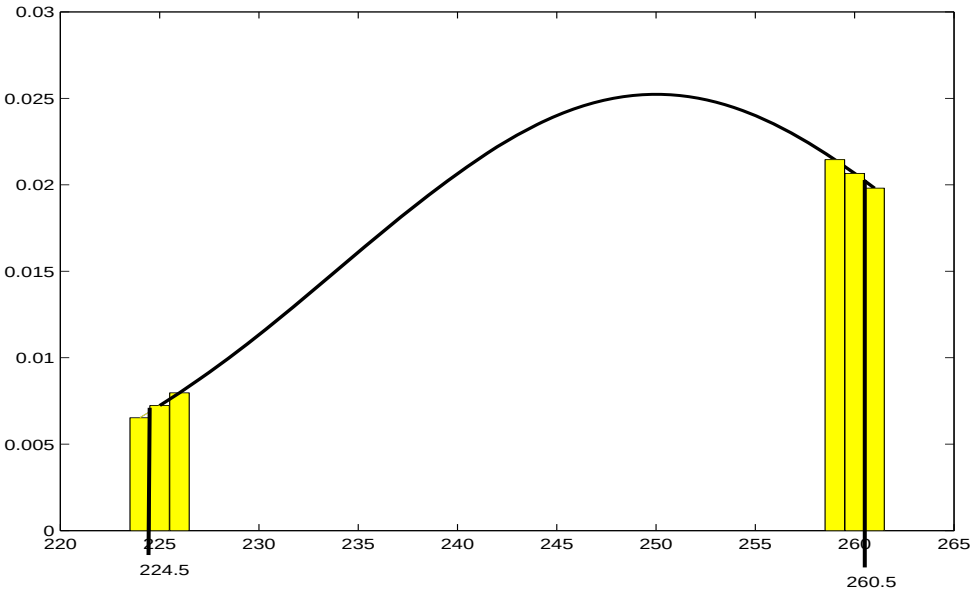


Figure 9.12:

Unless making this adjustment greatly complicates the solution, it is preferable to make this "**continuity correction**".

Notes:

(1) A continuity correction should not be applied when approximating a continuous distribution by the normal distribution. Since it involves going halfway to the next possible value of x, there would be no adjustment to make if x takes real values.

(2) Rather than trying to guess or remember when to add .5 and when to subtract .5, it is often helpful to sketch a histogram and shade the bars we wish to include. It should then be obvious which value to use.

(3) When you are approximating a probability such as $P(X = 50)$ where X is Binomial$(100, 0.5)$ it is **essential** to use the continuity correction because without it, we obtain the silly approximation $P(X = 50) \simeq 0$.

Example: Normal approximation to the Poisson Distribution

Let X be a random variable with a Poisson(λ) distribution and suppose λ is large. For the moment suppose that λ is an integer and recall that if we add λ independent Poisson random variables, each with parameter 1, then the sum has the Poisson distribution with parameter λ. In general, a Poisson

random variable with large expected value can be written as the sum of a large number of independent random variables, and so the central limit theorem implies that it must be close to normally distributed. We can prove this using moment generating functions. In Section 7.5 we found the moment generating function of a Poisson random variable X

$$M_X(t) = e^{-\lambda+\lambda e^t}.$$

Then the standardized random variable is

$$Z = \frac{X-\lambda}{\sqrt{\lambda}}$$

and this has moment generating function

$$\begin{aligned} M_Z(t) &= E(e^{Zt}) = E(e^{t\left(\frac{X-\lambda}{\sqrt{\lambda}}\right)}) \\ &= e^{-t\sqrt{\lambda}}E(e^{Xt/\sqrt{\lambda}}) \\ &= e^{-t\sqrt{\lambda}}M_X(t/\sqrt{\lambda}) \end{aligned}$$

This is easier to work with if we take logarithms,

$$\begin{aligned} \ln(M_Z(t)) &= -t\sqrt{\lambda} - \lambda + \lambda e^{t/\sqrt{\lambda}} \\ &= \lambda(e^{t/\sqrt{\lambda}} - 1 - \frac{t}{\sqrt{\lambda}}). \end{aligned}$$

Now as $\lambda \to \infty$,

$$\frac{t}{\sqrt{\lambda}} \to 0$$

and

$$e^{t/\sqrt{\lambda}} = 1 + \frac{t}{\sqrt{\lambda}} + \frac{1}{2}\frac{t^2}{\lambda} + o(\lambda^{-1})$$

so

$$\begin{aligned} \ln(M_Z(t)) &= \lambda(e^{t/\sqrt{\lambda}} - 1 - \frac{t}{\sqrt{\lambda}}). \\ &= \lambda(\frac{t^2}{2\lambda} + o(\lambda^{-1})) \\ &\to \frac{t^2}{2} \text{ as } \lambda \to \infty. \end{aligned}$$

Therefore the moment generating function of the standardized Poisson random variable Z approaches $e^{t^2/2}$, the moment generating function of the standard normal and this implies that the Poisson distribution approaches the normal as $\lambda \to \infty$.

Example: Suppose $X \sim Poisson(\lambda)$. Use the normal approximation to approximate

$$P(X > \lambda).$$

Compare this approximation with the true value when $\lambda = 9$.

Solution We have verified above that the moment generating function of the Poisson(λ) distribution approaches $e^{t^2/2}$, the moment generating function of the standard normal distribution as $\lambda \to \infty$. This implies that the cumulative distribution function of the standardized random variable

$$Z_\lambda = \frac{X - \lambda}{\sqrt{\lambda}}$$

(note: identify $E(X)$ and $Var(X)$ in the above standardization) approaches the cumulative distribution function of a **standard normal** random variable Z. In particular, without a continuity correction,

$$P(X \leq \lambda) = P(Z_\lambda \leq 0) \to P(Z \leq 0) = \frac{1}{2} \text{ as } \lambda \to \infty.$$

Computing the true value when $\lambda = 9$,

$$P(X > 9) = 1 - P(X \leq 9) = 1 - e^{-9} - 9e^{-9} - \frac{9^2}{2!}e^{-9} - ... - \frac{9^9}{9!}e^{-9} = 1 - 0.5874 = 0.4126.$$

There is a considerable difference here between the true value 0.4126 and the normal approximation $\frac{1}{2}$ since the value of $\lambda = 9$ is still quite small. However, if we use the continuity correction when we apply the normal approximation,

$$P(X > 9) = P(X > 9.5) = P(Z_\lambda > \frac{0.5}{3}) \to P(Z > 0.1667) = 0.4338$$

which is much closer to the true value 0.4126.

Normal approximation to the Binomial Distribution

It is well-known that the binomial distribution, at least for large values of n, resembles a bell-shaped or normal curve. The most common demonstration of this is with a mechanical device common in science museums called a "Galton board" or "Quincunx"[44] which drop balls through a mesh of equally spaced pins (see Figure 9.13 and the applet at `http://javaboutique.internet.com/BallDrop/`). Notice that if balls either go to the right or left at each of the 8 levels of pins, independently of the movement of the other balls, then $X =$number of moves to right has a $Bin(8, \frac{1}{2})$ distribution. If the balls are dropped from location 0 (on the $x-$axis) then the ball eventually rests at location $2X - 8$ which is approximately normally distributed since X is approximately normal.
The following result is easily proved using the Central Limit Theorem.

[44]The word comes from Latin quinque (five) unicia (twelve) and means five twelfths.

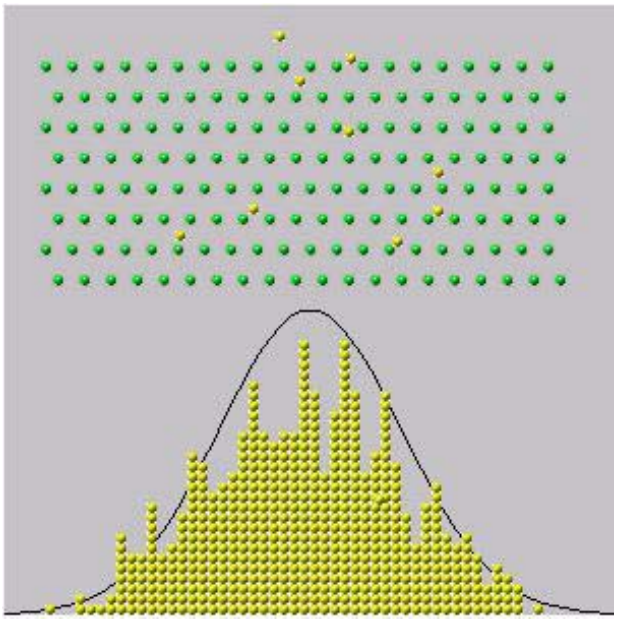


Figure 9.13: A "Galton Board" or "Quincunx"

Theorem 38 *Let X have a binomial distribution, $Bi(n,p)$. Then for n large, the random variable*

$$W = \frac{X - np}{\sqrt{np(1-p)}} \qquad \textit{is approximately } N(0,1)$$

Proof: We use indicator variables $X_i (i = 1, \ldots, n)$ where $X_i = 1$ if the ith trial in the binomial process is an "S" outcome and 0 if it is an "F" outcome. Then $X = \sum_{i=1}^{n} X_i$ and we can use the CLT. Since

$$\mu = E(X_i) = p, \text{ and } \sigma^2 = \text{Var}(X_i) = p(1-p)$$

we have that as $n \to \infty$

$$\frac{\sum X_i - np}{\sqrt{np(1-p)}} = \frac{X - np}{\sqrt{np(1-p)}}$$

is $N(0,1)$, as stated. □

An alternative proof uses moment generating functions and is essentially a proof of this particular case of the Central Limit Theorem. Recall that the moment generating function of the binomial random variable X is

$$M_X(t) = (1 - p + pe^t)^n.$$

As we did with the standardized Poisson random variable, we can show with some algebraic effort that the moment generating function of W

$$E(e^{Wt}) \to e^{t^2/2} \text{ as } n \to \infty$$

proving that the standardized binomial random variable W approaches the standard normal distribution.

Remark: We can write the normal approximation either as $W \sim N(0,1)$ or as $X \sim N(np, np(1-p))$.

Remark: The continuity correction method can be used here. The following numerical example illustrates the procedure.

Example: If (i) $X \sim Bi(n = 20, p = .4)$, use the theorem to find the approximate probability $P(4 \leq X \leq 12)$ and (ii) if $X \sim Bi(100, .4)$ find the approximate probability $P(34 \leq X \leq 48)$. Compare the answer with the exact value in each case.

Solution (i) By the theorem above, $X \sim N(8, 4.8)$ approximately. Without the continuity correction,

$$\begin{aligned} P(4 \leq X \leq 12) &= P\left(\tfrac{4-8}{\sqrt{4.8}} \leq \tfrac{X-9}{\sqrt{4.8}} \leq \tfrac{12-8}{\sqrt{4.8}}\right) \\ &\doteq P(-1.826 \leq Z \leq 1.826) = 0.932 \end{aligned}$$

where $Z \sim N(0,1)$. Using the continuity correction method, we get

$$\begin{aligned} P(4 \leq X \leq 12) &\doteq P\left(\tfrac{3.5-8}{\sqrt{4.8}} \leq Z \leq \tfrac{12.5-8}{\sqrt{4.8}}\right) \\ &= P(-2.054 \leq Z \leq 2.054) = 0.960 \end{aligned}$$

The exact probability is $\sum_{x=4}^{12} \binom{20}{x}(.4)^x(.6)^{20-x}$, which (using the R function $pbinom(\ \ )$) is 0.963. As expected the continuity correction method gives a more accurate approximation.
(ii) $X \sim N(40, 24)$ approximately so without the continuity correction

$$\begin{aligned} P(34 \leq X \leq 48) &\simeq P\left(\frac{34-40}{\sqrt{24}} \leq Z \leq \frac{48-40}{\sqrt{24}}\right) \\ &\simeq P(-1.225 \leq Z \leq 1.633) \\ &\simeq 0.9488 - (1 - 0.8897) = 0.8385 \end{aligned}$$

With the continuity correction

$$\begin{aligned} P(34 \leq X \leq 48) &\simeq P\left(\frac{33.5-40}{\sqrt{24}} \leq Z \leq \frac{48.5-40}{\sqrt{24}}\right) \\ &\simeq P(-1.327 \leq Z \leq 1.735) \\ &\simeq 0.9586 - (1 - 0.9076) = 0.866 \end{aligned}$$

The exact value, $\sum_{x=34}^{48} f(x)$, equals 0.866 (correct to 3 decimals). The error of the normal approximation decreases as n increases, but it is a good idea to use the continuity correction when it is convenient. For example if we are using a normal approximation to a discrete distribution like the binomial which takes integer values and the standard deviation of the binomial is less than $10,$ then the continuity correction makes a difference of $0.5/10 = 0.05$ to the number we look up in the table. This can result in a difference in the probability of up to around 0.02. If you are willing to tolerate errors in probabilities of that magnitude, your rule of thumb might be to use the continuity correction whenever the standard deviation of the integer-valued random variable being approximated is less than 10.

Example: Let p be the proportion of Canadians who think Canada should adopt the US dollar.

a) Suppose 400 Canadians are randomly chosen and asked their opinion. Let X be the number who say yes. Find the probability that the proportion, $\frac{X}{400}$, of people who say yes is within 0.02 of p, if p is 0.20.

b) Find the number, n, who must be surveyed so there is a 95% chance that $\frac{X}{n}$ lies within 0.02 of p. Again suppose p is 0.20.

c) Repeat (b) when the value of p is unknown.

Solution:

a) $X \sim Bi\,(400, .2)$. Using the normal approximation we take

$X \sim$ Normal with mean $np = (400)(.2) = 80$ and variance $np(1-p) = (400)(.2)(.8) = 64$

If $\frac{X}{400}$ lies within $p \pm .02$, then $.18 \leq \frac{X}{400} \leq .22$, so $72 \leq X \leq 88$. Thus, we find

$$\begin{aligned} P\,(72 \leq X \leq 88) &\doteq P\left(\frac{71.5-80}{\sqrt{64}} < Z < \frac{88.5-80}{\sqrt{64}}\right) \\ &= P\,(-1.06 < Z < 1.06) = .711 \end{aligned}$$

b) Since n is unknown, it is difficult to apply a continuity correction, so we omit it in this part. By the normal approximation,

$$X \sim N\,(np = .2n, np(1-p) = .16n)$$

Therefore,

$$\frac{X}{n} \sim N\left(\frac{0.2n}{n} = 0.2, \frac{0.16n}{n^2} = \frac{0.16}{n}\right) \quad (\text{Recall Var}\,(aX) = a^2\,\text{Var}\,(X))$$

$P\left(.18 \leq \frac{X}{n} \leq .22\right) = .95$ is the condition we need to satisfy. This gives

$$P\left(\frac{.18-.2}{\sqrt{\frac{.16}{n}}} \leq Z \leq \frac{.22-.2}{\sqrt{\frac{.16}{n}}}\right) = .95$$
$$P\left(-.05\sqrt{n} \leq Z \leq .05\sqrt{n}\right) = 0.95$$

Therefore, $F\left(.05\sqrt{n}\right) = .975$ and so $.05\sqrt{n} = 1.9600$ giving $n = 1536.64$. In other words, we need to survey 1537 people to be at least 95% sure that $\frac{X}{n}$ lies within .02 either side of p.

c) Now using the normal approximation to the binomial, approximately $X \sim N\left(np, np(1-p)\right)$ and so

$$\frac{X}{n} \sim N\left(p, \frac{p(1-p)}{n}\right)$$

We wish to find n such that

$$\begin{aligned} 0.95 &= P\left(p - .02 \leq \frac{X}{n} \leq p + .02\right) \\ &= P\left(\frac{p-.02-p}{\sqrt{\frac{p(1-p)}{n}}} \leq Z \leq \frac{p+.02-p}{\sqrt{\frac{p(1-p)}{n}}}\right) \\ &= P\left(\frac{-.02}{\sqrt{\frac{p(1-p)}{n}}} \leq Z \leq \frac{.02}{\sqrt{\frac{p(1-p)}{n}}}\right) \end{aligned}$$

As is part (b),

$$F\left(\frac{.02}{\sqrt{\frac{p(1-p)}{n}}}\right) = .975,$$
$$\frac{.02\sqrt{n}}{\sqrt{p(1-p)}} = 1.96$$

Solving for n,

$$n = \left(\frac{1.96}{.02}\right)^2 p(1-p)$$

Unfortunately this does not give us an explicit expression for n because we don't know p. The way out of this dilemma is to find the maximum value $\left(\frac{1.96}{.02}\right)^2 p(1-p)$ could take. If we choose n this large, then we can be sure of having the required precision in our estimate, $\frac{X}{n}$, for any p. It's easy to see that $p(1-p)$ is a maximum when $p = \frac{1}{2}$. Therefore we take

$$n = \left(\frac{1.96}{.02}\right)^2 \left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = 2401$$

i.e., if we survey 2401 people we can be 95% sure that $\frac{X}{n}$ lies within .02 of p, regardless of the value of p.

Remark: This method is used when poll results are reported in the media: you often see or hear that "this poll is accurate to with 3 percent 19 times out of 20". This is saying that n was big enough so that $P(p - .03 \leq X/n \leq p + .03)$ was 95%. (This requires n of about 1067.)

Problems:

9.6.1 Tomato seeds germinate (sprout to produce a plant) independently of each other, with probability 0.8 of each seed germinating. Give an expression for the probability that at least 75 seeds out of 100 which are planted in soil germinate. Evaluate this using a suitable approximation.

9.6.2 A metal parts manufacturer inspects each part produced. 60% are acceptable as produced, 30% have to be repaired, and 10% are beyond repair and must be scrapped. It costs the manufacturer $10 to repair a part, and $100 (in lost labour and materials) to scrap a part. Find the approximate probability that the total cost associated with inspecting 80 parts will exceed $1200.

9.7 Problems on Chapter 9

9.1 The diameters X of spherical particles produced by a machine are randomly distributed according to a uniform distribution on [.6,1.0] (cm). Find the distribution of Y, the volume of a particle.

9.2 A continuous random variable X has p.d.f.

$$f(x) = k(1 - x^2) \qquad -1 \leq x \leq 1.$$

(a) Find k and the c.d.f. of X. Graph $f(x)$ and the c.d.f.

(b) Find the value of c such that $P(-c \leq X \leq c) = .95$.

9.3 a) When people are asked to make up a random number between 0 and 1, it has been found that the distribution of the numbers, X, has p.d.f. close to

$$f(x) = \begin{cases} 4x; & 0 < x \leq 1/2 \\ 4\,(1-x)\,; & \frac{1}{2} < x < 1 \end{cases}$$

(rather than the $U[0,1]$ distribution which would be expected). Find the mean and variance of X.

b) For 100 "random" numbers from the above distribution find the probability their sum lies between 49.0 and 50.5.

c) What would the answer to (b) be if the 100 numbers were truly $U[0,1]$?

9.4 Let X have p.d.f. $f(x) = \frac{1}{20}; \quad -10 < x < 10$, and let $Y = \frac{X+10}{20}$. Find the p.d.f. of Y.

9.5 A continuous random variable X which takes values between 0 and 1 has probability density function

$$f(x) = (\alpha + 1)\, x^{\alpha}; \quad 0 < x < 1$$

a) For what values of α is this a p.d.f.? Explain.

b) Find $P\left(X \leq \frac{1}{2}\right)$ and $E(X)$

c) Find the probability density function of $T = 1/X$.

9.6 The magnitudes of earthquakes in a region of North America can be modelled by an exponential distribution with mean 2.5 (measured on the Richter scale).

(a) If 3 earthquakes occur in a given month, what is the probability that none exceed 5 on the Richter scale?

(b) If an earthquake exceeds 4, what is the probability it also exceeds 5?

9.7 A certain type of light bulb has lifetimes that follow an exponential distribution with mean 1000 hours. Find the median lifetime (that is, the lifetime x such that 50% of the light bulbs fail before x).

9.8 The examination scores obtained by a large group of students can be modelled by a normal distribution with a mean of 65% and a standard deviation of 10%.

(a) Find the percentage of students who obtain each of the following letter grades:

$$A(\geq 80\%),\ B(70-80\%),\ C(60-70\%),\ D(50-60\%),\ F(<50\%)$$

(b) Find the probability that the average score in a random group of 25 students exceeds 70%.

(c) Find the probability that the average scores of two distinct random groups of 25 students differ by more than 5%.

9.9 The number of liters X that a filling machine in a water bottling plant deposits in a nominal two liter bottle follows a normal distribution $N(\mu, \sigma^2)$, where $\sigma = .01$ (liters) and μ is the setting on the machine.

(a) If $\mu = 2.00$, what is the probability a bottle has less than 2 liters of water in it?

(b) What should μ be set at to make the probability a bottle has less than 2 liters be less than .01?

9.10 A turbine shaft is made up of 4 different sections. The lengths of those sections are independent and have normal distributions with μ and σ: (8.10, .22), (7.25, .20), (9.75, .24), and (3.10, .20). What is the probability an assembled shaft meets the specifications $28 \pm .26$?

9.11 Let $X \sim G(9.5, 2)$ and $Y \sim N(-2.1, 0.75)$ be independent.

Find:

(a) $P(9.0 < X < 11.1)$

(b) $P(X + 4Y > 0)$

(c) a number b such that $P(X > b) = .90$.

9.12 The amount, A, of wine in a bottle $\sim N\left(1.05l, .0004l^2\right)$ (Note: l means liters.)

a) The bottle is labelled as containing $1l$. What is the probability a bottle contains less than $1l$?

b) Casks are available which have a volume, V, which is $N(22l, .16l^2)$. What is the probability the contents of 20 randomly chosen bottles will fit inside a randomly chosen cask?

9.13 In problem 8.18, calculate the probability of passing the exam, both with and without guessing if (a) each p_i = .45; (b) each $p_i = .55$.
What is the best strategy for passing the course if (a) $p_i = .45$ (b) $p_i = .55$?

9.14 Suppose that the diameters in millimeters of the eggs laid by a large flock of hens can be modelled by a normal distribution with a mean of 40 mm. and a variance of 4 mm^2. The wholesale selling price is 5 cents for an egg less than 37 mm in diameter, 6 cents for eggs between 37 and 42 mm, and 7 cents for eggs over 42 mm. What is the average wholesale price per egg?

9.15 In a survey of n voters from a given riding in Canada, the proportion $\frac{x}{n}$ who say they would vote Conservative is used to estimate p, the probability a voter would vote P.C. (x is the number of Conservative supporters in the survey.) If Conservative support is actually 16%, how large should n be so that with probability .95, the estimate will be in error at most .03?

9.16 When blood samples are tested for the presence of a disease, samples from 20 people are pooled and analysed together. If the analysis is negative, none of the 20 people is infected. If the pooled sample is positive, at least one of the 20 people is infected so they must each be tested separately; i.e., a total of 21 tests is required. The probability a person has the disease is .02.

a) Find the mean and variance of the number of tests required for each group of 20.

b) For 2000 people, tested in groups of 20, find the mean and variance of the total number of tests. What assumption(s) has been made about the pooled samples?

c) Find the approximate probability that more than 800 tests are required for the 2000 people.

9.17 Suppose 80% of people who buy a new car say they are satisfied with the car when surveyed one year after purchase. Let X be the number of people in a group of 60 randomly chosen new car buyers who report satisfaction with their car. Let Y be the number of satisfied owners in a second (independent) survey of 62 randomly chosen new car buyers. Using a suitable approximation, find $P(|X - Y| \geq 3)$. A continuity correction is expected.

9.18 Suppose that the unemployment rate in Canada is 7%.

(a) Find the approximate probability that in a random sample of 10,000 persons in the labour force, the number of unemployed will be between 675 and 725 inclusive.

(b) How large a random sample would it be necessary to choose so that, with probability $.95$, the proportion of unemployed persons in the sample is between 6.9% and 7.1%?

9.19 **Gambling.** Your chances of winning or losing money can be calculated in many games of chance as described here.

Suppose each time you play a game (or place a bet) of $1 that the probability you win (thus ending up with a profit of $1) is .49 and the probability you lose (meaning your "profit" is -$1) is .51

(a) Let X represent your profit after n independent plays or bets. Give a normal approximation for the distribution of X.

(b) If $n = 20$, determine $P(X \geq 0)$. (This is the probability you are "ahead" after 20 plays.) Also find $P(X \geq 0)$ if $n = 50$ and $n = 100$. What do you conclude?

Note: For many casino games (roulette, blackjack) there are bets for which your probability of winning is only a little less than .5. However, as you play more and more times, the probability you lose (end up "behind") approaches 1.

(c) Suppose now you are the casino. If all players combined place $n = 100,000$ $1 bets in an evening, let X be your profit. Find the value c with the property that $P(X > c) = .99$. Explain in words what this means.

9.20 **Gambling: Crown and Anchor.** Crown and Anchor is a game that is sometimes played at charity casinos or just for fun. It can be played with a "wheel of fortune" or with 3 dice, in which each die has its 6 sides labelled with a crown, an anchor, and the four card suits club, diamond, heart and spade, respectively. You bet an amount (let's say $1) on one of the 6 symbols: let's suppose you bet on "heart". The 3 dice are then rolled simultaneously and you win $\$t$ if t hearts turn up ($t = 0, 1, 2, 3$).

(a) Let X represent your profits from playing the game n times. Give a normal approximation for the distribution of X.

(b) Find (approximately) the probability that $X > 0$ if (i) $n = 10$, (ii) $n = 50$.

9.21 **Binary classification**. Many situations require that we "classify" a unit of some type as being one of two types, which for convenience we will term Positive and Negative. For example, a diagnostic test for a disease might be positive or negative; an email message may be spam or not spam; a credit card transaction may be fraudulent or not. The problem is that in many cases we cannot tell for certain whether a unit is Positive or Negative, so when we have to decide which a unit is, we may make errors. The following framework helps us to deal with these problems.

For a randomly selected unit from the population being considered, define the indicator random variable

$$Y = I(\text{unit is Positive})$$

Suppose that we cannot know for certain whether $Y = 0$ or $Y = 1$ for a given unit, but that we can get a measurement X with the property that

$$\text{If } Y = 1, \qquad X \sim N(\mu_1, \sigma_1^2)$$

$$\text{If } Y = 0, \qquad X \sim N(\mu_0, \sigma_0^2)$$

where $\mu_1 > \mu_0$. We now decide to classify units as follows, based on their measurement X: select some value d between μ_0 and μ_1, and then

- if $X \geq d$, classify the unit as Positive
- if $X < d$, classify the unit as Negative

(a) Suppose $\mu_0 = 0,\ \mu_1 = 10,\ \sigma_0 = 4,\ \sigma_1 = 6$ and $d = 5$. Find the probability that

(i) If a unit is really Positive, they are wrongly classified as Negative. (This is called the "false negative" probability.)

(ii) If a unit is really Negative, they are wrongly classified as Positive. (This is called the "false positive" probability.)

(b) Repeat the calculations if $\mu_0 = 0,\ \mu_1 = 10$ as in (a), but $\sigma_1 = 3,\ \sigma_2 = 3$. Explain in plain English why the false negative and false positive misclassification probabilities are smaller than in (a).

9.22 **Binary classification and spam detection**. The approach in the preceding question can be used for problems such as spam detection, which was discussed earlier in Problems 4.17 and 4.18. Instead of using binary features as in those problems, suppose that for a given email message we compute a measure X, designed so that X tends to be high for spam messages and low for regular (non-spam) messages. (For example X can be a composite measure based on the presence or absence of certain words in a message, as well as other features.) We will treat X as a continuous random variable.

Suppose that for spam messages, the distribution of X is approximately $N(\mu_1, \sigma_1^2)$, and that for regular messages, it is approximately $N(\mu_0, \sigma_0^2)$, where $\mu_1 > \mu_0$. This is the same setup as for Problem 9.21. We will filter spam by picking a value d, and then filtering any message for which $X \geq d$. The trick here is to decide what value of d to use.

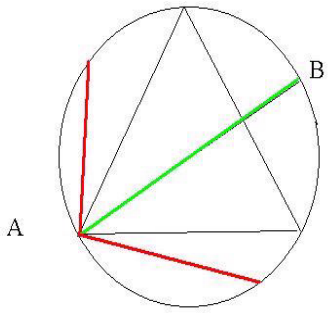


Figure 9.14: Bertrand's Paradox

(a) Suppose that $\mu_0 = 0,\ \mu_1 = 10,\ \sigma_1 = 3,\ \sigma_2 = 3$. Calculate the probability of a false positive (filtering a message that is regular) and a false negative (not filtering a message that is spam) under each of the three choices (i) $d = 5$ (ii) $d = 4$ (iii) $d = 6$.

(b) What factors would determine which of the three choices of d would be best to use?

9.23 **Random chords of a circle**. Given a circle, find the probability that a chord chosen at random be longer than the side of an inscribed equilateral triangle. For example in Figure 9.14, the line joining A and B satisfies the condition, the other lines do not. This is called Bertrand's paradox (see the Java applet at http://www.cut-the-knot.org/bertrand.shtml) and there various possible solutions, depending on exactly how you interpret the phrase "a chord chosen at random". For example, since the only important thing is the position of the second point relative to the first one, we can fix the point A and consider only the chords that emanate from this point. Then it becomes clear that 1/3 of the outcomes (those with angle with the tangent at that point between 60 and 120 degrees) will result in a chord longer than the side of an equilateral triangle. But a chord is fully determined by its midpoint. Chords whose length exceeds the side of an equilateral triangle have their midpoints inside a smaller circle with radius equal to 1/2 that of the given one. If we choose the midpoint of the chord at random and uniformly from the points within the circle, what is the probability that corresponding chord has length greater than the side of the triangle? Can you think of any other interpretations which lead to different answers?

9.24 **A model for stock returns**. A common model for stock returns is as follows: the number of trades N of stock XXX in a given day has a Poisson distribution with parameter λ. At each trade, say the i'th trade, the change in the price of the stock is X_i and has a normal distribution with mean 0 and variance σ^2, say and these changes are independent of one another and independent of N. Find the moment generating function of the total change in stock price over the day. Is this a distribution that you recognise? What is its mean and variance?

9.25 Let $X_1, X_2, ..., X_n$ be independent random variable with a Normal distribution having mean 1, and variance 2. Find the moment generating function for

(a) X_1

(b) $X_1 + X_2$

(c) $S_n = X_1 + X_2 + ... + X_n$

(d) $n^{-1/2}(S_n - n)$

9.26* **Challenge problem:** Suppose $U_1, U_2, ...$ is a sequence of independent $U[0, 1]$ random variables. For a given number k, define the random variable

$$N = \min\{n; \sum_{i=1}^{n} U_i \geq k\}$$

What is the expected value of N? How would you approximate this expected value if k were large?

10. Solutions to Section Problems

3.1.1 (a) Each student can choose in 4 ways and they each get to choose.

(i) Suppose we list the points in S in a specific order, for example (choice of student A, choice of student B, choice of student C) so that the point $(1,2,3)$indicates A chose section1, B chose section 2 and C chose section 3. Then S looks like

$$\{(1,1,1),(1,1,2),(1,1,3),...\}$$

Since each student can choose in 4 ways regardless of the choice of the other two students, by the multiplication rule S has $4 \times 4 \times 4 = 64$ points.

(ii) To satisfy the condition, the first student can choose in 4 ways and the others then only have 1 section they can go in. Therefore the probability they are all in the same section is $\frac{4\times1\times1}{64} = 1/16$.

(iii) To satisfy the condition, the first to pick has 4 ways to choose, the next has 3 sections left, and the last has 2 sections left. Therefore the probability they are all in different sections is $\frac{4\times3\times2}{64} = 3/8$.

(iiii) To satisfy the condition, each has 3 ways to choose a section. Therefore the probability there is no-one in section 1 is $\frac{3\times3\times3}{64} = 27/64$

(b) (i) Now S has n^s points, each a sequence like $(1,2,3,2,...)$ of length s.

(ii) P(all in same section) = $n \times 1 \times 1 \times ... \times 1/n^s = 1/n^{s-1}$.

(iii) P(different sections) = $n(n-1)(n-2)...(n-s+1)/n^s = \frac{n^{(s)}}{n^s}$.

(iiii) P(nobody in section 1)=$(n-1)(n-1)(n-1)...(n-1)/n^s = \frac{(n-1)^s}{n^s}$.

3.1.2 (a) There are 26 ways to choose each of the 3 letters, so in all the letters can be chosen in $26\times26\times26$ ways. If all letters are the same, there are 26 ways to choose the first letter, and only 1 way to choose the remaining 2 letters. So P(all letters the same) is $\frac{26\times1\times1}{26^3} = 1/26^2$.

(b) There are $10\times10\times10$ ways to choose the 3 digits. The number of ways to choose all even digits is $4\times4\times4$. The number of ways to choose all odd digits is $5\times5\times5$. Therefore P(all even or all odd) $= \frac{4^3+5^3}{10^3} = .189$.

3.1.3 (a) There are 35 symbols in all (26 letters + 9 numbers). The number of different 6-symbol passwords is $35^6 - 26^6$ (we need to subtract off the 26^6 arrangements in which only letters are used, since there must be at least one number). Similarly, we get the number of 7-symbol and 8-symbol passwords as $35^7 - 26^7$ and $35^8 - 26^8$. The total number of possible passwords is then

$$(35^6 - 26^6) + (35^7 - 26^7) + (35^8 - 26^8).$$

(b) Let N be the answer to part (a) (the total no. of possible passwords). Assuming you never try the same password twice, the probability you find the correct password within the first 1,000 tries is $1000/N$.

3.1.4 There are 7! different orders

(a) We can stick the even digits together in 3! orders. This block of even digits plus the 4 odd digits can be arranged in 5! orders. Therefore P(even together) = $\frac{3!5!}{7!} = 1/7$.

(b) For even at ends, there are 3 ways to fill the first place, and 2 ways to fill the last place and 5! ways to arrange the middle 5 digits. For odd at ends there are 4 ways to fill the first place and 3 ways to fill the last place and 5! ways to arrange the middle 5 digits. P(even or odd at ends) = $\frac{(3)(2)(5!)+(4)(3)(5!)}{7!} = \frac{3}{7}$.

3.1.5 The number of arrangements in S is $\frac{9!}{3!2!}$

(a) E at each end gives $\frac{7!}{2!}$ arrangements of the middle 7 letters. L at each end gives $\frac{7!}{3!}$ arrangements of the middle 7 letters. Therefore P(same letter at ends) = $\frac{\frac{7!}{2!}+\frac{7!}{3!}}{\frac{9!}{3!2!}} = \frac{1}{9}$.

(b) The $X, C$and N can be "stuck" together in 3! ways to form a single unit. We can then arrange the $3E$'s, $2L$'s, T, and (XCN) in $\frac{7!}{3!2!}$ways. Therefore $P(XCN$ together) = $\frac{\frac{7!}{3!2!}\times 3!}{\frac{9!}{3!2!}} = \frac{1}{12}$.

(c) There is only 1 way to arrange the letters in the order CEEELLNTX. Therefore P(alphabetical order) = $\frac{1}{\frac{9!}{3!2!}} = \frac{12}{9!}$

3.2.1 (a) The 8 cars can be chosen in $\binom{160}{8}$ ways. We can choose x with faulty emission controls and $(8-x)$ with good ones in $\binom{35}{x}\binom{125}{8-x}$ ways. Therefore P(at least 3 faulty) = $\frac{\sum_{x=3}^{8}\binom{35}{x}\binom{125}{8-x}}{\binom{160}{8}}$ since we need $x = 3$ or 4 or .... or 8.

(b) This assumes all $\binom{160}{8}$ combinations are equally likely. This assumption probably doesn't hold since the inspector would tend to select older cars or those in bad shape.

3.2.2 (a) The first 6 finishes can be chosen in $\binom{15}{6}$ ways. Choose 4 from numbers 1,2, ..., 9 in $\binom{9}{4}$ ways and 2 from numbers 10, ..., 15 in $\binom{6}{2}$ ways. Therefore P(4 single digits in top 6) = $\frac{\binom{9}{4}\binom{6}{2}}{\binom{15}{6}} = \frac{54}{143}$.

(b) Need 2 single digits and 2 double digit numbers in 1st4 and then a single digit. This occurs in $\binom{9}{2}\binom{6}{2} \times 7$ ways. Therefore

$$P(5^{\text{th}} \text{ is } 3^{\text{rd}} \text{ single digit}) = \frac{\binom{9}{2}\binom{6}{2} \times 7}{\binom{15}{4} \times 11} = \frac{36}{143}$$

(since we can choose 1st 4 in $\binom{15}{4}$ ways and then have 11 choices for the 5^{th})

Alternate Solution: There are $15^{(5)}$ ways to choose the first 5 in order. We can choose in order, 2 double digit and 3 single digit finishers in $6^{(2)}9^{(3)}$ ways, and then choose which 2 of the first 4 places have double digit numbers in $\binom{4}{2}$ ways. Therefore $P(5^{\text{th}}$is 3^{rd}single digit) =$\frac{6^{(2)}9^{(3)}\binom{4}{2}}{15^{(5)}} = \frac{36}{143}$.

(c) Choose 13 in 1 way and the other 6 numbers in $\binom{12}{6}$ ways. (from 1,2, ..., 12). Therefore P(13 is highest) = $\frac{\binom{12}{6}}{\binom{15}{7}} = \frac{28}{195}$.

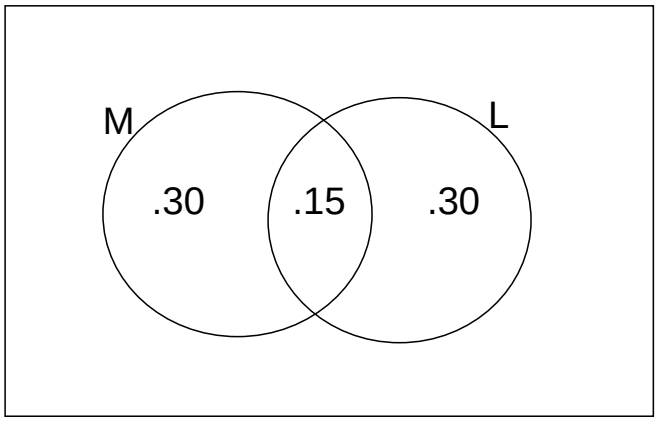
Alternate Solution: From the $\binom{13}{7}$ ways to choose 7 numbers from 1,2, ..., 13 subtract the $\binom{12}{7}$ which don't include 13 (i.e. all 7 chosen from 1,2, ..., 12). Therefore P(13 is highest) = $\frac{\binom{13}{7}-\binom{12}{7}}{\binom{15}{7}} = \frac{28}{195}$.

4.1.1 Let $R = \{\text{rain}\}$and $T = \{\text{temp.} > 22°C\}$, and draw a Venn diagram. Then

$$P(T\overline{R}) = .4 \times .2 = .08$$
$$P(\overline{T}R) = .7 \times .8 = .56$$

(Note that the information that 40% of days with temp $> 22°$ have no rain is not needed to solve the question). Therefore $P(R\overline{T}) = 24\%$. This result is to be expected since 80% of days have a high temperature $\leq$ $22°C$ and 30% of these days have rain.

4.1.2 $P(ML) = .15$, $P(M) = .45$, $P(L) = .45$ (see the Figure below)



The region outside the circles represents females to the right. To make $P(S) = 1$. we need $P(FR) = .25$.

4.2.1 (a)

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \\ &= 1 - .1 - [P(AC) + P(BC) - P(ABC)] \\ &= 0.9 - P(AC \cup BC) \end{aligned}$$

Therefore $P(A \cup B \cup C) = .9$ is the largest value, and this occurs when$P(AC \cup BC) = 0$.

(b) If each point in the sample space has strictly positive probability then if $P(AC \cup BC) = 0$, then $AC = \varphi$ and $BC = \varphi$ so that A and C are mutually exclusive and B and C are mutually exclusive. Otherwise we cannot make this determination. While A and C could be mutually exclusive, it can't be determined for sure.

4.2.2

$$\begin{aligned} P(A \cup B) = P(A \text{ or } B \text{ occur}) &= 1 - P(A \text{ doesn't occur AND } B \text{ doesn't occur}) \\ &= 1 - P(\bar{A}\bar{B}). \end{aligned}$$

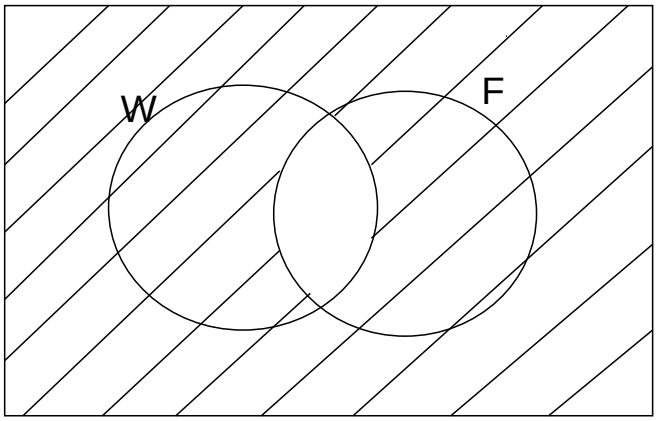
Alternatively, (look at a Venn diagram), $S = (A \cup B) \cup (\bar{A}\bar{B})$is a partition, so $P(S) = 1 \Rightarrow P(A \cup B) + P(\bar{A}\bar{B}) = 1$.

4.3.1 (a) Points giving a total of 9 are: $(3, 6), (4, 5), (5, 4)$ and $(6, 3)$. The probabilities are $(.1)(.3) = .03$ for $(3, 6)$ and for $(6, 3)$, and $(.2)(.2) = .04$ for $(4, 5)$ and for $(5, 4)$. Therefore $P\{$(3, 6) or (4, 5) or (5, 4) or (6, 3)$\} = .03 + .04 + .04 + .03 = .14$.

(b) There are $\binom{4}{1}$ arrangements with 1 nine and 3 non-nines. Each arrangement has probability $(.14)(.86)^3$.
Therefore P(nine on 1 of 4 repetitions) = $\binom{4}{1}(.14)(.86)^3 = .3562$.

4.3.2 Let W= {at least 1 woman} and F= {at least 1 French speaking student}.

$P(WF) = 1-P(\overline{WF}) = 1-P(\overline{W}\cup\overline{F}) = 1-\left[P(\overline{W}) + P(\overline{F}) - P(\bar{W}\bar{F})\right]$ (see figure below)



Venn diagram, Problem 4.3.2

But $P(\overline{WF}) = P$(no woman and no French speaking student)$= P$(all men who don't speak French)
P(woman who speaks French) =P(woman)P(French|woman)= $.45 \times .20 = .09$.
From Venn diagram,P(man without French$) = .49$.

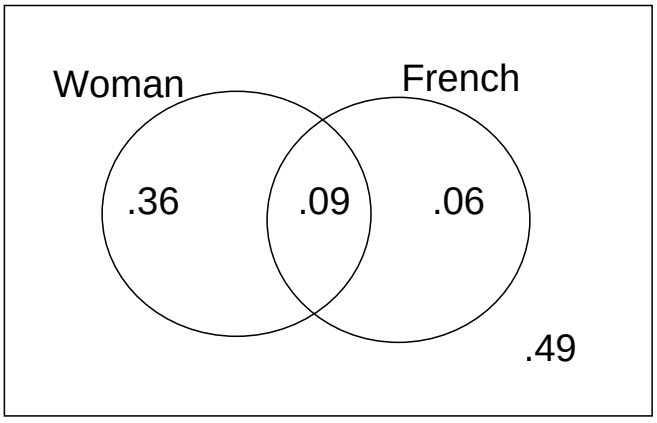


Figure 10.1: P(woman who speaks french)

$$P(\overline{W}\,\overline{F}) = (.49)^{10} \text{ and } P(\overline{W}) = (.55)^{10}; P(\overline{F}) = (.85)^{10}$$

Therefore $P(WF) = 1 - \left[(.55)^{10} + (.85)^{10} - (.49)^{10}\right] = 0.8014.$

4.3.3 From a Venn diagram: (1) $P(\overline{A}B) = P(B) - P(AB)$ (2) $P(\overline{A}\,\overline{B}) = P(\overline{A \cup B})$

$P(\overline{A}\,\overline{B}) = P(\overline{A})P(\overline{B})$
$\Leftrightarrow P(\overline{A \cup B}) = P(\overline{A})P(\overline{B}) \Leftrightarrow P(\overline{A \cup B}) = P(\overline{A})P(\overline{B})$
$\Leftrightarrow 1 - P(A \cup B) = P(\overline{A})P(\overline{B})$
$\Leftrightarrow 1 - [P(A) + P(B) - P(AB)] = P(\overline{A})\,[1 - P(B)]$
$\Leftrightarrow [1 - P(A)] - [P(B) - P(AB)] = P(\overline{A}) - P(\overline{A})P(B)$
$\Leftrightarrow P(\overline{A}) - P(\overline{A}B) = P(\overline{A}) - P(\overline{A})P(B)$
$\Leftrightarrow P(\overline{A})P(B) = P(\overline{A}B)$

Therefore $\overline{A}$ and $\overline{B}$ are independent iff $\overline{A}$ and B are independent.

4.4.1 Let$B$= $\{\text{bus}\}$and L= $\{\text{late}\}$.

$P(B|L) = \frac{P(BL)}{P(L)} = \frac{P(L|B)P(B)}{P(L|B)P(B)+P(L|\bar{B})P(\bar{B})} = \frac{(.3)(.2)}{(.3)(.2)+(.7)(.1)} = 6/13.$

4.4.2 Let F= $\{\text{fair}\}$and H= $\{5 \text{ heads}\}$

$$P(F|H) = \frac{P(FH)}{P(H)} = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|\bar{F})P(\bar{F})}$$
$$= \frac{(\frac{3}{4})\binom{6}{5}(\frac{1}{2})^6}{(\frac{3}{4})\binom{6}{5}(\frac{1}{2})^6 + (\frac{1}{4})\binom{6}{5}(.8)^5(.2)^1} = 0.4170$$

4.4.3 Let H= { defective headlights}, M= {defective muffler}

$$P(M/H) = \frac{P(MH)}{P(H)} = \frac{P(MH)}{P(MH \cup \overline{M}H)} = \frac{.1}{.1 + .15} = .4$$

5.1.1 We need $f(x) \geq 0$ and $\sum_{x=0}^{2} f(x) = 1$

$$9c^2 + 9c + c^2 = 10c^2 + 9c = 1$$

$$\text{Therefore} \quad 10c^2 + 9c - 1 = 0$$

$$(10c - 1)(c + 1) = 0$$

$$c = 1/10 \text{ or } -1$$

But if $c = -1$ we have $f(1) < 0$..which is impossible. Therefore $c = .1$

5.1.2 We are arranging $Y\ F\ O\ O\ O$ where Y= {you}, F= {friend}, O = {other}. There are $\frac{5!}{3!} = 20$ distinct arrangements.

$X = 0$: $Y\ F\ O\ O\ O, \cdots, O\ O\ O\ Y F$ has 4 arrangements with Y first and 4 with F first.
$X = 1$: $YOFOO, \cdots, OOYOF$ has 3 arrangements with Y first and 3 with F first.
$X = 2$: $YOOFO, OYOOF$ has 2 with Y first and 2 with F.
$X = 3$: $YOOOF$ has 1 with Y first and 1 with F.

x	0	1	2	3
$f(x)$	.4	.3	.2	.1
$F(x)$	.4	.7	.9	1

5.3.1 (a) Using the hypergeometric distribution,

$$f(0) = \frac{\binom{d}{0}\binom{12-d}{7}}{\binom{12}{7}}$$

d	0	1	2	3
$f(0)$	1	$5/12$	$5/33$	$5/110$

(b) While we could find none tainted if d is as big as 3, it is not likely to happen. This implies the box is not likely to have as many as 3 tainted tins.

5.3.2 Considering order, there are $N^{(n)}$ points in S. We can choose which x of the n selections will have "success" in $\binom{n}{x}$ ways. We can arrange the x "successes" in their selected positions in $r^{(x)}$ ways and the $(n-x)$ "failures" in the remaining positions in $(N-r)^{(n-x)}$ ways.

Therefore $f(x) = \frac{\binom{n}{x} r^{(x)} (N-r)^{(n-x)}}{N^{(n)}}$ with x ranging from max $(0, n-(N-r))$ to min (n, r)

5.4.1 (a) Using hypergeometric, with $N = 130, r = 26, n = 6$,

$$f(2) = \frac{\binom{26}{2}\binom{104}{4}}{\binom{130}{6}} \qquad (= .2506)$$

(b) Using binomial as an approximation,

$$f(2) = \binom{6}{2}\left(\frac{26}{130}\right)^2\left(\frac{104}{130}\right)^4 = 0.2458$$

5.4.2 (a) P(fail twice)
$= P(A)P$(fail twice $|A) + P(B)P$(fail twice $|B) = \left(\frac{1}{2}\right)\binom{10}{2}(.1)^2(.9)^8 + \left(\frac{1}{2}\right)\binom{10}{2}(.05)^2(.95)^8 = .1342$.
Where A= { camera A is picked } and B= { camera B is picked }. This assumes shots are independent with a constant failure probability.

(b) $P(A|$failed twice) = $\frac{P(A \text{ and } fail \text{ } twice)}{P(fail \text{ } twice)} = \frac{\left(\frac{1}{2}\right)\binom{10}{2}(.1)^2(.9)^8}{.1342} = .7219$

5.5.1 We need $(x - 25)$ "failures" before our 25th "success".

$$f(x) = \binom{x-1}{x-25}(.2)^{25}(.8)^{x-25} \text{ or } \binom{x-1}{24}(.2)^{25}(.8)^{x-25};\ x = 25, 26, 27, ...$$

5.5.2 (a) In the first $(x + 17)$ selections we need to get $x$defective (use hypergeometric distribution) and then we need a good one on the $(x + 18)^{\text{th}}$ draw.

$$\text{Therefore } f(x) = \frac{\binom{200}{x}\binom{2300}{17}}{\binom{2500}{x+17}} \times \frac{2283}{2500 - (x + 17)};\ \ x = 0, 1, \cdots, 200.$$

(b) Since 2500 is large and we're only choosing a few of them, we can approximate the hypergeometric portion of $f(x)$ using binomial

$$f(2) \doteq \binom{19}{2}\left(\frac{200}{2500}\right)^2\left(1 - \frac{200}{2500}\right)^{17} \times \frac{2283}{2481} = .2440.$$

5.6.1 Using geometric,

$$P(x \text{ not leaky found before first leaky}) = (0.7)^x(0.3) = f(x)$$

$$\begin{aligned} P(X \geq n - 1) = &\ f(n - 1) + f(n) + f(n + 1) + ... \\ &= (0.7)^{n-1}(0.3) + (0.7)^n(0.3) + (0.7)^{n+1}(0.3) + ... \\ &= \frac{(.7)^{n-1}(.3)}{1 - .7} = (.7)^{n-1} = 0.05 \end{aligned}$$

$$(n - 1)\log .7 = \log .05;\ \text{ so } n = 9.4$$

At least 9.4 cars means 10 or more cars must be checked. Therefore $n = 10$.

5.7.1 (a) Let X be the number who don't show. Then

$$X \sim Bi(122, .03)$$

$$\begin{aligned} P(\text{not enough seats}) &= P(X = 0 \text{ or } 1) \\ &= \binom{122}{0}(.03)^0(.97)^{122} + \binom{122}{1}(.03)^1(.97)^{121} \\ &= 0.1161 \end{aligned}$$

(To use a Poisson approximation we need p near 0. That is why we defined "success" as not showing up).

For Poisson, $\mu = np = (122)(.03) = 3.66$

$$f(0) + f(1) = e^{-3.66} + 3.66e^{-3.66} = 0.1199$$

(b) Binomial requires all passengers to be independent as to showing up for the flight, and that each passenger has the same probability of showing up. Passengers are not likely independent since people from the same family or company are likely to all show up or all not show. Even strangers arriving on an earlier incoming flight would not miss their flight independently if the flight was delayed. Passengers may all have roughly the same probability of showing up, but even this is suspect. People travelling in different fare categories or in different classes (e.g. charter fares versus first class) may have different probabilities of showing up.

5.8.1 (a)

$$\lambda = 3, \;\; t = 2\frac{1}{2}, \;\; \mu = \lambda t = 7.5$$

$$f(6) = \frac{7.5^6 e^{-7.5}}{6!} = 0.1367$$

(b)

$$\begin{aligned} P(2 \text{ in 1st minute} | 6 \text{ in } 2\frac{1}{2} \text{ minutes}) &= \frac{P(2 \text{ in 1st min. and 6 in } 2\frac{1}{2} \text{ min.})}{P(6 \text{ in } 2\frac{1}{2} \text{ min})} \\ &= \frac{P(2 \text{ in 1st min. and 4 in last } 1\frac{1}{2} \text{ min})}{P(6 \text{ in } 2\frac{1}{2} \text{ min.})} \\ &= \frac{\left(\frac{3^2 e^{-3}}{2!}\right)\left(\frac{4.5^4 e^{-4.5}}{4!}\right)}{\left(\frac{7.5^6 e^{-7.5}}{6!}\right)} \\ &= \binom{6}{2}\left(\frac{3}{7.5}\right)^2\left(\frac{4.5}{7.5}\right)^4 = .3110 \end{aligned}$$

Note this is a binomial probability function.

5.8.2 Assuming the conditions for a Poisson process are met, with lines as units of "time":

(a) $\lambda = .02$ per line; $t = 1$line; $\mu = \lambda t = .02$

$$f(0) = \frac{\mu^0 e^{-\mu}}{0!} = e^{-.02} = .9802$$

(b) $\mu_1 = 80 \times .02 = 1.6$; $\mu_2 = 90 \times .02 = 1.8$

$$\left[\frac{\mu_1^2 e^{-\mu_1}}{2!}\right] \left[\frac{\mu_2^2 e^{-\mu_2}}{2!}\right] = .0692$$

5.9.1 Consider a 1 minute period with no occurrences as a "success". Then Xhas a geometric distribution. The probability of "success" is

$$f(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda}.$$

$$\text{Therefore } f(x) = (e^{-\lambda})(1 - e^{-\lambda})^{x-1};\ x = 1, 2, 3, ...$$

(There must be $(x - 1)$ failures before the first success.)

5.9.2 (a) $\mu = 3 \times 1.25 = 3.75$

$$f(0) = \frac{3.75^0 e^{-3.75}}{0!} = .0235$$

(b) $\left(1 - e^{-3.75}\right)^{14} e^{-3.75}$, using a geometric distribution

(c) Using a binomial distribution

$$f(x) = \binom{100}{x} \left(e^{-3.75}\right)^x \left(1 - e^{-3.75}\right)^{100-x}$$

Approximate by Poisson with $\mu = np = 100e^{-3.75} = 2.35$. $f(x) \doteq e^{-2.35} \frac{2.35^x}{x!}$($n$ large, p small). Thus, $P(X \geq 4) = 1 - P(X \leq 3) = 1 - .789 = .211$.

7.3.1 There are 10 tickets with all digits identical. For these there is only 1 prize. There are 10×9 ways to pick a digit to occur twice and a different digit to occur once. These can be arranged in $\frac{3!}{2!\,1!} = 3$ different orders; i.e. there are 270 tickets for which 3 prizes are paid. There are $10 \times 9 \times 8$ways to pick 3 different digits in order. For each of these 720 tickets there will be 3! prizes paid.

The organization takes in \$1,000.

$$\text{Therefore } E(\text{Profit}) = \left[(1000 - 200) \times \frac{10}{1000}\right] + \left[(1000 - 600) \times \frac{270}{1000}\right]$$

$$+ \left[(1000 - 1200) \times \frac{720}{1000}\right] = -\$28$$

i.e., on average they lose \$28.

7.4.1 Let them sell n tickets. Suppose X show up. Then $X \sim Bi(n, .97)$. For the binomial distribution, $\mu = E(X) = np = .97n$
If $n \leq 120$, the revenues will be $100X$, and the expected revenue is $E(100X) = 100E(X) = 97n$. This is maximized for $n = 120$. Therefore the maximum expected revenue is \$11,640. For $n = 121$, revenues are $100X$, less \$500 if all 121 show up. i.e. the expected revenue is,

$$100 \times 121 \times .97 - 500\ f(121) = 11,737 - 500(.97)^{121} = \$11,724.46.$$

For $n = 122$, revenues are $100X$, less \$500 if 121 show up, less \$1000 if all 122 show. i.e. the expected revenue is

$$100 \times 122 \times .97 - 500\binom{122}{121}(.97)^{121}(.03) - 1000(.97)^{122} = \$11,763.77$$

For $n = 123$, revenues are $100X$, less \$500 if 121 show, less \$1,000 if 122 show, less \$1500 if all 123 show. i.e. the expected revenue is

$$100 \times 123 \times .97 - 500\binom{123}{121}(.97)^{121}(.03)^2 - 1000\binom{123}{122}(.97)^{122}(.03) - 1500(.97)^{123}$$
$$= \$11,721.13$$

Therefore they should sell 122 tickets.

7.4.2 (a) Let X be the number of words needing correction and let T be the time to type the passage. Then $X \sim Bi(450, .04)$ and $T = 450 + 15X$. X has mean $np = 18$ and variance $np(1-p) = 17.28$.
$E(T) = E(450 + 15X) = 450 + 15E(X) = 450 + (15)(18) = 720$
Var(T)= Var$(450 + 15X) = 15^2$Var$(X) = 3888$.

(b) At 45 words per minute, each word takes $1\frac{1}{3}$seconds. $X \sim Bi(450, .02)$and $T = \left(450 \times 1\frac{1}{3}\right) + 15X = 600 + 15X$
$E(X) = 450 \times .02 = 9$; $E(T) = 600 + (15)(9) = 735$, so it takes longer on average.

8.1.1 (a) The marginal probability functions are:

x	0	1	2
$f_1(x)$	.3	.2	.5

and

y	0	1	2
$f_2(y)$	.3	.4	.3

Since $f_1(x)\ f_2(y) \neq f(x,y)\ for\ all\ (x,y)$
Therefore X and Y are not independent. e.g. $f_1(1)\ f_2(1) = 0.08 \neq 0.05$

(b)

$$f(y|X=0) = \frac{f(0,y)}{f_1(0)} = \frac{f(0,y)}{.3}$$

y	0	1	2
$f(y\|X=0)$	.3	.5	.2

(c)

d	-2	-1	0	1	2
$f(d)$	.06	.24	.29	.26	.15

(e.g.$P(D=0) = f(0,0) + f(1,1) + f(2,2)$)

8.1.2

$$f(y|x) = \frac{f(x,y)}{f_1(x)}$$

$$\begin{aligned} f(x,y) &= P(y \text{ calls})P(x \text{ sales}|y \text{ calls}) \\ &= \left(\tfrac{20^y e^{-20}}{y!}\right)\left(\tbinom{y}{x}(.2)^x(.8)^{y-x}\right) \\ &= \tfrac{(20^{y-x})(.8)^{y-x}}{(y-x)!} \cdot \tfrac{(20)^x(.2)^x}{x!}e^{-20}; \text{ for } x = 0,1,2,... \text{ and } y = x, x+1, x+2, ... \end{aligned}$$

(y starts at x since no. of calls $\geq$ no. of sales).

$$\begin{aligned} f_1(x) = \sum_{y=x}^{\infty} f(x,y) &= \frac{[(20)(.2)]^x}{x!}e^{-20}\sum_{y=x}^{\infty}\frac{[(20)(.8)]^{y-x}}{(y-x)!} \\ &= \frac{4^x e^{-20}}{x!}\left[\frac{16^0}{0!} + \frac{16^1}{1!} + \frac{16^2}{2!} + ...\right] = \frac{4^x e^{-20}}{x!} \cdot e^{16} \\ &= \frac{4^x e^{-20}}{x!} \cdot e^{16} = \frac{4^x e^{-4}}{x!} \end{aligned}$$

Therefore $f(y|x) = \dfrac{\frac{16^{y-x}}{(y-x)!}\ \frac{4^x}{x!}\ e^{-20}}{\frac{4^x e^{-4}}{x!}} = \dfrac{16^{y-x}e^{-16}}{(y-x)!}$; $y = x, x+1, x+2, ...$

8.1.3

$$f(x,y) = f(x)f(y) = \binom{x+k-1}{x}\binom{y+\ell-1}{y}p^{k+\ell}(1-p)^{x+y}$$

$$\begin{aligned} f(t) &= \sum_{x=0}^{t} f(x, y=t-x) \\ &= \sum_{x=0}^{t}\binom{x+k-1}{x}\binom{t-x+\ell-1}{t-x}p^{k+\ell}(1-p)^t \\ &= \sum_{x=0}^{t}(-1)^x\binom{-k}{x}(-1)^{t-x}\binom{-\ell}{t-x}p^{k+\ell}(1-p)^t \\ &= (-1)^t p^{k+\ell}(1-p)^t\sum_{x=0}^{t}\binom{-k}{x}\binom{-\ell}{t-x} \\ &= (-1)^t p^{k+\ell}(1-p)^t\binom{-k-\ell}{t} \text{ using the hypergeometric identity} \\ &= \binom{t+k+\ell-1}{t}p^{k+\ell}(1-p)^t;\ \ t=0,1,2,\cdots \end{aligned}$$

using the given identity on $(-1)^t\binom{-k-\ell}{t}$. (T has a negative binomial distribution)

8.2.1 (a) Use a multinomial distribution.

$$f(3,11,7,4) = \frac{25!}{3!\ 11!\ 7!\ 4!}\ (.1)^3(.4)^{11}(.3)^7(.2)^4$$

(b) Group C's and D's into a single category.

$$f(3,11,11) = \frac{25!}{3!\ 11!\ 11!}(.1)^3(.4)^{11}(.5)^{11}$$

(c) Of the 21 non D's we need $3A$'s, 11 B's and $7C$'s. The (conditional) probabilities for the non-D's are: 1/8 for A, 4/8 for B, and $3/8$ for C.
(e.g. $P(A|\overline{D}) = P(A)/P(\overline{D}) = .1/.8 = 1/8$)
Therefore $f(3\ A's,\ 11B's\ 7C's|4D's) = \frac{21!}{3!11!7!}(\frac{1}{8})^3(\frac{4}{8})^{11}(\frac{3}{8})^7$.

8.2.2 $\mu = .6 \times 12 = 7.2$

$p_1 = P$(fewer than 5 chips) = $\sum_{x=0}^{4}\frac{7.2^x e^{-7.2}}{x!}$

$p_2 = P$(more than 9 chips) = $1 - \sum_{x=0}^{9}\frac{7.2^x e^{-7.2}}{x!}$

(a) $\binom{12}{3}p_1^3(1-p_1)^9$

(b) $\frac{12!}{3!7!2} p_1^3\, p_2^7\, (1-p_1-p_2)^2$

(c) Given that 7 have >9 chips, the remaining 5 are of 2 types - under 5 chips, or 5 to 9 chips

$$P(<5|\leq 9 \text{ chips}) = \frac{P(<5 \text{ and } \leq 9)}{P(\leq 9)} = \frac{p_1}{1-p_2}.$$

Using a binomial distribution,

P(3 under 5|7 over 9) = $\binom{5}{3}\left(\frac{p_1}{1-p_2}\right)^3\left(1-\frac{p_1}{1-p_2}\right)^2$

8.4.1

x	0	1	2
$f_1(x)$	.2	.5	.3

y	0	1
$f_2(y)$	.3	.7

$$E(X) = (0\times .2) + (1\times .5) + (2\times .3) = 1.1$$
$$E(Y) = (0\times .3) + (1\times .7) = .7$$
$$E(X^2) = (0^2\times .2) + (1^2\times .5) + (2^2\times .3) = 1.7;$$
$$E(Y^2) = .7$$
$$Var(X) = 1.7 - 1.1^2 = .49$$
$$Var(Y) = .7 - .7^2 = .21$$
$$E(XY) = (1\times 1\times .35) + (2\times 1\times .21) = .77$$
$$Cov(X,Y) = .77 - (1.1)(.7) = 0$$
$$\text{Therefore } \rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = 0$$

While $\rho = 0$ indicates X and Y may be independent (and indeed are in this case), it does not prove that they are independent. It only indicates that there is no linear relationship between X and Y.

8.4.2

(a)

x	2	4	6
$f_1(x)$	3/8	3/8	1/4

y	-1	1
$f_2(y)$	$\frac{3}{8}+p$	$\frac{5}{8}-p$

$E(X) = \left(2\times\frac{3}{8}\right) + \left(4\times\frac{3}{8}\right) + \left(6\times\frac{1}{4}\right) = 15/4;\ E(Y) = -\frac{3}{8} - p + \frac{5}{8} - p = \frac{1}{4} - 2p;$

$E(XY) = \left(-2\times\frac{1}{8}\right) + \left(-4\times\frac{1}{4}\right) + \cdots + \left(6\times\left(\frac{1}{4}-p\right)\right) = \frac{5}{4} - 12p$

$\text{Cov}(X,Y) = 0 = E(XY) - E(X)E(Y) \Rightarrow \frac{5}{4} - 12p = \frac{15}{16} - \frac{15}{2}p$

Therefore $p = 5/72$

(b) If X and Y are independent then Cov$(X, Y) = 0$, and so p must be 5/72. But if $p = 5/72$ then

$$f_1(2)f_2(-1) = \frac{3}{8} \times \frac{4}{9} = \frac{1}{6} \neq f(2, -1)$$

Therefore X and Y cannot be independent for any value of p

8.5.1

$x =$	0	1	2
$f_1(x) =$	0.5	0.3	0.2

$$E(X) = (0 \times .5) + (1 \times .3) + 2 \times .2) = 0.7$$

$$E(X^2) = (0^2 \times .5) + (1^2 \times .3) + (2^2 \times .2) = 1.1$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 0.61$$

$$E(XY) = \sum_{\text{all } x,y} xyf(x, y) \text{and this has only two non-zero terms}$$
$$= (1 \times 1 \times 0.2) + (2 \times 1 \times .15) = 0.5$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.01$$
$$\text{Var}(3X - 2Y) = 9\text{Var}(X) + (-2)^2\text{Var}(Y) + 2(3)(-2)\text{Cov}(X, Y)$$
$$= 9(.61) + 4(.21) - 12(.01) = 6.21$$

8.5.2 Let $X_i = \begin{cases} 0, & \text{if the ith pair is alike} \\ 1, & \text{if the ith pair is unlike} \end{cases}, i = 1, 2, ..., 24.$

$$E(X_i) = \sum_{x_i=0}^{1} x_i f(x_i) = 1f(1) = P(\text{ON OFF } \cup \text{ OFF ON})$$

$$= (0.6)(0.4) + (0.4)(0.6) = 0.48$$

$$E(X_i^2) = E(X_i) = .48 \text{ (for } X_i = 0 \text{ or } 1)$$

$$\text{Var } (X_i) = .48 - (.48)^2 = .2496$$

Consider a pair which has no common switch such as X_1, X_3. Since X_1 depends on switch 1&2 and X_3 on switch 3&4 and since the switches are set independently, X_1 and X_3 are independent and so $cov(X_1, X_3) = 0$. In fact all pairs are independent if they have no common switch, but

may not be independent if the pairs are adjacent. In this case, for example, since X_iX_{i+1} is also an indicator random variable,

$$\begin{aligned} E(X_iX_{i+1}) &= P(X_iX_{i+1}=1) \\ &= P(\text{ON OFF ON } \cup \text{ OFF ON OFF}) \\ &= (.6)(.4)(.6)+(.4)(.6)(.4) = .24 \end{aligned}$$

Therefore

$$\text{Cov}\ (X_i,\ X_{i+1}) = E(X_iX_{i+1}) - E(X_i)E(X_{i+1})$$

$$= 0.24 - (.48)^2 = .0096$$

$$E(\sum X_i) = \sum_{i=1}^{24} E(X_i) = 24 \times .48 = 11.52$$

$$\text{Var}(\sum X_i) = \sum_{i=1}^{24} \text{Var}(X_i) + 2\sum_{i=1}^{23} \text{Cov}\ (X_i,\ X_{i+1}) = (24 \times .2496) + (2 \times 23 \times .0096)$$

$$=\ 6.432$$

8.5.3

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_x\sigma_y} = 0.5$$

$$Cov(X,Y) = 0.5\sqrt{1.69 \times 4} = 1.3$$

$$Var(U) = Var(2X-Y) = 4\sigma_X^2 + \sigma_Y^2 - 4Cov(X,Y) = 5.56$$

Therefore $s.d.(U) = 2.36$

8.5.4

$$\begin{aligned} Cov\,(X_{i-1}, X_i) &= Cov\,(Y_{i-2}+Y_{i-1}, Y_{i-1}+Y_i) \\ &= Cov\,(Y_{i-2}, Y_{i-1}) + Cov\,(Y_{i-2}, Y_i) + Cov\,(Y_{i-1}, Y_{i-1}) + Cov\,(Y_{i-1}, Y_i) \\ &= 0 + 0 + Var\,(Y_{i-1}) + 0 = \sigma^2 \end{aligned}$$

Also, $Cov\,(X_i, X_j) = 0$ for $j \neq i \pm 1$ and $Var(X_i) = Var(Y_{i-1}) + Var(Y_i) = 2\sigma^2$

$Var\,(\sum X_i) = \sum Var(X_i) + 2\sum_{i=2}^{n} Cov\,(X_{i-1}, X_i) = n(2\sigma^2) + 2(n-1)\sigma^2 = (4n-2)\sigma^2$

8.5.5 Using X_i as defined, $E(X_i) = \sum_{x_i=0}^{1} x_i f(x_i) = f(1) = E\left(X_i^2\right)$ since $X_i = X_i^2$

$E(X_1) = E(X_{24}) = .9$ since only 1 cut is needed

$E(X_2) = E(X_3) = \cdots = E(X_{23}) = .9^2 = .81$ since 2 cuts are needed.

Var(X_1)= Var$(X_{24}) = .9 - .9^2 = .09$

Var(X_2)= Var$(X_3) = \cdots =$ Var$(X_{23}) = .81 - .81^2 = .1539$

Cov $(X_i, X_j) = 0$if $j \neq i \pm 1$ since there are no common pieces and cuts are independent.

$E(X_i X_{i+1}) = \sum x_i x_{i+1} f(x_i, x_{i+1}) = f(1,1)$

(product is 0 if either x_i or x_{i+1} is a 0)

$$= \begin{cases} .9^2 & \text{for } i = 1 \text{ or } 23 \text{.....2 cuts needed} \\ .9^3 & \text{for } i = 2, .., 22 \text{.....3 cuts needed} \end{cases}$$

$$\text{Cov}(X_i, X_{i+1}) = E(X_i X_{i+1}) - E(X_i)E(X_{i+1})$$

$$= \begin{cases} .9^2 - (.9)(.9^2) = .081 & \text{for } i = 1 \text{ or } 23 \\ .9^3 - (.9^2)(.9^2) = .0729 & \text{for } i = 2, \cdots, 22 \end{cases}$$

$$E\left(\sum_{i=1}^{24} X_i\right) = \sum_{i=1}^{24} E(X_i) = (2 \times .9) + (22 \times .81) = 19.62$$

$$Var\left(\sum_{i=1}^{24} X_i\right) = \sum_{i=1}^{24} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

$$= (2 \times .09) + (22 \times .1539) + 2[(2 \times .081) + (21 \times .0729)] = 6.9516$$

Therefore s.d.$(\sum X_i) = \sqrt{6.9516} = 2.64$

9.1.1 (a) $\int_{-1}^{1} kx^2 dx = k\frac{x^3}{3}|_{-1}^{1} = \frac{2k}{3} = 1$

Therefore $k = 3/2$

(b) $F(x) = \begin{cases} 0; \text{ for } x \leq -1 \\ \int_{-1}^{x} \frac{3}{2}x^2 dx = \frac{x^3}{2}|_{-1}^{x} = \frac{x^3}{2} + \frac{1}{2}; \text{ for} -1 < x < 1 \\ 1; \text{ for } x \geq 1 \end{cases}$

(c) $P(-.1 < X < .2) = F(.2) - F(-.1) = .504 - .4995 = .0045$

(d)

$$E(X) = \int_{-1}^{1} x \times \frac{3}{2}x^2 dx = \frac{3}{2}\int_{-1}^{1} x^3 dx = \frac{3}{8}x^4|_{-1}^{1} = 0$$

$$E(X^2) = \int_{-1}^{1} x^2 \frac{3}{2}x^2 dx = \frac{3}{10}x^5|_{-1}^{1} = 3/5$$

Therefore $Var(X) = E(X^2) - \mu^2 = 3/5$

(e)

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \left[\frac{(\sqrt{y})^3}{2} + \frac{1}{2}\right] - \left[\frac{(-\sqrt{y})^3}{2} + \frac{1}{2}\right] = y^{3/2}$$

Therefore $f(y) = \frac{d}{dy}\, F_Y(y) = \frac{3}{2}\,\sqrt{y}$ for $0 \le y < 1$ and is 0 otherwise.

9.1.2 (a) $F(\infty) = 1 = \lim\limits_{x\to\infty} \frac{kx^n}{1+x^n} = \lim\limits_{x\to\infty} \frac{k}{\frac{1}{x^n}+1} = k.$ Therefore $k = 1$

(b) $f(x) = \frac{d}{dx}F(x) = \frac{nx^{n-1}}{(1+x^n)^2}$; for $x > 0$.

(c) Let m be the median. Then $F(m) = .5 = \frac{m^n}{1+m^n}$. Therefore $m^n = 1$ and so the median is 1

9.2.1 $F(x) = \int_{-1}^{x} \frac{3}{2}x^2 dx = \frac{x^3+1}{2}$. If $y = F(x) = \frac{x^3+1}{2}$ is a random number between $0$and 1, then $x = (2y-1)^{1/3}$. For $y = .27125$we get $x = (-.4574)^{1/3} = -.77054$.

9.3.1 Let the time to disruption be X.

$$\text{Then } P(X \le 8) = F(8) = 1 - e^{-8/\theta} = .25$$

Therefore $e^{-8/\theta} = .75$. Take natural logs giving $\theta = -\frac{8}{\ln\ .75} = 27.81$hours.

9.3.2 (a) $F(x) = P$(distance $\le x$) = $1 - P$(distance $> x$) $= 1 - P$(0 flaws or 1 flaw within radius x)

The number of flaws has a Poisson distribution with mean $\mu = \lambda\pi x^2$.

$$F(x) = 1 - \frac{\mu^0 e^{-\mu}}{0!} - \frac{\mu^1 e^{-\mu}}{1!} = 1 - e^{-\lambda\pi x^2}\left(1 + \lambda\pi x^2\right)$$
$$f(x) = \frac{d}{dx}F(x) = 2\lambda^2\pi^2 x^3 e^{-\lambda\pi x^2} \text{ for } x > 0$$

(b) $\mu = E(X) = \int_0^\infty x 2\lambda^2\pi^2 x^3 e^{-\lambda\pi x^2} = \int_0^\infty 2\lambda^2\pi^2 x^4 e^{-\lambda\pi x^2} dx$. Let $y = \lambda\pi x^2$. Then $dy = 2\lambda\pi x dx$, so $dx = \frac{dy}{2\sqrt{\lambda\pi y}}$

$$\mu = \int_0^\infty 2y^2 e^{-y} \frac{dy}{2\sqrt{\lambda\pi y}} = \frac{1}{\sqrt{\lambda\pi}} \int_0^\infty y^{3/2} e^{-y} dy$$
$$= \frac{1}{\sqrt{\lambda\pi}}\Gamma\left(\frac{5}{2}\right) = \frac{1}{\sqrt{\lambda\pi}}\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$
$$= \frac{1}{\sqrt{\lambda\pi}}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}{\sqrt{\lambda\pi}} = \frac{3}{4\sqrt{\lambda}}$$

9.5.1 (a) $P\left(8.4 < X < 12.2\right) = P\left(\frac{8.4-10}{2} < Z < \frac{12.2-10}{2}\right)$.

= $P(-.8 < Z < 1.1)$
= $F(1.1) - F(-.8)$
= $F(1.1) - [1 - F(.8)]$
= $.8643 - (1 - .7881) = .6524$

(see Figure 10.2)

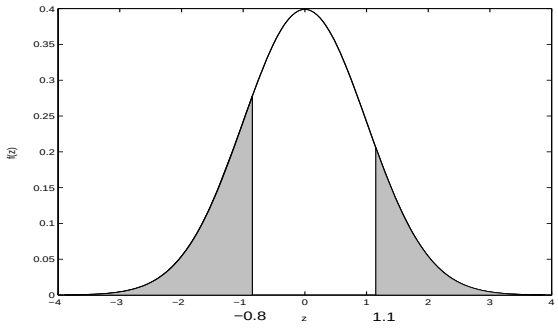


Figure 10.2:

(b) $2Y - X$ is normally distributed with mean $2(3) - 10 = -4$, and variance $2^2(100) + (-1)^2(4) = 404$.

$$\begin{aligned} P(2Y > X) &= P(2Y - X > 0) \\ &= P(Z > \frac{0-(-4)}{\sqrt{404}} = .20) \\ &= P(Z > .20) \\ &= 1 - F(.20) = 1 - .5793 = .4207 \end{aligned}$$

(c) $\overline{Y}$ is normally distributed with mean 3, and variance $\frac{100}{25} = 4$. Therefore $P(\overline{Y} < 0) = P(Z < \frac{0-3}{2} = -1.5) = P(Z > 1.5) = 1 - F(1.5) = 1 - .9332 = .0668$

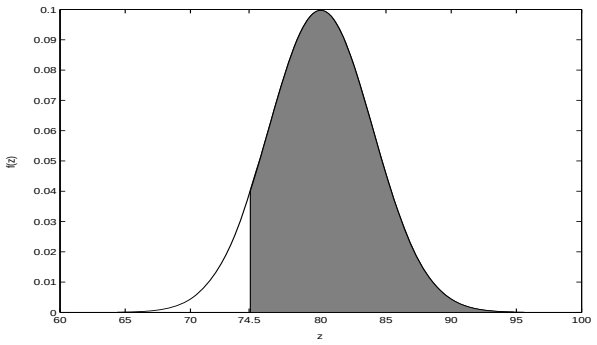


Figure 10.3:

9.5.2

$$\begin{aligned} P\left(|X-\mu|<\sigma\right) &= P\left(-\sigma<X-\mu<\sigma\right) = P\left(-1<Z<1\right) \\ &= F(1)-[1-F(1)] = .8413-(1-.8413) = 68.26\% \text{ (about 2/3)} \\ P\left(|X-\mu|<2\sigma\right) &= P\left(-2\sigma<X-\mu<2\sigma\right) = P\left(-2<Z<2\right) \\ &= F(2)-[1-F(2)] = .9772-(1-.9772) = 95.44\% \text{ (about 95\%)} \end{aligned}$$

Similarly, $P\left(|X-\mu|<3\sigma\right) = P\left(-3<Z<3\right) = 99.73\%$ (over 99%)

9.5.3 (a) $2X-Y$ is normally distributed with mean $2(5)-7=3$,variance $2^2(4)+9=25$.

$$\begin{aligned} P\left(|2X-Y|>4\right) &= P(2X-Y>4)+P(2X-Y<-4) \\ &= P\left(Z>\frac{4-3}{5}=.20\right)+P\left(Z<\frac{-4-3}{5}=-1.40\right) \\ &= .42074+.08076=.5015 \end{aligned}$$

(b) $P\left(\left|\overline{X}-5\right|<0.1\right) = P\left(|Z|<\frac{0.1}{2/\sqrt{n}}\right) = .98$ (since ...$\overline{X} \sim N(5,4/n)$). Therefore $F\left(\frac{0.1}{2/\sqrt{n}}\right)=.99$.
Therefore $.05\sqrt{n}=2.3263$ and $n=2164.7$ so we take $n=2165$ observations.

9.6.1 Let X be the number germinating. Then $X \sim Bi(100,.8)$.
$P(X\geq 75)=\sum\limits_{x=75}^{100}\binom{100}{x}(.8)^x(.2)^{100-x}$.
Approximate using a normal distribution with $\mu=np=80$ and $\sigma^2=np(1-p)=16$.

$$\begin{aligned} P(X \geq 75) &\simeq P(X > 74.5) \text{ (see Figure } 10.3) \\ &= P(Z > \frac{74.5 - 80}{4} = -1.375) \\ &\simeq F(1.38) = .9162 \end{aligned}$$

Possible variations on this solution include calculating $F(1.375)$ as $\frac{F(1.37)+F(1.38)}{2}$ and realizing that $X \leq 100$ means

$$P(X \geq 75) \simeq P(74.5 < X < 100.5)$$

However,

$$P(X \geq 100.5) \simeq P(Z > \frac{100.5 - 80}{4} = 5.125) \simeq 0$$

so we get the same answer as before.

9.6.2 Let X_i be the cost associated with inspecting part i

$$\begin{aligned} E(X_i) &= (0 \times .6) + (10 \times .3) + (100 \times .1) = 13 \\ E\left(X_i^2\right) &= \left(0^2 \times .6\right) + \left(10^2 \times .3\right) + \left(100^2 \times .1\right) = 1030 \\ Var(X_i) &= 1030 - 13^2 = 861 \end{aligned}$$

By the central limit theorem $\sum_{i=1}^{80} X_i$ is Normal with mean $80 \times 13 = 1040$ and variance $80 \times 861 = 68,880$ approximately. Since $\sum X_i$ increases in \$10 increments,

$$P(\sum X_i > 1200) \simeq P\left(Z > \frac{1205 - 1040}{\sqrt{68,880}} = 0.63\right) = 0.2643$$

Answers to End of Chapter Problems

Chapter 2:

12.1 (a) Label the profs A, B, C and D.

$$S = \{AA, AB, AC, AD, BA, BB, BC, BD, CA, CB, CC, CD, DA, DB, DC, DD\}$$

(b) 1/4

2.2 (a) $\{HHH,\ HHT,\ HTH,\ THH,\ HTT,\ THT,\ TTH,\ TTT\}$
(b) $\frac{1}{4}$;

2.3 $S = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5), (2,1), (3,1), (4,1,), ...(5,4)\}$; probability consecutive $= 0.4$;

2.4 (c) $\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, 0$

2.5 (a) $\frac{8}{27}, \frac{1}{27}, \frac{2}{9}$ (b) $\frac{(n-1)^3}{n^3}, \frac{(n-2)^3}{n^3}, \frac{n^{(3)}}{n^3}$ (c) $\frac{(n-1)^r}{n^r}, \frac{(n-2)^r}{n^r}, \frac{n^{(r)}}{n^r}$

2.6 (a) .018 (b) .020 (c) 18/78 = .231

2.7 (b) .978

Chapter 3:

1. 3.1 (a) 4/7 (b) 5/42 (c) 5/21;

   3.2 (a) (i) $\frac{(n-1)^r}{n^r}$ (ii) $\frac{n^{(r)}}{n^r}$

   (b) All n^r outcomes are equally likely. That is, all n floors are equally likely to be selected, and each passenger's selection is unrelated to each other person's selection. Both assumptions are doubtful since people may be travelling together (e.g. same family) and the floors may not have equal traffic (e.g. more likely to use the stairs for going up 1 floor than for 10 floors);

   3.3 (a) 5/18 (b) 5/72;

3.4 $\frac{\binom{4}{2}\binom{12}{4}\binom{36}{7}}{\binom{52}{13}}$

3.5 (a) 1/50,400 (b) 7/45;

3.6 (a) 1/6 (b) 0.12;

3.7 Values for $r = 20, 40$ and 60 are .589, .109 and .006.

3.8 (a) $\frac{1}{n}$ (b) $\frac{2}{n}$

3.9 $\frac{1+3+\cdots+(2n-1)}{\binom{2n+1}{3}} = \frac{n^2}{\binom{2n+1}{3}}$

3.10 (a) (i) .0006 (ii) .0024 (b) $\frac{10^{(4)}}{10^4} = .504$

3.11 (a) $\binom{6}{2}\binom{19}{3}/\binom{25}{5}$ (b) 15

3.12 (a) $1/\binom{49}{6}$ (b) $\binom{6}{5}\binom{43}{1}/\binom{49}{6}$ (c) $\binom{6}{4}\binom{43}{2}/\binom{49}{6}$ (d) $\binom{6}{3}\binom{43}{3}/\binom{49}{6}$

3.13

$$\text{(a) } 1 - \frac{\binom{48}{3}}{\binom{50}{3}} \qquad \text{(b) } 1 - \frac{\binom{45}{2}}{\binom{47}{2}} \qquad \text{(c)} \frac{\binom{48}{3}}{\binom{50}{5}}$$

3.14 By the binomial theorem $\sum_{x=0}^{n} \binom{n}{x} a^x = (1+a)^n$

Differentiate with respect to a on both sides:

$\sum_{x=0}^{n} x\binom{n}{x} a^{x-1} = n(1+a)^{n-1}$. Multiply by a to get $\sum_{x=0}^{n} x\binom{n}{x} a^x = na(1+a)^{n-1}$

Let $a = \left(\frac{p}{1-p}\right)$. Then $\sum_{x=0}^{n} x\binom{n}{x}\left(\frac{p}{1-p}\right)^x = n\left(\frac{p}{p-1}\right)\left(1+\frac{p}{1-p}\right)^{n-1} = \frac{np}{(1-p)^n}(1)^{n-1}$

Multiply by $(1-p)^n$:

$$\sum_{x=0}^{n} x\binom{n}{x}\left(\frac{p}{1-p}\right)^x (1-p)^n = \sum_{x=0}^{n} x\binom{n}{x} p^x (1-p)^{n-x} = \frac{np}{(1-p)^n}(1-p)^n = np$$

3.15 Let $Q = \{\text{heads on quarter}\}$ and $D = \{\text{heads on dime}\}$. Then

$$\begin{aligned} P(\text{Both heads at same time}) &= P(QD \cup \overline{Q}\,\overline{D}QD \cup \overline{Q}\,\overline{D}\,\overline{Q}\,\overline{D}\,QD \cup \cdots) \\ &= (.6)(.5) + (.4)(.5)(.6)(.5) + (.4)(.5)(.4)(.5)(.6)(.5) + \cdots \\ &= \frac{(.6)(.5)}{1-(.4)(.5)} = 3/8 \text{ (using } a + ar + ar^2 + \cdots = \frac{a}{1-r} \text{ with } r = (.4)(.5)) \end{aligned}$$

3.16

3.17

Chapter 4:

1. 4.1 $0.75,\ 0.6,\ 0.65,\ 0,\ 1,\ 0.35,\ 1$

4.2 $P(A) = 0.01$, $P(B) = 0.72$, $P(C) = (0.9)^3$, $P(D) = (.5)^3$, $P(E) = (0.5)^2$

4.3 $P(A|\overline{B}) = \frac{P(A\overline{B})}{P(\overline{B})} = \frac{P(A)-P(AB)}{1-P(B)} = \frac{P(A)-P(B)P(A|B)}{1-P(B)} = \frac{0.3-0.4(0.5)}{1-0.4} = \frac{1}{6}$

4.4 (a) 0.0576 (b) 0.4305 (c) 0.0168 (d) 0.5287

4.5 0.44

4.6 0.7354

4.7 (a) 0.3087 (b) 0.1852;

4.8

4.9 0.342

4.10 (a) 0.1225, 0.175 (b) 0.395

4.11 $(\frac{f}{F}) = (\frac{m}{M})$

4.12

4.13

4.14 (a) $\frac{1}{30} + \frac{4P}{5}$ (b) $p = \frac{(30x/n)-1}{24}$ (c) $\frac{24p}{1+24p}$

4.15 0.9, 0.061, 0.078

4.16 (a) 0.024 (b) 8 on any one wheel and 1 on the others

4.17 (a) 0.995 and 0.005 (b) 0.001

4.18

(a) 0.99995

(b) 0.99889

(c) $0.2 + 0.1 + 0.1 - (.2)(.1) - (.2)(.1) - (.1)(.1) + (.2)(.1)(.1) = 0.352$

4.19 (a) $\frac{r}{r+1999}$; $0.005, 0.0148, 0.0476$ (b) 2.1%

Chapter 5:

1. 5.1 (a) .623, .251; for males, .408, .103 (b) .166

5.2 (a) $f(0) = 0, f(x) = 2^{-x}$ $(x = 1, 2, \dots)$ (b) $f(5) = \frac{1}{32}$; $P(X \geq 5) = \frac{1}{16}$

5.3

5.4 $\frac{p(1-p)^r}{1-(1-p)^4}$; $r = 0, 1, 2, 3$

5.5 (a) .0800 (b) .171 (c) .00725

5.6 (a) .010 (b) .864

5.7 (a)$\frac{4}{15}$ (b) $\binom{74}{y}\binom{76}{12-y}/\binom{150}{12}$ (c) .0176

5.8 0.9989

5.9 (a) .0758 (b) .0488 (c) $\binom{10}{y}(e^{-10\lambda})^y(1-e^{-10\lambda})^{10-y}$ (d) $\lambda = .12$

5.10 (a) .0769 (b) 0.2019; 0.4751

5.11 (a) 0.2753 (b) 0.1966 (c) 0.0215

5.12 (b) enables us to approximate hypergeometric distribution by binomial distribution when n is large and p near 0.

5.13 (a) $1 - \left[\sum_{x=0}^{k-1} \frac{\lambda^x e^{-\lambda}}{x!}\right]^n$ (b) (Could probably argue for other answers also). Budworms probably aren't distributed at a uniform rate over the forest and may not occur singly

5.14 (a) .2048 (b) .0734 (c) .428 (d) .1404

5.15 $\frac{\binom{35}{x}\binom{70}{7}}{\binom{105}{x+7}}\frac{63}{98-x}$; $x = 0, 1, \cdots, 35$

5.16 (a) .004264; .006669 (b) .0032 (c) (i) $\binom{1399}{11}(.004264)^{12}(.995736)^{1388}$ (ii) 9.336×10^{-5} On the first 1399 attempts we essentially have a binomial distribution with $n = 1399$ (large) and $p = .004264$ (near 0)

5.17 (a) $\binom{n}{x}\left(e^{-0.96}\right)^x\left(1-e^{-0.96}\right)^{n-x}$; $x = 0, 1, \cdots, n$ (b) $\lambda \leq 0.866$ bubbles per m^2

5.18 0.5;

x	0	1	2	3	4	5
$f(x)$	0	.05	.15	.05	.25	.5

; 0.3

5.19 (a) $(1-p)^y$ (b) $Y = 0$ (c) $p/[1-(1-p)^3]$ (d) $P(R = r) = \frac{p(1-p)^r}{1-(1-p)^3}$ for $r = 0, 1, 2$

5.20 (a) .555 (b) .809; .965 (c) .789; .946 (d) $n = 1067$

5.21 (a) $\binom{x-1}{999}(.3192^{1000})(.6808^{x-1000})$ (b) .002, .051, .350, .797 (c) $\binom{3200}{y}(.3192^y)(.6808^{3200-y})$; .797

Chapter 7:

1. 7.1 2.775; 2.574375

   7.2 -$3

   7.3 $16.90

   7.4 (a) 3 cases (b) 32 cases

   7.5 (a) - 10/37 dollars in both cases (b) .3442; 0.4865

   7.6 $.94

   7.7 (b) $n + \frac{n}{k} - n(1-p)^k$, which gives $1.01n, 0.249n, 0.196n$ for $k = 1, 5, 10$

7.8 50

7.9 (a) $\frac{p}{1-(1-p)e^t}$ for $t < -\ln(1-p)$; (b) $\frac{1-p}{p}; \frac{1-p}{p^2}$

7.10

7.11 (a) Expand $M(t)$ in a power series in powers of e^t, i.e.

$$M_X(t) = \frac{1}{3}e^t + \frac{2}{9}e^{2t} + \frac{4}{27}e^{3t} + \frac{8}{81}e^{4t} + \frac{16}{243}e^{5t} + ...$$

and this converges for

$$|\frac{2}{3}e^t| < 1 \text{ or } t < \ln(\frac{3}{2}).$$

Then $P(X = j)$ =coefficient of $e^{jt} = \frac{1}{3}(\frac{2}{3})^{j-1}, j = 1, 2, ...$

(b) Similarly

$$M_X(t) = e^{-2} + 2e^{-2}e^t + 2e^{-2}e^{2t} + \frac{4}{3}e^{-2}e^{3t} + \frac{2}{3}e^{-2}e^{4t} + \frac{4}{15}e^{-2}e^{5t} + ...$$

Then $P(X = j) = e^{-2}\frac{2^j}{j!}, j = 0, 1, ...$

7.12 $M(t) = \frac{1}{b-a+1}\sum_{x=a}^{b} e^{xt} = \frac{e^{at}-e^{(b+1)t}}{(1-e^t)(b-a+1)}$. $E(X) = M'(0) = \frac{1}{b-a+1}\sum_{x=a}^{b} x$, $E(X^2) = M''(0) = \frac{1}{b-a+1}\sum_{x=a}^{b} x^2$

7.13 (a) $M(t) = 0.25 + 0.5e^t + 0.25e^{2t}$

(b) $M^{(k)}(0), k = 1, 2, ..., 6$. $\frac{1}{2} + \frac{1}{4}$

(c) $p_0 = 1/4, p_1 = 1/2, p_2 = 1/4$

(d) Note that for given values of the mean $E(X) = \mu_1$, $E(X^2) = \mu_2$, there is a unique solution to the equations $p_0 + p_1 + p_2 = 1, p_1 + 2p_2 = \mu_1, p_1 + 4p_2 = \mu_2$

7.14 If X is Bin(13, $\frac{1}{2}$) then $g(S_{13}) = \max(2X - 18, 0)$ and

$$E[g(S_{13})] = \frac{\binom{13}{10} + 2\binom{13}{11} + 3\binom{13}{12} + 4\binom{13}{13}}{2^{12}} = \frac{485}{4096}$$

Chapter 8:

1. 8.1 (a) no $f(1,0) \neq f_1(1)f_2(0)$ (b) 0.3 and 1/3

8.2 (a) mean = 0.15, variance = 0.15

8.3 (a) No (b) 0.978 (c) .05

8.4 (a) $\frac{(x+y+9)!}{x!y!9!}\, p^x\, q^y(1-p-q)^{10}$ for $x, y = 0, 1, 2, ...$

(b) $\binom{x+y+9}{y}\, q^y(1-q)^{x+10}$; $y = 0, 1, 2, ...$; 6 .0527

8.5 (b) - .10 dollars , (c) $d = .95/n$

8.7 (a)
$$\frac{\binom{5}{x}\binom{3}{2-x}\binom{5-x}{y-x}\binom{1+x}{2+x-y}}{\binom{8}{2}\binom{6}{2}};\ \text{for } x = 0, 1, 2, ;\ \ y = max(1, x), x + 1, x + 2;$$

(b) note e.g. that $f_1(0) \neq 0; f_2(3) \neq 0$, but $f(0, 3) = 0$

8.8 (a) $\frac{\binom{2}{x}\binom{1}{y}\binom{7}{3-x-y}}{\binom{10}{3}};\ \ x = 0, 1, 2,$ and $y = 0, 1$

(b) $f_1(x) = \binom{2}{x}\binom{8}{3-x}/\binom{10}{3};\ \ x = 0, 1, 2;$

$$f_2(y) = \binom{1}{y}\binom{9}{3-y}/\binom{10}{3};\ \ y = 0, 1$$

(c) 49/120 and 1/2

8.9 (a) $k\frac{2^x e^2}{x!};\ x = 0, 1, 2, ...$ (b) e^{-4}

(c) Yes. $f(x, y) = f_1(x)f_2(y)$

(d) $\frac{4^t e^{-4}}{t!};\ t = 0, 1, 2, \ldots$

8.10 (b) .468

8.11 (a) $\frac{40!}{(10!)^4}\left(\frac{3}{16}\right)^{20}\left(\frac{5}{16}\right)^{20}$

(b) $\binom{40}{16}(1/2)^{40}$

(c) $\binom{16}{10}\left(\frac{3}{8}\right)^{10}\left(\frac{5}{8}\right)^{6}$

8.12 (a) 1.76 (b) $P(Y = y) = \sum_{x=y}^{8}\binom{x}{y}(\frac{1}{2})^x f(x);\ \ E(Y) = 0.88 = \frac{1}{2}E(X)$

8.13 (a) Multinomial (b) .4602 (c) $5700

8.15 207.867

8.16 (a) $Bi(n, p + q)$ (b) $n(p + q)$ and $n(p + q)(1 - p - q)$ (c) $-npq$

8.17 (a) $\mu_U = 2, \mu_V = 0,\ \ \sigma_U^2 = \sigma_V^2 = 1$ (b) 0

(c) no. e.g. $P(U = 0) \neq 0; P(V = 1) \neq 0; P(U = 0 \text{ and } V = 1) = 0$

8.19 -1

8.20 (a) 1.22 (b) 17.67%

8.21 $p^3(4 + p); 4p^3(1 - p^3) + p^4(1 - p^4) + 8p^5(1 - p^2)$

8.22 Suppose P is $N \times N$ and let $\mathbf{1}$ be a column vector of ones of length n. Consider the probability vector corresponding to the discrete uniform distribution $\pi = \frac{1}{N}\mathbf{1}$. Then

$\pi^T P = \frac{1}{N}\mathbf{1}^T P = \frac{1}{N}(\sum_i P_{i1}, \sum_i P_{i2}, ... \sum_i P_{iN}) = \frac{1}{N}\mathbf{1}^T = \pi^T$ since P is doubly stochastic.

Therefore π is a stationary distribution of the Markov chain.

8.23 The transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

from which, solving $\pi^T P = \pi^T$ and rescaling so that the sum of the probabilities is one, we obtain $\pi^T = (0.4, 0.45, 0.15)$, the long run fraction of time spent in cities A,B,C respectively.

8.24 By arguments similar to those in section 8.3, the limiting matrix has rows all identically π^T where the vector π^T are the stationary probabilities satisfying $\pi^T P = \pi^T$ and

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The solution is $\pi^T = (0.1, 0.6, 0.3)$ and the limit is

$$\begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}$$

8.25 With $Z = X + Y$,

$$\begin{aligned} M_Z(t) = Ee^{t(X+Y)} &= E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t) = \exp(-\lambda_1 + \lambda_1 e^t)\exp(-\lambda_2 + \lambda_2 e^t) \\ &= \exp(-(\lambda_1 + \lambda_2) + (\lambda_1 + \lambda_2)e^t) \end{aligned}$$

and since this is the MGF of a Poisson$(\lambda_1 + \lambda_2)$ distribution, this must be the distribution of Z.

8.26 If today is raining, the probability of Rain, Nice, Snow three days from now is obtainable from the first row of the matrix P^3, i.e. $(0.406\ 0.203\ \ 0.391)$. The probabilities of the three states in five days, given (1) today is raining (ii) today is nice (iii) today is snowing are the three rows of the matrix P^5. In this case call rows are identical to three decimals; they are all equal the equilibrium distribution $\pi^T = (0.400\ 0.200\ 0.400)$.

(b) If $a>b$, and both parties raise then the probability B wins is

$$\frac{13-b}{2(13-a)} < \frac{1}{2}$$

and the probability A wins is 1 minus this or $\frac{13-2a+b}{2(13-a)}$. If $a \leq b$, then the probability A wins is

$$\frac{13-a}{2(13-b)}.$$

8.27 (a) In the special case *b=1,* count the number of possible pairs (i, j) for which $A = i \geq a$ and $B = j > A$.

1	$(A = 12)$
2	$(A = 11)$
:	
$13 - a$	$(A = a)$
$\frac{(13-a)(13-a+1)}{2}$	Total

This leads to

$$P(B > A, A \geq a) = \frac{(13-a)(13-a+1)}{2(13^2)}$$

Similarly, since the number of pairs (A, B) for which $A \geq a$, and $B < a$ is $(13 - a + 1)(a - 1)$, we have

$$\begin{aligned} P(A > B, A \geq a) &= P(A > B, A \geq a, B \geq a) + P(A > B, A \geq a, B < a) \\ &= \frac{(13-a)(13-a+1)}{2(13^2)} + \frac{(13-a+1)(a-1)}{13^2} = \frac{(14-a)(a+11)}{2(13^2)} \end{aligned}$$

Therefore, in case *b=1,* the expected winnings of A are

$$\begin{aligned} &-1P(\text{B raises, A does not}) - 6P(\text{both raise, B wins}) + 6P(\text{both raise, A wins}) \\ &= -1P(A < a) - 6P(B > A, A \geq a) + 6P(A > B, A \geq a) \\ &= -1 \times \frac{a-1}{13} - 6 \times \frac{(13-a)(13-a+1)}{2(13^2)} + 6 \times \frac{(14-a)(a+11)}{2(13^2)} \\ &= -\frac{6}{169}a^2 + \frac{77}{169}a - \frac{71}{169} = -\frac{1}{169}(a-1)(6a-71) \end{aligned}$$

:whose maximum (over real a) is at $77/12$ and over integer a, at 6 or 7. For *a=1,2,...,13* this gives expected winnings of *0, 0.38462, 0.69231, 0.92308, 1.0769, 1.1538, 1.1538, 1.0769, 0.92308, 0.69231, 0.38462, 0, -0.46154* respectively, and the maximum is for *a=6* or 7.

(b) We want $P(A > B, A \geq a, B \geq b)$. Count the number of pairs (i, j) for which $A \geq a$ and $B \geq b$ and $A > B$. Assume that $b \leq a$.

1	$(B = 12)$
2	$(B = 11)$
:	:
$13 - a$	$(B = a)$
$(a - b)(13 - a + 1)$	$(b \leq B < a)$

for a total of

$$\frac{(13-a)(13-a+1)}{2} + (a-b)(13-a+1) = \frac{1}{2}(14-a)(13+a-2b)$$

and

$$P(A > B, A \geq a, B \geq b) = \frac{(14-a)\,(13+a-2b)}{2(13^2)}$$

Similarly

$$P(A < B, A \geq a, B \geq b) = P(A < B, A \geq a) = \frac{(13-a)(13-a+1)}{2(13^2)}$$

Therefore the expected return to A (still assuming $b \leq a$) is

$$\begin{aligned}
&-1P(A < a, B \geq b) + 1P(A \geq a, B < b) \\
&+6P(A > B, A \geq a, B \geq b) - 6P(A < B, A \geq a, B \geq b) \\
&\quad = -1\frac{(a-1)(13-b+1)}{13^2} + 1\frac{(b-1)(13-a+1)}{13^2} \\
&\quad +6\frac{(14-a)\,(13+a-2b)}{2(13^2)} - 6\frac{(13-a)(13-a+1)}{2(13^2)} \\
&\quad = \frac{1}{13^2}\,(71-6a)\,(a-b)
\end{aligned}$$

If $b > a$ then the expected return to B is obtained by switching the role of a, b above, namely

$$\frac{1}{13^2}\,(71-6b)\,(b-a)$$

and so the expected return to A is

$$\frac{1}{13^2}\,(71-6b)\,(a-b)$$

In general, then the expected return to A is

$$\frac{1}{13^2}\,(71-6\max(a,b))\,(a-b)$$

(c) By part (b), A's possible expected profit per game for *a=1,2,...,13* and $b = 11$ is

$$\frac{1}{13^2}\,(71-6\max(a,11))\,(a-11) = -\frac{6}{13^2}\left(\max(a,11) - \frac{71}{6}\right)(a-11)$$

For $a = 1, 2, ...13$ these are, respectively,-0.2959, -0.2663, -0.2367, -0.2071, -0.1775, -0.1479, -0.1183,-0.0888, -0.0592, -0.0296, 0, -0.0059, -0.0828. There is no strategy that provides a positive expected return. The optimal is the break-even strategy *a=11*. (Note: in this two-person zero-sum game, *a= 11* and *b=11* is a minimax solution)

8.28 i. The permutation X_{j+1} after $j+1$ requests depends only on the permutation X_j before and the record requested at time $j+1$. Thus the new state depends only only the old state X_t (without knowing the previous states) and the record currently requested.

ii. For example the long-run probability of the state (i, j, k) is, with $q_i = p_i/(1 - p_i)$,

$$q_i p_j$$

iii. The probability that record j is in position $k = 1, 2, 3$ is,

$$p_j \text{ for } k = 1, \;\; (Q - q_j)p_j, \text{ for } k = 2, \;\; 1 - p_j(1 + Q - q_j) \text{ for } k = 3$$

where $Q = \sum_{i=1}^{3} q_i$. The expected cost of accessing a record in the long run is

$$\sum_{j=1}^{3} \{p_j^2 + 2p_j^2(Q - q_j) + 3p_j[1 - p_j(1 + Q - q_j)]\} \tag{10.12}$$

Substitute $p_1 = 0.1, p_2 = 0.3, p_3 = 0.6$ so $q_1 = \frac{1}{9}, q_2 = \frac{3}{7}, q_3 = \frac{6}{4}$ and $Q = \frac{1}{9} + \frac{3}{7} + \frac{6}{4} = 2.0397$ and (10.12) is 1.7214.

iv. If they are in random order, the expected cost$= 1(\frac{1}{3}) + 2(\frac{1}{3}) + 3(\frac{1}{3}) = 2$. If they are ordered in terms of decreasing p_j, expected cost is $p_3^2 + 2p_2^2 + 3p_1^2 = 0.57$

8.29 Let J =index of maximum. $P(J = j) = 1/N$, for $j = 1, 2, ..., N$. Let A =" your strategy chooses the maximum". A occurs only $J > k$ and if $\max\{X_i; k < i < J\} < \max\{X_i; 1 \le i \le k\}$. Given $J = j > k,$ the probability of this is the probability that the maximum $\max\{X_i; 1 \le i < j\}$ occurs among the first k values, i.e. the probability is $k/(j - 1)$. Therefore,

$$\begin{aligned} P(A) &= \sum_j P(A|J = j)P(J = j) = \sum_{j=k+1}^{N} P(A|J = j)\frac{1}{N} \\ &= \sum_{j=k+1}^{N} \frac{k}{j-1}\frac{1}{N} = \frac{k}{N}\{\frac{1}{k} + \frac{1}{k+1} + ... + \frac{1}{N-1}\} \\ &\approx \frac{k}{N}\ln(\frac{N}{k}) \end{aligned}$$

Note that the value of x maximizing $x \ln(1/x)$ is $x = e^{-1} \approx 0.37$ so roughly, the best k is Ne^{-1} . The probability that you select the maximum is approximately $e^{-1} \approx 0.37$.

8.30 The optimal weights are

$$w_1 = \frac{1}{c\sigma_1^2}, w_2 = \frac{1}{c\sigma_2^2}, w_3 = \frac{1}{c\sigma_3^2} \text{ where } c = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}$$

and $\sigma_1 = 0.2, \quad \sigma_2 = 0.3, \quad \sigma_3 = 0.4$

Chapter 9:

1. 9.1 $f(y) = (\frac{5}{6})(\frac{6}{\pi})^{\frac{1}{3}} y^{-\frac{2}{3}}$ for $.036\pi \leq y \leq \frac{\pi}{6}$

   9.2 (a) $k = .75;\ F(x) = .75\left(\frac{2}{3} + x - \frac{x^3}{3}\right)$ for $-1 \leq x \leq 1$
   (b) Find c such that $c^3 - 3c + 1.9 = 0$. This gives $c = .811$

   9.3 (a) 1/2, 1/24 (b) 0.2828 (c) 0.2043

   9.4 $f(y) = 1;\ \ 0 < y < 1$

   9.5 (a) $\alpha > -1$ (b) $0.5^{\alpha+1}, \frac{\alpha+1}{\alpha+2}$ (c) $\frac{\alpha+1}{t^{\alpha+2}};\ \ 1 < t < \infty$

   9.6 (a) $(1 - e^{-2})^3$ (b) $e^{-.4}$

   9.7 $1000 \log 2 = 693.14$

   9.8 (a) .0668, .2417, .3829, .2417, .0668 (b) .0062 (c) .0771

   9.9 (a) .5 (b) $\mu \geq 2.023$

   9.10 0.4134

   9.11 (a) .3868 (b) .6083 (c) 6.94

   9.12 (a) .0062 (b) .9927

   9.13 (a) .2327, .1841 (b) .8212, .8665; Guess if $p_i = 0.45$, don't guess if $p_i = 0.55$

   9.14 6.092 cents

   9.15 574

   9.16 (a) 7.6478, 88.7630
   (b) 764.78, 8876.30, people within pooled samples are independent and each pooled sample is independent of each other pooled sample.
   (c) 0.3520

   9.17 0.5969

   9.18 (a) .6728 (b) 250,088

   9.19 (a) $X \sim N(-.02n, .9996n)$
   (b) $P(X \geq 0) = .4641, .4443, .4207$ (using table) for $n = 20, 50, 100$ The more you play, the smaller your chance of winning.
   (c) 1264.51
   With probability .99 the casino's profit is at least \$1264.51.

   9.20 (a) X is approximately $N\left(-\frac{n}{2}, \frac{5n}{12}\right)$ (b) (i)$P(X > 0) = P(Z > 2.45) = 0.0071$. (ii) $P(X > 0) = P(Z > 5.48) \simeq 0$.

9.21 (a) (i) .202 (ii) .106 (b) .0475, .0475

9.22 (a) False positive probabilities are $P(Z > \frac{d}{3}) = 0.0475, 0.092, 0.023$ for Z standard normal and $d = 5, 4, 6$ in (i), (ii), (iii). False negative probabilities are $P(Z < \frac{d-10}{3}) =$.0475, 0.023, 0.092 for Z standard normal and $d = 5, 4, 6$ in (i), (ii), (iii). (b)The factors are the security (proportion of spam in email) and proportion of legitimate messages that are filtered out.

9.23

9.24 Let $Y =$total change over day. Given $N = n$, Y has a Normal$(0, n\sigma^2)$ distribution and therefore

$$E[e^{tY}|N = n] = \exp(n\sigma^2 t^2/2)$$

$$M_Y(t) = E[e^{tY}] = \sum_n E[e^{tY}|N = n]P(N = n) = e^{-\lambda}\sum_n \exp(n\sigma^2 t^2/2)\frac{\lambda^n}{n!}$$

$$= e^{-\lambda}\sum_n \frac{(e^{\sigma^2 t^2/2}\lambda)^n}{n!} = \exp(-\lambda + e^{\sigma^2 t^2/2}\lambda)$$

Not a MGF in this course at least. The mean is $M'_Y(0) = 0$ and the variance is $M"_Y(0) = \lambda\sigma^2$.

9.25 i. $\exp(t + t^2)$

ii. $\exp(2t + 2t^2)$

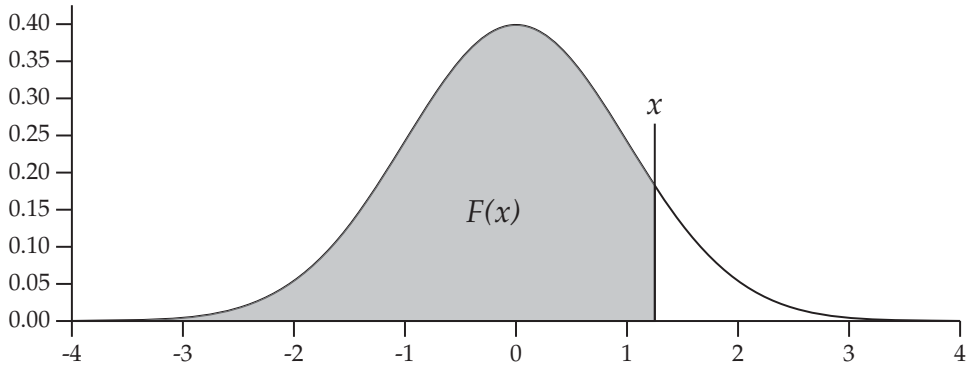
iii. $\exp(nt + nt^2)$

iv. $\exp(t^2)$

Summary of Distributions

Discrete				
Notation and Parameters	Probability function $f(x)$	**Mean**	**Variance**	Moment generating function $M_X(t)$
Binomial(n,p) $0<p<1, q=1-p$	$\binom{n}{x} p^x q^{n-x}$ $x=0,1,2,\ldots,n$	np	npq	$(pe^t+q)^n$
Bernoulli(p) $0<p<1, q=1-p$	$p^x(1-p)^{1-x}$ $x=0,1$	p	$p(1-p)$	(pe^t+q)
Negative Binomial(k,p) $0<p<1, q=1-p$	$\binom{x+k-1}{x} p^k q^x$ $x=0,1,2,\ldots$	$\frac{kq}{p}$	$\frac{kq}{p^2}$	$\left(\frac{p}{1-qe^t}\right)^k$
Geometric(p) $0<p<1, q=1-p$	pq^x $x=0,1,2,\ldots$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\left(\frac{p}{1-qe^t}\right)$
Hypergeometric(N,r,n) $r<N, n<N$	$\frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $x=0,1,2,\ldots \min(r,n)$	$\frac{nr}{N}$	$n\frac{r}{N}(1-\frac{r}{N})\frac{N-n}{N-1}$	intractible
Poisson(λ) $\lambda>0$	$\frac{e^{-\lambda}\lambda^x}{x!}$ $x=0,1,\ldots$	λ	λ	$e^{\lambda(e^t-1)}$
Continuous	**p.d.f.** $f(x)$	**Mean**	**Variance**	Moment generating function $M_X(t)$
Uniform(a,b)	$f(x)=\frac{1}{b-a}, \quad a<x<b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Exponential(θ) $0<\theta$	$f(x)=\frac{1}{\theta}e^{-x/\theta}, \quad 0<x$	θ	θ^2	$\frac{1}{1-\theta t}, \quad t<1/\theta$
Normal(μ,σ^2) $-\infty<\mu<\infty, \sigma^2>0$	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}$ $-\infty<x<\infty$	μ	σ^2	$e^{\mu t+\sigma^2 t^2/2}$

Probabilities for Standard Normal *N(0,1)* Distribution



This table gives the values of $F(x)$ for $x \geq 0$

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56750	0.57142	0.57534
0.2	0.57926	0.58317	0.58706	0.59095	0.59484	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983

This table gives the values of $F^{-1}(p)$ for $p \geq 0.50$

p	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.5	0.0000	0.0251	0.0502	0.0753	0.1004	0.1257	0.1510	0.1764	0.2019	0.2275
0.6	0.2533	0.2793	0.3055	0.3319	0.3585	0.3853	0.4125	0.4399	0.4677	0.4959
0.7	0.5244	0.5534	0.5828	0.6128	0.6433	0.6745	0.7063	0.7388	0.7722	0.8064
0.8	0.8416	0.8779	0.9154	0.9542	0.9945	1.0364	1.0803	1.1264	1.1750	1.2265
0.9	1.2816	1.3408	1.4051	1.4758	1.5548	1.6449	1.7507	1.8808	2.0537	2.3263