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Solutions of Liouville equations with non-trivial profile in dimension 4

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CHAPTER 1

Introduction

Liouville equations are a class of elliptic nonlinear partial differential equations of the form

$$(-\Delta)^n \varphi(x) = \mathrm{e}^{\varphi(x)}, \quad x \in \mathbb{R}^{2n},$$

for $n \in \mathbb{N}$.[1] This family of equations plays a fundamental role in many problems of Conformal Geometry and Mathematical Physics, as they govern the transformation laws for some curvatures. For example, the 2-dimensional equation provides the structure of metrics with constant Gaussian curvature which are conformal to the restriction of the Euclidean metric to a 2D surface. In Mathematical Physics, Liouville equations appear for example in the description of mean field vorticity in steady flows ([**7**], [**11**]), Chern-Simons vortices in superconductivity or Electroweak theory ([**33**], [**35**]). Moreover, they also arise naturally when dealing with functional determinants, which play an essential role in modern Quantum Physics and String theory [**28**]. The 2-dimensional Liouville equation was also taken as an example by David Hilbert in the formulation of the "nineteenth problem" [**22**].

The interest in Liouville equations particularly renewed after the introduction of Q-curvature (see Appendix A) and many authors studied non-trivial solutions of this class of problems. Classification results for solutions with with finite "volume" $V := \int \exp(u)$ were found in [**12**] (for the 2D case) and [**26**] (for the 4D case). Explicitly, solutions with finite volume in $\mathbb{R}^4$ have been constructed in [**34**] (a generalization in which one can fix also the asymptotic behavior of the solution was proved in [**27**]).

The case in which the integral of the solution is not finite, though, is still quite unexplored. In my Master's thesis [**1**], I proved the existence of non-trivial solutions with infinite volume in dimension 2, using basic bifurcation theory. In that setting, solutions could be found almost explicitly, allowing us to use the Simple Eigenvalue Bifurcation Theorem [**2**, Theorem 5.4.1]. Unfortunately, already in dimension 4 we do not have such an explicit description of the problem, so that the existence of non-trivial solutions in higher dimensions with infinite volume remained an open question. The goal of this work is to prove the same result in $\mathbb{R}^4$. The philosophy of the proof will still be finding a trivial solution in some lower dimension and then using some bifurcation theorem to prove the existence of a non-trivial solution which is a perturbation of the trivial one. Nonetheless, as everything will be implicit, the proof will have to resort to quite different techniques. Specifically, the underlying framework will be the one provided by Degree Theory (see for instance [**29**]).

[1] We will deal only with spaces of even dimension. For the odd-dimensional case, which is much more difficult as it involves the fractional Laplacian, see [**23**].

In Chapter 2 we will prove the existence of an infinite-volume solution of the four dimensional Liouville equation not depending on the fourth variable, i.e. a solution of $\Delta^2 u = \mathrm{e}^u$ in $\mathbb{R}^3$ (Theorem 2.2). We will call this the *trivial solution*. Then, in Chapter 3, we will prove that this solution "bifurcates", giving a non-trivial solution, again with infinite volume (Theorem 3.7).

Of course, this is not the only way to find non-trivial solutions: indeed, not even the choice of the trivial solution is unique! For instance, we could choose to look for a trivial solution not depending on the las *two* variables (i.e., a solution of $\Delta^2 u = \mathrm{e}^u$ in $\mathbb{R}^2$) and then find bifurcations of that. The reason behind choosing to have a trivial solution in $\mathbb{R}^3$ is that in this way we have to deal with just one parameter when using bifurcation results.

REMARK. Notice that choosing to have a trivial solution in dimension 1 leads nowhere. Indeed, we would be looking for solutions of the ODE

$$u^{(4)}(x) = \mathrm{e}^{u(x)}, \quad \forall x \in \mathbb{R}.$$

Integrating this equation, one immediately finds

$$u^{(3)}(y) - u^{(3)}(x) = \int_x^y \mathrm{e}^{u(s)}\,\mathrm{d}s > 0, \quad \forall x, y \in \mathbb{R}, x \neq y,$$

which means that $\lim_{x\to-\infty} u^{(3)}(x) < \lim_{x\to+\infty} u^{(3)}(x)$. But this is a contradiction: as we are requiring that u goes to $-\infty$ both at $-\infty$ and $+\infty$, indeed, we should have $\lim_{x\to-\infty} u^{(3)}(x) \geq 0$ and $\lim_{x\to+\infty} u^{(3)}(x) \leq 0$.

CHAPTER 2

Trivial solution

Our first goal is to show that there exists at least one solution of

$$\begin{cases} \Delta^2 u = \mathrm{e}^u \text{ in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} \mathrm{e}^{u(x)}\,\mathrm{d}x < +\infty \end{cases} . \tag{1}$$

The proof will be done in two steps and will look in particular for solutions of the integral form of (1), namely solutions of the form

$$u(x) = -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\mathrm{e}^{u(y)}\,\mathrm{d}y.$$

Observe that a u with finite volume satisfying this last expression is a solution of (1). Indeed, a fundamental solution to Δ^2 is $G(x) = -\frac{1}{8\pi}|x|$ (see [**13**]). I am much in debt with Dr. Ali Hyder for the big suggestions he gave me for this part.

Lemma 2.1. *Let*

$$X := \left\{ u \in C^0(\mathbb{R}^3) \,\middle|\, u \textit{ is radially symmetric and } \|u\| < +\infty \right\},$$

where $\|u\| := \sup_{x\in\mathbb{R}^3} \frac{|u(x)|}{1+|x|}$. *Then for every* $\varepsilon > 0$ *there exist* $u_\varepsilon \in X$ *such that*

$$u_\varepsilon(x) = -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)}\,\mathrm{d}y. \tag{2}$$

Proof. First of all, observe that X, endowed with the norm $\|\cdot\|$, is a well-defined Banach space. Define then

$$\begin{aligned} T_\varepsilon :\ & X \longrightarrow X \\ & u \longmapsto \bar{u}_\varepsilon, \quad \bar{u}_\varepsilon(x) := -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u(y)}\,\mathrm{d}y \end{aligned} .$$

T_ε is well defined. Take in fact $u \in X$, then $\bar{u}_\varepsilon \in C^0(\mathbb{R}^3)$ thanks to the Lebesgue Dominated Convergence Theorem. Moreover, $\bar{u}_\varepsilon$ is clearly radial because of the radial invariance of the Lebesgue integral: indeed, if $A \in SO(3)$, then

$$\begin{aligned} \bar{u}_\varepsilon(Ax) &= -\frac{1}{8\pi}\int_{\mathbb{R}^3} |Ax-y|\,\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u(y)}\,\mathrm{d}y \\ &= -\frac{1}{8\pi}\int_{\mathbb{R}^3} |A(x-y)|\,\mathrm{e}^{-\varepsilon|Ay|^2}\mathrm{e}^{u(Ay)}|\det A|\,\mathrm{d}y \\ &= -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\,\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u(y)}\,\mathrm{d}y = \bar{u}_\varepsilon(x). \end{aligned}$$

Finally,

$$
\begin{aligned}
|\bar{u}_\varepsilon(x)| &= \frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u(y)}\,\mathrm{d}y \\
&\le \frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{\|u\|(1+|y|)}\,\mathrm{d}y \\
&\le \frac{1}{8\pi}\int_{\mathbb{R}^3} |y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{\|u\|(1+|y|)}\,\mathrm{d}y + |x|\frac{1}{8\pi}\int_{\mathbb{R}^3} \mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{\|u\|(1+|y|)}\,\mathrm{d}y \\
&\le C_1 + C_2|x| \le \bar{C}(1+|x|),
\end{aligned}
$$

so that $\|\bar{u}_\varepsilon\| < +\infty$. Hence $T_\varepsilon(u) = \bar{u}_\varepsilon \in X$.

We now show that T_ε is compact. Take a bounded sequence $\{u_n\}_n \subset X$, $\|u_n\| \le C < +\infty$ for all $n \in \mathbb{N}$. We want to show that then $\{T_\varepsilon(u_n)\}_n$ admits a converging subsequence. The idea is to use Arzelà-Ascoli's Theorem on the sequence $\left\{\frac{T_\varepsilon(u_n)}{1+|x|}\right\}_n$. First,

$$
\begin{aligned}
\frac{|T_\varepsilon(u_n)|}{1+|x|} &= \frac{1}{8\pi(1+|x|)}\int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_n(y)}\,\mathrm{d}y \\
&\le \frac{1}{8\pi(1+|x|)}\int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{C(1+|x|)}\,\mathrm{d}y \\
&\le \frac{C_1+C_2|x|}{8\pi(1+|x|)} \le \bar{C} < +\infty,
\end{aligned}
$$

for any $x \in \mathbb{R}^3$ and $n \in \mathbb{N}$, so that $\left\{\frac{T_\varepsilon(u_n)}{1+|x|}\right\}_n$ is uniformly bounded. Moreover,

$$
\begin{aligned}
\left|\frac{T_\varepsilon(u_n(x))}{1+|x|} - \frac{T_\varepsilon(u_n(y))}{1+|y|}\right| &= \frac{1}{8\pi}\left|\int_{\mathbb{R}^3}\left(\frac{|x-z|}{1+|x|} - \frac{|y-z|}{1+|y|}\right)\mathrm{e}^{-\varepsilon|z|^2}\mathrm{e}^{u_n(z)}\,\mathrm{d}z\right| \\
&\le \frac{1}{8\pi}\int_{\mathbb{R}^3}\left|\frac{|x-z|}{1+|x|} - \frac{|y-z|}{1+|y|}\right|\mathrm{e}^{-\varepsilon|z|^2}\mathrm{e}^{u_n(z)}\,\mathrm{d}z \\
&\le \left(\frac{1}{8\pi}\int_{\mathbb{R}^3}(2+|z|)\mathrm{e}^{-\varepsilon|z|^2}\mathrm{e}^{C(1+|z|)}\,\mathrm{d}z\right)|x-y|
\end{aligned}
$$

for any $x, y \in \mathbb{R}^3$ and $n \in \mathbb{N}$, so that $\left\{\frac{T_\varepsilon(u_n(x))}{1+|x|}\right\}_n$ is equicontinuous. The last inequality, in particular follows from the triangular inequality. As one clearly has

$$
\begin{aligned}
|y| &\le |x| + |y-x|, \\
|x-z| &\le |x-y| + |y-z|, \\
|z| &\le |x| + |z-x|, \\
|x-z| &\le |x| + |z|,
\end{aligned}
$$

we obtain indeed

$$\begin{aligned}\left|\frac{|x-z|}{1+|x|}-\frac{|y-z|}{1+|y|}\right| &= \left|\frac{|x-z|-|y-z|+|y||x-z|-|x||y-z|}{(1+|x|)(1+|y|)}\right| \\ &\leq \frac{|x-y|+(|x|+|y-x|)|x-y|-|x||y-z|}{(1+|x|)(1+|y|)} \\ &\leq \frac{|x-y|+|x||x-y|+|x-z||x-y|}{(1+|x|)(1+|y|)} \\ &\leq \left(\frac{1+2|x|}{(1+|x|)(1+|y|)}+\frac{|z|}{(1+|x|)(1+|y|)}\right)|x-y| \\ &\leq \left(\frac{2}{(1+|y|)}+\frac{|z|}{(1+|x|)(1+|y|)}\right)|x-y| \\ &\leq (2+|z|)|x-y|.\end{aligned}$$

By Arzelà-Ascoli's Theorem, then, $\left\{\frac{T_\varepsilon(u_n(x))}{1+|x|}\right\}_n$ admits a subsequence which converges uniformly. Thus, $\{T_\varepsilon(u_n)\}_n$ admits a converging subsequence in $(X,\|\cdot\|)$ and therefore T is a compact operator.

Next, we prove that T has a fixed point using Schaefer's Fixed Point Theorem (see for example [**36**]). Let $u \in X$ satisfy $u = tT_\varepsilon(u)$ for some $0 \leq t \leq 1$, then

$$u(x) = -\frac{t}{8\pi}\int_{\mathbb{R}^3}|x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u(y)}\,\mathrm{d}y \leq 0.$$

Consequently

$$|u(x)| \leq \frac{t}{8\pi}\int_{\mathbb{R}^3}|x-y|\mathrm{e}^{-\varepsilon|y|^2}\,\mathrm{d}y \leq C(1+|x|)$$

and therefore $\|u\| \leq C$. That means that the set $\{u \in X \mid u = tT_\varepsilon(u), 0 \leq t \leq 1\}$ is bounded in $(X,\|\cdot\|)$: by Schaefer's Theorem then T_ε has a fixed point in X. □

THEOREM 2.2. *u_ε converges to some u in $(X,\|\cdot\|)$ as ε goes to 0, with u satisfying*

$$u(x) = -\frac{1}{8\pi}\int_{\mathbb{R}^3}|x-y|e^{u(y)}\,\mathrm{d}y$$

(hence being a solution of (1) *with the desired properties).*

PROOF. First of all, we check that u_ε is monotone decreasing for each $\varepsilon > 0$. Indeed, write the integral in u_ε in polar coordinates (with a slight abuse of notation)

$$\begin{aligned}u_\varepsilon(r) &= -\frac{1}{8\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{s=0}^{+\infty}\sqrt{r^2-2rs\cos\theta+s^2}\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}s^2\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= -\frac{1}{12r}\int_0^{+\infty}(r^2-2rs\cos\theta+s^2)^{\frac{3}{2}}\Big|_{\theta=0}^{\theta=\pi}\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}s\,\mathrm{d}s \\ &= -\frac{1}{12r}\int_0^{+\infty}\left[(r+s)^3-|r-s|^3\right]\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}s\,\mathrm{d}s \\ &= -\frac{1}{6r}\int_0^r s^2(3r^2+s^2)\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \frac{1}{6}\int_r^{+\infty}s(r^2+3s^2)\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s.\end{aligned}$$

In the previous computation we set $y = (s\sin\theta\cos\varphi, s\sin\theta\sin\varphi, s\cos\theta)$ and we chose $x = (r, 0, 0)$ (recall that we have already checked the radial invariance). Now take a derivative in r:

$$\begin{aligned} u_\varepsilon'(r) &= \frac{1}{6r^2}\int_0^r s^2(3r^2+s^2)\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \frac{2}{3}r^3\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} \\ &\quad - \int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s + \frac{2}{3}r^3\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} \\ &\quad - \frac{1}{3}\int_r^{+\infty} rs\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s \\ &= \int_0^r \underbrace{\frac{s^2-3r^2}{6r^2}}_{<0} s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \frac{r}{3}\int_r^{+\infty} s\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)} < 0. \end{aligned}$$

Hence u_ε is monotone decreasing for all $\varepsilon > 0$.

Applying Pohozaev's identity to (2) one gets

$$\int_{\mathbb{R}^3}\left(u_\varepsilon(x) + 6 - 4\varepsilon|x|^2\right)\mathrm{e}^{-\varepsilon|x|^2}\mathrm{e}^{u_\varepsilon(x)}\,\mathrm{d}x = 0.$$

Since u_ε is monotone decreasing and continuous, we must have $u_\varepsilon(0) > -6$ (otherwise the previous integral would be strictly negative). Hence $-6 < u_\varepsilon(0) < 0$: applying that to (2) we get

$$\left|\int_{\mathbb{R}^3}|y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y\right| \le 6$$

and thus

$$|\Delta u_\varepsilon(0)| = \frac{1}{4\pi}\left|\int_{\mathbb{R}^3}\frac{1}{|y|}\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y\right| \le C < +\infty. \tag{3}$$

By Green's formula indeed

$$\Delta u_\varepsilon(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}.$$

We now check that Δu_ε is monotone increasing for each $\varepsilon > 0$. Indeed, using again polar coordinates as before,

$$\begin{aligned} (\Delta u_\varepsilon)(r) &= -\frac{1}{4\pi}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{s=0}^{+\infty}\frac{\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}s^2}{\sqrt{r^2-2rs\cos\theta+s^2}}\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= -\frac{1}{2r}\int_0^{+\infty}\sqrt{r^2-2rs\cos\theta+s^2}\Big|_{\theta=0}^{\theta=\pi} s\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s \\ &= -\frac{1}{2r}\int_0^{+\infty}\left[(r+s)-|r-s|\right]s\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s \\ &= -\frac{1}{r}\int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - \int_r^{+\infty}s\,\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s, \end{aligned}$$

one sees that the derivative in r is positive:

$$\begin{aligned} (\Delta u_\varepsilon)'(r) &= \frac{1}{r^2}\int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s - r\,\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} + r\,\mathrm{e}^{-\varepsilon r^2}\mathrm{e}^{u_\varepsilon(r)} \\ &= \frac{1}{r^2}\int_0^r s^2\mathrm{e}^{-\varepsilon s^2}\mathrm{e}^{u_\varepsilon(s)}\,\mathrm{d}s > 0. \end{aligned}$$

Now, by monotonicity of Δu_ε and because $\Delta u_\varepsilon < 0$ and (3) hold, we have $\|\Delta u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \le C < \infty$. Therefore u_ε goes to some radial u in $C^4_{\mathrm{loc}}(\mathbb{R}^3)$, because of elliptic estimates.

At this point it suffices to check that there exists some $\delta > 0$, independent on ε, such that $u_\varepsilon(x) \le \delta(1 - |x|)$ for all $\varepsilon > 0$. Indeed, that shows that the limit grows at most linearly and that

$$u(x) = -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y| \mathrm{e}^{u(y)}\,\mathrm{d}y.$$

Indeed,

$$\left| |x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\right| \le |x-y|\mathrm{e}^{\delta(1-|y|)} \in L^1(\mathbb{R}^3),$$

so that by Lebesgue's Dominated Convergence Theorem

$$\begin{aligned} u(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) &= -\frac{1}{8\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y \\ &= -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y. \end{aligned}$$

Let us check then that such a $\delta > 0$ exists. Observe preliminarly that $|u_\varepsilon(x)| \le \|u_\varepsilon\|(1+|x|)$ and $u_\varepsilon(x) < 0$ for all $x \in \mathbb{R}^3$ imply that $u_\varepsilon(x) \ge -\|u_\varepsilon\|(1+|x|)$ for all $x \in \mathbb{R}^3$. Therefore

$$\begin{aligned} -\|u_\varepsilon\|(1+|x|) \le u_\varepsilon(x) &= -\frac{1}{8\pi}\int_{\mathbb{R}^3} |x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y \\ &\le -\frac{1}{8\pi}\int_{|y|<1} |x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{u_\varepsilon(y)}\,\mathrm{d}y \\ &\le -\frac{1}{8\pi}\int_{|y|<1} |x-y|\mathrm{e}^{-\varepsilon|y|^2}\mathrm{e}^{-\|u_\varepsilon\|(1+|y|)}\,\mathrm{d}y \\ &\le -\frac{1}{8\pi}\left(\int_{|y|<1} |x-y|\,\mathrm{d}y\right)\mathrm{e}^{-2\|u_\varepsilon\|-1}. \end{aligned}$$

Now, if for the sake of contradiction we suppose that $\|u_\varepsilon\|$ goes to zero, on the left hand side we would have something going pointwise to zero, while on the right hand side we would have something going poinwise to some strictly negative function of x, which is a contradiction. Hence it is true that there exists some $C > 0$ such that $\|u_\varepsilon\| \le C$ for all $\varepsilon > 0$. Thus

$$u_\varepsilon(x) \le -\frac{C}{8\pi}\int_{|y|<1} |x-y|\,\mathrm{d}y \le \delta(1-|x|),$$

for some $\delta > 0$. This completes the proof. □

To sum up, we have just showed the existence of a solution u_0 to the Liouville equation in $\mathbb{R}^4$ which is radial with linear decay in the first three coordinates and does not depend on the last one.

Remark. Observe that, if $u_1(x)$ is a solution of $\Delta^2 u = \mathrm{e}^u$, then

$$u_\mu(x) = u_1(\mu x) + 4\log\mu$$

is a solution as well. Therefore, actually, we have shown the existence of a whole family of trivial solutions. Once a trivial solution with the aforementioned properties u_1 is fixed, u_λ can be characterized equivalently by its volume $\int_{\mathbb{R}^3} \mathrm{e}^{u_\lambda}$, its value in 0 or its asymptotic behavior.

CHAPTER 3

Non-trivial solution

Now that we have a family of trivial solutions we can start looking at bifurcations. We will restrict our problem to the strip $S_\lambda := \mathbb{R}^3 \times (0, \lambda)$ and find the values of λ for which the solution is not unique. For these values of λ, we will have then non-trivial solutions in the strip S_λ. Extending them to the whole plane $\mathbb{R}^4$ by reflection and using elliptic regularity, this will give a non-trivial solution with infinite volume.

DEFINITION. We say that λ^* is a *bifurcation point* for F (from the trivial solution) if there is a sequence of solutions $(\lambda_n, u_n)_{n\in\mathbb{N}} \subset \mathbb{R}\times X$, with $u_n \neq 0$ for each $n \in \mathbb{N}$, that converges to $(\lambda^*, 0)$.

Observe that then, in order to prove non-uniqueness, it suffices to find a bifurcation point λ^*. To this end, we will use Krasnosel'skii's Index Theorem:

THEOREM 3.1. [**25**, Theorem 56.2] *Let A be a completely continuous operator and assume that λ^* is a point where the index of $u-A(\lambda, u)$ changes. Then λ^* is a bifurcation point for equation $u = A(\lambda, u)$.*

To start, we need to fix the spaces of functions we are working in. Denote $u(x_1, x_2, x_3, x_4) = u(x, x_4)$ (i.e., $x = (x_1, x_2, x_3)$) and define

$$X_\lambda := \left\{ u \in C^{4,\alpha}(S_\lambda) \;\middle|\; \begin{array}{l} \frac{\partial}{\partial x_4} u(x,0) = \frac{\partial}{\partial x_4} u(x,\lambda) = 0 \quad \forall x \in \mathbb{R}^3, \\ u \text{ radial in } x, \\ \left|\langle x\rangle^{\frac{1}{2}} u\right|_{4,\alpha,S_\lambda} + \left|\langle x\rangle^{\frac{5}{2}} \Delta u\right|_{2,\alpha,S_\lambda} \\ \qquad + \left|\langle x\rangle^{\frac{9}{2}} \Delta^2 u\right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right\},$$

where $\langle x\rangle := \sqrt{1+x^2}$. Define also

$$Y_\lambda := \left\{ f \in C^{0,\alpha}(S_\lambda) \;\middle|\; \begin{array}{l} \frac{\partial}{\partial y} f(x,0) = \frac{\partial}{\partial y} f(x,\lambda) = 0 \quad \forall x \in \mathbb{R}^3, \\ f \text{ radial in } x, \\ \left|\langle x\rangle^{\frac{9}{2}} f\right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right\}.$$

Recalling that the interior Hölder spaces are Banach spaces ([**18**, Problem 5.2]), it can be checked that both X_λ and Y_λ are Banach spaces when endowed, respectively, with the norms

$$\|u\|_{X_\lambda} := \left|\langle x\rangle^{\frac{1}{2}} u\right|_{4,\alpha,S_\lambda} + \left|\langle x\rangle^{\frac{5}{2}} \Delta u\right|_{2,\alpha,S_\lambda} + \left|\langle x\rangle^{\frac{9}{2}} \Delta u\right|_{0,\alpha,S_\lambda}$$

(the only point here is to show that $\Delta u_n \to f = \Delta u$ and $\Delta^2 u_n \to g = \Delta^2 u$, but this true because u_n converges in C^4) and

$$\|f\|_{Y_\lambda} := \left|\langle x\rangle^{\frac{9}{2}} f\right|_{0,\alpha,S_\lambda}.$$

Observe moreover that the functions in X_λ grow at most like $|x|^{-\frac{1}{2}}$, while those in Y_λ grow at most like $|x|^{-\frac{9}{2}}$, and thus $X_\lambda, Y_\lambda \subset L^2(S_\lambda)$. Notice also that perturbing the trivial solution with functions in X_λ preserves the growth at infinity[2].

Our problem is then finding zeros of the following functional:

$$\begin{aligned} F_\lambda : X_\lambda &\longrightarrow Y_\lambda \\ u &\longmapsto \Delta^2(u_0+u) - \mathrm{e}^{u_0+u} = \Delta^2 u - \mathrm{e}^{u_0}(\mathrm{e}^u - 1). \end{aligned}$$

At this point one should notice that the equation we get in this way is not in the form of Krasnosel'skii's Theorem 3.1. Indeed, the operator is not in the required form $I - K$, with K compact. Nonetheless, one can overcome this hurdle as follows. Suppose that we can invert the operator $\Delta^2 : X_\lambda \to Y_\lambda$. Then, instead of

$$\Delta^2 u - \mathrm{e}^{u_0}(\mathrm{e}^u - 1) = 0, \tag{4}$$

one could consider the equation

$$u - \Delta^{-2}(\mathrm{e}^{u_0}(\mathrm{e}^u - 1)) = 0. \tag{5}$$

Notice that u is a solution of the original equation if and only if it is a solution of this second equation (because we are assuming that Δ^2 is invertible). Hence, instead of solving (4), we will deal with (5). The advantage obtained in this way is that now one can show that (5) is in the form required, namely $\Delta^{-2} \circ F_\lambda : X_\lambda \to X_\lambda$ is a compact perturbation of the identity.

1. Invertibility of Laplacian and compactness

The first thing we need to prove is that $\Delta^{-2} \circ F_\lambda : X_\lambda \to X_\lambda$ is well defined.

LEMMA 3.2. *$\Delta^2 : X_\lambda \to Y_\lambda$ is invertible. Consequently, $\Delta^{-2} \circ L : X_\lambda \to X_\lambda$ is well defined.*

PROOF. Let

$$Z_\lambda := \left\{ w \in C^{2,\alpha}(S_\lambda) \;\middle|\; \begin{array}{l} \dfrac{\partial}{\partial y} w(x,0) = \dfrac{\partial}{\partial y} w(x,\lambda) = 0 \quad \forall x \in \mathbb{R}^3, \\ w \text{ radial in } x, \\ \left|\langle x\rangle^{\frac{9}{2}} w\right|_{2,\alpha,S_\lambda} + \left|\langle x\rangle^{\frac{5}{2}} \Delta w\right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right\}.$$

Notice that Δ maps X_λ to Z_λ and maps Z_λ to Y_λ, by construction. Therefore, it will be enough to prove that (with a slight abuse of notation) both $\Delta : X_\lambda \to Z_\lambda$ and $\Delta : Z_\lambda \to Y_\lambda$ are invertible.

Similarly to X_λ and Y_λ, Z_λ is a Banach space contained in $L^2(S_\lambda)$. Consider first $\Delta : X_\lambda \to Z_\lambda$: it is a linear self-adjoint operator (with respect

[2]Here the essential hypothesis is that the functions go to zero at infinity. Therefore, the choice of the power $-\frac{1}{2}$ is quite arbitrary and could be replaced by any power $\varepsilon < 0$.

to the L^2 product), so that it suffices to show that $\ker\Delta \subseteq X_\lambda$ is trivial. Indeed, this will imply that Δ is injective and surjective, because the operator is self-adjoint and thus both the image and the cokernel are contained in the domain, by elliptic regularity. But then Δ will be a bijective map between the Banach spaces X_λ and Z_λ which is continuous by construction, so that it will have a continuous inverse by the Open Mapping Theorem [**6**, Corollary 2.7].

To begin with, suppose that $u(x, x_4) = u_k(x)\cos\left(\frac{k\pi}{\lambda}x_4\right) \in \ker\Delta$, where x stands for (x_1, x_2, x_3). Write the Laplacian as $\Delta = \Delta_x + \frac{\partial^2}{\partial x_4^2}$, where Δ_x is the Laplacian in the first three coordinates only. Then one gets

$$\begin{aligned} 0 &= \Delta\left(u_k(x)\cos\left(\frac{k\pi}{\lambda}x_4\right)\right) \\ &= \left(\Delta_x u_k(x) - 2\frac{k^2\pi^2}{\lambda^2}u_k(x)\right)\cos\left(\frac{k\pi}{\lambda}x_4\right), \end{aligned}$$

so that $u \in \ker\Delta$ if and only if $\left(\Delta_x - \frac{k^2\pi^2}{\lambda^2}\right)u_k(x) = 0$. Now, if $k = 0$, then the equation becomes $\Delta_x u_0(x) = 0$. Recalling that $u_0(x)$ is radial, the problem reduces to solving

$$w''(r) + \frac{2}{r}w'(r) = 0, \qquad r > 0,$$

where $u_0(x) = w(|x|)$. Solutions of this equation are functions of the form $w(r) = a + \frac{b}{r}$, so that the only way $u_0(x) = w(|x|)$ satisfies the growth requirements at infinity and continuity at the origin is to have $a = b = 0$, i.e $u_0 = 0$.

Let now $k \neq 0$, so that we have the equation $\Delta_x u_k(x) = \frac{k^2\pi^2}{\lambda^2}u_k(x)$. Suppose that u_k attains a maximum at $x = x_M \in \mathbb{R}^3$, then $\Delta_x u_k(x_M) \leq 0$ implies that $u_k(x) \leq u_k(x_M) \leq 0$ for all $x \in \mathbb{R}^3$. If instead u_k has no interior maximum, then it must be at infinity, and thus $u_k(x) \leq 0$ for all $x \in \mathbb{R}^3$, because according to our choice of X_λ we have $u_k(x) \to 0$ as $|x| \to +\infty$. Similar considerations with the minimum of u_k lead to $u_k(x) \geq 0$ for all $x \in \mathbb{R}^3$. But then $u_k = 0$ also for $k \neq 0$.

Given now any $u(x, x_4)$ in the kernel of the Laplacian, the previous argument shows then that all coefficients of its Fourier expansion in x_4 are forced to be zero. Therefore, $u = 0$ and thus $\ker\Delta = 0$, proving that $\Delta : X_\lambda \to Z_\lambda$ is invertible with continuous inverse.

The same argument proves that $\Delta : Z_\lambda \to Y_\lambda$ is invertible with continuous inverse, so that $\Delta^2 = \Delta \circ \Delta : X_\lambda \to Y_\lambda$ is invertible with continuous inverse. ☐

As for compactness, notice that the operator $\Delta^{-2} \circ F_\lambda$ already has the form $I - K$, with $K(u) = \Delta^{-2}(\mathrm{e}^{u_0}(\mathrm{e}^u - 1))$.

Lemma 3.3. *$K : X_\lambda \to X_\lambda$ is a compact operator.*

Proof. Using the same notation of the previous Lemma, we have $\Delta^{-2}f = G * f$. Here, the important property of G is that it grows slower than the exponential as $|x| \to +\infty$ (see again [**13**]). Then the result follows from repeated application of Arzelà-Ascoli's Theorem.

Let indeed $\{u_k\}_k$ be a bounded sequence in X_λ, i.e. $\|u_k\|_{X_\lambda} \leq C$ for all $k \in \mathbb{N}$ for some $C < +\infty$ independent on k. Then, for any $x \in S_\lambda$,

$$|K(u_k)(x)| = \left| \int_{S_\lambda} G(x-y) \mathrm{e}^{u_0(y)} (\mathrm{e}^{u(y)} - 1) \,\mathrm{d}y \right| \leq C_x,$$

because $\mathrm{e}^{u_0(y)}$ goes to zero as $\mathrm{e}^{-|y|}$ for $|y| \to +\infty$ (so, it goes to zero much faster than how all other terms go to infinity), making the integral converge for all $x \in S_\lambda$. Therefore, $\{K(u_k)\}_k$ is uniformly bounded. Similarly, for all $x \in S_\lambda$, one has

$$|\nabla K(u_k)(x)| \leq \int_{S_\lambda} |\nabla G(x-y)| \mathrm{e}^{u_0(y)} \left| \mathrm{e}^{u(y)} - 1 \right| \mathrm{d}y \leq C'_x,$$

again because $\mathrm{e}^{u_0(y)}$ goes to zero as $\mathrm{e}^{-|y|}$ and all the other terms do not grow exponentially to infinity. Hence, $\{K(u_k)\}_k$ is also equicontinuous and thus, by Arzelà-Ascoli, it converges up to subsequences in the C^0 norm.

The same argument applied to the derivatives of $K(u_k)$ gives convergence in all C^j norms (because we can make the derivative fall on G and apply the same argument as before), and thus in particular in the $C^{4,\alpha}$ norm. Moreover, we have already seen that all the $K(u_k)$'s satisfy the growth requirements, so that $\{K(u_k)\}_k$ has a converging subsequence in the X_λ norm, proving compactness of K. □

Remark. The exact same argument proves also that the linearization $\Delta^{-2} \circ L$ has the form $I - K$ with K a compact operator.

2. Change of index and bifurcation

Now that the setup of our problem is complete, we can move to actually proving bifurcation. As explained at the beginning of this chapter, we plan to use Krasnosel'skii's Bifurcation Theorem 3.1, so that we need to prove that the index of the operator $\Delta^{-2} \circ F_\lambda$ changes for some λ, i.e. that the dimension of the negative space of its linearization $\Delta^{-2} \circ L_\lambda$ changes for some λ.

Lemma 3.4. *The linearized operator*

$$L_\lambda[v] := \Delta^2 v - e^{u_0} v$$

admits an eigenfunction not depending on x_4 and with negative eigenvalue.

Proof. Observe that L_λ can be restricted to an operator

$$\tilde{L}_\lambda := L_\lambda|_{X_\lambda^0} : X_\lambda^0 \to Y_\lambda,$$

where X_λ^0 is the subset of X_λ of functions not depending on x_4. The functions in X_λ^0 are then actually functions of $\mathbb{R}^3$, so in the rest of this proof we will just drop the dependence on x_4.

We construct a function $v : \mathbb{R}^3 \to \mathbb{R}$ that is compactly supported, radial and such that $\langle \tilde{L}_\lambda v, v \rangle_{L^2} < 0$.

Define

$$f(r) := \begin{cases} 0 & \text{if } r \le 1 \\ \mathrm{e}^{-\frac{1}{(r-1)^2}}\,\mathrm{e}^{-\frac{1}{(r-2)^2}} & \text{if } 1 < r < 2 \\ 0 & \text{if } r \ge 2 \end{cases}$$

and take

$$v(x_1, x_2, x_3) := \frac{1}{A}\int_{\sqrt{x_1^2+x_2^2+x_3^2}}^{+\infty} f(s)\,\mathrm{d}s,$$

with

$$A := \int_0^{+\infty} f(s)\,\mathrm{d}s.$$

Notice that v is bounded with compact support, so that it belongs to the space of functions X_λ^0. Observe also that v is constantly equal to 1 if $r := \sqrt{x_1^2+x_2^2+x_3^2} < 1$ and is identically 0 outside the ball $B^3(0,2)$, so that its Laplacian is different from zero only in the annulus $1 \le r \le 2$. Therefore, setting

$$V\left(\sqrt{x_1^2+x_2^2+x_3^2}\right) := v(x_1,x_2,x_3),$$

one gets

$$\int_{S_\lambda} (\Delta v)^2\,\mathrm{d}x = \lambda \int_1^2 \left(V''(r) + \frac{2}{r}V'(r)\right)^2 4\pi r^2\,\mathrm{d}r = C < +\infty.$$

Fix now a trivial solution u_1, as found in Chapter 2. Recall that we thus have the family $\{u_\mu\}_\mu$ of trivial solutions (as the functions in this family do not depend on x_4, they be tought as a functions on $\mathbb{R}^3$). We will then show that we can choose $u_0 \in \{u_\mu\}_\mu$ so that

$$\int_{S_\lambda} \mathrm{e}^{u_0(x)}[v(x)]^2\,\mathrm{d}x$$

is sufficiently large. In fact

$$\begin{aligned}\int_{S_\lambda} \mathrm{e}^{u_\mu(x)}[v(x)]^2\,\mathrm{d}x &= \lambda \int_{\mathbb{R}^3} \mu^4 \mathrm{e}^{u_1(\mu x)}\,\mathrm{d}x \\ &= \lambda \int_{\mathbb{R}^3} \mu^4 \mathrm{e}^{u_1(y)} v^2\left(\frac{y}{\mu}\right)\frac{\mathrm{d}y}{\mu^3} \ge \lambda\mu \int_{|y|\le\mu} \mathrm{e}^{u_1(y)}\, \underbrace{v^2\left(\frac{y}{\mu}\right)}_{1}\,\mathrm{d}y \\ &= \lambda\mu \int_{|y|\le\mu} \mathrm{e}^{u_1(y)}\,\mathrm{d}y \underset{\mu\to+\infty}{\longrightarrow} +\infty\end{aligned}$$

Summing up, if we fix $u_0 := u_\mu$ with μ sufficiently large, then $\langle \tilde{L}_\lambda v, v\rangle_{L^2} < 0$, with v the function defined before.

Now, as $X_\lambda^0 \subset L^2(\mathbb{R}^3)$, we get that $\tilde{L}_\lambda$ is self-adjoint and semibounded (in the L^2 norm):

$$\begin{aligned}\langle \tilde{L}_\lambda v, v\rangle_{L^2} &= \int_{\mathbb{R}^3} \left[\left(\Delta^2 v(x)\right) v(x) - \mathrm{e}^{u_0(x)} v^2(x)\right]\mathrm{d}x \\ &= \int_{\mathbb{R}^3} \left[(\Delta v(x))^2 - \mathrm{e}^{u_0(x)} v^2(x)\right]\mathrm{d}x \\ &\ge -\int_{\mathbb{R}^3} \mathrm{e}^{u_0(x)} v^2(x)\,\mathrm{d}x \ge -\mathrm{e}^{u_0(0)}\|v\|_{L^2(\mathbb{R}^3)}\,.\end{aligned}$$

Hence we have that the lowest eigenvalue for $\tilde{L}_\lambda$ is

$$\nu_0 = \min_{u \neq 0} \frac{\langle \tilde{L}_\lambda u, u\rangle_{L^2}}{\|u\|_{L^2}^2} \leq \frac{\langle \tilde{L}_\lambda v, v\rangle_{L^2}}{\|v\|_{L^2}^2} < 0$$

(see for example [**21**, Theorem 11.4]). Notice that ν_0 is an eigenvalue of $L_\lambda : X_\lambda \to Y_\lambda$ as well (possibly not the first one) because $X_\lambda^0 \subset X_\lambda$ and that a corresponding eigenfunction $v_0 \in X_\lambda^0$ is an eigenfunction of L_λ (extending it trivially in x_4). Note that v_0 does not depend on x_4 and $\nu_0 < 0$, as required. □

This negative eigenfunction v_0 for L_λ (not depending on x_4) is negative also for $\Delta^{-2} \circ L_\lambda$ (even if not necessarily an eigefunction):

$$\langle (\Delta^{-2} \circ L_\lambda) w_0, w_0\rangle_{L^2} = \mu_0 \langle \Delta^{-2} w_0, w_0\rangle_{L^2} = \mu_0 \left\| \Delta^{-1} w_0 \right\|_{L^2}^2 < 0,$$

where the last equality comes from self-adjointness of Δ^{-1}, while the strict inequality comes from the fact that, if $\left\|\Delta^{-1} w_0\right\|_{L^2} = 0$, then $\|w_0\|_{L^2} = 0$, a contradiction.

Remark. As this function v_0 does not depend on the last variable x_4, it is independent on the choice of λ. Therefore, in what follows, even if the family of functions that we construct depends on λ, the way we construct it does not. For ease of notation, we will not explicitly indicate the dependence of the family of functions on λ.

We now want to prove that we have a family of linearly independent functions in the negative space of L_λ that gets bigger as λ get larger. For a fixed λ, consider the functions

$$v_k(x_1, x_2, x_3, x_4) := v_0(x_1, x_2, x_3) \cos\left(\frac{2\pi k}{\lambda} x_4\right).$$

Observe that the v_k's are orthogonal and satisfy Neumann conditions on ∂S_λ, so that they belong to X_λ.

Lemma 3.5. *The number of values of k for which $\langle (\Delta^{-2} \circ L_\lambda) v_k, v_k\rangle_{L^2} < 0$ is finite for all $\lambda > 0$ and goes to infinity as $\lambda \to +\infty$.*

Proof. Recall first that we already know that $\langle (\Delta^{-2} \circ L_\lambda) v_0, v_0\rangle_{L^2} < 0$. We also have

$$\begin{aligned} \langle (\Delta^{-2} \circ L_\lambda) v_k, v_k\rangle_{L^2} &= \langle v_k - \Delta^{-2}(\mathrm{e}^{u_0} v_k), v_k\rangle_{L^2} \\ &= \|v_k\|_{L^2}^2 - \langle \Delta^{-2}(\mathrm{e}^{u_0} v_k), v_k\rangle_{L^2} \\ &= \|v_0\|_{L^2}^2 - \langle \mathrm{e}^{u_0} v_k, \Delta^{-2} v_k\rangle_{L^2}, \end{aligned} \tag{6}$$

where $\|v_k\|_{L^2} = \|v_0\|_{L^2}$ because $\int_0^\lambda \cos^2\left(\frac{2\pi k}{\lambda} x_4\right) \mathrm{d}x_4 = 1$.

We now have to compute $\Delta^{-2} v_k$ explicitly. Set $w = \Delta^{-2} v_k$, i.e. w is the solution of $\Delta^2 w = v_k$. Decomposing w into its Fourier modes in the x_4 variable we get

$$w(x, x_4) = \sum_{n=0}^{+\infty} w_k(x) \cos\left(\frac{2\pi k}{\lambda} x_4\right)$$

and thus

$$v_0(x)\cos\left(\frac{2\pi k}{\lambda}x_4\right) = \sum_{n=0}^{+\infty}\left[\left(\Delta_x - \frac{2\pi n}{\lambda}\right)^2 w_k(x)\right]\cos\left(\frac{2\pi n}{\lambda}x_4\right).$$

For the same maximum principle argument of the previous section, the operators $\left(\Delta_x - \frac{2\pi n}{\lambda}\right)^2$ are invertible, and so we get

$$\Delta^{-2}v_k(x,x_4) = \left[\left(\Delta_x - \frac{2\pi k}{\lambda}\right)^{-2} v_0(x)\right]\cos\left(\frac{2\pi k}{\lambda}x_4\right).$$

Plugging this into (6), we then obtain

$$\langle(\Delta^{-2}\circ L_\lambda)v_k, v_k\rangle_{L^2} = \|v_0\|_{L^2}^2 - \left\langle \mathrm{e}^{u_0}v_0, \left(\Delta_x - \frac{2\pi k}{\lambda}\right)^{-2} v_0\right\rangle_{L^2}.$$

Consider now the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ defined as

$$f(t) := \|v_0\|_{L^2}^2 - \left\langle \mathrm{e}^{u_0}v_0, (\Delta_x - t)^{-2}v_0\right\rangle_{L^2}.$$

This is a continuous map, because the map that takes an invertible operator to its inverse is continuous in the operator norm [**2**, Prop. 2.1.1]. By construction

$$f(0) = \|v_0\|_{L^2}^2 - \left\langle \mathrm{e}^{u_0}v_0, \Delta^{-2}v_0\right\rangle_{L^2} = \langle(\Delta^{-2}\circ L_\lambda)v_0, v_0\rangle_{L^2} < 0,$$

so that $f(t) < 0$ for all $t < \varepsilon$ for some $\varepsilon > 0$. Notice that the number of k's for which $t = \frac{2\pi k}{\lambda} < \varepsilon$ grows with λ. Therefore, to conclude that the number of k's such that $\langle(\Delta^{-2}\circ L_\lambda)v_k, v_k\rangle_{L^2} < 0$ grows with λ, it suffices to prove that for all fixed $\lambda > 0$ there are only finitely many k's for which $\langle(\Delta^{-2}\circ L_\lambda)v_k, v_k\rangle_{L^2} < 0$. This is equivalent to say that there exists $M > 0$ such that $f(t) > 0$ for all $t > M$. Specifically, we are going to prove that $f(t) \to \|v_0\|^2 > 0$ as $t \to +\infty$.

Fix indeed $u \in X_\lambda$ independent on x_4, then $\left\|(\Delta_x - t)^2 u\right\|_{C^0} \sim |u|_{C^0}t^2$ as $t \to +\infty$ (recall that $|\Delta_x^2 u|_{C^0}, |\Delta_x u|_{C^0} < +\infty$ are fixed). Then

$$\left\|(\Delta_x - t)^{-2}v_0\right\|_{C^0} \to 0, \quad t \to +\infty,$$

decreasing like t^{-2}. Recalling that $\mathrm{e}^{u_0}v_0 \in L^1$, we conclude then that

$$\left|\left\langle \mathrm{e}^{u_0}v_0, (\Delta_x - t)^{-2}v_0\right\rangle_{L^2}\right| \le \|\mathrm{e}^{u_0}v_0\|_{L^1}\left\|(\Delta_x - t)^{-2}v_0\right\|_{C^0} \to 0$$

as $t \to +\infty$. Thus $f(t) \to \|v_0\|_{L^2}^2$ when $t \to +\infty$, as wanted. □

LEMMA 3.6. *The dimension of the negative space of* $\Delta^{-2}\circ L_\lambda$ *grows as* λ *grows.*

PROOF. Fix λ and suppose that $v_0, \dots, v_r$ are the negative functions for L_λ of Lemma 3.5.

Notice first that, if v_i, v_j are negative for $\Delta^{-2}\circ L_\lambda$, then any their linear combination $\alpha v_i + \beta v_j$ is negative as well. Indeed: if $i = j$ this is trivial, while if $i \neq j$ then

$$\begin{aligned}\langle \Delta^{-2} \circ L_\lambda(\alpha v_i + \beta v_j), \alpha v_i + \beta v_j \rangle_{L^2} \\ = \alpha^2 \langle (\Delta^{-2} \circ L_\lambda) v_i, v_i \rangle_{L^2} + \beta^2 \langle (\Delta^{-2} \circ L_\lambda) v_j, v_j \rangle_{L^2} \\ + \alpha\beta \langle (\Delta^{-2} \circ L_\lambda) v_i, v_j \rangle_{L^2} + \alpha\beta \langle (\Delta^{-2} \circ L_\lambda) v_j, v_i \rangle_{L^2}\end{aligned}$$

and

$$\begin{aligned}&\langle (\Delta^{-2} \circ L_\lambda) v_i, v_j \rangle_{L^2} \\ &= \langle v_i, v_j \rangle_{L^2} \\ &\quad - \left\langle \mathrm{e}^{u_0} v_0(x) \cos\left(\frac{2\pi i}{\lambda} x_4\right), \left[\left(\Delta_x - \frac{2\pi j}{\lambda}\right)^{-2} v_0(x)\right] \cos\left(\frac{2\pi j}{\lambda}\right) \right\rangle_{L^2} = 0,\end{aligned}$$

because $\int_0^\lambda \cos\left(\frac{2\pi i}{\lambda} x_4\right) \cos\left(\frac{2\pi j}{\lambda} x_4\right) \mathrm{d}x_4 = 0$ for $i \neq j$, so that

$$\begin{aligned}\langle \Delta^{-2} \circ L_\lambda(\alpha v_i + \beta v_j), \alpha v_i + \beta v_j \rangle_{L^2} \\ = \alpha^2 \langle (\Delta^{-2} \circ L_\lambda) v_i, v_i \rangle_{L^2} + \beta^2 \langle (\Delta^{-2} \circ L_\lambda) v_j, v_j \rangle_{L^2} < 0.\end{aligned}$$

Then, by Rayleigh Min-Max principle [**16**, Theorem 12.1] the first r eigenvalues of L_λ are negative: for $n \leq r$ we have

$$\begin{aligned}\nu_n &= \min_{\varphi_1, \dots, \varphi_n} \max\{\langle L_\lambda \varphi, \varphi \rangle_{L^2} \mid \varphi \in \operatorname{span}(\varphi_1, \dots, \varphi_n),\ \|\varphi\|_{L^2} = 1\} \\ &\leq \max\{\langle L_\lambda \varphi, \varphi \rangle_{L^2} \mid \varphi \in \operatorname{span}(v_1, \dots, v_n),\ \|\varphi\|_{L^2} = 1\} < 0.\end{aligned}$$

As for each of this eigenvalues ν_k (counted with multiplicity) there is an eigenfunction linearly independent to the previous ones, we conclude that the dimension of the negative space of $\Delta^{-2} \circ L_\lambda$ must grow with λ. Indeed, Lemma 3.5 tells us that the number of elements in the negative family of functions grows with λ, and thus, for bigger λ, Rayleigh Min-Max principle gives us more negative eigenvalues. □

Lemma 3.6, along with the fact that $\Delta^{-2} \circ L = I - K$ (where I is the identity and K is a compact operator), shows that there must be some value λ^* of the parameter λ for which the number of negative eigenvalues changes. Indeed, notice that f is an eigenfunction of $-K$ with eigenvalue μ if and only if f is an eigenfunction of $I - K$ with eigenvalue $1 + \mu$. Recalling that the spectrum of the compact operator $-K$ is bounded and accumulating at most in 0 [**21**, Theorem 6.16], we then get that the spectrum of $I - K$ is bounded and accumulating at most in 1. In both cases, this implies that there are only finitely many negative eigenvalues, for each fixed value of λ. But, according to Lemma 3.6, the number of negative eigenvalues goes to infinity as $\lambda \to +\infty$. Consequently, there must be some λ^* for which the number of negative eigenvalues changes. But then at λ^* the dimension of the space of negative eigenfunctions grows, so that λ^* is a point of changing index for the operator $\Delta^{-2} \circ F_\lambda$.

Now we can finally apply Krasnosel'skii's Bifurcation Theorem 3.1 to the equation $\Delta^{-2} \circ F_\lambda(u) = 0$. Indeed, we proved that $\Delta^{-2} \circ F_\lambda$ has the form $I - K$ with K compact, and that λ^* is a point of changing index. Hence, λ^* is a point of bifurcation for $\Delta^{-2} \circ F_\lambda$. This means that there is a non-trivial solution of $\Delta^{-2} \circ F_\lambda(u) = 0$ for some λ which is a perturbation of u_0, i.e. a zero of F_λ on S_λ. We can then extend u to the whole $\mathbb{R}^4$ by reflecting it along the boundaries of the strip and repeating this procedure. Weighted elliptic regularity then implies that this is still a solution of the Liouville equation (apply twice the results found in [**1**, Chapter 3]). As this solution is periodic and non-zero, it must then have infinite volume $\int_{\mathbb{R}^4} \mathrm{e}^u = +\infty$. We have then proved the following result.

THEOREM 3.7. *The Liouville equation $\Delta^2 u = e^u$ in $\mathbb{R}^4$ admits a non-trivial solution $u \in C^{4,\alpha}(\mathbb{R}^4)$ with infinite volume.*

3. Conclusions and further research

This work shows the existence of non-trivial solutions of the Liouville equation in $\mathbb{R}^4$ with infinite volume. As already remarked, though, this is far from being the only way of finding such solutions. For instance, one could try to follow the same procedure with a trivial solution in $\mathbb{R}^2$ and two parameters, but even this does not exhaust all possibilities. Another option, indeed, might be to bifurcate radially. Moreover, similarly to what appens for Delaunay surfaces [**15**] (whose equations are somehow analogous to the ones we are considering), we might be able to connect the non-trivial solutions of Theorem 3.7 to the spherical solution by some global bifurcation theorem, maybe following the ideas of [**14**].

Another aspect worth consideration is, of course, going to higher dimensions. As the methods emplyed here are quite general, it seems reasonable to believe that the same argument we followed can be quite easily generalized to any even dimension.

Finally, other lines of research aimed at finding more general non-trivial solutions with infinite volume are possible. For instance, we might be able to "glue" the oscillating solutions obtained from the bifurcation into more complex solutions, in a similar manner to the one used to construct Delaunay k-noids starting from Delaunay unduloids and nodoids. In this way we could then obtain non-trivial solutions with infinite volume that are not a direct result of a bifurcation from a cylindrical solution.

APPENDIX A

Q-curvature

In this appendix we will briefly explain how Liouville equations arise in Differential Geometry. In 1985 Thomas P. Branson introduced the concept of *Q-curvature* [**3**], a quantity that turned out to be very important in many contexts and that can be regarded as a generalization of the Gaussian curvature. For example, Q-curvature appears naturally when studying the *functional determinant* of conformally covariant operators[3], which plays an essential role both in Functional Analysis and in Theoretical Physics. Indeed, for example on a four-manifold, given a conformally covariant operator A_g (like the conformal Laplacian or the Paneitz operator [**30**]) and a conformal factor w, one has

$$\log \frac{\det A_{\hat{g}}}{\det A_g} = \gamma_1(A)F_1[w] + \gamma_2(A)F_2[w] + \gamma_3(A)F_3[w],$$

where $\gamma_1(A)$, $\gamma_2(A)$ and $\gamma_3(A)$ are real numbers (see [**5**]). In particular, $\hat{g} = \mathrm{e}^{2w}g$ is a critical point of F_2 if and only if the Q-curvature corresponding to $\hat{g}$ is constant (see [**20**] and the references therein).

Q-curvature appears also as the 0-th order term of the GJMS-operator in the ambient metric construction [**17**] and can be related to the Poincaré metric in one higher dimension via an "holographic formula" [**19**]. GJMS-operators, in turn, play an important role in Physics, as their definition extends to Lorentzian manifolds: they are generalizations of the Yamabe operator and the conformally covariant powers of the wave operator on Minkowski space [**24**]. Moreover, the integral of the Q-curvature satisfies the so-called Chern-Gauss-Bonnet formula [**24**], which links the integral of some function of the Q-curvature to the Euler characteristic of the manifold (as the Gauss-Bonnet formula did with the Gaussian curvature). In $\mathbb{R}^4$, that equation can tell us whether a metric is normal and, in that case, is strictly related to the behavior of the isoperimetric ratios [**10**].

In what follows, we will present only the 2 and 4-dimensional cases. A generic definition of Q-curvature can be found in [**4**] and explicit formulas

[3]Given an operator A with spectrum $\{\lambda_j\}_j$, one can formally define its *determinant* as $\prod_j \lambda_j$. This is divergent, in general, so one should perform some sort of "regularization" of the definition. Define then the *Zeta function* as

$$\zeta(s) := \sum_j \lambda_j^{-s} = \sum_j \mathrm{e}^{-s\log\lambda_j}.$$

One can show by means of Weyl's asymptotic law (see for example [**32**, Chapter 11]) that this defines an analytic function for $\Re(s) > n/2$ if A is the Laplace-Beltrami opeator. Moreover, one can meromorphically extend ζ so that it becomes regular at $s = 0$ (see [**31**]). Taking the derivative, one has $\zeta'(0) := -\sum_j \log\lambda_j = -\log\det A$, so that $\det A := \exp(-\zeta'(0))$. For more details we refer to [**28**], [**9**], [**20**] and the references therein.

in [**24**]. In dimension 2 the Q-curvature is essentially the usual Gaussian curvature (see [**9**] for a more complete introduction in both 2, 4 and higher dimensions). We just want to point out that in this case, if we conformally rescale the metric, $\hat{g}_{ij} = \mathrm{e}^{2\varphi} g_{ij}$ for some smooth function φ on M, then

$$R_{\hat{g}} = \mathrm{e}^{-2\varphi}(R_g - 2\Delta f),$$

where $R_{\hat{g}}$ and R_g denotes, respectively, the scalar curvatures of $\hat{g}$ and g. Specifically, if g is an Euclidean metric, then we recover the 2D Liouville equation

$$\Delta\varphi(u,v) + 2K\mathrm{e}^{\varphi(u,v)} = 0.$$

In dimension 4 things start to become more interesting.

DEFINITION. Let (M,g) be a 4-dimensional Riemannian manifold. Let Ric_g be its Ricci curvature, R_g its scalar curvature and Δ_g its Laplace-Beltrami operator. The *Q-curvature* of M is defined as

$$Q_g := -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right).$$

Conformally rescaling the metric, $\hat{g}_{ij} = \mathrm{e}^{2\varphi} g_{ij}$ for some smooth function φ on M, then the Q-curvature transforms as follows

$$P_g\varphi + 2Q_g = 2Q_{\hat{g}}\mathrm{e}^{4\varphi} \tag{7}$$

(see for example [**8**, Chapter 4]), where P_g is the *Paneitz operator*

$$P_g\varphi := \Delta_g^2\varphi + \mathrm{div}_g\left(\frac{2}{3}R_g g - 2\mathrm{Ric}_g\right)\mathrm{d}\varphi$$

introduced in 1983 by Stephen M. Paneitz [**30**].

Observe that, if we take $M = \mathbb{R}^4$ and g equal to the standard Euclidean metric and consider $\hat{g}$ conformal to g and such that $Q_{\hat{g}} \equiv \bar{Q} \in \mathbb{R}$, then equation (7) becomes

$$\Delta_g^2\varphi = 2\bar{Q}\mathrm{e}^{4\varphi}.$$

Setting $u := 4\varphi$ and $\bar{Q} = \frac{1}{8}$ and taking into account that the Laplace-Beltrami operator in $\mathbb{R}^4$ endowed with the Euclidean metric is the standard Laplacian, we finally end up with the 4-dimensional Liouville equation

$$\Delta^2 u(x) = \mathrm{e}^{u(x)}, \quad x \in \mathbb{R}^4.$$

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