Solutions of Liouville equations with non-trivial profile in dimensions 2 and 4

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Abstract

We prove the existence of a family of non-trivial solutions of the Liouville equation in dimensions two and four with infinite volume. These solutions are perturbations of a finite-volume solution of the same equation in one dimension less. In particular, they are periodic in one variable and decay linearly to $-\infty$ in the other variables. In dimension two, we also prove that the periods are arbitrarily close to $\pi k, k \in \mathbb{N}$ (from the positive side). The main tool we employ is bifurcation theory in weighted Hölder spaces.

1. Introduction

Liouville equations are a class of elliptic nonlinear partial differential equations of the form

$$(-\Delta)^n \varphi(x) = \mathrm{e}^{\varphi(x)}, \quad x \in \mathbb{R}^{2n}, \tag{1.1}$$

for $n \in \mathbb{N}$.[1] This family of equations plays a fundamental role in many problems of Conformal Geometry and Mathematical Physics, governing the transformation laws for some curvatures. For example, the 2-dimensional equation (which is often called *classical Liouville equation*) provides the structure of metrics with constant Gaussian curvature. Indeed, let g be a metric on a surface M: if we conformally rescale the metric as $\hat{g}_{ij} = \mathrm{e}^{2\varphi} g_{ij}$, for some smooth function φ on M, then

$$R_{\hat{g}} = \mathrm{e}^{-2\varphi}(R_g - 2\Delta f),$$

where $R_{\hat{g}}$ and R_g denote, respectively, the Gaussian curvatures of $\hat{g}$ and g. Specifically, if g is an Euclidean metric, then we obtain the 2D Liouville equation

$$\Delta\varphi(u,v) + 2K\mathrm{e}^{\varphi(u,v)} = 0.$$

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[1]We will deal only with spaces of even dimension. For the odd-dimensional case, which is much more difficult as it involves the fractional Laplacian, we refer to [1].

In Mathematical Physics, Liouville equations appear for example in the description of mean field vorticity in steady flows ([2], [3]), Chern-Simons vortices in superconductivity or Electroweak theory ([4], [5]). Moreover, they also arise naturally when dealing with functional determinants, which play an essential role in modern Quantum Physics and String theory [6]. The 2-dimensional Liouville equation was also taken as an example by David Hilbert in the formulation of the "nineteenth problem" [7].

Liouville equations, therefore, have been extensively studied in the past years, and many authors tried to find their non-trivial solutions. For the classical Liouville equation, for instance, solutions in $\mathbb{R}^2$ with finite *volume* $V := \int \exp(u)$ have been completely classified:

Theorem 1.1 (CHEN - LI [8]). *Every solution of*

$$\begin{cases} \Delta u + \mathrm{e}^u = 0 & in \ \mathbb{R}^2 \\ \int_{\mathbb{R}^2} \mathrm{e}^{u(x)} \, \mathrm{d}x < +\infty \end{cases}$$

is radially symmetric with respect to some point in $\mathbb{R}^2$. Specifically, it assumes the form

$$u(x) = \phi_{\lambda,x^0}(x) = \frac{\log(32\lambda^2)}{(4 + \lambda^2|x - x^0|^2)^2}$$

for some $\lambda > 0$ and $x^0 \in \mathbb{R}^2$.

The infinte-volume case, though, is still quite unexplored. The first main result of this paper is the following.

Theorem 1.2. *The classical Liouville equation $\Delta u + \mathrm{e}^u = 0$ in $\mathbb{R}^2$ admits a class of non-trivial solutions $u \in C^{2,\alpha}(\mathbb{R}^2)$ with infinite volume. These solutions tend symmetrically to $-\infty$ along the x direction and are periodic in the y direction.*

More precisely, for every $k \in \mathbb{N}_{>0}$, we have a one-dimensional family of non-trivial solutions of the form

$$u_{k,\varepsilon}(x, y) = \log(2\operatorname{sech}(x)) + \varepsilon \operatorname{sech}(x) \cos(\pi k y) + O(\varepsilon^2),$$

for small $\varepsilon \in \mathbb{R}$.

Remark. The period appearing in the second part of the previous theorem is actually arbitrary, being the equation scale-invariant (see the end of the proof of Theorem 1.2). The important fact to keep in mind is that we have a collection of one-dimensional families of non-trivial solutions for which the period of the first-order perturbation is given by increasing multiples of some fixed positive number.

As one can tell from the statement, the result relies on the fact that there exists a class of solutions in dimension 1, that can be used to build a "trivial" solution with infinite volume in dimension 2 (i.e. a solution that does not actually depend on all variables). Specifically, the family of solutions in dimension 1 is

$$\log\left[c_1 - c_1 \tanh^2\left(\frac{1}{2}\sqrt{2c_1\left(c_2 + x\right)^2}\right)\right], \quad c_1 > 0, \quad c_2 \in \mathbb{R}.$$

Extending them trivially along y gives the class of *1-dimensional solutions.*
Notice that Theorem 1.2 can actually be generalized to perturbations of any
element in this class simply by changing variables.

After we have a 1-dimensional solution, a bifurcation theorem comes into
play: the *Bifurcation from the simple eigenvalue Theorem* 2.2, in this case.
The main difficulty here is to prove that the linearized operator is Fredholm
of index 0, which is accomplished via some growth arguments (Lemma 3.1)
and weighted elliptic regularity theory (Lemma 3.2). In particular, a weighted
version of Schauder's estimates (Theorem 2.8) is necessary for the proof. We
also find that the bifurcation is supercritical and, as a byproduct, we correct
the formula for non-transcritical bifurcations (Proposition 2.4). Section 3 deals
with the classical Liouville equation.

Remark. In real dimension 2, one might be able to find non-trivial solutions of
(1.1) in terms of meromophic functions. For instance, on a simply connected
domain $\Omega \subseteq \mathbb{C}$, the general solution is given by

$$u(z, \bar{z}) = \log \left(4 \frac{|\partial f(z)/\partial z|}{(1 + K|f(z)|^2)^2} \right),$$

where f is any meromorphic function such that $\frac{\partial f}{\partial z}(z) \neq 0$ for all $z \in \Omega$ and
f has at most simple poles in Ω (see [9] – see also [10] for a classification of
solutions with finite volume in the upper half-plane). Notice though that this
only applies to real dimension 2, and does not extend to higher dimension, which
instead we are able to treat here with the above bifurcation method.

1.1. Higher-dimensional Liouville equations

The role that Liouville equations have in Conformal Geometry does not
hold only for constant Gaussian curvature surfaces. Indeed, more generally, the
generalized Liouville equation (1.1) plays also a fundamental part in constant
Q-curvature manifolds.

In 1985 Thomas P. Branson introduced the concept of *Q*-curvature [11], a
quantity that turned out to be very important in many contexts and that can
be regarded as a generalization of the Gaussian curvature. For example, *Q*-
curvature appears naturally when studying the *functional determinant* of con-
formally covariant operators, which is a fundamental concept of both Functional
Analysis and Theoretical Physics[2]. For example, on a four-manifold, given a

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[2]Given an operator A with spectrum $\{\lambda_j\}_j$, one can formally define its *determinant* as
$\prod_j \lambda_j$. This is divergent, in general, so one should perform some sort of "regularization" of
the definition. Define then the *Zeta function* as

$$\zeta(s) := \sum_j \lambda_j^{-s} = \sum_j \mathrm{e}^{-s \log \lambda_j}.$$

One can show by means of Weyl's asymptotic law (see for example [12, Chapter 11]) that this
defines an analytic function for $\Re(s) > n/2$ if A is the Laplace-Beltrami opeator. Moreover,
one can meromorphically extend ζ so that it becomes regular at $s = 0$ [13]. Taking the

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conformally covariant operator A_g (like the conformal Laplacian or the Paneitz operator [16]) and a conformal factor w, one has

$$\log \frac{\det A_{\hat{g}}}{\det A_g} = \gamma_1(A)F_1[w] + \gamma_2(A)F_2[w] + \gamma_3(A)F_3[w],$$

where $\gamma_1(A)$, $\gamma_2(A)$ and $\gamma_3(A)$ are real numbers [17]. In particular, $\hat{g} = e^{2w}g$ is a critical point of F_2 if and only if the Q-curvature corresponding to $\hat{g}$ is constant (see [18], [15] and the references therein).

Q-curvature appears also as the 0-th order term of the GJMS-operator in the ambient metric construction [19] and can be related to the Poincaré metric in one higher dimension via an "holographic formula" [20]. GJMS-operators, in turn, play an important role in Physics, as their definition extends to Lorentzian manifolds: they are generalizations of the Yamabe operator and the conformally covariant powers of the wave operator on Minkowski's space [21]. Moreover, the integral of the Q-curvature satisfies the so-called Chern-Gauss-Bonnet formula [21], which links the integral of some function of the Q-curvature to the Euler characteristic of the manifold (as the Gauss-Bonnet formula did with the Gaussian curvature). In $\mathbb{R}^4$, the integral of Q-curvature can tell us whether a metric is normal and, in that case, is strictly related to the behavior of the isoperimetric ratios [22].

In what follows, we will deal only with the 2- and 4-dimensional cases. A more complete introduction to 2, 4 and higher dimensional Q-curvature can be found in [14]. See [23] for a generic definition of Q-curvature and [21] for explicit formulas.

In dimension 2 the Q-curvature is the usual Gaussian curvature [14]. As we have already seen, this leads to the classical Lioville equation. In dimension 4, this quantity starts to become more interesting.

Definition. Let (M, g) be a 4-dimensional Riemannian manifold. Let Ric_g be its Ricci curvature, R_g its scalar curvature and Δ_g its Laplace-Beltrami operator. The *Q-curvature* of M is defined as

$$Q_g := -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2\right).$$

Conformally rescaling the metric, $\hat{g}_{ij} = e^{2\varphi}g_{ij}$ for some smooth function φ on M, then the Q-curvature transforms as follows

$$P_g\varphi + 2Q_g = 2Q_{\hat{g}}e^{4\varphi} \tag{1.2}$$

(see for example [24, Chapter 4]), where P_g is the *Paneitz operator* [16]

$$P_g\varphi := \Delta_g^2\varphi + \mathrm{div}_g\left(\frac{2}{3}R_g g - 2\mathrm{Ric}_g\right)d\varphi.$$

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derivative, one has $\zeta'(0) := -\sum_j \log \lambda_j = -\log \det A$, so that $\det A := \exp(-\zeta'(0))$. For more details we refer to [6], [14], [15] and the references therein.

Observe that, if we take $M = \mathbb{R}^4$ and g equal to the standard Euclidean metric and consider $\hat{g}$ conformal to g and such that $Q_{\hat{g}} \equiv \bar{Q} \in \mathbb{R}$, then equation (1.2) becomes

$$\Delta_g^2 \varphi = 2\bar{Q} e^{4\varphi}.$$

Setting $u := 4\varphi$ and $\bar{Q} = \frac{1}{8}$ and taking into account that the Laplace-Beltrami operator in $\mathbb{R}^4$ endowed with the Euclidean metric is the standard Laplacian, we finally end up with the 4-dimensional Liouville equation

$$\Delta^2 u(x) = e^{u(x)}, \quad x \in \mathbb{R}^4.$$

As for dimension 2, a complete classification is known for solutions of the 4-dimensional Liouville equation that have finite volume:

Theorem 1.3 (LIN [25])**.** *Suppose that u is a solution of*

$$\begin{cases} \Delta^2 u = 6e^{4u} & \text{in } \mathbb{R}^4 \\ e^{4u} \in L^1(\mathbb{R}^4) \end{cases},$$

then the following statements hold.

1. *After an orthorgonal transformation, $u(x)$ can be represented by*

$$u(x) = \frac{3}{4\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) e^{4u(y)} \, dy - \sum_{j=1}^4 a_j(x_j - x_j^0)^2 + c_0$$

$$= -\sum_{j=1}^4 a_j(x_j - x_j^0)^2 - \alpha \log|x| + c_0 + O(|x|^{-\tau})$$

for some $\tau > 0$ and for large $|x|$, where $a_j \geq 0$ and c_0 are constants, and $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0) \in \mathbb{R}^4$. Moreover, if $a_i \neq 0$ for all i, then u is symmetric with respect to the hyperplane $\{x \mid x_i = x_i^0\}$. If $a_1 = a_2 = a_3 = a_4 \neq 0$, then u is radially symmetric with respect to x^0.

2. *Let*

$$\alpha = \frac{3}{4\pi^2} \int_{\mathbb{R}^4} e^{4u(y)} \, dy,$$

then $\alpha \leq 2$. Moreover, if $\alpha = 2$, then

$$u(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}.$$

Non-standard solutions with finite volume in $\mathbb{R}^4$ have been constructed explicitly also in [26].

In Section 4 we will show that there exists a non-trivial solution with infinite volume also in dimension 4. Specifically, we will prove the following theorem.

Theorem 1.4. *The Liouville equation $\Delta^2 u = e^u$ in $\mathbb{R}^4$ admits a non-trivial solution $u \in C^{4,\alpha}(\mathbb{R}^4)$ with infinite volume. In particular, this solution is radial in the first three coordinates (x_1, x_2, x_3), decaying to $-\infty$ as $O(|(x_1, x_2, x_3)|)$ (i.e. at most linearly), and is periodic in the last coordinate x_4.*

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Again, the philosophy is: proving the existence of a finite-volume solution in dimension 3, extending that solution trivially to $\mathbb{R}^4$ and then using a bifurcation theorem to prove that there is a non-trivial solution which is the perturbation of the 3-dimensional one. In this case, the existence of a 3-dimensional solution already (i.e. a solution of $\Delta^2 u = \mathrm{e}^u$ in $\mathbb{R}^3$) is not completely obvious and could be obtained only implicitly (Subsection 4.1). As for the non-trivial solution, the bifurcation theorem we employed is the more sophisticated *Krasnosel'skii's Index Theorem* 2.3. The reason that prevents us from using simpler bifurcation results as in the case of dimension 2 is that all the functions involved in higher dimension cannot be expressed explicitly. Hence, even if Theorem 1.2 and Theorem 1.4 look quite similar, the two proofs will be notably different, with the second having to resort to more abstract machinery, such as degree theory.

1.2. Some perspectives

The argument exposed in this work seems far from being the only way of finding non-trivial solutions of the Liouville equation. For instance, in four dimensions, one could try to follow the same procedure starting with a 2-dimensional solution and two parameters[3], but even this might not exhaust all possibilities. Another option, indeed, might be to bifurcate radially. Moreover, similarly to what happens for Delaunay surfaces [27] (whose equations are somehow analogous to the ones we are considering), we might be able to connect the non-trivial solutions of Theorem 1.4 to the radial solution by some global bifurcation theorem, maybe following the ideas of [28].

Another aspect which is worth considering is, of course, going to higher dimensions. As the methods employed in Section 4 are quite general, it seems reasonable to believe that the same argument can be generalized to any even dimension. Similarly to dimensions 2 and 4, the finite-volume case has already been classified in the works of L. Martinazzi [29], [30] and [31] (with A. Hyder). The existence of large volume solutions has been studied by the same author also in [32].

Finally, other lines of research aimed at finding more general non-trivial solutions with infinite volume are possible. For instance, we might be able to "glue" the oscillating solutions obtained from the bifurcation into more complex solutions, in a manner similar to the one used to construct Delaunay k-noids

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[3]Observe that choosing to start with a 1-dimensional solution leads nowhere. Indeed, we would be looking for solutions of the ODE

$$u^{(4)}(x) = \mathrm{e}^{u(x)}, \quad \forall x \in \mathbb{R}.$$

Integrating this equation, one immediately finds

$$u^{(3)}(y) - u^{(3)}(x) = \int_x^y \mathrm{e}^{u(s)} \, \mathrm{d}s > 0, \quad \forall x, y \in \mathbb{R}, x \neq y,$$

which means that $\lim_{x \to -\infty} u^{(3)}(x) < \lim_{x \to +\infty} u^{(3)}(x)$. But this is a contradiction: as we are asking u to have finite volume we need that u goes to $-\infty$ both at $-\infty$ and $+\infty$, and so $\lim_{x \to -\infty} u^{(3)}(x) \geq 0$ and $\lim_{x \to +\infty} u^{(3)}(x) \leq 0$.

starting from Delaunay unduloids and nodoids. In this way we could then obtain non-trivial solutions with infinite volume that are not a direct result of a bifurcation from a cylindrical solution. For an example of this phenomenon in a PDE context, see [33].

2. Preliminaries

We list here some known theorems used in the proofs of the main results of this article.

2.1. Bifurcation theory

The results in this section are well known and can be found for instance in [34] and [35]. The only new contribution is Subsection 2.1.1, in which we find a correct version of the formula discriminating between subcritical and supercritical bifurcations.

Definition. Let X and Y be Banach spaces, and $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. We say that λ^* is a *bifurcation point* for F (from the trivial solution $u \equiv 0$) if there is a sequence of solutions $(\lambda_n, u_n)_{n \in \mathbb{N}} \subset \mathbb{R} \times X$ of $F(\lambda_n, u_n) = 0$, with $u_n \neq 0$ for each $n \in \mathbb{N}$, that converges to $(\lambda^*, 0)$.

A direct consequence of the Implicit Function Theorem is the following necessary condition for bifurcation.

Proposition 2.1. *A necessary condition for λ^* to be a bifurcation point for F is that $F_u(\lambda^*, 0) := \frac{\partial F}{\partial u}(\lambda^*, 0)$ is not invertible.*

Observe that, in order to prove non-uniqueness, it suffices to find a bifurcation point λ^*. In the 2D case, this will be achieved by means of the vary basic *Bifurcation from the simple eigenvalue Theorem* [34, Theorem 5.4.1]:

Theorem 2.2. *Let X and Y be Banach spaces, $F \in C^2(\mathbb{R} \times X, Y)$ be such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Let λ^* be such that*

$$L := F_u(\lambda^*, 0) \equiv \frac{\partial F}{\partial u}(\lambda^*, 0)$$

has one-dimensional kernel

$$V = \{tu^* \mid t \in \mathbb{R}\}$$

and closed codimension 1 range

$$R = \{y \in Y \mid \langle \psi, y \rangle = 0\},$$

for some $\psi \in Y^*, \psi \neq 0$ (where $\langle \psi, y \rangle$ denotes the evaluation of $\psi \in Y^*$ at $y \in Y$). Letting $M := F_{u\lambda}(\lambda^*, 0)$, assume moreover that $M[u^*] \notin R$. Then λ^* is a bifurcation point for F. In addition, the set of non-trivial solutions of $F = 0$ is, near $(\lambda^*, 0)$, a unique C^1 cartesian curve with parametric representation on V.

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Unfortunately, this very simple result has no application in the 4D case, because it does not seem possible to have explicit solutions of the differential equations involved. The result will be proved anyway using the more sophisticated *Krasnosel'skii's Index Theorem* [36, Theorem 56.2]:

Theorem 2.3. *Let A be a completely continuous operator and assume that λ^* is a point where the index of $u - A(\lambda, u)$ changes, that is a point at which the dimension of the negative space of the linearized operator $L[v] := v - A_u(\lambda^*, 0)[v]$ changes. Then λ^* is a bifurcation point for equation $u = A(\lambda, u)$.*

2.1.1. Shape of bifurcation

In the relatively simple 2D case, we can know something about the shape of the bifurcating branch from the explicit computations on the solutions of the linearized problem.

Definition. Let λ^* be as in Theorem 2.2. We say that the bifurcation is *supercritical* (*subcritical*) if $\lambda \geq \lambda^*$ ($\lambda \leq \lambda^*$) along the bifurcating branch, in some ball around $(\lambda^*, 0)$. If, along the bifurcating solution, λ assumes values both bigger and smaller than λ^* in all balls around $(\lambda^*, 0)$, then the bifurcation is said to be *transcritical*.

Proposition 2.4. *In the setting of Theorem 2.2, suppose that F is C^∞. Then*

1. *If $\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle \neq 0$, then the bifurcation is transcritical.*
2. *If instead $\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle = 0$, let $u_2 \in L^{-1}\left(-\frac{1}{2}F_{uu}(\lambda^*, 0)[u^*, u^*]\right)$ and consider*

$$\lambda_2 = -\frac{\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u_2] \rangle + \frac{1}{6}\langle \psi, F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] \rangle}{\langle \psi, F_{\lambda, u}(\lambda^*, 0)[u^*] \rangle}.$$

   If $\lambda_2 > 0$, then the bifurcation is supercritical; while, if $\lambda_2 < 0$, then the bifurcation is subcritical.

Remark. The formula for λ_2 that can be found in many books, like [35, (I.6.11)] or [34, (5.4.7)], is not correct because it misses the term containing u_2. Observe that in general u_2 is different from 0 because of (2.1).

Proof. 1. This is well-known and can be found for instance in [34, 5.4.3.iv]. We summarize the proof here for convenience.

We compute the first terms of the Taylor expansion of F centered in $(\lambda^*, 0)$: Let $\lambda_0 := \lambda^*$ and $u_1 := u^*$, then

$$0 = F(\lambda(t), u(t)) = F(\lambda_0 + \lambda_1 t + \lambda_2 t^2 + O(t^3), u_1 t + u_2 t^2 + O(t^3))$$
$$= F(\lambda_0, 0) + (F_u(\lambda_0, 0)[u_1] + \lambda_1 F_\lambda(\lambda_0, 0))t$$
$$+ \left(F_u(\lambda_0, 0)[u_2] + \frac{1}{2}F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_2 F_\lambda(\lambda_0, 0)\right.$$
$$\left. + \lambda_1 F_{\lambda u}(\lambda_0, 0)[u_1] + \frac{1}{2}\lambda_1^2 F_{\lambda\lambda}(\lambda_0, 0)\right) t^2 + O(t^3).$$

Hence we have:
$$F(\lambda_0, 0) = 0,$$
which is true by hypothesis;
$$F_u(\lambda_0, 0)[u_1] + \lambda_1 F_\lambda(\lambda_0, 0) = 0,$$
which is also true because $F_u(\lambda_0, 0)[u_1] = F_u(\lambda_0, 0)[u^*] = 0$ by hypothesis and $F_\lambda(\lambda_0, 0) = 0$ because $F(\lambda, 0) \equiv 0$ for all λ;

$$0 = F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_2 F_\lambda(\lambda_0, 0)$$
$$+ \lambda_1 F_{\lambda u}(\lambda_0, 0)[u_1] + \frac{1}{2} \lambda_1^2 F_{\lambda\lambda}(\lambda_0, 0)$$
$$= F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_1 F_{\lambda u}(\lambda_0, 0)[u_1],$$

again because $F(\lambda, 0) \equiv 0$ for all λ. Applying ψ to that last equality, where coker $F_u(\lambda^*, 0) = \operatorname{coker} L = \langle \psi \rangle$, one gets

$$0 = \frac{1}{2} \langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle + \lambda_1 \langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle,$$

from which we obtain

$$\lambda_1 = -\frac{1}{2} \frac{\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u^*] \rangle}{\langle \psi, F_{\lambda u}(\lambda^*, 0)[u^*] \rangle}$$

(the fraction is well defined because by the hypothesis of Theorem 2.2 we already know that the denominator can't be 0, see also [34, 5.4.3.iv] or [35, (I.6.3)]). If $\lambda_1 \neq 0$ we have then a transcritical bifurcation, proving (1).

2. Assume now that $\lambda_1 = 0$. Again, in general, one has the following expansion

$$0 = F(\lambda(t), u(t)) = F(\lambda_0 + \lambda_2 t^2 + \lambda_3 t^3 + O(t^3), u_1 t + u_2 t^2 + u_3 t^3 + O(t^3))$$
$$= F(\lambda_0, 0) + F_u(\lambda_0, 0)[u_1] t$$
$$+ \left(F_u(\lambda_0, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda_0, 0)[u_1, u_1] + \lambda_2 F_\lambda(\lambda_0, 0) \right) t^2$$
$$+ \left(F_u(\lambda_0, 0)[u_3] + F_{uu}(\lambda_0, 0)[u_1, u_2] + \frac{1}{6} F_{uuu}(\lambda_0, 0)[u_1, u_1, u_1] \right.$$
$$\left. + \lambda_3 F_\lambda(\lambda_0, 0) + \lambda_2 F_{\lambda,u}(\lambda_0, 0)[u_1] \right) t^3 + O(t^4).$$

As before, the first and the second summands are already known to be 0. The third term gives us the following condition:

$$F_u(\lambda^*, 0)[u_2] + \frac{1}{2} F_{uu}(\lambda^*, 0)[u^*, u^*] = 0. \tag{2.1}$$

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The fourth term, instead, is the one from which we would like to extract the value of λ_2:

$$F_u(\lambda^*, 0)[u_3] + F_{uu}(\lambda^*, 0)[u^*, u_2]$$
$$+ \frac{1}{6} F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] + \lambda_2 F_{\lambda, u}(\lambda^*, 0)[u^*] = 0,$$

which leads to

$$\lambda_2 = -\frac{\langle \psi, F_{uu}(\lambda^*, 0)[u^*, u_2] \rangle + \frac{1}{6} \langle \psi, F_{uuu}(\lambda^*, 0)[u^*, u^*, u^*] \rangle}{\langle \psi, F_{\lambda, u}(\lambda^*, 0)[u^*] \rangle} \qquad (2.2)$$

(again, observe that the fraction is well defined because the denominator is not 0).

At least implicitly, then, one can find λ_2. Indeed, $L : X \to Y$ factors through

$$\tilde{L} : \frac{X}{\langle u^* \rangle} \to \{y \in Y \mid \langle \psi, y \rangle = 0\},$$

which is invertible, so that we get

$$u_2 = \tilde{L}^{-1} \left(-\frac{1}{2} F_{uu}(\lambda^*, 0)[u^*, u^*] \right)$$

to substitute in (2.2). Observe indeed that, applying ψ to (2.1), one immediately gets that

$$F_{uu}(\lambda^*, 0)[u^*, u^*] \in \{y \in Y \mid \langle \psi, y \rangle = 0\}.$$

Moreover, notice that

$$\langle \psi, F_{uu}(\lambda^*, 0)[u^*, v] \rangle = 0, \quad \forall v \in \langle u^* \rangle$$

implies also that the λ_2 obtained in this way is well defined.

If $\lambda_2 > 0$ we get a supercritical bifurcation, while if $\lambda_2 < 0$ we have a subcritical bifurcation, proving (2). $\qquad \square$

2.2. Elliptic regularity

Let L be a linear partial differential operator of order 2 defined in an open subset Ω of $\mathbb{R}^n$, $n \geq 2$. Assume that L can be written using the standard cartesian coordinates (and Einstein's notation) as follows

$$Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u,$$

with $a^{ij} = a^{ji}$ for each $1 \leq i, j \leq n$.

Theorem 2.5. *[37, 4.2.1] If L is an elliptic operator in $\Omega \subset \mathbb{R}^n$ open and if the coefficients of L are of class C^∞ in Ω, then A is hypoelliptic in Ω, namely: If u is a distribution in an open subset Ω_1 of Ω and if Lu is of class C^∞ in Ω_1, then u is of class C^∞ in Ω_1.*

Definition. Let $\Omega \subset \mathbb{R}^n$ be open, k be a non-negative integer and $\alpha \in (0,1)$. Set $d_x := \text{dist}(x, \partial\Omega)$ and $d_{x,y} := \min\{d_x, d_y\}$ and consider $u \in C^k(\Omega)$. We say that $u \in C_*^{k,\alpha}(\Omega)$ if its *interior Hölder norm* is finite, namely

$$|u|_{k,\alpha,\Omega}^* := |u|_{k,\Omega}^* + [u]_{k,\alpha,\Omega}^* < +\infty,$$

where

$$|u|_{k,\Omega}^* := \sum_{j=0}^k [u]_{j,\Omega}^* := \sum_{j=0}^k \sup_{\substack{x \in \Omega \\ |\beta|=j}} d_x^j |D^\beta u(x)|$$

and

$$[u]_{k,\alpha,\Omega} := \sup_{\substack{x,y \in \Omega \\ |\beta|=k}} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

One can prove that $C_*^{k,\alpha}(\Omega)$, equipped with the interior norm, is a Banach space (see for example [38, Problem 5.2]). In order to simplify our notation, from now on we will drop the subscript $*$ and just write $C^{k,\alpha}(\Omega) := C_*^{k,\alpha}(\Omega)$.

In order to state the Schauder's interior estimates we also need to introduce the following norms. Let σ be a real number and define

$$|u|_{k,\alpha,\Omega}^{(\sigma)} := |u|_{k,\Omega}^{(\sigma)} + [u]_{k,\alpha,\Omega}^{(\sigma)} < +\infty,$$

where

$$|u|_{k,\Omega}^{(\sigma)} := \sum_{j=0}^k [u]_{j,\Omega}^{(\sigma)} := \sum_{j=0}^k \sup_{\substack{x \in \Omega \\ |\beta|=j}} d_x^{j+\sigma} |D^\beta u(x)|$$

and

$$[u]_{k,\alpha,\Omega}^{(\sigma)} := \sup_{\substack{x,y \in \Omega \\ |\beta|=k}} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

One can check the following [38, 6.11].

Proposition 2.6. *Let $\sigma + \tau \geq 0$, then*

$$|fg|_{0,\alpha,\Omega}^{(\sigma+\tau)} \leq |f|_{0,\alpha,\Omega}^{(\sigma)} |g|_{0,\alpha,\Omega}^{(\tau)}.$$

The basic Schauder's interior estimates are provided by the following

Theorem 2.7. *[38, 6.2] Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\Omega)$ be a bounded solution in Ω of*

$$Lu = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u = f,$$

where $f \in C^{0,\alpha}(\Omega)$ and there are positive constants λ, Λ such that the coefficients satisfy

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

and

$$|a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \Lambda.$$

Then there exists a constant $C > 0$ not depending on u and f such that

$$|u|^*_{2,\alpha,\Omega} \leq C \left(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha,\Omega} \right).$$

The following is a generalization of Theorem 2.7 to the case in which we consider "weighted" Hölder norms. Let $w \in C^{k,\alpha}(\Omega), w > 0$ be the weight. Define, according to the notation explained before, the space

$$C^{k,\alpha}_w(\Omega) := \left\{ u \in C^k(\Omega) \Big| |wu|^*_{k,\alpha,\Omega} < +\infty \right\}.$$

Theorem 2.8. (WEIGHTED SCHAUDER'S ESTIMATES) *Let Ω be an open subset of $\mathbb{R}^n$ and let $w \in C^{2,\alpha}(\Omega), w > 0$ be a weight such that there exists a positive constant K such that*

$$|\partial_j \log w|^{(1)}_{0,\alpha,\Omega}, \left|\frac{\partial_i \partial_j w}{w}\right|^{(2)}_{0,\alpha,\Omega} \leq K, \quad \forall i, j.$$

Let $u \in C^{2,\alpha}_w(\Omega)$ be a solution in Ω of

$$Lu = a^{ij}(x)\partial_i \partial_j u + b^i(x)\partial_i u + c(x)u = f,$$

where $f \in C^{0,\alpha}_w(\Omega)$ and there are positive constants λ, Λ such that the coefficients satisfy

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

and

$$|a^{ij}|^{(0)}_{0,\alpha,\Omega}, |b^i|^{(1)}_{0,\alpha,\Omega}, |c|^{(2)}_{0,\alpha,\Omega} \leq \Lambda.$$

Assume moreover that wu is bounded. Then there exists a constant $C > 0$ not depending on u and f such that

$$|u|^*_{2,\alpha,\Omega;w} \leq C \left(|u|_{0,\Omega;w} + |f|^{(2)}_{0,\alpha,\Omega;w} \right),$$

where $|u|^*_{2,\alpha,\Omega;w} := |wu|^*_{2,\alpha,\Omega}, |u|_{0,\Omega;w} := |wu|_{0,\Omega}$ and $|f|^{(2)}_{0,\alpha,\Omega;w} := |wf|^{(2)}_{0,\alpha,\Omega}.$

For the proof, see Appendix A.

3. Dimension 2

We want to find some non-trivial solutions of the 2-dimensional Liouville equation

$$\Delta u(x,y) + e^{u(x,y)} = 0, \qquad \forall (x,y) \in \mathbb{R}^2,$$

To use a bifurcation theorem, we first need to find a 1-dimensional solution, i.e. a solution depending only on one variable, say x, and constant in the other. The equation then becomes the ordinary differential equation

$$u''(x) + e^{u(x)} = 0, \qquad \forall x \in \mathbb{R}^2,$$

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which admits the family of solutions

$$\log\left[c_1 - c_1 \tanh^2\left(\frac{1}{2}\sqrt{2c_1\left(c_2 + x\right)^2}\right)\right], \quad c_1 \geq 0, \quad c_2 \in \mathbb{R}.$$

Observe that the two parameters account only for a translation and a dilation of the solution, so we can simply fix them. For our convenience, we choose $c_1 = 2$ and $c_2 = 0$ (namely, we are requiring that the solution is even and we are fixing its volume), getting

$$u_0(x, y) := \log[2(1 - \tanh^2(|x|))] = \log(2 \operatorname{sech}^2(x)).$$

We now need to specify the setting we are working in. As we know that the 1-dimensional solution does not depend on the y variable, we could just choose to restrict the problem to the subsets

$$S_\lambda := \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, y \in (0, \lambda)\}, \quad \lambda > 0.$$

In this way, we get a natural parameter for our problem: the witdth λ of the strip S_λ. We now need to fix the Banach spaces: we ask that the solution goes to $-\infty$ as $|x| \to +\infty$. Consequently, the growth must be at most linear (the equation essentially says that at ∞ the second derivative must be 0). In particular, one might ask that the perturbation is bounded by a function that grows slower than $|x|$, like $\sqrt{|x|}$. Define then

$$X_\lambda := \left\{ u \in C^{2,\alpha}(S_\lambda) \left|
\begin{array}{l}
\dfrac{\partial}{\partial y}u(x, 0) = \dfrac{\partial}{\partial y}u(x, \lambda) = 0 \quad \forall x \in \mathbb{R}, \\[2mm]
u(-x, y) = u(x, y) \quad \forall(x, y) \in S_\lambda, \\[2mm]
\left\| \langle x \rangle^{-\frac{1}{2}} u \right\|_{2,\alpha,S_\lambda} + \left\| \langle x \rangle^{\frac{3}{2}} \Delta u \right\|_{0,\alpha,S_\lambda} < +\infty
\end{array}
\right. \right\}$$

and

$$Y_\lambda := \left\{ u \in C^{0,\alpha}(S_\lambda) \left|
\begin{array}{l}
u(-x, y) = u(x, y) \quad \forall(x, y) \in S_\lambda, \\[2mm]
\left\| \langle x \rangle^{\frac{3}{2}} f \right\|_{0,\alpha,S_\lambda} < +\infty
\end{array}
\right. \right\},$$

where $\langle x \rangle := \sqrt{1 + x^2}$. Recalling that the interior Hölder spaces of Definition 2.2 are Banach spaces ([38, Problem 5.2]), it can be easily checked that both X_λ and Y_λ are Banach spaces when endowed, respectively, with the norms

$$\|u\|_{X_\lambda} := \left\| \langle x \rangle^{-\frac{1}{2}} u \right\|_{2,\alpha,S_\lambda} + \left\| \langle x \rangle^{\frac{3}{2}} \Delta u \right\|_{0,\alpha,S_\lambda}$$

and

$$\|f\|_{Y_\lambda} := \left\| \langle x \rangle^{\frac{3}{2}} f \right\|_{0,\alpha,S_\lambda}.$$

Observe moreover that the functions in X_λ grow at most as $\sqrt{|x|}$, while those in Y_λ grow at most as $|x|^{-\frac{3}{2}}$.

Our problem is then finding the zeros of the following function:

$$\tilde{F} : X_\lambda \longrightarrow Y_\lambda$$
$$u \longmapsto \Delta(u_0 + u) + e^{u_0 + u} = \Delta u + e^{u_0}(e^u - 1).$$

Remark. Once we have a non-trivial solution on the strip, we can obtain a non-trivial solution on the whole plane by extending the solution reflecting it repeatedly along the borders of the strip. Continuity is automatic by construction and Neumann conditions assure that this is still a weak solution of the problem. Bootstrapping and weighted elliptic regularity (Theorem 2.8) make this also a strong solution.

Lemma 3.1. *A necessary condition for λ to be a bifurcation point is that $\lambda = \pi k$ for some $k \in \mathbb{N}_{>0}$. Moreover, in that case we have*

$$\ker L = \{t \operatorname{sech}(x) \cos(y) \mid t \in \mathbb{R}\}.$$

Proof. Recall that by Proposition 2.1 a necessary condition to have a bifurcation on S_λ is that the linearized operator in u_0 on the strip S_λ has a non-trivial kernel.

The linearization of $\tilde{F}(u) = \Delta u + e^{u_0}(e^u - 1)$ at 0 is

$$L(v)(x, y) = \Delta v(x, y) + 2 \operatorname{sech}^2(x) v(x, y).$$

We might first want to look for solutions of $Lv = 0$ having the form

$$v(x, y) = w_1(x) w_2(y),$$

hence satisfying

$$w_1''(x) w_2(y) + w_1(x) w_2''(y) + 2 \operatorname{sech}^2(x) w_1(x) w_2(y) = 0.$$

By separation of variables we get $w_2(y) = A \cos(\mu y) + B \sin(\mu y)$. Imposing Neumann boundary conditions we get $B = 0$ and $\mu = \frac{\pi j}{\lambda}$, $j \in \mathbb{Z}$. Hence, we can directly look for solutions of the form

$$v_j(x, y) = \cos\left(\frac{\pi j}{\lambda} y\right) \tilde{v}_j(x),$$

where $j \in \mathbb{N}$ is fixed, which means that we need to solve the problem:

$$-\tilde{v}_j''(x) - 2 \operatorname{sech}^2(x) \tilde{v}_j(x) = -\left(\frac{\pi j}{\lambda}\right)^2 \tilde{v}_j(x), \quad \forall x \in \mathbb{R}, \forall j \in \mathbb{N}.$$

Observe that the last equation is a stationary Schrödinger equation, with a so-called *Pöschl-Teller potential* [39]. Some lenghty but not difficult considerations on the growth of Legendre functions lead to

$$\tilde{v}_j \not\equiv 0 \iff -\left(\frac{\pi j}{\lambda}\right)^2 = -1 \iff j = \frac{\lambda}{\pi}$$

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(see Appendix B). We then get that the only non-zero element of the family is

$$v_{\frac{\lambda}{\pi}}(x,y) = A \operatorname{sech}(x) \cos(y).$$

From the discussion above, we conclude moreover that the Fourier series in y of a generic solution $v = v(x,y)$ of $Lv = 0$ has only one non-zero summand at most, $v_{\frac{\lambda}{\pi}}$. Thus, as both the Fourier series and its series of second derivatives trivially converge, we get that the only non-zero solution of $Lv = 0$ is $v_{\frac{\lambda}{\pi}}$ itself.

Summing up, we found that the only values of λ for which we can expect to have a bifurcation are the points πj, with $j \in \mathbb{N}_{>0}$. For these values of λ, in particular, the operator L has one dimensional kernel:

$$\ker L = \{t \operatorname{sech}(x) \cos(y) \mid t \in \mathbb{R}\}.$$

$\square$

Lemma 3.1 restricts the set of candidate bifurcation points to just $\lambda = \pi k$ for $k \in \mathbb{N}_{>0}$. In the following we will prove that all of them are actually bifurcation points (and that the bifurcations are all supercritical).

Lemma 3.2. *The linearized operator L is Freholm of index 0.*

Proof. Observe first that $Y_\lambda \subset L^2(S_\lambda)$, so that

$$\operatorname{codim} L = \dim\{f \in Y_\lambda \mid \langle Lu, f \rangle_{L^2} = 0 \quad \forall u \in X_\lambda\}.$$

Indeed, if $[f] \in \operatorname{coker} L$, then $[f]$ has a representative which is perpendicular to $R(L)$ (just take its projection on $R(L)^\perp$), while if f is in the set on the right hand side then f cannot be parallel to any element of $R(L)$ (otherwise the scalar product with such an element would not be zero) and thus cannot be in $R(L)$, which is a vector subspace of Y_λ because L is linear.

Let then $f \in Y_\lambda$ be such that $\langle Lu, f \rangle_{L^2} = 0$ for each $u \in X_\lambda$. By Theorem 2.5, we have that f is C^∞ on S_λ.

We now only need to prove that L is symmetric on the elements f in Y_λ such that $\langle Lu, f \rangle_{L^2} = 0$ for each $u \in X_\lambda$. Take $u \in X_\lambda$ and $f \in Y_\lambda \cap C^2(S_\lambda) \subset L^2(S_\lambda)$ such that $Lf = 0$. Then

$$\begin{aligned}
\langle Lu, f \rangle_{L^2} &= \int_{S_\lambda} \left(\Delta u + \mathrm{e}^{u_0} u \right) f \\
&= \int_{(0,\lambda) \times \mathbb{R}} \left(\tfrac{\partial^2 u}{\partial x^2}(x,y) + \tfrac{\partial^2 u}{\partial y^2}(x,y) + 2\operatorname{sech}^2(x) u(x,y) \right) f(x,y) \, \mathrm{d}x \, \mathrm{d}y.
\end{aligned} \tag{3.1}$$

The last summand inside the integral is clearly symmetric. Thus, we can just look at the first two. Theorem 2.8 applied to $Lf = 0$ gives

$$\left| \langle x \rangle^{\frac{3}{2}} f \right|_{2,\alpha,S_\lambda} \le C' \left| \langle x \rangle^{\frac{3}{2}} f \right|_{0,\alpha,S_\lambda} \le C \tag{3.2}$$

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for some constant $C > 0$ (recall that $f \in Y_\lambda$). Hence, both the first order and the second order partial derivatives in x decay as $|x|^{-\frac{3}{2}}$ as $|x| \to \infty$. Indeed

$$\frac{\partial}{\partial x}\left(\langle x\rangle^{\frac{3}{2}} f(x,y)\right) = \frac{3}{2}x\langle x\rangle^{-\frac{1}{2}} f(x,y) + \langle x\rangle^{\frac{3}{2}}\frac{\partial f}{\partial x}(x,y)$$

and consequently, making use of (3.2),

$$
\sup_{(x,y)\in S_\lambda}\left|\langle x\rangle^{\frac{3}{2}}\tfrac{\partial f}{\partial x}(x,y)\right| \leq C + \sup_{(x,y)\in S_\lambda}\left|\frac{3}{2}x\langle x\rangle^{-\frac{1}{2}} f(x,y)\right| \\
\leq C + \frac{3}{2}\sup_{(x,y)\in S_\lambda}\left|\langle x\rangle^{\frac{3}{2}} f(x,y)\right| \leq C_1
\tag{3.3}
$$

for some $C_1 > 0$ (which can be computed explicitly). Analogously, the second derivative turns out to be

$$\frac{\partial^2}{\partial x^2}\left(\langle x\rangle^{\frac{3}{2}} f(x,y)\right)$$
$$= \frac{3}{4}(x^2+2)\langle x\rangle^{-\frac{5}{2}} f(x,y) + 3x\langle x\rangle^{-\frac{1}{2}}\frac{\partial f}{\partial x}(x,y) + \langle x\rangle^{\frac{3}{2}}\frac{\partial^2 f}{\partial x^2}(x,y).$$

With estimates similar to the above and using (3.3) we then find

$$\sup_{(x,y)\in S_\lambda}\left|\langle x\rangle^{\frac{3}{2}}\tfrac{\partial^2 f}{\partial x^2}(x,y)\right| \leq C_2 \tag{3.4}$$

for some $C_2 > 0$ (which, again, can be computed explicitly).

A first consequence of (3.3) and (3.4), together with the Hölder inequality, is that in what follows we can always apply Fubini's Theorem. Let us look then at the first summand in (3.1).

$$\int_{S_\lambda}\frac{\partial^2 u}{\partial x^2}(x,y)f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^\lambda\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}(x,y)f(x,y)\right) - \frac{\partial u}{\partial x}(x,y)\frac{\partial f}{\partial x}(x,y)\right]\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^\lambda\left[\frac{\partial u}{\partial x}(x,y)f(x,y)\right]_{x=-\infty}^{+\infty}\mathrm{d}x - \int_{S_\lambda}\frac{\partial u}{\partial x}(x,y)\frac{\partial f}{\partial x}(x,y)\,\mathrm{d}x\,\mathrm{d}y \ ,$$
$$= -\int_0^\lambda\left[u(x,y)\frac{\partial f}{\partial x}(x,y)\right]_{x=-\infty}^{+\infty}\mathrm{d}x + \int_{S_\lambda}u(x,y)\frac{\partial^2 f}{\partial x^2}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{S_\lambda}u(x,y)\frac{\partial^2 f}{\partial x^2}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

where, in order to pass from the 3-rd to the 4-th to the 5-th line, we used the fact that both $\frac{\partial u}{\partial x}f$ and $u\frac{\partial f}{\partial x}$ decay as $|x|^{-2}$ as $|x| \to \infty$ (the estimate for $\frac{\partial u}{\partial x}f$ is obtained in the same way as in (3.3)).

As for the second summand, the computations are similar but easier, as we can directly exploit the fact that we are imposing Neumann conditions on the

boundary of S_λ (which, recall, is $\mathbb{R} \times (0, \lambda)$). Hence, again the boundary terms go away while integrating by parts and so, putting all together, we obtain that L is symmetric.

By the same argument of the Lemma 3.1, we get then that, if $\lambda \neq \pi k$ for some $k \in \mathbb{N}_{>0}$, then coker L is trivial. Instead, if $\lambda = \pi k$ for some $k \in \mathbb{N}_{>0}$, then there exists only one family of solutions of $\langle Lu, f \rangle = 0, \forall u \in X_\lambda$ satisfying Neumann conditions on ∂S_λ and belonging to Y_λ, namely $\langle u^* \rangle \subset X_\lambda \cap Y_\lambda$. Consequently, codim $L = 1$. In both cases, L is Freholm of index 0, as wanted. □

Proof of Theorem 1.2. We just have to apply Theorem 2.2. The change of variables that changes y into λy and leaves x unchanged transforms our original problem into one to which we can apply Theorem 2.2:

$$F(\lambda, u) = \frac{\partial^2 u}{\partial x^2} + \frac{1}{\lambda^2} \frac{\partial^2 u}{\partial y^2} + e^{u_0} \left(e^u - 1 \right),$$

where $F : (0, +\infty) \times X_1 \to Y_1$.

Take then $\lambda^* = \pi j$, with $j \in \mathbb{N}_{>0}$. First,

$$F_u(\lambda^*, 0)[v] = \frac{\partial^2 v}{\partial x^2} + \frac{1}{(\lambda^*)^2} \frac{\partial^2 v}{\partial y^2} + e^{u_0} v$$

is Fredholm, because so it was before the change of variables. Secondly,

$$M[v] := F_{u,\lambda}(\lambda^*, 0)[1, v] = -\frac{1}{(\lambda^*)^3} \frac{\partial^2 v}{\partial y^2}$$

and, as we know that

$$v_0(x, y) = \operatorname{sech}(x) \cos(y) \in \ker L,$$

we have that

$$\ker F_u(\lambda^*, 0) = \langle u^* \rangle$$

with

$$u^*(x, y) := R_{\lambda^*}(v_0)(x, y) = \operatorname{sech}(x) \cos(\lambda^* y) \in X_1.$$

Hence

$$M[u^*] = \frac{2}{\lambda^*} u^*(x, y) \in \langle u^* \rangle,$$

so that $M[u^*] \notin R$.

According to Theorem 2.2, then, $\lambda^* = \pi j$ for $j \in \mathbb{N}_{>0}$ are bifurcation points for F and, consequently, also for $\tilde{F}$ (which is nothing less than F written using different coordinates).

Finally, Proposition 2.4 gives that all bifurcations are supercritical. This means that the period of the non-trivial solutions we find in this way is actually slightly bigger than πk (for $\lambda = \pi k$ the trivial branch and the non-trivial branch meet, so that they identify the same trivial solution). Notice though that we

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can actually make the period be exactly πk by replacing the non-trivial solution u of period $(1 + \varepsilon)\pi k$ with

$$u\left(\frac{x}{1 + \varepsilon}, \frac{y}{1 + \varepsilon}\right) + 2\log\frac{1}{1 + \varepsilon},$$

which is still a solution of $\Delta u + \mathrm{e}^u = 0$, of period πk. $\qquad\square$

4. Dimension 4

In this section, we prove the same result of the previous section in the 4D case. The main tool we use is Krasnosel'skii's Theorem 2.3.

4.1. 3-dimensional solutions

Our first goal is to show that there exists at least one solution of

$$\begin{cases} \Delta^2 u = \mathrm{e}^u \text{ in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} \mathrm{e}^{u(x)} \, \mathrm{d}x < +\infty \end{cases} . \tag{4.1}$$

The proof will be done in two steps and we will look in particular for solutions of the integral form of (4.1), namely solutions of

$$u(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{u(y)} \, \mathrm{d}y.$$

Observe that a function u with finite volume satisfying this last expression is a solution of (4.1). Indeed, a fundamental solution of Δ^2 is $G(x) = -\frac{1}{8\pi}|x|$ (see [40]).

Lemma 4.1. *Let*

$$X := \left\{ u \in C^0(\mathbb{R}^3) \, \middle| \, u \text{ is radially symmetric and } \|u\| < +\infty \right\},$$

where $\|u\| := \sup_{x \in \mathbb{R}^3} \frac{|u(x)|}{1 + |x|}$. *Then for every* $\varepsilon > 0$ *there exist* $u_\varepsilon \in X$ *such that*

$$u_\varepsilon(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u_\varepsilon(y)} \, \mathrm{d}y. \tag{4.2}$$

Proof. First of all, observe that X, endowed with the norm $\|\cdot\|$, is a well-defined Banach space. Define then

$$T_\varepsilon : X \longrightarrow X$$
$$u \longmapsto \bar{u}_\varepsilon, \quad \bar{u}_\varepsilon(x) := -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)} \, \mathrm{d}y \quad .$$

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T_ε is well defined. Take in fact $u \in X$, then $\bar{u}_\varepsilon \in C^0(\mathbb{R}^3)$ by the Lebesgue Dominated Convergence Theorem. Moreover, $\bar{u}_\varepsilon$ is clearly radial because of the radial invariance of the Lebesgue integral: indeed, if $A \in SO(3)$, then

$$
\begin{aligned}
\bar{u}_\varepsilon(Ax) &= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |Ax - y|\, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)}\, \mathrm{d}y \\
&= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |A(x-y)|\, \mathrm{e}^{-\varepsilon|Ay|^2} \mathrm{e}^{u(Ay)} |\det A|\, \mathrm{d}y \\
&= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)}\, \mathrm{d}y = \bar{u}_\varepsilon(x).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
|\bar{u}_\varepsilon(x)| &= \frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)}\, \mathrm{d}y \\
&\leq \frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{\|u\|(1+|y|)}\, \mathrm{d}y \\
&\leq \frac{1}{8\pi} \int_{\mathbb{R}^3} |y| \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{\|u\|(1+|y|)}\, \mathrm{d}y + |x| \frac{1}{8\pi} \int_{\mathbb{R}^3} \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{\|u\|(1+|y|)}\, \mathrm{d}y \\
&\leq C_1 + C_2|x| \leq \bar{C}(1+|x|),
\end{aligned}
$$

so that $\|\bar{u}_\varepsilon\| < +\infty$. Hence $T_\varepsilon(u) = \bar{u}_\varepsilon \in X$.

We now show that T_ε is compact. Take a bounded sequence $\{u_n\}_n \subset X$, $\|u_n\| \leq C < +\infty$ for all $n \in \mathbb{N}$. We want to show that then $\{T_\varepsilon(u_n)\}_n$ admits a converging subsequence. The idea is to use Arzelà-Ascoli's Theorem on the sequence $\left\{ \frac{T_\varepsilon(u_n)}{1+|x|} \right\}_n$. First,

$$
\begin{aligned}
\frac{|T_\varepsilon(u_n)|}{1+|x|} &= \frac{1}{8\pi(1+|x|)} \int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u_n(y)}\, \mathrm{d}y \\
&\leq \frac{1}{8\pi(1+|x|)} \int_{\mathbb{R}^3} |x-y|\, \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{C(1+|y|)}\, \mathrm{d}y \\
&\leq \frac{C_1 + C_2|x|}{8\pi(1+|x|)} \leq \bar{C} < +\infty,
\end{aligned}
$$

for any $x \in \mathbb{R}^3$ and $n \in \mathbb{N}$, so that $\left\{ \frac{T_\varepsilon(u_n)}{1+|x|} \right\}_n$ is uniformly bounded. Moreover,

$$
\begin{aligned}
\left| \frac{T_\varepsilon(u_n(x))}{1+|x|} - \frac{T_\varepsilon(u_n(y))}{1+|y|} \right| &= \frac{1}{8\pi} \left| \int_{\mathbb{R}^3} \left(\frac{|x-z|}{1+|x|} - \frac{|y-z|}{1+|y|} \right) \mathrm{e}^{-\varepsilon|z|^2} \mathrm{e}^{u_n(z)}\, \mathrm{d}z \right| \\
&\leq \frac{1}{8\pi} \int_{\mathbb{R}^3} \left| \frac{|x-z|}{1+|x|} - \frac{|y-z|}{1+|y|} \right| \mathrm{e}^{-\varepsilon|z|^2} \mathrm{e}^{u_n(z)}\, \mathrm{d}z \\
&\leq \left(\frac{1}{8\pi} \int_{\mathbb{R}^3} (2+|z|) \mathrm{e}^{-\varepsilon|z|^2} \mathrm{e}^{C(1+|z|)}\, \mathrm{d}z \right) |x-y|
\end{aligned}
$$

for any $x, y \in \mathbb{R}^3$ and $n \in \mathbb{N}$, so that $\left\{ \frac{T_\varepsilon(u_n(x))}{1+|x|} \right\}_n$ is equicontinuous. The last inequality, in particular follows from the triangular inequality: as one clearly

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has

$$|y| \leq |x| + |y - x|,$$
$$|x - z| \leq |x - y| + |y - z|,$$
$$|z| \leq |x| + |z - x|,$$
$$|x - z| \leq |x| + |z|,$$

we obtain indeed

$$
\begin{aligned}
\left| \frac{|x - z|}{1 + |x|} - \frac{|y - z|}{1 + |y|} \right| &= \left| \frac{|x - z| - |y - z| + |y||x - z| - |x||y - z|}{(1 + |x|)(1 + |y|)} \right| \\
&\leq \frac{|x - y| + (|x| + |y - x|)|x - y| - |x||y - z|}{(1 + |x|)(1 + |y|)} \\
&\leq \frac{|x - y| + |x||x - y| + |x - z||x - y|}{(1 + |x|)(1 + |y|)} \\
&\leq \left(\frac{1 + 2|x|}{(1 + |x|)(1 + |y|)} + \frac{|z|}{(1 + |x|)(1 + |y|)} \right) |x - y| \\
&\leq \left(\frac{2}{(1 + |y|)} + \frac{|z|}{(1 + |x|)(1 + |y|)} \right) |x - y| \\
&\leq (2 + |z|)|x - y|.
\end{aligned}
$$

By Arzelà-Ascoli's Theorem, $\left\{ \frac{T_\varepsilon(u_n(x))}{1 + |x|} \right\}_n$ admits a subsequence which converges uniformly. Thus, $\{T_\varepsilon(u_n)\}_n$ admits a converging subsequence in $(X, \|\cdot\|)$ and therefore T is a compact operator.

Next, we prove that T has a fixed point using Schaefer's Fixed Point Theorem (see for example [41]). Let $u \in X$ satisfy $u = tT_\varepsilon(u)$ for some $0 \leq t \leq 1$, then

$$u(x) = -\frac{t}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{-\varepsilon|y|^2} \mathrm{e}^{u(y)} \, \mathrm{d}y \leq 0.$$

Consequently

$$|u(x)| \leq \frac{t}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{-\varepsilon|y|^2} \, \mathrm{d}y \leq C(1 + |x|)$$

and therefore $\|u\| \leq C$. That means that the set $\{u \in X \mid u = tT_\varepsilon(u), 0 \leq t \leq 1\}$ is bounded in $(X, \|\cdot\|)$: by Schaefer's Theorem then T_ε has a fixed point in X. $\quad\square$

Theorem 4.2. *u_ε converges to some u in $(X, \|\cdot\|)$ as ε goes to 0, with u satisfying*

$$u(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \mathrm{e}^{u(y)} \, \mathrm{d}y$$

(hence being a solution of (4.1) *with the desired properties).*

Before proving the Theorem, we need to establish the following Pohozaev-type identity [42].

Lemma 4.3. *Let u_ε be a solution of* (4.2), *namely*

$$u_\varepsilon(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| \, e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy.$$

Then

$$\int_{\mathbb{R}^3} \left(u_\varepsilon(x) + 6 - 4\varepsilon|x|^2 \right) e^{-\varepsilon|x|^2} e^{u_\varepsilon(x)} \, dx = 0.$$

Proof of Lemma 4.3. This proof follows the ideas of [43].

Differentiating (4.2) one gets

$$x \cdot \nabla u_\varepsilon(x) = -\frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{x \cdot (x - y)}{|x - y|} e^{-\varepsilon|y|^2 + u_\varepsilon(y)} \, dy.$$

Multiplying both sides by $e^{-\varepsilon|x|^2 + u_\varepsilon(x)}$ and integrating on x we obtain

$$\int_{\mathbb{R}^3} \left(x \cdot \nabla u_\varepsilon(x) \right) e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \, dx$$
$$= -\frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x \cdot (x - y)}{|x - y|} e^{-\varepsilon|y|^2 + u_\varepsilon(y)} e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \, dy \, dx. \tag{4.3}$$

Integrating by parts the LHS one gets

$$\begin{aligned}
(\text{LHS}) &= \int_{\mathbb{R}^3} \sum_{j=1}^{3} x_j \frac{\partial u_\varepsilon(x)}{\partial x_j} e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \, dx \\
&= \sum_{j=1}^{3} \int_{\mathbb{R}^3} x_j e^{-\varepsilon|x|^2} \frac{\partial}{\partial x_j} \left(e^{u_\varepsilon(x)} \right) dx \\
&= -\sum_{j=1}^{3} \int_{\mathbb{R}^3} \frac{\partial}{\partial x_j} \left(x_j e^{-\varepsilon|x|^2} \right) e^{u_\varepsilon(x)} \, dx \\
&= -\sum_{j=1}^{3} \int_{\mathbb{R}^3} \left(e^{-\varepsilon|x|^2} - 2\varepsilon x_j^2 e^{-\varepsilon|x|^2} \right) e^{u_\varepsilon(x)} \, dx \\
&= -3 \int_{\mathbb{R}^3} e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \, dx + \int_{\mathbb{R}^3} 2|x|^2 \varepsilon e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \, dx.
\end{aligned} \tag{4.4}$$

For the RHS, instead, notice that $x = \frac{(x+y)+(x-y)}{2}$. Then

$$\begin{aligned}
(\text{RHS}) &= \frac{1}{2} \int_{\mathbb{R}^3} e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \left(-\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon|y|^2 + u_\varepsilon(y)} \, dy \right) dx \\
&\quad - \frac{1}{16\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(x + y) \cdot (x - y)}{|x - y|} e^{-\varepsilon|x|^2 + u_\varepsilon(x) - \varepsilon|y|^2 + u_\varepsilon(y)} \, dy \, dx \quad (4.5) \\
&= \frac{1}{2} \int_{\mathbb{R}^3} u_\varepsilon(x) e^{-\varepsilon|x|^2 + u_\varepsilon(x)} \, dx,
\end{aligned}$$

where the second term in the sum vanishes because the integrand is skew-symmetric in (x, y).

Plugging (4.4) and (4.5) in (4.3) one finally gets

$$\int_{\mathbb{R}^3} \left(u_\varepsilon(x) + 6 - 4\varepsilon|x|^2 \right) e^{-\varepsilon|x|^2} e^{u_\varepsilon(x)} \, dx = 0.$$

$\square$

Proof of Theorem 4.2. First of all, we check that u_ε is radially monotone decreasing for each $\varepsilon > 0$. Indeed, write the integral in u_ε in polar coordinates (with a slight abuse of notation)

$$\begin{aligned}
u_\varepsilon(r) &= -\frac{1}{8\pi} \int\limits_{\varphi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{s=0}^{+\infty} \sqrt{r^2 - 2rs\cos\theta + s^2} e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s^2 \sin\theta \, ds \, d\theta \, d\varphi \\
&= -\frac{1}{12r} \int_0^{+\infty} (r^2 - 2rs\cos\theta + s^2)^{\frac{3}{2}} \Big|_{\theta=0}^{\theta=\pi} e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s \, ds \\
&= -\frac{1}{12r} \int_0^{+\infty} \left[(r+s)^3 - |r-s|^3 \right] e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s \, ds \\
&= -\frac{1}{6r} \int_0^r s^2 (3r^2 + s^2) e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds - \frac{1}{6} \int_r^{+\infty} s(r^2 + 3s^2) e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds.
\end{aligned}$$

In the previous computation we set $y = (s\sin\theta\cos\varphi, s\sin\theta\sin\varphi, s\cos\theta)$ and we chose $x = (r, 0, 0)$ (recall that we have already checked the radial invariance). Now take a derivative in r:

$$u_\varepsilon'(r) = \int_0^r \underbrace{\frac{s^2 - 3r^2}{6r^2}}_{<0} s^2 e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds - \frac{r}{3} \int_r^{+\infty} s e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds < 0.$$

Hence u_ε is radially monotone decreasing for all $\varepsilon > 0$.

By Lemma 4.3 one has

$$\int_{\mathbb{R}^3} \left(u_\varepsilon(x) + 6 - 4\varepsilon|x|^2 \right) e^{-\varepsilon|x|^2} e^{u_\varepsilon(x)} \, dx = 0.$$

Hence, since u_ε is radially monotone decreasing and continuous, for any $\varepsilon > 0$ we must have $u_\varepsilon(0) > -6$ (because otherwise the integrand would be strictly negative, making the previous integral non-zero). Hence $-6 < u_\varepsilon(0) < 0$: applying that to (4.2) we get

$$\left| \int_{\mathbb{R}^3} |y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy \right| \leq 6$$

and thus

$$\left| \Delta u_\varepsilon(0) \right| = \frac{1}{4\pi} \left| \int_{\mathbb{R}^3} \frac{1}{|y|} e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy \right| \leq C < +\infty. \tag{4.6}$$

By Green's formula indeed

$$\Delta u_\varepsilon(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy.$$

We now check that Δu_ε is monotone increasing for each $\varepsilon > 0$. Indeed, using again polar coordinates as before,

$$
\begin{aligned}
(\Delta u_\varepsilon)(r) &= -\frac{1}{4\pi} \int\limits_{\varphi=0}^{2\pi} \int\limits_{\theta=0}^{\pi} \int\limits_{s=0}^{+\infty} \frac{e^{-\varepsilon s^2} e^{u_\varepsilon(s)} s^2}{\sqrt{r^2 - 2rs\cos\theta + s^2}} \sin\theta \, ds \, d\theta \, d\varphi \\
&= -\frac{1}{2r} \int_0^{+\infty} \sqrt{r^2 - 2rs\cos\theta + s^2} \Big|_{\theta=0}^{\theta=\pi} s \, e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds \\
&= -\frac{1}{2r} \int_0^{+\infty} \left[(r+s) - |r-s|\right] s \, e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds \\
&= -\frac{1}{r} \int_0^r s^2 e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds - \int_r^{+\infty} s \, e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds,
\end{aligned}
$$

one sees that the derivative in r is positive:

$$(\Delta u_\varepsilon)'(r) = \frac{1}{r^2} \int_0^r s^2 e^{-\varepsilon s^2} e^{u_\varepsilon(s)} \, ds > 0.$$

Now, by monotonicity of Δu_ε and because $\Delta u_\varepsilon < 0$ and (4.6) hold, we have $\|\Delta u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq C < \infty$. Therefore u_ε converges to some radial u in $C^4_{\text{loc}}(\mathbb{R}^3)$, because of elliptic estimates.

We now check that $\|u_\varepsilon\| \geq C$ for some new constant $C > 0$ independent on ε. Observe preliminarly that $|u_\varepsilon(x)| \leq \|u_\varepsilon\| (1+|x|)$ and $u_\varepsilon(x) < 0$ for all $x \in \mathbb{R}^3$ imply that $u_\varepsilon(x) \geq -\|u_\varepsilon\| (1+|x|)$ for all $x \in \mathbb{R}^3$. Therefore

$$
\begin{aligned}
-\|u_\varepsilon\| (1+|x|) \leq u_\varepsilon(x) &= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x-y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy \\
&\leq -\frac{1}{8\pi} \int_{|y|<1} |x-y| e^{-\varepsilon|y|^2} e^{u_\varepsilon(y)} \, dy \\
&\leq -\frac{1}{8\pi} \int_{|y|<1} |x-y| e^{-\varepsilon|y|^2} e^{-\|u_\varepsilon\|(1+|y|)} \, dy \\
&\leq -\frac{1}{8\pi} \left(\int_{|y|<1} |x-y| \, dy \right) e^{-2\|u_\varepsilon\|-1}.
\end{aligned}
$$

Now, if for the sake of contradiction we suppose that $\|u_\varepsilon\|$ goes to zero, on the left hand side we would have something going pointwise to zero, while on the right hand side we would have something going poinwise to some strictly negative function of x, which is a contradiction. Hence there exists some $C > 0$ such that $\|u_\varepsilon\| \geq C$ for all $\varepsilon > 0$.

We can now check that the limit u satisfies the properties of Theorem 4.2. First, as for every $x \in \mathbb{R}^3$ we have

$$\left| |x - \cdot| e^{-\varepsilon |\cdot|^2} e^{u_\varepsilon(\cdot)} \right| \leq |x - \cdot| e^{-C(1+|\cdot|)} \in L^1(\mathbb{R}^3),$$

by Lebesgue's Dominated Convergence Theorem we get

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = -\frac{1}{8\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon |y|^2} e^{u_\varepsilon(y)} \, \mathrm{d}y$$
$$= -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x - y| e^{u(y)} \, \mathrm{d}y,$$

so that u is a solution of (4.1). Then, again using the fact that $\|u_\varepsilon\| \geq C$ (so that $u_\varepsilon(x) \leq -C(1 + |x|)$ uniformly in $\varepsilon > 0$) and Dominated Convergence, we get

$$u(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x) = -\frac{1}{8\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon |y|^2} e^{u_\varepsilon(y)} \, \mathrm{d}y$$
$$\geq -\frac{1}{8\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} |x - y| e^{-\varepsilon |y|^2} e^{-C(1+|y|)} \, \mathrm{d}y$$
$$\geq -\frac{1}{8\pi} \int_{\mathbb{R}^3} |x| e^{-C(1+|y|)} \, \mathrm{d}y - \frac{1}{8\pi} \int_{\mathbb{R}^3} |y| e^{-C(1+|y|)} \, \mathrm{d}y \geq -\bar{C}(1 + |x|),$$

for some constant $\bar{C} > 0$. Thus $\|u\| \leq \bar{C}$ (recall that $u(x) \leq 0$ for all $x \in \mathbb{R}^3$), so that $u \in X$. This completes the proof. $\qquad \square$

Remark. Observe that, if $u_1(x)$ is a solution of $\Delta^2 u = e^u$, then

$$u_\mu(x) = u_1(\mu x) + 4 \log \mu$$

is a solution as well. Therefore, actually, we have shown the existence of a whole family of 3-dimensional solutions. Once a 3-dimensional solution with the aforementioned properties u_1 is fixed, u_λ can be characterized equivalently by its volume $\int_{\mathbb{R}^3} e^{u_\lambda}$, its value in 0 or its asymptotic behavior.

4.2. Non-trivial solutions

Now that we have a family of 3-dimensional solutions we can start looking at bifurcations. Similarly to what we did in dimension 2, we will restrict our problem to the strip $S_\lambda := \mathbb{R}^3 \times (0, \lambda)$ and find the values of λ for which the solution is not unique. For these values of λ, we will have then non-trivial solutions in the strip S_λ. Extending them to the whole plane $\mathbb{R}^4$ by reflection and using elliptic regularity, this will give a non-trivial solution with infinite volume.

Recall that, in order to prove non-uniqueness, it suffices to find a bifurcation point λ^* and that, by Theorem 2.3, we need to find a value of the parameter for which the index of the operator associated to the differential equation changes.

To start, we need to fix the spaces of functions we are working in. Write $u(x_1, x_2, x_3, x_4) = u(x, x_4)$ (i.e., $x = (x_1, x_2, x_3)$) and define

$$X_\lambda := \left\{ u \in C^{4,\alpha}(S_\lambda) \middle| \begin{array}{l} \dfrac{\partial}{\partial x_4} u(x,0) = \dfrac{\partial}{\partial x_4} u(x,\lambda) = 0 \quad \forall x \in \mathbb{R}^3, \\[2mm] \dfrac{\partial^3}{\partial x_4^3} u(x,0) = \dfrac{\partial^3}{\partial x_4^3} u(x,\lambda) = 0 \quad \forall x \in \mathbb{R}^3, \\[2mm] u \text{ radial in } x, \\[2mm] \left| \langle x \rangle^{\frac{1}{2}} u \right|_{4,\alpha,S_\lambda} + \left| \langle x \rangle^{\frac{5}{2}} \Delta u \right|_{2,\alpha,S_\lambda} \\[2mm] \quad + \left| \langle x \rangle^{\frac{9}{2}} \Delta^2 u \right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right\},$$

where $\langle x \rangle := \sqrt{1 + x^2}$. Define also

$$Y_\lambda := \left\{ f \in C^{0,\alpha}(S_\lambda) \middle| \begin{array}{l} f \text{ radial in } x, \\[2mm] \left| \langle x \rangle^{\frac{9}{2}} f \right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right\}.$$

Similarly to dimension 2, it can be checked that both X_λ and Y_λ are Banach spaces when endowed, respectively, with the norms

$$\|u\|_{X_\lambda} := \left| \langle x \rangle^{\frac{1}{2}} u \right|_{4,\alpha,S_\lambda} + \left| \langle x \rangle^{\frac{5}{2}} \Delta u \right|_{2,\alpha,S_\lambda} + \left| \langle x \rangle^{\frac{9}{2}} \Delta^2 u \right|_{0,\alpha,S_\lambda}$$

and

$$\|f\|_{Y_\lambda} := \left| \langle x \rangle^{\frac{9}{2}} f \right|_{0,\alpha,S_\lambda}.$$

Observe moreover that the functions in X_λ grow at most like $|x|^{-\frac{1}{2}}$, while those in Y_λ grow at most like $|x|^{-\frac{9}{2}}$, and thus $X_\lambda, Y_\lambda \subset L^2(S_\lambda)$. Notice also that perturbing the 3-dimensional solution with functions in X_λ preserves the growth at infinity[4].

Our problem is then finding zeros of the following functional:

$$F_\lambda : X_\lambda \longrightarrow Y_\lambda$$
$$u \longmapsto \Delta^2(u_0 + u) - e^{u_0 + u} = \Delta^2 u - e^{u_0}(e^u - 1),$$

where u_0 is one element of the family of 3-dimensional solutions found in the previous Subsection (extended trivially in the fourth direction).

At this point one should notice that the equation we get in this way is not in the form of Krasnosel'skii's Theorem 2.3. Indeed, the operator is not in the required form $I - K$, with K compact. Nonetheless, one can overcome this

---

[4]Here the essential hypothesis is that the functions go to zero at infinity. Therefore, the choice of the power $-\frac{1}{2}$ is quite arbitrary and could be replaced by any power $\varepsilon < 0$.

obstacle as follows. Suppose that we can invert the operator $\Delta^2 : X_\lambda \to Y_\lambda$. Then, instead of

$$\Delta^2 u - e^{u_0}(e^u - 1) = 0, \tag{4.7}$$

one could consider the equation

$$u - \Delta^{-2}(e^{u_0}(e^u - 1)) = 0. \tag{4.8}$$

Notice that u is a solution of the original equation if and only if it is a solution of this second equation (because we are assuming that Δ^2 is invertible). Hence, instead of solving (4.7), we will deal with (4.8). The advantage is that now one can show that (4.8) is in the required form, namely $\Delta^{-2} \circ F_\lambda : X_\lambda \to X_\lambda$ is a compact perturbation of the identity.

4.2.1. Invertibility of bi-Laplacian and compactness

Lemma 4.4. $\Delta^2 : X_\lambda \to Y_\lambda$ *is invertible. Consequently,* $\Delta^{-2} \circ L : X_\lambda \to X_\lambda$ *is well defined.*

Proof. Let

$$Z_\lambda := \left\{ w \in C^{2,\alpha}(S_\lambda) \left| \begin{array}{l} \dfrac{\partial}{\partial x_4} w(x,0) = \dfrac{\partial}{\partial x_4} w(x,\lambda) = 0 \quad \forall x \in \mathbb{R}^3, \\[2mm] w \text{ radial in } x, \\[2mm] \left| \langle x \rangle^{\frac{9}{2}} w \right|_{2,\alpha,S_\lambda} + \left| \langle x \rangle^{\frac{5}{2}} \Delta w \right|_{0,\alpha,S_\lambda} < +\infty \end{array} \right. \right\}.$$

Notice that Δ maps X_λ to Z_λ and maps Z_λ to Y_λ, by construction. Therefore, it will be enough to prove that (with a slight abuse of notation) both $\Delta : X_\lambda \to Z_\lambda$ and $\Delta : Z_\lambda \to Y_\lambda$ are invertible.

Similarly to X_λ and Y_λ, Z_λ is a Banach space contained in $L^2(S_\lambda)$. Consider first $\Delta : X_\lambda \to Z_\lambda$: it is a linear symmetric operator (with respect to the L^2 product), so that it suffices to show that $\ker \Delta \subseteq X_\lambda$ is trivial. Indeed, this will imply that Δ is injective and surjective, because the operator is symmetric and thus both the image and the cokernel are contained in the domain, by elliptic regularity. But then Δ will be a bijective map between the Banach spaces X_λ and Z_λ which is continuous by construction, so that it will have a continuous inverse by the Open Mapping Theorem [44, Corollary 2.7].

To begin with, suppose that $u(x, x_4) = u_k(x) \cos\left(\frac{k\pi}{\lambda} x_4\right) \in \ker \Delta$, where x stands for (x_1, x_2, x_3). Write the Laplacian as $\Delta = \Delta_x + \frac{\partial^2}{\partial x_4^2}$, where Δ_x is the Laplacian in the first three coordinates only. Then one gets

$$0 = \Delta \left(u_k(x) \cos\left(\frac{k\pi}{\lambda} x_4\right) \right)$$

$$= \left(\Delta_x u_k(x) - 2\frac{k^2\pi^2}{\lambda^2} u_k(x) \right) \cos\left(\frac{k\pi}{\lambda} x_4\right),$$

so that $u \in \ker \Delta$ if and only if $\left(\Delta_x - \frac{k^2 \pi^2}{\lambda^2} \right) u_k(x) = 0$. Now, if $k = 0$, then the equation becomes $\Delta_x u_0(x) = 0$. By the Maximum Principle and the growth requirements at infinity, we must then have $u_0 = 0$.

Let now $k \neq 0$, so that we have the equation $\Delta_x u_k(x) = \frac{k^2 \pi^2}{\lambda^2} u_k(x)$. Suppose that u_k attains a maximum at $x = x_M \in \mathbb{R}^3$, then $\Delta_x u_k(x_M) \leq 0$ implies that $u_k(x) \leq u_k(x_M) \leq 0$ for all $x \in \mathbb{R}^3$. If instead u_k has no interior maximum, then it must be at infinity, and thus $u_k(x) \leq 0$ for all $x \in \mathbb{R}^3$, because according to our choice of X_λ we have $u_k(x) \to 0$ as $|x| \to +\infty$. Similar considerations with the minimum of u_k lead to $u_k(x) \geq 0$ for all $x \in \mathbb{R}^3$. But then $u_k = 0$ also for $k \neq 0$.

Given now any $u(x, x_4)$ in the kernel of the Laplacian, the previous argument shows then that all coefficients of its Fourier expansion in x_4 are forced to be zero. Therefore, $u = 0$ and thus $\ker \Delta = 0$, proving that $\Delta : X_\lambda \to Z_\lambda$ is invertible with continuous inverse.

The same argument proves that $\Delta : Z_\lambda \to Y_\lambda$ is invertible with continuous inverse, so that $\Delta^2 = \Delta \circ \Delta : X_\lambda \to Y_\lambda$ is invertible with continuous inverse. $\square$

As for compactness, notice first that the operator $\Delta^{-2} \circ F_\lambda$ already has the form $I - K$, with $K(u) = \Delta^{-2}(\mathrm{e}^{u_0}(\mathrm{e}^u - 1))$.

Lemma 4.5. $K : X_\lambda \to X_\lambda$ *is a compact operator.*

Proof. Using the same notation of the previous Lemma, we have $\Delta^{-2} f = G * f$. Here, the important property of G is that it grows slower than the exponential as $|x| \to +\infty$ (see again [40]). Then the result follows from repeated application of Arzelà-Ascoli's Theorem.

Let indeed $\{u_k\}_k$ be a bounded sequence in X_λ, i.e. $\|u_k\|_{X_\lambda} \leq C$ for all $k \in \mathbb{N}$ for some $C < +\infty$ independent on k. Then, for any $x \in S_\lambda$,

$$|K(u_k)(x)| = \left| \int_{S_\lambda} G(x - y) \mathrm{e}^{u_0(y)} (\mathrm{e}^{u(y)} - 1) \, \mathrm{d}y \right| \leq C_x,$$

because $\mathrm{e}^{u_0(y)}$ goes to zero as $\mathrm{e}^{-|y|}$ for $|y| \to +\infty$ (so, it goes to zero much faster than how all other terms go to infinity), making the integral converge for all $x \in S_\lambda$. Therefore, $\{K(u_k)\}_k$ is uniformly bounded. Similarly, for all $x \in S_\lambda$, one has

$$\left| \nabla K(u_k)(x) \right| \leq \int_{S_\lambda} |\nabla G(x - y)| \mathrm{e}^{u_0(y)} \left| \mathrm{e}^{u(y)} - 1 \right| \mathrm{d}y \leq C_x',$$

again because $\mathrm{e}^{u_0(y)}$ goes to zero as $\mathrm{e}^{-|y|}$ and all the other terms do not grow exponentially to infinity. Hence, $\{K(u_k)\}_k$ is also equicontinuous and thus, by Arzelà-Ascoli, it converges up to subsequences in the C^0 norm.

The same argument applied to the derivatives of $K(u_k)$ gives convergence in all C^j norms (because we can make the derivative fall on G and apply the same argument as before), and thus in particular in the $C^{4,\alpha}$ norm. Moreover, we have already seen that all the $K(u_k)$'s satisfy the growth requirements, so that

$\{K(u_k)\}_k$ has a converging subsequence in the X_λ norm, proving compactness of K. □

Remark. The same argument proves also that the linearization $\Delta^{-2} \circ L$ has the form $I - K$ with K a compact operator.

4.2.2. Change of index and bifurcation

Now that the setup of our problem is complete, we can move to actually proving bifurcation. As mentioned at the beginning of this Section, we plan to use Krasnosel'skii's Bifurcation Theorem 2.3, so that we need to prove that the index of the operator $\Delta^{-2} \circ F_\lambda$ changes for some λ, i.e. that the dimension of the negative space of its linearization $\Delta^{-2} \circ L_\lambda$ changes for some λ.

Lemma 4.6. *The linearized operator*

$$L_\lambda[v] := \Delta^2 v - e^{u_0} v$$

admits an eigenfunction not depending on x_4 and with negative eigenvalue.

Proof. Observe that L_λ can be restricted to an operator

$$\tilde{L}_\lambda := L_\lambda|_{X_\lambda^0} : X_\lambda^0 \to Y_\lambda,$$

where X_λ^0 is the subset of X_λ of functions not depending on x_4. The elements of X_λ^0 are then actually functions of $\mathbb{R}^3$, so in the rest of this proof we will just drop the dependence on x_4.

We construct a function $v : \mathbb{R}^3 \to \mathbb{R}$ that is compactly supported, radial and such that $\langle \tilde{L}_\lambda v, v \rangle_{L^2} < 0$.

Define

$$f(r) := \begin{cases} 0 & \text{if } r \leq 1 \\ e^{-\frac{1}{(r-1)^2}} e^{-\frac{1}{(r-2)^2}} & \text{if } 1 < r < 2 \\ 0 & \text{if } r \geq 2 \end{cases}$$

and take

$$v(x_1, x_2, x_3) := \frac{1}{A} \int_{\sqrt{x_1^2+x_2^2+x_3^2}}^{+\infty} f(s)\,\mathrm{d}s,$$

with

$$A := \int_0^{+\infty} f(s)\,\mathrm{d}s.$$

Notice that v is bounded with compact support, so that it belongs to the space of functions X_λ^0. Observe also that v is constantly equal to 1 if $r := \sqrt{x_1^2 + x_2^2 + x_3^2} < 1$ and is identically 0 outside the ball $B^3(0,2)$, so that its Laplacian is different from zero only in the annulus $1 \leq r \leq 2$. Therefore, setting

$$V\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) := v(x_1, x_2, x_3),$$

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one gets

$$\int_{S_\lambda} (\Delta v)^2 \, \mathrm{d}x = \lambda \int_1^2 \left(V''(r) + \frac{2}{r} V'(r) \right)^2 4\pi r^2 \, \mathrm{d}r = C < +\infty.$$

Fix now a 3-dimensional solution u_1, as found in Subsection 4.1. Recall that we thus have the family $\{u_\mu\}_\mu$ of 3-dimensional solutions (as the functions in this family do not depend on x_4, they can be tought as a functions on $\mathbb{R}^3$). We will then show that we can choose $u_0 \in \{u_\mu\}_\mu$ so that

$$\int_{S_\lambda} \mathrm{e}^{u_0(x)} [v(x)]^2 \, \mathrm{d}x$$

is sufficiently large. In fact

$$\int_{S_\lambda} \mathrm{e}^{u_\mu(x)} [v(x)]^2 \, \mathrm{d}x = \lambda \int_{\mathbb{R}^3} \mu^4 \mathrm{e}^{u_1(\mu x)} \, \mathrm{d}x$$

$$= \lambda \int_{\mathbb{R}^3} \mu^4 \mathrm{e}^{u_1(y)} v^2 \left(\frac{y}{\mu} \right) \frac{\mathrm{d}y}{\mu^3} \geq \lambda \mu \int_{|y| \leq \mu} \mathrm{e}^{u_1(y)} \underbrace{v^2 \left(\frac{y}{\mu} \right)}_{1} \, \mathrm{d}y$$

$$= \lambda \mu \int_{|y| \leq \mu} \mathrm{e}^{u_1(y)} \, \mathrm{d}y \xrightarrow[\mu \to +\infty]{} +\infty$$

Summing up, if we fix $u_0 := u_\mu$ with μ sufficiently large, then $\langle \tilde{L}_\lambda v, v \rangle_{L^2} < 0$, with v the function defined before.

Now, as $X_\lambda^0 \subset L^2(\mathbb{R}^3)$, we get that $\tilde{L}_\lambda$ is self-adjoint and semibounded (in the L^2 norm):

$$\langle \tilde{L}_\lambda v, v \rangle_{L^2} = \int_{\mathbb{R}^3} \left[\left(\Delta^2 v(x) \right) v(x) - \mathrm{e}^{u_0(x)} v^2(x) \right] \mathrm{d}x$$

$$= \int_{\mathbb{R}^3} \left[(\Delta v(x))^2 - \mathrm{e}^{u_0(x)} v^2(x) \right] \mathrm{d}x$$

$$\geq - \int_{\mathbb{R}^3} \mathrm{e}^{u_0(x)} v^2(x) \, \mathrm{d}x \geq -\mathrm{e}^{u_0(0)} \|v\|_{L^2(\mathbb{R}^3)}.$$

Hence we have that the lowest eigenvalue for $\tilde{L}_\lambda$ with eigenfunction in X_λ^0 (note that $v \in X_\lambda^0$ and recall that X_λ^0 is a Banach space) is

$$\nu_0 = \min_{u \in X_\lambda^0 \setminus \{0\}} \frac{\langle \tilde{L}_\lambda u, u \rangle_{L^2}}{\|u\|_{L^2}^2} \leq \frac{\langle \tilde{L}_\lambda v, v \rangle_{L^2}}{\|v\|_{L^2}^2} < 0$$

(see for example [45, Theorem 11.4]). Notice that ν_0 is an eigenvalue of $L_\lambda : X_\lambda \to Y_\lambda$ as well (possibly not the first one) because $X_\lambda^0 \subset X_\lambda$ and that a corresponding eigenfunction $v_0 \in X_\lambda^0$ is an eigenfunction of L_λ (extending it trivially in x_4). Note that v_0 does not depend on x_4 and $\nu_0 < 0$, as required. $\quad\square$

This negative eigenfunction v_0 for L_λ (not depending on x_4) is negative also for $\Delta^{-2} \circ L_\lambda$ (even if not necessarily an eigefunction):

$$\langle (\Delta^{-2} \circ L_\lambda) w_0, w_0 \rangle_{L^2} = \mu_0 \langle \Delta^{-2} w_0, w_0 \rangle_{L^2} = \mu_0 \left\| \Delta^{-1} w_0 \right\|_{L^2}^2 < 0,$$

where the last equality comes from self-adjointness of Δ^{-1}, while the strict inequality comes from the fact that, if $\left\| \Delta^{-1} w_0 \right\|_{L^2} = 0$, then $\|w_0\|_{L^2} = 0$, a contradiction.

Remark. As this function v_0 does not depend on the last variable x_4, it is independent on the choice of λ. Therefore, in what follows, even if the family of functions that we construct depends on λ, the way we construct it does not. To simplify our notation, we will not explicitly indicate the dependence of the family of functions on λ.

We now want to prove that we have a family of linearly independent functions in the negative space of L_λ that gets bigger as λ get larger. For a fixed λ, consider the functions

$$v_k(x_1, x_2, x_3, x_4) := v_0(x_1, x_2, x_3) \cos\left(\frac{2\pi k}{\lambda} x_4\right).$$

Observe that the v_k's are orthogonal, satisfy Neumann conditions on ∂S_λ and their third normal derivatives on ∂S_λ vanish, so that they belong to X_λ.

Lemma 4.7. *The number of values of k for which $\langle (\Delta^{-2} \circ L_\lambda) v_k, v_k \rangle_{L^2} < 0$ is finite for all $\lambda > 0$ and goes to infinity as $\lambda \to +\infty$.*

Proof. Recall first that we already know that $\langle (\Delta^{-2} \circ L_\lambda) v_0, v_0 \rangle_{L^2} < 0$. We also have

$$
\begin{aligned}
\langle (\Delta^{-2} \circ L_\lambda) v_k, v_k \rangle_{L^2} &= \langle v_k - \Delta^{-2}(e^{u_0} v_k), v_k \rangle_{L^2} \\
&= \|v_k\|_{L^2}^2 - \langle \Delta^{-2}(e^{u_0} v_k), v_k \rangle_{L^2} \\
&= \|v_0\|_{L^2}^2 - \langle e^{u_0} v_k, \Delta^{-2} v_k \rangle_{L^2},
\end{aligned}
\tag{4.9}
$$

where $\|v_k\|_{L^2} = \|v_0\|_{L^2}$ because $\int_0^\lambda \cos^2\left(\frac{2\pi k}{\lambda} x_4\right) dx_4 = 1$.

We now have to compute $\Delta^{-2} v_k$ explicitly. Set $w = \Delta^{-2} v_k$, i.e. w is the solution of $\Delta^2 w = v_k$. Decomposing w into its Fourier modes in the x_4 variable we get

$$w(x, x_4) = \sum_{n=0}^{+\infty} w_n(x) \cos\left(\frac{2\pi n}{\lambda} x_4\right)$$

and thus, taking Δ^2 of both sides,

$$v_0(x) \cos\left(\frac{2\pi k}{\lambda} x_4\right) = \sum_{n=0}^{+\infty} \left[\left(\Delta_x - \frac{2\pi n}{\lambda} \right)^2 w_n(x) \right] \cos\left(\frac{2\pi n}{\lambda} x_4\right).$$

For the same Maximum Principle argument of Subsection 4.2.1, the operators $\left(\Delta_x - \frac{2\pi n}{\lambda}\right)^2$ are invertible, and so we get

$$\Delta^{-2} v_k(x, x_4) = \left[\left(\Delta_x - \frac{2\pi k}{\lambda}\right)^{-2} v_0(x)\right] \cos\left(\frac{2\pi k}{\lambda} x_4\right).$$

Plugging this into (4.9), we then obtain

$$\langle (\Delta^{-2} \circ L_\lambda) v_k, v_k \rangle_{L^2} = \|v_0\|_{L^2}^2 - \left\langle e^{u_0} v_0, \left(\Delta_x - \frac{2\pi k}{\lambda}\right)^{-2} v_0 \right\rangle_{L^2}.$$

Consider now the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ defined as

$$f(t) := \|v_0\|_{L^2}^2 - \left\langle e^{u_0} v_0, (\Delta_x - t)^{-2} v_0 \right\rangle_{L^2}.$$

This is a continuous map, because the map that takes an invertible operator to its inverse is continuous in the operator norm [34, Prop. 2.1.1]. By construction

$$f(0) = \|v_0\|_{L^2}^2 - \left\langle e^{u_0} v_0, \Delta^{-2} v_0 \right\rangle_{L^2} = \langle (\Delta^{-2} \circ L_\lambda) v_0, v_0 \rangle_{L^2} < 0,$$

so that $f(t) < 0$ for all $t < \varepsilon$ for some $\varepsilon > 0$. Notice that the number of k's for which $t = \frac{2\pi k}{\lambda} < \varepsilon$ grows with λ. Therefore, to conclude that the number of k's such that $\langle (\Delta^{-2} \circ L_\lambda) v_k, v_k \rangle_{L^2} < 0$ grows with λ, it suffices to prove that for all fixed $\lambda > 0$ there are only finitely many k's for which $\langle (\Delta^{-2} \circ L_\lambda) v_k, v_k \rangle_{L^2} < 0$. This is equivalent to saying that there exists $M > 0$ such that $f(t) > 0$ for all $t > M$. Specifically, we are going to prove that $f(t) \to \|v_0\|^2 > 0$ as $t \to +\infty$.

Fix indeed $u \in X_\lambda$ independent on x_4, then $\left\|(\Delta_x - t)^2 u\right\|_{C^0} \sim |u|_{C^0} t^2$ as $t \to +\infty$ (recall that $|\Delta_x^2 u|_{C^0}, |\Delta_x u|_{C^0} < +\infty$ are fixed). Then

$$\left\|(\Delta_x - t)^{-2} v_0\right\|_{C^0} \to 0, \quad t \to +\infty,$$

decreasing like t^{-2}. Recalling that $e^{u_0} v_0 \in L^1$, we conclude then that

$$\left|\left\langle e^{u_0} v_0, (\Delta_x - t)^{-2} v_0 \right\rangle_{L^2}\right| \leq \|e^{u_0} v_0\|_{L^1} \left\|(\Delta_x - t)^{-2} v_0\right\|_{C^0} \to 0$$

as $t \to +\infty$. Thus $f(t) \to \|v_0\|_{L^2}^2$ when $t \to +\infty$, as wanted. $\qquad\square$

Lemma 4.8. *The dimension of the negative space of $\Delta^{-2} \circ L_\lambda$ goes to $+\infty$ as $\lambda \to +\infty$.*

Proof. Fix λ and suppose that $v_0, \ldots, v_r$ are the negative functions for L_λ of Lemma 4.7.

Notice first that, if v_i, v_j are negative for $\Delta^{-2} \circ L_\lambda$, then any their linear combination $\alpha v_i + \beta v_j$ is negative as well. Indeed: if $i = j$ this is trivial, while

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if $i \neq j$ then

$$
\begin{aligned}
\langle \Delta^{-2} &\circ L_\lambda(\alpha v_i + \beta v_j), \alpha v_i + \beta v_j \rangle_{L^2} \\
&= \alpha^2 \langle (\Delta^{-2} \circ L_\lambda) v_i, v_i \rangle_{L^2} + \beta^2 \langle (\Delta^{-2} \circ L_\lambda) v_j, v_j \rangle_{L^2} \\
&\quad + \alpha\beta \langle (\Delta^{-2} \circ L_\lambda) v_i, v_j \rangle_{L^2} + \alpha\beta \langle (\Delta^{-2} \circ L_\lambda) v_j, v_i \rangle_{L^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle (\Delta^{-2} &\circ L_\lambda) v_i, v_j \rangle_{L^2} \\
&= \langle v_i, v_j \rangle_{L^2} \\
&\quad - \left\langle e^{u_0} v_0(x) \cos\left(\frac{2\pi i}{\lambda} x_4\right), \left[\left(\Delta_x - \frac{2\pi j}{\lambda}\right)^{-2} v_0(x) \right] \cos\left(\frac{2\pi j}{\lambda}\right) \right\rangle_{L^2} = 0,
\end{aligned}
$$

because $\int_0^\lambda \cos\left(\frac{2\pi i}{\lambda} x_4\right) \cos\left(\frac{2\pi j}{\lambda} x_4\right) dx_4 = 0$ for $i \neq j$, so that

$$
\begin{aligned}
\langle \Delta^{-2} &\circ L_\lambda(\alpha v_i + \beta v_j), \alpha v_i + \beta v_j \rangle_{L^2} \\
&= \alpha^2 \langle (\Delta^{-2} \circ L_\lambda) v_i, v_i \rangle_{L^2} + \beta^2 \langle (\Delta^{-2} \circ L_\lambda) v_j, v_j \rangle_{L^2} < 0.
\end{aligned}
$$

Then, by Rayleigh's Min-Max Principle [46, Theorem 12.1] the first r eigenvalues of L_λ are negative: for $n \leq r$ we have

$$
\begin{aligned}
\nu_n &= \min_{\varphi_1, \ldots, \varphi_n} \max\{ \langle L_\lambda \varphi, \varphi \rangle_{L^2} \mid \varphi \in \text{span}(\varphi_1, \ldots, \varphi_n), \|\varphi\|_{L^2} = 1 \} \\
&\leq \max\{ \langle L_\lambda \varphi, \varphi \rangle_{L^2} \mid \varphi \in \text{span}(v_1, \ldots, v_n), \|\varphi\|_{L^2} = 1 \} < 0.
\end{aligned}
$$

As for each of this eigenvalues ν_k (counted with multiplicity) there is an eigenfunction linearly independent to the previous ones, we conclude that the dimension of the negative space of $\Delta^{-2} \circ L_\lambda$ must grow with λ. Indeed, Lemma 4.7 tells us that the number of elements in the negative family of functions grows with λ, and thus, as λ increases, Rayleigh's Min-Max Principle gives us more negative eigenvalues. $\qquad\square$

Proof of Theorem 1.4. Lemma 4.8, along with the fact that $\Delta^{-2} \circ L = I - K$ (where I is the identity and K is a compact operator), shows that there must be some value λ^* of the parameter λ for which the number of negative eigenvalues changes. Indeed, notice that f is an eigenfunction of $-K$ with eigenvalue μ if and only if f is an eigenfunction of $I - K$ with eigenvalue $1 + \mu$. Recalling that the spectrum of the compact operator $-K$ is bounded and accumulating at most in 0 [45, Theorem 6.16], we then get that the spectrum of $I - K$ is bounded and accumulating at most in 1. This implies that there are only finitely many negative eigenvalues, for each fixed value of λ. But, according to Lemma 4.8, the number of negative eigenvalues goes to infinity as $\lambda \to +\infty$. Consequently, there must be some λ^* for which the number of negative eigenvalues increases. But then at λ^* the dimension of the space of negative eigenfunctions grows, so that λ^* is a point of changing index for the operator $\Delta^{-2} \circ F_\lambda$.

Now we can finally apply Krasnosel'skii's Bifurcation Theorem 2.3 to the equation $\Delta^{-2} \circ F_\lambda(u) = 0$. Indeed, we proved that $\Delta^{-2} \circ F_\lambda$ has the form $I - K$ with K compact, and that λ^* is a point of changing index. Hence, λ^* is a point of bifurcation for $\Delta^{-2} \circ F_\lambda$. This means that there is a non-trivial solution of $\Delta^{-2} \circ F_\lambda(u) = 0$ for some λ which is a perturbation of u_0, i.e. a zero of F_λ on S_λ. We can then extend u to the whole $\mathbb{R}^4$ by reflecting it along the boundaries of the strip and repeating this procedure. Continuity is then assured by construction, the accordance of first and third normal derivatives comes from the definition of the space X_λ, and accordance of second derivatives is again given by our construction (the operator $\partial_{x_4}^2$ is indeed invariant under the reflection $x_4 \leftrightarrow -x_4$). Therefore, this is a weak solution and thus a strong solution, by bootstrapping and applying twice Theorem 2.8. As this solution is periodic and non-zero, it must then have infinite volume $\int_{\mathbb{R}^4} e^u = +\infty$. $\qquad\square$

Appendix A. Proof of Weighted Schauder's Estimates Theorem 2.8

Let u be a distributional solution of

$$Lu = a^{ij}(x)\partial_i\partial_j u + b^i(x)\partial_i u + c(x)u = f, \tag{A.1}$$

where L is a uniformly elliptic operator. Take $w \in C^{2,\alpha}(\Omega), w > 0$ and set $v := wu$. We have

$$\partial_i v(x) = \partial_i w(x)u(x) + w(x)\partial_i u(x) \tag{A.2}$$

and

$$\begin{aligned}\partial_i\partial_j v(x) = {} & \partial_i\partial_j w(x)u(x) + \partial_i w(x)\partial_j u(x) \\ & + \partial_j w(x)\partial_i u(x) + w(x)\partial_i\partial_j u(x).\end{aligned} \tag{A.3}$$

Contracting (A.3) with $a^{ij}(x)$ and using (A.1) we get

$$\begin{aligned}& a^{ij}(x)\partial_i\partial_j v(x) + \left(w(x)b^j(x) - 2a^{ij}(x)\partial_i w(x)\right)\partial_j u(x) \\ & + \left(c(x)w(x) - a^{ij}(x)\partial_i\partial_j w(x)\right)u(x) = w(x)f(x) =: g(x).\end{aligned} \tag{A.4}$$

where we also use the fact that, by hypothesis, $a^{ij} = a^{ji}$.

According to the definition of v and to equation (A.2), we know that:

- $u(x) = \dfrac{v(x)}{w(x)}$ (recall that w is never 0),

- $\partial_i u(x) = \dfrac{1}{w(x)}\left(\partial_i v(x) - \dfrac{\partial_i w(x)}{w(x)}v(x)\right)$.

Hence we get

$$\begin{aligned}& a^{ij}(x)\partial_i\partial_j v(x) + \left(b^j(x) - 2a^{ij}(x)\partial_i \log w(x)\right)\partial_j v(x) \\ & + \Big(2a^{ij}\partial_i \log w(x)\partial_j \log w(x) \\ & \quad - a^{ij}(x)\frac{\partial_i\partial_j w(x)}{w(x)} - b^j(x)\partial_j \log w(x) + c(x)\Big)v(x) \\ & \quad = w(x)f(x) =: g(x).\end{aligned} \tag{A.5}$$

Observe now that what we got in this way is still a uniformly elliptic equation, as the highest order coefficients are still the same.

Assume then that the hypothesis on the coefficients of L given by Theorem 2.7 hold. In order to apply the Schauder's estimates on equation (A.5), then, it suffices to check that there exists some positive constant $\tilde{\Lambda}$ such that

$$|\tilde{b}^i|^{(1)}_{0,\alpha,\Omega}, |\tilde{c}|^{(2)}_{0,\alpha,\Omega} \le \tilde{\Lambda},$$

where $\tilde{b}^i$ and $\tilde{c}$ are, respectively, the 1-st and 0-th order coefficients of the new equation (A.5). Moreover, we will need $g \in C^{0,\alpha}(\Omega)$, which is precisely $f \in C^{0,\alpha}_w(\Omega)$.

We first deal with the first order coefficient. By the triangular inequality we have

$$|\tilde{b}^i|^{(1)}_{0,\alpha,\Omega} = |b^i - 2a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega}$$
$$\le |b^i|^{(1)}_{0,\alpha,\Omega} + 2|a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega},$$

hence, it suffices to show that $|a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega}$ is finite. By hypothesis we know that $|a^{ij}|^{(0)}_{0,\alpha,\Omega} \le \Lambda$. Consequently, by Proposition 2.6, we get that

$$|a^{ij}\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le |a^{ij}|^{(0)}_{0,\alpha,\Omega}|\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le \Lambda|\partial_j \log w|^{(1)}_{0,\alpha,\Omega}.$$

Thus, it suffices to require that

$$|\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le C_1 < +\infty, \quad \forall j.$$

We now move to the 0-th order coefficient. Again by the triangular inequality

$$|\tilde{c}|^{(2)}_{0,\alpha,\Omega} = \left|c - b^j\partial_j \log w - a^{ij}\frac{\partial_i\partial_j w}{w} + 2a^{ij}\partial_i \log w\partial_j \log w\right|^{(2)}_{0,\alpha,\Omega}$$

$$\le |c|^{(2)}_{0,\alpha,\Omega} + |b^j\partial_j \log w|^{(2)}_{0,\alpha,\Omega} + \left|a^{ij}\frac{\partial_i\partial_j w}{w}\right|^{(2)}_{0,\alpha,\Omega} + 2|a^{ij}\partial_i \log w\partial_j \log w|^{(2)}_{0,\alpha,\Omega}.$$

By hypothesis, the first summand in this last expression is bounded by Λ. Moreover, again by Proposition 2.6, we get

$$|b^j\partial_j \log w|^{(2)}_{0,\alpha,\Omega} \le |b^j|^{(1)}_{0,\alpha,\Omega}|\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le \Lambda C_1,$$

$$|a^{ij}\partial_i \log w\partial_j \log w|^{(2)}_{0,\alpha,\Omega} \le |a^{ij}|^{(0)}_{0,\alpha,\Omega}|\partial_i \log w|^{(1)}_{0,\alpha,\Omega}|\partial_j \log w|^{(1)}_{0,\alpha,\Omega} \le \Lambda C_1^2$$

and

$$\left|a^{ij}\frac{\partial_i\partial_j w}{w}\right|^{(2)}_{0,\alpha,\Omega} \le |a^{ij}|^{(2)}_{0,\alpha,\Omega}\left|\frac{\partial_i\partial_j w}{w}\right|^{(2)}_{0,\alpha,\Omega} \le \Lambda\left|\frac{\partial_i\partial_j w}{w}\right|^{(2)}_{0,\alpha,\Omega},$$

meaning that it suffices to require that

$$\left|\frac{\partial_i\partial_j w}{w}\right|^{(2)}_{0,\alpha,\Omega} \le C_2 < +\infty, \quad \forall i,j.$$

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Appendix B. Computations on Legendre functions

We solve the equation

$$-\tilde{v}_j''(x) - 2\operatorname{sech}^2(x)\tilde{v}_j(x) = -\left(\frac{\pi j}{\lambda}\right)^2 \tilde{v}_j(x), \quad \forall x \in \mathbb{R}, \forall j \in \mathbb{N}.$$

for fixed $j \in \mathbb{N}$. First, if $j = 0$ this becomes

$$\tilde{v}_0''(x) + 2\operatorname{sech}^2(x)\tilde{v}_0(x) = 0,$$

which has general solution

$$\tilde{v}_0(x) = c_1 \tanh(x) + c_2 \left(-\frac{1}{2}\tanh(x)\log\frac{1 - \tanh(x)}{1 + \tanh(x)} - 1 \right).$$

Now, notice that the first summand is odd[5] and the second is even. Thus, $c_1 = 0$. Moreover, we also have $c_2 = 0$, as the second summand grows linearly at infinity (i.e., faster than the requirements). Hence, we can already exclude the possibility of having elements in $\ker L$ with $j = 0$.

Let then $j > 0$ and make the substitution $y = \tanh(x)$:

$$\left[\left(1 - y^2\right)\tilde{v}_j'(y) \right]' + 2\tilde{v}_j(y) - \frac{1}{1 - y^2}\left(\frac{\pi j}{\lambda}\right)^2 \tilde{v}_j(y) = 0.$$

We get then a Legendre equation with integer degree $l = 1$ and with order $\mu = \frac{2\pi j}{\lambda}$. A general solution is then given by a linear combination of first and second order Legendre functions:

$$\tilde{v}_j(x) = A\,P_1^{\frac{\pi j}{\lambda}}(\tanh(x)) + B\,Q_1^{\frac{\pi j}{\lambda}}(\tanh(x)).$$

Actually, not all the values of A and B are admissible, as we shall immediately see. The following expansions can be found, for instance, on [47].

Suppose first that $B = 0$ and consider thus $P_1^{\frac{\pi j}{\lambda}}$ only. It is known that

$$P_1^{\mu}(y) \underset{y\to 1^-}{\sim} \frac{1}{\Gamma(1 - \mu)}\left(\frac{2}{1 - y}\right)^{\frac{\mu}{2}}$$

---

[5]Indeed, it is the x derivative of the 1-dimensional solution u_0, which is even. Observe that the x derivative of a solution of the original equation is always a solution of the linearized equation. In fact, as u_0 is a solution of the original equation

$$\Delta u_0(x, y) + e^{u_0(x,y)} = 0,$$

taking the derivative in x of this expression one gets

$$\Delta\frac{\partial u_0}{\partial x}(x, y) + e^{u_0(x,y)}\frac{\partial u_0}{\partial x}(x, y) = 0,$$

which means that $\frac{\partial u_0}{\partial x}$ is a solution of the linearized problem in u_0.

for $\mu \notin \mathbb{N}$. Therefore, for such values of μ,

$$P_1^\mu(\tanh(x)) \sim_{x \to +\infty} C e^{\mu x},$$

meaning that such a solution cannot lead to functions in the space X_λ. Hence, we know that, if $B = 0$, μ must be an integer. We now recall that, if μ is an integer and $\mu > l$, then $P_l^\mu \equiv 0$. Consequently, if $B = 0$, the only non-trivial solution is the one with $\mu = 1$, namely $1 = \frac{\pi j}{\lambda}$. Explicitly:

$$\tilde{v}_j(x) = \begin{cases} A \operatorname{sech}(x) & \text{if } j = \frac{\lambda}{\pi} \\ 0 & \text{otherwise} \end{cases}$$

(notice that $\operatorname{sech}(x)$ is even).

Suppose now that $A = 0$ and consider $Q_1^{\frac{\pi j}{\lambda}}$ only. It is known that

$$Q_1^\mu(y) \underset{y \to 1^-}{\sim} \frac{1}{2} \cos(\mu\pi)\Gamma(\mu) \left(\frac{2}{1-y}\right)^{\frac{\mu}{2}}$$

for $\mu \neq \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. As before, then, for such values of μ we have that

$$Q_1^\mu(\tanh(x)) \sim_{x \to +\infty} C e^{\mu x}.$$

Hence, in order to have functions in X_λ, we must require that $\mu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. In this case the expansion at 1^- becomes

$$Q_1^\mu(y) \underset{y \to 1^-}{\sim} (-1)^{\mu + \frac{1}{2}} \frac{\pi\Gamma(\mu+2)}{2\Gamma(\mu+1)\Gamma(2-\mu)} \left(\frac{1-y}{2}\right)^{\frac{\mu}{2}}$$

if $1 \pm \mu = l \pm \mu \neq -1, -2, -3, \ldots$ (which is trivially true). Therefore, the function stays bounded for $x \to +\infty$ (i.e. $y \to 1^-$). Nonetheless,

$$Q_1^\mu(y) = -\cos((1+\mu)\pi)Q_1^\mu(-y) - \frac{\pi}{2}\sin((1+\mu)\pi)P_1^\mu(-y)$$

immediately shows that the function blows-up as $y \to -1^+$, i.e. as $x \to -\infty$. In this way we have excluded all the possible μ's and we can therefore assess that, in order to have a non-trivial $v_j \in X_\lambda$, it must be $A \neq 0$.

We finally have to check that there are no combinations of $A, B \neq 0$ that lead to solutions in X_λ. Observe first that, according to what we said before

- $\mu \in \mathbb{N}$ implies that $A\,P_1^\mu(\tanh(x)) + B\,Q_1^\mu(\tanh(x))$ blows-up exponentially at both $+\infty$ and $-\infty$ (P is finite and Q blows-up as before);

- $\mu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ implies that $A\,P_1^\mu(\tanh(x)) + B\,Q_1^\mu(\tanh(x))$ blows-up exponentially at $-\infty$ (Q is finite and P blows-up as before);

so that we can choose from the beginning $\mu \neq \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$. From the expansion for $y \to 1^-$ we know that:

$$A\,P_1^\mu(y) + B\,Q_1^\mu(y) \underset{y \to 1^-}{\sim} \left[\frac{A}{\Gamma(1-\mu)} + \frac{B}{2}\cos(\mu\pi)\Gamma(\mu)\right] \left(\frac{2}{1-y}\right)^{\frac{\mu}{2}},$$

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so we need
$$A = -\frac{\Gamma(\mu)\Gamma(1-\mu)}{2}\cos(\mu\pi)B.$$

We now turn to the expansions for $y \to -1^+$. We have that

$$Q_1^\mu(y) = -\cos((1+\pi)\pi)Q_1^\mu(-y) - \frac{\pi}{2}\sin((1+\mu)\pi)P_1^\mu(-y)$$

$$\underset{y\to-1^+}{\sim} -\frac{1}{2}\cos((1+\mu)\pi)\cos(\mu\pi)\Gamma(\mu)\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}} +$$

$$-\frac{\pi}{2}\sin((1+\mu)\pi)\frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}}$$

and

$$P_1^\mu(y) = -\frac{2}{\pi}\sin((1+\mu)\pi)Q_1^\mu(-y) + \cos((1+\mu)\pi)P_1^\mu(-y)$$

$$\underset{y\to-1^+}{\sim} -\frac{2}{\pi}\sin((1+\mu)\pi)\frac{1}{2}\cos(\mu\pi)\Gamma(\mu)\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}} +$$

$$+ \cos((1+\mu)\pi)\frac{1}{\Gamma(1-\mu)}\left(\frac{2}{1+y}\right)^{\frac{\mu}{2}}.$$

We hence need to check whether it is possible to have (writing the full expansion for $AP_1^\mu + BQ_1^\mu$ and substituting the value of A we found before)

$$-\frac{\Gamma(\mu)\Gamma(1-\mu)}{2}\cos(\mu\pi)\left[-\frac{\Gamma(\mu)}{\pi}\sin((1+\mu)\pi)\cos(\mu\pi) + \frac{\cos((1+\mu)\pi)}{\Gamma(1-\mu)}\right] +$$

$$+ \left[-\frac{\Gamma(\mu)}{2}\cos((1+\mu)\pi)\cos(\mu\pi) - \frac{\pi}{2}\frac{\sin((1+\mu)\pi)}{\Gamma(1-\mu)}\right] = 0$$

for some μ. This is the only case, indeed, for which the solution does not grow exponentially as $x \to -\infty$ ($y \to -1^+$). This equation in μ can be simplified to

$$-\Gamma(1-\mu)\Gamma(\mu)\cos(\pi\mu)\cot(\pi\mu)\left[\Gamma(\mu)\Gamma(1-\mu)\sin((1+\mu)\pi) + 2\pi\right] = \pi^2$$

and one can check numerically that it does not exist a $\mu \in \mathbb{R}_{>0}$ that satisfies this last expression.

Hence we get

$$\tilde{v}_j \not\equiv 0 \iff -\left(\frac{\pi j}{\lambda}\right)^2 = -1 \iff j = \frac{\lambda}{\pi}.$$

Acknowledgments

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