Degeneration techniques in complex geometry

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Abstract of the Dissertation

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In 2009, B. Berndtsson proved a theorem on the positivity of direct image bundles of positive line bundles. Berndtsson's theorem has been successfully used to give radically new proofs of some fundamental theorems in the part of complex geometry often referred to as L^2 methods. Among these is a proof of the L^2 extension theorem with sharp estimates. This thesis is a step towards determining how much of the classical L^2 theory can be recovered by this technique. The main contribution is a new proof of a Skoda-type L^2 division theorem.

Contents

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Introduction

In a 2009 *Annals of Mathematics* article [4], B. Berndtsson proved two remarkable theorems about the variation of Hilbert spaces of holomorphic sections of line bundles on a family of complete Kähler manifolds. In very broad strokes, Berndtsson's theorems say that, given either a trivial family whose fiber is a bounded domain in a Stein manifold or a smooth proper family, and a semi-positively curved line bundle L on the total space of the family, the direct image (in this case locally trivial) has a metric with Nakano-positive curvature.

More precisely, fix a domain $\Omega \subset \mathbb{C}^m$ and consider the following two situations:

(1) X is a bounded domain in some Stein manifold and $L \to X$ is a holomorphic line bundle, $p : X \times \Omega \to X$ is the projection to the first factor, $\mathrm{e}^{-\phi}$ is a metric for $p^*L \to X \times \Omega$ that is smooth up to the vertical boundary of $X \times \Omega$. For each $\tau \in \Omega$, define $\mathcal{H}_\tau$ to be the Hilbert space of holomorphic sections of $L \otimes K_X \to X$ that are L^2-integrable with respect to the metric $\mathrm{e}^{-\phi_\tau}$ obtained from the metric $\mathrm{e}^{-\phi}|_{X\times\{\tau\}}$ after the identification $X \cong X \times \{\tau\}$. For each $\tau \in \Omega$ the sub-spaces $\mathcal{H}_\tau \subseteq H^0(X, L \otimes K_X)$ are independent of $\tau \in \Omega$, so they form a trivial infinite-rank vector bundle $\mathcal{H} \to \Omega$, with non-trivial metric given by the L^2 inner product determined by $\mathrm{e}^{-\phi_\tau}$.

(2) X is Kähler, $p : X \to \Omega$ is a proper holomorphic submersion with fiber X_τ over $\tau \in \Omega$, and $L \to X$ is a holomorphic line bundle with a smooth Hermitian metric $\mathrm{e}^{-\phi}$. For each $\tau \in \Omega$, define $\mathcal{H}_\tau$ to be the vector space of holomorphic sections of $L|_{X_\tau} \otimes K_{X_\tau}$,

endowed with the L^2 inner product induced by $e^{-\phi}|_{p^{-1}(\tau)}$. As a consequence of the L^2 extension theorem, the $\mathcal{H}_\tau$ fit together to form a finite-rank vector bundle $\mathcal{H} \to \Omega$ with metric given by the L^2 inner product.

Then Berndtsson's first and second theorems can be summarized as follows.

Theorem 1 ([4, Theorems 1.1 and 1.2])**.** *In both situations* (1) *and* (2)*, if* $e^{-\phi}$ *has* (*strictly*) *positive curvature, then the curvature of the metric induced on* $\mathcal{H} \to \Omega$ *by the* L^2 *inner product is* (*strictly*) *positive in the sense of Nakano.*

In the trivial fibration case, the main idea is to realize $\mathcal{H}$ as a subbundle of the bundle $\mathcal{L}$ of smooth L^2-integrable sections. The curvature of $\mathcal{L}$ is easy to compute, and the curvature of $\mathcal{H}$ is obtained from that of $\mathcal{L}$ by a formula of Griffiths for the curvature of subbundles. The bound on the curvature of $\mathcal{H}$ is then a consequence of the Hörmander–Skoda Theorem on the solution of the $\bar{\partial}$-equation with L^2 estimate.

Aside from the intrinsic beauty of Berndtsson's results, Theorem 1 is particularly fascinating because it can be used to obtain new proofs of some fundamental results in complex analytic geometry; proofs that are based on the monotonicity of certain degenerations into situations in which the results are obvious, and that reveal an unexpected underlying convexity. In fact, the full extent of the method is not yet known, and the central goal of the present thesis is to better understand what can be accomplished with Berndtsson's method.

Remarkably, Berndtsson's first theorem can be used to give degeneration-based proofs of Suita's conjecture [9, 7], of the L^2 extension theorem with sharp constants [7], and of the strong openness conjecture [5, 6, 41], while Berndtsson's second theorem gives partial evidence towards Griffiths's conjecture on the existence of Griffiths-positive metrics on ample vector bundles, and more generally provides ways of constructing Nakano-positive metrics for vector bundles [4].

The main contribution of our thesis is a new degeneration proof of the following classical Skoda-type L^2 division theorem. Let X be a Stein manifold and let $E, G \to X$ be holomorphic line bundles with Hermitian metrics $\mathfrak{h}_E, \mathfrak{h}_G$, respectively. Fix holomorphic sections $h_1, \ldots, h_r$ of $E^* \otimes G$ and $1 < \alpha < \frac{r+1}{r-1}$. Given a section g of $G \otimes K_X \to X$ with suitable L^2 estimates, a natural question is whether we can find holomorphic sections $f_1, \ldots, f_r$ of $E \otimes K_X \to X$ such that $g = \sum_{i=1}^r h_i f_i$ and having good L^2 estimates.

Theorem 2 (**L^2 division** [1])**.** *Assume that the curvature of $\mathfrak{h}_E$ is bounded below by $\frac{\alpha(r-1)}{\alpha(r-1)+1}$ times the curvature of $\mathfrak{h}_G$, and that*

$$\int_X \frac{\langle \mathfrak{h}_G, g \wedge \bar{g} \rangle}{(\mathfrak{h}_G \otimes \mathfrak{h}_E^*)(h, \bar{h})^{\alpha(r-1)+1}} < +\infty.$$

Then there are holomorphic sections $f_1, \ldots, f_r$ of $E \otimes K_X \to X$ such that $g = \sum_{i=1}^r h_i f_i$ and

$$\int_X \frac{\langle \mathfrak{h}_E, f \wedge \bar{f} \rangle}{(\mathfrak{h}_G \otimes \mathfrak{h}_E^*)(h, \bar{h})^{\alpha(r-1)}} \leq r \frac{\alpha}{\alpha - 1} \int_X \frac{\langle \mathfrak{h}_G, g \wedge \bar{g} \rangle}{(\mathfrak{h}_G \otimes \mathfrak{h}_E^*)(h, \bar{h})^{\alpha(r-1)+1}}.$$

Skoda's original theorem, which is stated for holomorphic functions and was motivated as an L^2 version of the corona problem, has many remarkable applications in algebraic geometry such as Briançon–Skoda's theorem [54], effective versions of the Nullstellensatz [12, 33, 23], and Y.-T. Siu's proof of deformation invariance of plurigenera [49] and approach to finite generation of the canonical ring [50].

If the number of generators r is at most $\dim X + 1$, Theorem 2 almost recovers the classical line bundle version of Skoda's L^2 division theorem. We say "almost" because in Theorem 2 we have $\|f\|^2_{E^{\oplus r}} \leq \frac{r\alpha}{\alpha-1} \|g\|^2_G$ rather than Skoda's estimate $\|f\|^2_{E^{\oplus r}} \leq \frac{\alpha}{\alpha-1} \|g\|^2_G$ (cf. Corollary 1.8). Still, even though stronger results are known, the intent is to emphasize the technique used to prove Theorem 2: while the standard proof is based on functional analysis and the Bochner–Kodaira–Nakano identity, we instead obtain Theorem 2 by a degeneration argument based on Theorem 1.

The general philosophy is inspired by B. Berndtsson and L. Lempert's proof of the L^2 extension theorem, and by T. Ohsawa's proof of a Skoda-type division theorem as a corollary of the Ohsawa–Takegoshi L^2 extension theorem. Ohsawa indeed remarks that the division problem can be reformulated as an extension problem on the projectivizations of the dual bundles (see [43], [44] and [45, Section 3.2]). It is thus natural to wonder whether a Skoda-type theorem could be proved directly by a degeneration argument, and the present work answers in the affirmative.

The major difficulty in the proof of Theorem 2 lies in defining the correct family of metrics. The situation is complicated by the presence of some negatively curved terms, for which one needs an improved version of some tools employed in [7]. The improvement can in turn be used to generalize and simplify the proof of L^2 extension given by Berndtsson and Lempert.

Outline. Chapter 1 is a review of background material and of the classical L^2 theory: after recalling the notions of positivity for Hermitian metrics of holomorphic vector bundles, we review L. Hörmander's theorem on the solution of the $\bar{\partial}$-equation and the classical proofs of the L^2 extension theorem and the L^2 division theorem.

We present Berndtsson's two theorems in Chapter 2. In particular, we recall the proof of Berndtsson's theorem for trivial fibrations (which is the one needed for degeneration-based proofs), and explain how Berndtsson's theorem for proper fibrations is used to give partial evidence towards Griffiths's conjecture on positive metrics on ample vector bundles. We also show how a version of Berndtsson's theorem on trivial fibrations can be obtained from the L^2 extension theorem with sharp constants.

Chapter 3 explains how the L^2 extension theorem with sharp constants can be proved from Berndtsson's theorem on trivial fibrations via a degeneration argument. The proof is a slight generalization and simplification of the argument of B. Berndtsson and L. Lempert. In the same chapter we will also present a proof of the openness conjecture due to B. Berndtsson, that is based on Theorem 1.

Finally, the degeneration proof of Theorem 2 is explained in Chapter 4, following the author's work in [1].

Chapter 1

Classical L^2 theory

A very short survey of some fundamental results in the classical L^2 theory of complex geometry is presented. After recalling in Section 1.1 the various notions of positivity for Hermitian metrics on vector bundles, in Section 1.2 we sketch the proof of H. Skoda's version of the fundamental theorem of L. Hörmander on the solution of the $\bar\partial$-equation. Then we present the classical proofs of the L^2 extension theorem in Section 1.3, and of the L^2 division theorem in Section 1.4.

1.1 Notions of positivity

Let $V \to X$ be a holomorphic vector bundle, meaning that the projection map $\pi : V \to X$ is holomorphic and that every point $x \in X$ has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times W$, where W is a vector space with a smoothly varying Hermitian inner product. If W has infinite dimension, we moreover assume that the inner products are complete, i.e. that the fibers are Hilbert spaces. Fix a Hermitian metric g on X.

Let h be a Hermitian metric for $V \to X$. The Chern connection ∇ for h is the unique connection that is compatible with h and such that $\nabla^{0,1} = \bar\partial$. Its $(1,0)$-part is thus given by

$$\partial h(u, v) = h(\nabla^{1,0} u, v) + h(u, \bar\partial\, v).$$

The curvature of the Chern connection ∇ is the $\operatorname{End}(V)$-valued $(1,1)$-form $\Theta(h)$ defined by

$$\Theta(h)\alpha := (\nabla^{1,0}\,\bar\partial + \bar\partial\,\nabla^{1,0})\alpha.$$

Definition 1.1. We say that h has m-positive curvature in the sense of Demailly at a point $x \in X$ (and write $\Theta(h) >_m 0$) if there is $c > 0$ such that

$$h\left(\Theta(h)\sum_{i=1}^{m} u_j \otimes \zeta_j, \sum_{k=1}^{m} u_k \otimes \zeta_k\right) \geq c \sum_{j,k=1}^{m} h(u_j, u_k) g(\zeta_j, \zeta_k)$$

for all $u_1 \otimes \zeta_1, \ldots, u_m \otimes \zeta_m \in V_x \otimes T^{1,0}_{X,x}$. In particular, $\Theta(h)$ (or just h, with an abuse of notation) is said to be Griffiths-positive ($\Theta(h) >_{\mathrm{Griff}} 0$) if it is 1-positive, and it is said to be Nakano-positive ($\Theta(h) >_{\mathrm{Nak}} 0$) if it is m-positive for all m, or equivalently for $m = \min(\operatorname{rank} V, \dim X)$.

Clearly $(m+1)$-positivity implies m-positivity, and thus Nakano-positivity implies Griffiths-postivity. The notion of m-negativity can be defined analogously.

Let $t = (t_1, \ldots, t_n)$ be a system of local coordinates on an open chart $U \subseteq X$. Then the Chern connection is the collection of operators $\nabla^{1,0}_{\frac{\partial}{\partial t_j}}$ acting on smooth sections of $U \times W$ defined by

$$\partial_{t_j} h(u,v) = h\left(\nabla^{1,0}_{\frac{\partial}{\partial t_j}} u, v\right) + h\left(u, \partial_{\bar t_j} v\right),$$

and the curvature of the Chern connection is the $\operatorname{End}(V)$-valued $(1,1)$-form $\Theta(h) = \sum_{j,k=1}^{n} \Theta_{j\bar k}\, \mathrm{d}t_j \wedge \mathrm{d}\bar t_k$ whose coefficients are the commutators $\Theta_{j\bar k} := \left[\nabla^{1,0}_{\frac{\partial}{\partial t_j}}, \partial_{\bar t_k}\right]$. Then h is Griffiths-positive (at $x \in X$) if and only if there is $c > 0$ such that

$$\sum_{j,k=1}^{n} h_x(\Theta_{j\bar k} u, u) v_j \bar v_k \geq c h_x(u,u)|v|^2$$

for all $u \in V_x$ and $v \in \mathbb{C}^n$, and h is Nakano-positive (at $x \in X$) if and only if there is $c > 0$ such that

$$\sum_{j,k=1}^{n} h_x(\Theta_{j\bar{k}} u_j, u_k) \geq c \sum_{j=1}^{n} h_x(u_j, u_j)$$

for all n-tuples $(u_1, \ldots, u_n) \in V_x^{\oplus n}$.

The dual bundle of V is the vector bundle V^* whose fiber V_x^* is the Hilbert space dual of V_x. Once a metric is fixed, the Riesz Representation Theorem gives a conjugate-linear isometry $\mathcal{R} : V^* \to V$ defined by $\xi(u) = h_x(u, \mathcal{R}\xi)$ for all $u \in V_x$ and $\xi \in V_x^*$. Denoting by $\Theta^*_{j\bar{k}}$ the coefficients of the curvature of the dual metric h^* on V^*, one then computes

$$\sum_{j,k=1}^{n} h_x^*(\Theta^*_{j\bar{k}} \xi_j, \xi_k) = - \sum_{j,k=1}^{n} h_x(\Theta_{j\bar{k}}(\mathcal{R}\xi_k), (\mathcal{R}\xi_j))$$

for all n-tuples $(\xi_1, \ldots, \xi_n) \in V_x^{*\oplus n}$. Therefore h has Griffiths-positive curvature if and only if h^* has Griffiths-negative curvature, but the same does not hold for Nakano-positivity and Nakano-negativity, because the indices on the right-hand side are reversed.

Griffiths negativity has a useful characterization in terms of the plurisubharmonicity of the logarithm of norms of sections.

Proposition 1.2. *The following are equivalent for a Hermitian metric h for a holomorphic vector bundle $V \to X$:*

(i) h *is (strictly) Griffiths-negative,*

(ii) $\log h(\sigma, \sigma)$ *is a (strictly) plurisubharmonic function on* X *for all holomorphic sections* σ *of* $V \to X$,

(iii) $h(\sigma, \sigma)$ *is a (strictly) plurisubharmonic function on* X *for all holomorphic sections* σ *of* $V \to X$,

(iv) $\log h(\cdot, \cdot)$ *is a (strictly) plurisubharmonic function on the total space* V,

(v) $h(\cdot, \cdot)$ *is a (strictly) plurisubharmonic function on the total space* V.

Proof. We compute

$$\sqrt{-1}\,\partial\bar{\partial}\log h(\sigma,\sigma) = -\frac{h(\Theta(h)\sigma,\sigma)}{h(\sigma,\sigma)} + \frac{h(\sigma,\sigma)h(\nabla^{1,0}\sigma,\nabla^{1,0}\sigma) - h(\nabla^{1,0}\sigma,\sigma)\wedge h(\sigma,\nabla^{1,0}\sigma)}{h(\sigma,\sigma)^2}. \tag{1.1}$$

(i)⇔(ii) The second term of (1.1) is non-negative by the Cauchy–Schwarz inequality, so if h is (strictly) Griffiths-negative, then $\log h(\sigma,\sigma)$ is (strictly) plurisubharmonic.

Conversely, we can assume that V is a trivial vector bundle (with non-trivial metric), since the negativity of the curvature is a local property. Then given any vector $v \in V_x$ there is a holomorphic section σ_v of $V \to X$ such that $\sigma_v(x) = v$ and $\nabla^{1,0}\sigma_v(x) = 0$. Plugging this into (1.1) we obtain

$$\sqrt{-1}\,\partial\bar{\partial}\log h(\sigma_v,\sigma_v) = -\frac{h(\Theta(h)\sigma_v,\sigma_v)}{h(\sigma_v,\sigma_v)},$$

showing that if $\log h(\sigma,\sigma)$ is (strictly) plurisubharmonic for all sections σ of $V \to X$, then h has (strictly) negative curvature in the sense of Griffiths.

(ii)⇔(iii) Since the exponential function is strictly convex increasing, if $\log h(\sigma,\sigma)$ is (strictly) plurisubharmonic then $h(\sigma,\sigma)$ is (strictly) plurisubharmonic.

Conversely, assume that $h(\tilde{\sigma},\tilde{\sigma})$ is plurisubharmonic for all holomorphic sections $\tilde{\sigma}$ of $V \to X$. Fix such a section σ and a disk $D \subset X$, and assume that $\log h(\sigma,\sigma)|_{\partial D} \leq \Phi|_{\partial D}$ for some harmonic function on D. Write $\Phi = 2\,\mathbb{R}\mathrm{e}\,F$, where F is a holomorphic function on D. Taking exponentials we have $h(\sigma,\sigma)|_{\partial D} \leq |\,\mathrm{e}^F|^2|_{\partial D}$, i.e. $h(\mathrm{e}^{-F}\sigma,\mathrm{e}^{-F}\sigma)|_{\partial D} \leq 1$. Since $h(\tilde{\sigma},\tilde{\sigma})$ is plurisubharmonic by assumption for all sections $\tilde{\sigma}$, it follows that $h(\mathrm{e}^{-F}\sigma,\mathrm{e}^{-F}\sigma) \leq 1$ on D, so that $\log h(\sigma,\sigma) \leq \Phi$ on D, proving that $\log h(\sigma,\sigma)$ is plurisubharmonic.

(ii)$\Rightarrow$(iv) Assume (ii). Fix a point v in the total space V and a direction ζ in the tangent space to V at v. We need to check that $\log h(\cdot,\cdot)$ is subharmonic when restricted to some curve C in the total space V tangent to ζ. If ζ projects to zero by the bundle map $V \to X$ we can choose C to be contained in the fiber of v and use the fact that the restriction of $\log h(\cdot,\cdot)$ to any fiber is automatically plurisubharmonic. If instead v does not project to the zero vector we can choose C contained in the graph of some section σ of $V \to X$, and then (iv) follows as before.

The converse follows directly from the fact that $\log h(\sigma,\sigma)$ being (strictly) plurisubharmonic function on X is equivalent to $\log h(\cdot,\cdot)$ being subharmonic when restricted to the graph of σ.

(iv)$\Leftrightarrow$(v) This is similar to [(ii)$\Leftrightarrow$(iii)] and is thus omitted. □

We conclude this Section by a useful formula of P. Griffiths on the curvature of subbundles. This formula will play a central role in the proof of Berndtsson's first theorem (Section 2.1).

Proposition 1.3 (**Griffiths's formula for the curvature of subbundles**). *Let E be a holomorphic subbundle of a holomorphic bundle F over a complex manifold M. Fix a metric h for F, endow E with the induced metric, and denote by $\pi^\perp$ the orthogonal projection of F to the orthogonal complement of E with respect to h. Then*

$$h(\Theta^F u, v) = h(\pi^\perp \nabla^{F1,0} u, \pi^\perp \nabla^{F1,0} v) + h(\Theta^E u, v)$$

for any two sections u, v of E.

Proof. Let $\pi : F \to E$ be the projection of F to E. For any two sections u, v of $E \to M$ we have

$$\partial h(u,v) = h(\nabla^{F1,0} u, v) + h(u, \bar\partial v) = h(\pi \nabla^{F1,0} u, v) + h(u, \bar\partial v),$$

so that $\nabla^{E1,0} = \pi\nabla^{F1,0}$. Let $\bar{\partial}\,\pi$ be defined by

$$\bar{\partial}(\pi\sigma) = (\bar{\partial}\,\pi)\sigma + \pi(\bar{\partial}\,\sigma)$$

and notice that $(\bar{\partial}\,\pi)u = 0$ when u is a section of $E \to M$. Then

$$\begin{aligned}\Theta^E u = [\nabla^{E1,0}, \bar{\partial}]u &= \pi\nabla^{F1,0}\,\bar{\partial}\,u + \bar{\partial}(\pi\nabla^{F1,0}u)\\ &= \pi\Theta^F u + (\bar{\partial}\,\pi)\nabla^{F1,0}u = \pi\Theta^F u + (\bar{\partial}\,\pi)\pi^{\perp}\nabla^{F1,0}u\\ &= \pi\Theta^F u - \pi\,\bar{\partial}(\pi^{\perp}\nabla^{F1,0}u),\end{aligned}$$

where the last line follows from differentiating $\pi\pi^{\perp} = 0$. Therefore

$$\begin{aligned}h(\Theta^E u, v) &= h(\pi\Theta^F u, v) - h(\pi\,\bar{\partial}(\pi^{\perp}\nabla^{F1,0}u), v)\\ &= h(\Theta^F u, v) - h(\bar{\partial}(\pi^{\perp}\nabla^{F1,0}u), v)\\ &= h(\Theta^F u, v) - h(\pi^{\perp}\nabla^{F1,0}u, \nabla^{F1,0}v)\\ &= h(\Theta^F u, v) - h(\pi^{\perp}\nabla^{F1,0}u, \pi^{\perp}\nabla^{F1,0}v),\end{aligned}$$

proving Griffiths's formula (the second to last equality holds by metric compatibility because $h(\pi^{\perp}\nabla^{F1,0}u, v) = 0$, since $v = \pi v$). □

1.2 The Hörmander–Skoda Theorem on the solution of the $\bar{\partial}$-equation

We now recall the following theorem on the solution of the $\bar{\partial}$-equation. This theorem is originally due to L. Hörmander [30, 31], and A. Andreotti and E. Vesentini [2]. Here we present a version due to H. Skoda [51].

Theorem 1.4 (**Hörmander–Skoda Theorem**). *Let X be a complete Kähler manifold of complex dimension n, and let $V \to X$ be a holomorphic vector bundle. Fix $p \geq 0, q \geq 1$ and a not necessarily complete Kähler form ω. Assume that V has a Hermitian metric*

h with whose operator $A = A^{p,q}_{h,\omega} = [\Theta(h), \Lambda_\omega]$ induced by curvature on $V \otimes \Lambda^{p,p} T^*_X$ is positively defined everywhere. Then for any V-valued (p,q)-form f such that $\bar\partial f = 0$ and

$$\int_X \langle A^{-1} f, f\rangle_{h,\omega} \omega^n < +\infty$$

there is a V-valued $(p, q-1)$-form u such that $\bar\partial u = f$ and

$$\int_X |u|^2_h \omega^n \le \int_X \langle A^{-1} f, f\rangle_{h,\omega} \omega^n < +\infty. \tag{1.2}$$

We will just present the main idea and refer to Demailly [15, 20] for further details and the complete proof.

Idea of the proof. First assume that ω is complete, then for any L^2-integrable V-valued (p,q) form σ with $\bar\partial \sigma \in L^2$ and $\bar\partial^* \sigma \in L^2$ in the sense of distribution there is a sequence of smooth forms σ_ν with compact support and such that $\sigma_\nu \to \sigma$, $\bar\partial \sigma_\nu \to \bar\partial \sigma$, and $\bar\partial^* \sigma_\nu \to \bar\partial^* \sigma$ in L^2. Consequently, the Bochner–Kodaira–Nakano Identity [10, 11, 32, 39] implies that

$$\left\|\bar\partial \sigma\right\|^2 + \left\|\bar\partial^* \sigma\right\|^2 \ge \int_X \langle A\sigma, \sigma\rangle_{h,\omega} \omega^n \tag{1.3}$$

holds for $\sigma \in L^2$ with $\bar\partial \sigma \in L^2$ and $\bar\partial^* \sigma \in L^2$.

Let $\mathcal{D}^{p,q}$ denote the set of smooth L^2-integrable V-valued (p,q)-forms with compact support. Fix $v \in \mathcal{D}^{p,q}$ and consider its decomposition $v = v_1 + v_2$ according to the orthogonal decomposition

$$L^2(X, V \otimes \Lambda^{p,q} T^*_X) = \ker \bar\partial \oplus (\ker \bar\partial)^\perp.$$

Since $f, v_1 \in \ker \bar\partial$, by the Cauchy–Schwarz inequality we have

$$\left|(f, v)_{L^2_{p,q}}\right|^2 = \left|(f, v_1)_{L^2_{p,q}}\right|^2 \le \int_X \langle A^{-1} f, f\rangle_{h,\omega} \omega^n \int_X \langle A v_1, v_1\rangle_{h,\omega} \omega^n.$$

According to (1.3) the second integral can be estimated by

$$\int_X \langle A v_1, v_1\rangle_{h,\omega}\omega^n \leq \left\|\bar\partial v_1\right\|^2 + \left\|\bar\partial^* v_1\right\|^2 = \left\|\bar\partial^* v_1\right\|^2 = \left\|\bar\partial^* v\right\|^2,$$

where the last equality follows from the fact that $v_2 \in (\ker\bar\partial)^\perp \subset \ker\bar\partial^*$. Hence

$$\left|(f,v)_{L^2_{p,q}}\right|^2 \leq \left(\int_X \langle A^{-1}f, f\rangle_{h,\omega}\omega^n\right)\left\|\bar\partial^* v\right\|^2$$

for every $v \in \mathcal{D}^{p,q}$. As a consequence, we have a well-defined bounded linear functional

$$\begin{aligned} \ell : \bar\partial^*(\mathcal{D}^{p,q}) &\longrightarrow \mathbb{C} \\ w = \bar\partial^* v &\longmapsto (f,v)_{L^2_{p,q}} \end{aligned}$$

whose norm is bounded by $\left(\int_X \langle A^{-1}f, f\rangle_{h,\omega}\omega^n\right)^{1/2}$. Extending ℓ by 0 in $\bar\partial^*(\mathcal{D}^{p,q})^\perp$ and applying the Riesz Representation Theorem, we find an element $u \in L^2(X, V\otimes\Lambda^{p,q}T^*_X)$ such that $(v,f)_{L^2_{p,q}} = (\bar\partial^* v, u)_{L^2_{p,q-1}}$ for all $v \in L^2(X, V\otimes\Lambda^{p,q-1}T^*_X)$ and

$$\int_X |u|^2_h\omega^n \leq \int_X \langle A^{-1}f, f\rangle_{h,\omega}\omega^n.$$

Hence $\bar\partial u = f$ in the sense of distributions and (1.2) is satisfied.

To pass to the case of ω not complete, one replaces ω by $\omega_\varepsilon := \omega + \varepsilon\theta$ with θ complete; we refer to Demailly for details [20]. □

The relevant case of Theorem 1.4 for the purposes of this thesis is $(p,q) = (n,1)$, and in such setting the hypothesis on the metric is that h has Nakano-positive curvature. If further V is a line bundle L with metric $e^{-\phi}$, then Theorem 1.4 says that, if $\sqrt{-1}\,\partial\bar\partial\phi > 0$, then for all $L\otimes K_X$-valued $(0,1)$-forms f such that $\bar\partial f = 0$ and

$$\int_X |f|^2_{\sqrt{-1}\,\partial\bar\partial\phi}\, e^{-\phi} < +\infty$$

there is a section u of $L \otimes K_X$ such that $\bar{\partial} u = f$ and

$$\int_X |u|^2 \, \mathrm{e}^{-\phi} \leq \int_X |f|^2_{\sqrt{-1}\,\partial\bar{\partial}\,\phi} \, \mathrm{e}^{-\phi} .$$

1.3 The L^2 extension theorem

In this section we explain the classical approach to the following extension problem. Let Z be a complex submanifold of some given Stein manifold X of complex dimension n, and let $L \to X$ be a sufficiently positive holomorphic line bundle. Given an L-valued holomorphic $(n, 0)$-form f along Z with suitable L^2 estimates, we want to find an L-valued holomorphic $(n, 0)$-form F on X satisfying $F|_X = f$ and with good estimates on its L^2 norm.

The first solution to the L^2 extension theorem was obtained by T. Ohsawa and K. Takegoshi in 1987 [46], albeit with a non-optimal estimate. In the years that followed, many generalizations and improvements have been achieved, to name a few: extension of top forms with values in vector bundles [36, 19], extension from submanifolds of higher codimension [36, 19, 42], and L^2 extension with optimal constant [8, 27].

For simplicity, here we present a statement of extension for line bundles from submanifold of codimension 1.

Theorem 1.5 (**L^2 extension**). *Let X be a Stein manifold of complex dimension n, and let $Z \subset X$ be an analytic hypersurface. Let $L_Z \to X$ be the holomorphic line bundle associated to Z, with $T \in H^0(X, L_Z)$ such that $Z = (T = 0)$ and $\mathrm{d}T|_Z$ generically non-zero. Assume moreover that L_Z carries a (singular) Hermitian metric $\mathrm{e}^{-\lambda}$ such that $\mathrm{e}^{-\lambda}|_Z \not\equiv +\infty$ and $\sup_X |T|^2 \, \mathrm{e}^{-\lambda} \leq 1$. Let $L \to X$ be a line bundle with (singular) Hermitian metric $\mathrm{e}^{-\varphi}$ such that*

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq 0 \quad \textit{and} \quad \sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \delta \sqrt{-1}\,\partial\bar{\partial}\,\lambda$$

for some $\delta > 0$. Then for any holomorphic section $f \in H^0(Z, L|_Z \otimes K_Z)$ such that

$$\|f\|^2_Z := \int_Z |f|^2 \, \mathrm{e}^{-\varphi} < +\infty$$

there is a holomorphic section $F \in H^0(X, L \otimes L_Z \otimes K_X)$ *such that* $F|_Z = f \wedge \mathrm{d}T$ *and*

$$\|F\|_X^2 := \int_X |F|^2 \,\mathrm{e}^{-\lambda-\varphi} \leq \pi \left(1 + \frac{1}{\delta}\right) \int_Z |f|^2 \,\mathrm{e}^{-\varphi} = \pi \left(1 + \frac{1}{\delta}\right) \|f\|_Z^2 \,.$$

The L^2 extension theorem has several applications in complex analytic and algebraic geometry; for instance Demailly's approximation of singular Hermitian metrics [16, 18], the proof of the strong openness conjecture by Q. Guan and X. Zhou [26], the proof of the deformation invariance of plurigenera [49, 47], and generalizations of the Nadel vanishing theorem [13]. It can also be used to prove the hard Lefschetz theorem for pseudoeffective line bundles [22], and it is instrumental in various result related to Fujita's conjecture [3, 17].

The main idea for the classical proof of Theorem 1.5 is to obtain a smooth extension and then correct it to a holomorphic one. Control on the norm of the extension is achieved by means of twisted estimates. To simplify the exposition, we will give up obtaining the optimal constant of Theorem 1.5, and we will mostly follow the ideas of [37]. See [8] and [27] for the proof with the optimal constant.

Proof. Fix the section f to be extended. By some standard reductions (cf. Subsection 3.1.1), we can assume that Z is smooth and that X is a relatively compact domain in some larger Stein manifold, to which all objects extend. We can also assume that the metrics are smooth. Then, since X is Stein, we automatically have an extension $\tilde{F}$ with finite L^2 norm, but we don't have any control on the L^2 norm of such solution (see Proposition 3.3 for a more detailed argument).

We will now modify $\tilde{F}$ to a smooth extension, and then correct this smooth extension to a holomorphic one by solving a twisted $\bar{\partial}$-equation. Let $t \in (0, 1)$ and $\chi \in C_c^\infty([0, 1))$ with

$$0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ on } [0, t] \quad \text{and} \quad |\chi'| \leq 1 + t.$$

Set $v := \log(|T|^2 \mathrm{e}^{-\lambda})$, $\chi_\varepsilon := \chi(\mathrm{e}^v/\varepsilon^2)$, and

$$\alpha_\varepsilon := \bar\partial(\chi_\varepsilon \tilde F) = \frac{1}{\varepsilon^2}\chi'(\mathrm{e}^v/\varepsilon^2)\tilde F \wedge \bar\partial \mathrm{e}^v = \tilde F \wedge \frac{2\,\mathrm{e}^{v/2}}{\varepsilon^2}\chi'(\mathrm{e}^v/\varepsilon^2)\,\bar\partial(\mathrm{e}^{v/2}).$$

Set also

$$\psi_\nu := \varphi + \log(\mathrm{e}^v + \nu^2) + \lambda + \nu^2\,\partial\bar\partial\,\rho,$$

so that $\mathrm{e}^{-\psi_\nu}$ is a metric for $L \otimes L_Z$. Here ρ is some bounded strictly plurisubharmonic function to be specified later on. Then for all $(L\otimes L_Z)$-valued $(n,1)$-forms u we estimate

$$\begin{aligned}
|(u,\alpha_\varepsilon)|^2 &= \left|\int_X \langle u,\alpha_\varepsilon\rangle_\omega \mathrm{e}^{-\psi_\nu}\right|^2 \le \left(\int_X |\langle u,\alpha_\varepsilon\rangle|\,\mathrm{e}^{-\psi_\nu}\right)^2\\
&= \left(\int_X \left|\left\langle u, \frac{2\,\mathrm{e}^{v/2}}{\varepsilon^2}\chi'(\mathrm{e}^v/\varepsilon^2)\,\bar\partial(\mathrm{e}^{v/2})\right\rangle\right| \mathrm{e}^{-\psi_\nu}\right)^2\\
&\le \left(\frac{1}{\delta}\int_X \left|\frac{\tilde F}{\varepsilon^2}\chi'(\mathrm{e}^v/\varepsilon^2)\right|^2 \frac{(\mathrm{e}^v+\varepsilon^2)^2}{\varepsilon^2}\,\mathrm{e}^{-\phi-\lambda}\right)\left(\delta\int_X \left|\left\langle u,\bar\partial(\mathrm{e}^{v/2})\right\rangle\right|^2 \frac{4\varepsilon^2}{(\mathrm{e}^v+\varepsilon^2)^2}\,\mathrm{e}^{-\psi_\nu}\right)\\
&\le C_\varepsilon\delta\int_X \left|\left\langle u,\bar\partial(\mathrm{e}^{v/2})\right\rangle\right|^2 \frac{4\varepsilon^2}{(\mathrm{e}^v+\varepsilon^2)^2}\,\mathrm{e}^{-\psi_\nu},
\end{aligned} \tag{1.4}$$

where

$$C_\varepsilon := \frac{4(1+t)^2}{\delta\varepsilon^2}\int_{\mathrm{e}^v\le\varepsilon^2} |\tilde F|^2\,\mathrm{e}^{-\phi-\lambda}$$

and the last inequality follows from $|\chi'| \le 1+t$ and $t\varepsilon^2 \le \mathrm{e}^v \le \varepsilon^2$ on the support of $\chi'(\mathrm{e}^v/\varepsilon^2)$. For later use notice that

$$\limsup_{\varepsilon\to 0} C_\varepsilon = \frac{8\pi(1+t)^2}{\delta}\int_Z |f|^2\,\mathrm{e}^{-\varphi}.$$

We now bound the last term in (1.4). We start by the following twisted Bochner–Kodaira–Nakano estimate.

Lemma 1.6. *Let X be a bounded pseudoconvex domain in some ambient Stein manifold Y of complex dimension n, and let $H \to Y$ be a holomorphic line bundle with smooth Hermitian metric $\mathrm{e}^{-\psi}$. Fix a Kähler metric g on Y. Let τ and A be positive functions on X, with $\tau \in C^2(X)$. Then for all H-valued $(n,1)$-forms in the domain of $\bar\partial$ and of the adjoint $\bar\partial^*_\psi$ the following inequality holds*

$$\int_X (\tau + A)|\,\bar\partial^*_\psi u|^2\,\mathrm{e}^{-\psi} + \int_X \tau|\,\bar\partial\, u|^2_g\,\mathrm{e}^{-\psi} \geq \int_X \left(\tau\,\partial\bar\partial\,\psi - \partial\bar\partial\,\tau - \frac{1}{A}\partial\tau \wedge \bar\partial\,\tau\right)_{i\bar j} g^{i\bar k} u_{\bar k}\overline{g^{j\bar l} u_{\bar l}}\,\mathrm{e}^{-\psi}\,.$$

The main idea for the proof of Lemma 1.6 is to apply the standard Bochner–Kodaira–Nakano identity [10, 11, 32, 39] to the twisted metric $\tau\,\mathrm{e}^{-\psi}$, dropping terms by positivity and Cauchy–Schwarz inequality (see Lemma 2.1 in [37]).

We will apply Lemma 1.6 to $H = L \otimes L_Z$ and $\mathrm{e}^{-\psi} = \mathrm{e}^{-\psi_\nu}$ as defined above. Notice that by the Poincaré–Lelong formula we have

$$\sqrt{-1}\,\partial\bar\partial\,v = 2\pi[Z] - \sqrt{-1}\,\partial\bar\partial\,\lambda,$$

where $[Z]$ denotes the current of integration along Z. Fix $\gamma > 1$ and define the function

$$a := \gamma - \delta\log(\mathrm{e}^v + \varepsilon^2) = \gamma - \delta\log(|T|^2\,\mathrm{e}^{-\lambda} + \varepsilon^2).$$

Then

$$-\,\partial\bar\partial\,a = \delta\,\partial\bar\partial\log(\mathrm{e}^v + \varepsilon^2) = \delta\partial\left(\frac{\bar\partial\,\mathrm{e}^v}{\mathrm{e}^v + \varepsilon^2}\right) = \delta\frac{\mathrm{e}^v}{\mathrm{e}^v + \varepsilon^2}\,\partial\bar\partial\,v + \delta\frac{4\varepsilon^2|\partial\,\mathrm{e}^{v/2}|^2}{(\mathrm{e}^v + \varepsilon^2)^2}$$
$$= \delta\frac{\mathrm{e}^v}{\mathrm{e}^v + \varepsilon^2}(2\pi[Z] - \partial\bar\partial\,\lambda) + \delta\frac{4\varepsilon^2|\partial\,\mathrm{e}^{v/2}|^2}{(\mathrm{e}^v + \varepsilon^2)^2} = -\delta\frac{\mathrm{e}^v}{\mathrm{e}^v + \varepsilon^2}\,\partial\bar\partial\,\lambda + \delta\frac{4\varepsilon^2|\partial\,\mathrm{e}^{v/2}|^2}{(\mathrm{e}^v + \varepsilon^2)^2},$$

where the last equality holds because $[Z]$ is supported on Z, where $\mathrm{e}^v = |T|^2\,\mathrm{e}^{-\lambda}$ vanishes.

We now take $\tau = a + h \circ a$ and $A = \frac{(1+h'\circ a)^2}{-h''\circ a}$, where $h : [0, +\infty) \to (0, +\infty)$ is a C^2 function that will be specified later. For the moment, since τ and A are required to be positive, we only note that we need $h' > -1$ and $h'' < 0$. Then we compute

$$-\partial\bar{\partial}\,\tau - \frac{1}{A}\partial\tau \wedge \bar{\partial}\,\tau = -(1 + h \circ a)\,\partial\bar{\partial}\,a.$$

Recall that

$$\psi_\nu = \varphi + \log(\mathrm{e}^v + \nu^2) + \lambda + \nu^2\,\partial\bar{\partial}\,\rho,$$

where we now specify ρ to be a bounded strictly plurisubharmonic function so that

$$\sqrt{-1}\,\partial\bar{\partial}\,\rho + \frac{1+\delta}{\mathrm{e}^v + \nu^2}\,\sqrt{-1}\,\partial\bar{\partial}\,\lambda \geq 0$$

for all $0 \leq \nu \leq 1$. Then

$$\begin{aligned}\partial\bar{\partial}\,\psi_\nu &= \partial\bar{\partial}\,\varphi + \partial\bar{\partial}\,\lambda - \frac{\mathrm{e}^v}{\mathrm{e}^v + \nu^2}\,\partial\bar{\partial}\,\lambda + 4\nu^2\frac{|\partial\,\mathrm{e}^{v/2}\,|^2}{(\mathrm{e}^v + \nu^2)^2} + \nu^2\,\partial\bar{\partial}\,\rho \\ &\geq \partial\bar{\partial}\,\varphi - \frac{\nu^2}{\mathrm{e}^v + \nu^2}\delta\,\partial\bar{\partial}\,\lambda + 4\nu^2\frac{|\partial\,\mathrm{e}^{v/2}\,|^2}{(\mathrm{e}^v + \nu^2)^2} \geq 0\end{aligned}$$

and

$$\partial\bar{\partial}\,\psi_\nu - \delta\,\partial\bar{\partial}\,\lambda \geq \partial\bar{\partial}\,\varphi - \delta\left(1 - \frac{\nu^2}{\delta(\mathrm{e}^v + \nu^2)}\right)\partial\bar{\partial}\,\lambda \geq 0.$$

Therefore

$$\begin{aligned}\tau\,\partial\bar{\partial}\,\psi_\nu - \partial\bar{\partial}\,\tau - \frac{1}{A}\partial\tau \wedge \bar{\partial}\,\tau = (a + h \circ a)\delta\frac{4\varepsilon^2|\partial\,\mathrm{e}^{v/2}\,|^2}{(\mathrm{e}^{v/2} + \nu^2)^2} \\ + \left(\tau - \frac{\mathrm{e}^v(1 + h' \circ a)}{\mathrm{e}^v + \varepsilon^2}\right)\partial\bar{\partial}\,\psi_\nu \\ + \frac{\mathrm{e}^v(1 + h' \circ a)}{\mathrm{e}^v + \varepsilon^2}\left(\partial\bar{\partial}\,\psi_\nu - \delta\,\partial\bar{\partial}\,\lambda\right).\end{aligned}$$

Now we choose $h(x) := 2 - x + \log(2\,\mathrm{e}^{x-1} - 1)$, so that

$$1 + h'(x) = \frac{2\,\mathrm{e}^{x-1}}{2\,\mathrm{e}^{x-1} - 1} \quad \text{and} \quad -h''(x) = \frac{2\,\mathrm{e}^{x-1}}{(2\,\mathrm{e}^{x-1} - 1)^2}.$$

Hence, since $a + h \circ a \geq a > 1$ for $\varepsilon > 0$ sufficiently small, we have

$$\tau - \frac{\mathrm{e}^{v}(1 + h' \circ a)}{\mathrm{e}^{v} + \varepsilon^2} \geq 2 + \log(2\,\mathrm{e}^{a-1} - 1) - \frac{2\,\mathrm{e}^{a-1}}{2\,\mathrm{e}^{a-1} - 1} > 0$$

and

$$\tau\, \partial\bar{\partial}\, \psi_\nu - \partial\bar{\partial}\, \tau - \frac{1}{A}\partial\tau \wedge \bar{\partial}\, \tau \geq \delta \frac{4\varepsilon^2 |\partial\, \mathrm{e}^{v/2}|^2}{(\mathrm{e}^{v/2} + \nu^2)^2}.$$

By Lemma 1.6 we then bound the last term in (1.4) by

$$\delta \int_X \left| \left\langle u, \bar{\partial}(\mathrm{e}^{v/2}) \right\rangle \right|^2 \frac{4\varepsilon^2}{(\mathrm{e}^{v} + \varepsilon^2)^2}\, \mathrm{e}^{-\psi_\nu} \leq \|\mathfrak{d}^* u\|^2_{\psi_\nu} + \|Su\|^2_{\psi_\nu},$$

where $\mathfrak{d}\beta := \bar{\partial}\left(\sqrt{\tau + A}\beta\right)$ and $Su = \sqrt{\tau}\, \bar{\partial}\, u$ are "twisted" $\bar{\partial}$ operators. Note that $S \circ \mathfrak{d} = 0$.

In this way we find that

$$|(u, \alpha_\varepsilon)|^2 \leq C_\varepsilon \left(\|\mathfrak{d}^* u\|^2_{\psi_\nu} + \|Su\|^2_{\psi_\nu} \right)$$

for all $(L \otimes L_Z)$-valued $(n, 1)$-forms u. Then by standard functional analysis, followed by sending $\nu \to 0$ and a standard weak-$*$ compactness diagonal sequence argument, we obtain a smooth section β_ε of $L \otimes L_Z \otimes K_X$ such that $\mathfrak{d}\beta_\varepsilon = \alpha_\varepsilon$ and

$$\int_X \frac{|\beta_\varepsilon|^2}{|T|^2}\, \mathrm{e}^{-\varphi} \leq C_\varepsilon.$$

Notice that the estimate forces $\beta_\varepsilon|_Z = 0$.

Now set $F_\varepsilon := \chi_\varepsilon \tilde{F} - \sqrt{\tau + A}\beta_\varepsilon$. Then $F_\varepsilon|_Z = \tilde{F}|_Z = f \wedge \mathrm{d}T$ and

$$\begin{aligned}
\int_X |F_\varepsilon|^2\, \mathrm{e}^{-\varphi-\lambda} &\leq o(1) + \int_X \mathrm{e}^{v}(\tau + A) \frac{|\beta_\varepsilon|^2}{|T|^2}\, \mathrm{e}^{-\varphi} \\
&\leq o(1) + \sup_X (\mathrm{e}^{v}(\tau + A)) C_\varepsilon \\
&\leq o(1) + 4\,\mathrm{e}^{\gamma-1} C_\varepsilon \underset{\varepsilon \sim 0}{\leq} o(1) + \frac{32\pi(1+t)^2\, \mathrm{e}^{\gamma-1}}{\delta} \int_Z |f|^2\, \mathrm{e}^{-\varphi}.
\end{aligned}$$

Up to passing to subsequences, we can let $\varepsilon \to 0$ and $t \to 0$, again by Alaoglu's Theorem and dominated convergence. By the sub-mean-value property we also have that the L^2 convergence of (a subsequence of) F_ε implies pointwise convergence, and thus we have $F \in H^0(X, L \otimes L_Z \otimes K_X)$ such that $F|_Z = f \wedge \mathrm{d}T$ and

$$\int_X |F|^2 \,\mathrm{e}^{-\varphi-\lambda} \le \frac{32\pi}{\delta} \int_Z |f|^2 \,\mathrm{e}^{-\varphi},$$

completing the proof of Theorem 1.5 (with non-optimal constant; again, see [8] and [27] for the proof with the optimal constant). □

1.4 The L^2 division theorem

Let X be a Stein manifold and fix holomorphic functions $h_1, \dots, h_r$ on X. Given another holomorphic function g with suitable L^2 estimates, the division problem asks whether one can find holomorphic functions $f_1, \dots, f_r$ such that $g = \sum_{j=1}^r h_j f_j$ and having good L^2 estimates.

A first solution to the division problem was found by H. Skoda in 1972 [52], and subsequent work of Skoda and Demailly generalized the result to generically surjective morphisms of vector bundles with sufficiently positive metrics [53, 15]. Here we present the following version of Skoda's division theorem, and its proof using classical L^2 methods. For the sake of simplicity, as well as similarity with Theorem 4.1, we only treat $(n, 0)$-forms; see [15, Théorème 6.2] and [20, Section 11] for the general treatment of (n, k)-forms. See also [56] for a different generalization of the L^2 division theorem.

Theorem 1.7 (**L^2 division**). *Let X be a Stein manifold and let $F, Q \to X$ be holomorphic vector bundles endowed with Hermitian metrics $\mathfrak{h}_F$ and $\mathfrak{h}_Q$, respectively. Let also $E \to X$ be a line bundle with metric $\mathrm{e}^{-\phi}$. Fix $h \in H^0(X, F^* \otimes Q)$ and $\alpha > 1$, and set*

$$q := \min(\operatorname{rank} F - \operatorname{rank} Q, \dim X).$$

Assume that $\Theta(\mathfrak{h}_F) \geq_q 0$ *and*

$$\sqrt{-1}\,\partial\bar{\partial}\,\phi \geq \alpha q \Theta(\det \mathfrak{h}_Q).$$

Then, for any holomorphic section $g \in H^0(X, Q \otimes E \otimes K_X)$ *such that*

$$\|g\|_Q^2 := \int_X \frac{\mathfrak{h}_Q(Hg, g)\,\mathrm{e}^{-\phi}}{\det(hh^*)^{\alpha q+1}} < +\infty,$$

there is a holomorphic section $f \in H^0(X, F \otimes E \otimes K_X)$ *such that* $g = h \otimes f$ *and*

$$\|f\|_F^2 := \int_X \frac{\mathfrak{h}_F(f, f)\,\mathrm{e}^{-\phi}}{\det(hh^*)^{\alpha q}} \leq \frac{\alpha}{\alpha - 1}\|g\|_Q^2\,.$$

Here $h^* : Q \to F$ *is the adjoint of* h *with respect to* $\mathfrak{h}_F$ *and* $\mathfrak{h}_Q$, *and* $H \in \mathrm{End}(Q)$ *is the endomorphism of* Q *whose matrix is the transposed comatrix of* hh^*.

By taking F to be the trivial vector bundle $\mathbb{C}^r$ and $Q = G \otimes E^*$ we obtain the following special case.

Corollary 1.8 (**Skoda's division theorem**). *Let* X *be a complete Kähler manifold of complex dimension* n *and let* $E, G \to X$ *be holomorphic line bundles with (singular) Hermitian metrics* $\mathrm{e}^{-\varphi}$ *and* $\mathrm{e}^{-\psi}$, *respectively. Fix* $h = (h_1, \ldots, h_r) \in H^0(X, (E^* \otimes G)^{\oplus r})$ *and* $\alpha > 1$. *Let* $q := \min(r-1, n)$ *and assume that*

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \frac{\alpha q}{\alpha q + 1}\sqrt{-1}\,\partial\bar{\partial}\,\psi.$$

Then, for any holomorphic section $g \in H^0(X, G \otimes K_X)$ *such that*

$$\|g\|_G^2 := \int_X \frac{|g|^2\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha q+1}} < +\infty,$$

there is a holomorphic section $f = (f_1, \dots, f_r) \in H^0(X, E^{\oplus r} \otimes K_X)$ *such that*

$$g = h \dot{\otimes} f := h_1 \otimes f_1 + \cdots + h_r \otimes f_r$$

and

$$\|f\|^2_{E^{\oplus r}} := \int_X \frac{|f|^2 \, \mathrm{e}^{-\varphi}}{(|h|^2 \, \mathrm{e}^{-\psi+\varphi})^{\alpha q}} \leq \frac{\alpha}{\alpha - 1} \|g\|^2_G \, .$$

For the proof of Theorem 1.7, we will follow the exposition of [20]. Theorem 1.7 is a consequence of the following result.

Theorem 1.9. *Let* X *be a Stein manifold and let* $F, Q \to X$ *be holomorphic vector bundles. Let* $\mathfrak{h}_F$ *be a Hermitian metric for* F*, and denote by* $\tilde{\mathfrak{h}}$ *the induced metric on* Q*. Fix* $h \in H^0(X, F^* \otimes Q)$ *and* $\alpha > 1$*. Let* $q := \min(\operatorname{rank} F - \operatorname{rank} Q, \dim X)$ *and assume that* $\Theta(\mathfrak{h}_F) \geq_q 0$ *and*

$$\sqrt{-1} \, \partial\bar{\partial} \, \phi \geq \alpha q \Theta(\det \tilde{\mathfrak{h}})$$

for some smooth metric $\mathrm{e}^{-\phi}$ *of some holomorphic line bundle* $E \to X$*. Then, for any holomorphic section* $g \in H^0(X, Q \otimes E \otimes K_X)$ *such that*

$$\|g\|^2_Q := \int_X \tilde{\mathfrak{h}}(g, g) \, \mathrm{e}^{-\phi} < +\infty,$$

there is a holomorphic section $f \in H^0(X, F \otimes E \otimes K_X)$ *such that* $g = h \otimes f$ *and*

$$\|f\|^2_F := \int_X \mathfrak{h}_F(f, f) \, \mathrm{e}^{-\phi} \leq \frac{\alpha}{\alpha - 1} \|g\|^2_Q \, .$$

Proof. Similar to the proof of L^2 extension, the main idea is to first find a special smooth solution and then correct it to a holomorphic one by adding a section of the kernel bundle.

To start, we reduce to the case of strict positivity of curvature: Since X is Stein, there is a strictly plurisubharmonic smooth function $\eta : X \to [0, +\infty)$. Then we can replace ϕ with $\phi + \frac{1}{j}\eta$, so that the inequality on curvature is strict. If Theorem 1.9 is true for strict positivity, then for every $j \in \mathbb{N}$ we get $f_j \in H^0(X, F \otimes E \otimes K_X)$ such that $g = h \otimes f_j$ and

$$\left\|f_j\right\|_F^2 \leq \frac{\alpha}{\alpha-1} \int_X \tilde{\mathfrak{h}}(g,g)\, \mathrm{e}^{-(\phi+j^{-1}\eta)} \leq \frac{\alpha}{\alpha-1}\|g\|_Q^2 .$$

By the sub-mean-value property of $|f_j|$ and the smoothness of ϕ, the sequence $\{f_j\}_j$ is locally uniformly bounded, so that by a diagonal argument involving Montel's theorem we can replace $\{f_j\}_j$ a subsequence (still denoted in the same way) that converges locally uniformly to some section $f \in H^0(X, F \otimes E \otimes K_X)$ satisfying $g = h \otimes f$. Moreover, for any j bigger than some fixed j_0 we get

$$\int_X \mathfrak{h}_F(f_j, f_j)\, \mathrm{e}^{-(\phi+j_0^{-1}\eta)} \leq \int_X \mathfrak{h}_F(f_j, f_j)\, \mathrm{e}^{-(\phi+j^{-1}\eta)} \leq \frac{\alpha}{\alpha-1}\|g\|_Q^2 .$$

Then we have a further subsequence, still denoted $\{f_j\}_j$, that converges in $L^2(\mathfrak{h}_F\, \mathrm{e}^{-(\phi+j_0^{-1}\eta)})$. We can then continue inductively by choosing $j_{k+1} > j_k$ and taking further subsequences. The diagonal subsequence then converges to f such that

$$f \in \bigcap_{k\geq 0} L^2(\mathfrak{h}_F\, \mathrm{e}^{-(\phi+j_k^{-1}\eta)}).$$

By the Monotone Convergence Theorem we conclude that

$$\|f\|_F^2 = \int_X \mathfrak{h}_F(f,f)\, \mathrm{e}^{-\phi} = \lim_{k\to+\infty} \int_X \mathfrak{h}_F(f,f)\, \mathrm{e}^{-(\phi+j_k^{-1}\eta)} \leq \frac{\alpha}{\alpha-1}\|g\|_Q^2 ,$$

as wanted.

Second, by working on the complement of a divisor $D \subset X$ containing the kernel of h, we can assume that h is surjective. Indeed, $X \setminus D$ is again Stein and, assuming that Theorem 1.9 holds for h surjective, we obtain $\tilde{f} \in H^0(X \setminus D, F \otimes E \otimes K_X)$ such that $g|_{X\setminus D} = h|_{X\setminus D} \otimes \tilde{f}$ and $\left\|\tilde{f}\right\|_F^2 \leq \frac{\alpha}{\alpha-1}\|g\|_Q^2$. By Riemann's Removable Singularities

Theorem, then $\tilde{f}$ extends to $f \in H^0(X, F \otimes E \otimes K_X)$. Since $X \setminus D$ is open and meets every component of X, by the identity principle $g = h \otimes f$ everywhere on X, and since D has measure 0, one has $\left\|\tilde{f}\right\|_F^2 \leq \frac{\alpha}{\alpha-1}\|g\|_Q^2$.

Assume then that the bound on the curvature of E is strict and that h is surjective. Then the kernel $\ker h$ is a holomorphic vector subbundle of F. Let $\iota : \ker h \to F$ be the inclusion morphism. Denote by $\iota^* : F \to \ker h$ and $h^* : Q \to F$ the adjoint morphisms of ι and h, respectively, with respect to the metric $\mathfrak{h}_F$ and the metrics $\mathfrak{h}_F|_{\ker h}$ and $\tilde{\mathfrak{h}}$, respectively.

Consider the smooth lifting h^*g of g to F. We aim to find f of the form $f = h^*g + \iota u$ for some $u \in L^2(X, \ker h \otimes E \otimes K_X)$. For such an f we have

$$\mathfrak{h}_F(f,f)\,\mathrm{e}^{-\phi} = \mathfrak{h}_F(h^*g, h^*g)\,\mathrm{e}^{-\phi} + \mathfrak{h}_F(\iota u, \iota u)\,\mathrm{e}^{-\phi} = \tilde{\mathfrak{h}}(g,g)\,\mathrm{e}^{-\phi} + \mathfrak{h}_F|_{\ker h}(u,u)\,\mathrm{e}^{-\phi}$$

at all points of X. Moreover since $\bar{\partial}\, g = 0$ we have

$$\bar{\partial}\, f = \bar{\partial}(h^*g) + \iota(\bar{\partial}\, u) = \iota(-\,\mathbf{II}^* \wedge g + \bar{\partial}\, u),$$

where $\mathbf{II} \in C^\infty(X, (\ker h)^* \otimes Q \otimes T_X^{*1,0})$ is the second fundamental form of $\ker h$ into F (see Lemma 10.2 in [20]). We then want to solve the equation $\bar{\partial}\, u = -\,\mathbf{II}^* \wedge g$.

Let $A := [\Theta(\mathfrak{h}_F|_{\ker h}), \Lambda_\omega]$ be the curvature operator induced on $\ker h \otimes T_X^{1,0}$ by the restriction of the metric $\mathfrak{h}_F$ to $\ker h$. We want to apply Theorem 1.4 with datum the $(n,1)$-form $-\,\mathbf{II}^* \wedge g$ to get a solution u with

$$\|u\|_F^2 \leq \int_X \langle A^{-1}(\mathbf{II}^* \wedge g), (\mathbf{II}^* \wedge g)\rangle_{\tilde{\mathfrak{h}},\omega},$$

in order to obtain f such that

$$\|f\|_F^2 \leq \int_X \tilde{\mathfrak{h}}(g,g) + \int_X \langle A^{-1}(\mathbf{II}^* \wedge g), (\mathbf{II}^* \wedge g)\rangle_{\tilde{\mathfrak{h}},\omega}.$$

We thus have to check that A is positive definite.

Lemma 1.10. *Under the assumptions of Theorem 1.9,*

$$\left|\langle \mathbf{II}^* \wedge g, v\rangle_{\mathfrak{h},\omega}\right|^2 < \frac{1}{\alpha - 1}\tilde{\mathfrak{h}}_x(g,g)\langle Av, v\rangle_{\mathfrak{h},\omega}$$

for all $v \in \ker h_x \otimes T_x^{1,0}$.

Proof. We have

$$\Theta(\mathfrak{h}_F|_{\ker h}) \geq_q \sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II}$$

and

$$\Theta(\det \tilde{\mathfrak{h}}) \geq \operatorname{tr}_Q\left(\sqrt{-1}\,\mathbf{II} \wedge \mathbf{II}^*\right) = \operatorname{tr}_{\ker h}\left(-\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II}\right)$$

(see Lemma 10.2 in [20]). Moreover, since $-\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II} \geq_{\mathrm{Grif}} 0$, we have

$$q\operatorname{tr}_{\ker h}\left(-\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II}\right) \otimes \mathrm{Id}_{\ker h} + \sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II} \geq_q 0$$

(see Lemma 10.16 in [20]). By the hypotheses on curvature in Theorem 1.9, it then follows that

$$\begin{aligned}
&\Theta(\mathfrak{h}_F|_{\ker h}) + \sqrt{-1}\,\partial\bar{\partial}\,\phi\,\mathrm{Id}_{\ker h} \\
&\qquad >_q \Theta(\mathfrak{h}_F|_{\ker h}) + \alpha q \Theta(\det \tilde{\mathfrak{h}}) \otimes \mathrm{Id}_{\ker h} \\
&\qquad \geq_q \sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II} + \alpha q \operatorname{tr}_{\ker h}\left(-\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II}\right) \otimes \mathrm{Id}_{\ker h} \\
&\qquad \geq_q -(\alpha - 1)\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II}
\end{aligned}$$

We therefore have

$$\langle Av, v\rangle_{\mathfrak{h},\omega} > (\alpha - 1)\langle -\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II} \wedge \Lambda_\omega v, v\rangle_{\mathfrak{h},\omega}$$

for all $v \in \ker h \otimes E \otimes T^{*1,0}_X$. Fix a point $x_0 \in X$ and an orthonormal basis $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ of $T^{*1,0}_{x_0}$. Write

$$\mathbf{II} = \sum_{j=1}^{n} \mathrm{d}z_j \otimes \mathbf{II}_j,$$

with $\mathbf{II}_j \in \ker h^*_{x_0} \otimes Q_{x_0}$. The adjoint of

$$\mathbf{II}^* \wedge \cdot = \sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathbf{II}^*_j \cdot$$

is the contraction $\mathbf{II} \lrcorner$ defined by

$$\mathbf{II} \lrcorner v = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} \lrcorner (\mathbf{II}_j\, v) = \sum_{j=1}^{n} -\sqrt{-1}\,\mathrm{d}z_j \wedge \Lambda_\omega(\mathbf{II}_j\, v) = -\sqrt{-1}\,\mathbf{II} \wedge \Lambda_\omega v.$$

Consequently $\langle -\sqrt{-1}\,\mathbf{II}^* \wedge \mathbf{II} \wedge \Lambda_\omega v, v\rangle_{\tilde{\mathfrak{h}},\omega} = \langle \mathbf{II} \lrcorner v, \mathbf{II} \lrcorner v\rangle_{\tilde{\mathfrak{h}},\omega}$ and thus

$$|\langle \mathbf{II}^* \wedge g, v\rangle_{\mathfrak{h},\omega}|^2 = |\langle g, \mathbf{II} \lrcorner v\rangle_{\tilde{\mathfrak{h}},\omega}|^2 \leq \tilde{\mathfrak{h}}(g,g)\langle \mathbf{II} \lrcorner v, \mathbf{II} \lrcorner v\rangle_{\tilde{\mathfrak{h}},\omega} < \frac{1}{\alpha - 1}\tilde{\mathfrak{h}}(g,g)\langle Av, v\rangle_{\tilde{\mathfrak{h}},\omega},$$

proving the statement. □

By Lemma 1.10, A is positive definite, so we can apply Theorem 1.4. Moreover, again by Lemma 1.10 applied to $v = A^{-1}(\mathbf{II}^* \wedge g)$, we also obtain

$$\|f\|^2_F \leq \frac{\alpha}{\alpha - 1} \int_X \tilde{\mathfrak{h}}(g,g),$$

proving Theorem 1.9. □

Proof of Theorem 1.7. Let $\mathfrak{h}_Q$ be the given metric for Q and $\tilde{\mathfrak{h}}$ the metric induced by $\mathfrak{h}_F$ on Q. Let $h^* : Q \to F$ the adjoint morphism with respect to $\mathfrak{h}_F$ and $\mathfrak{h}_Q$. Then, while $hh^* = \mathrm{Id}_Q$ does not hold anymore, the map $h^*(hh^*)^{-1} : Q \to F$ is a section of $h : F \to Q$ (in fact $hh^*(hh^*)^{-1} = \mathrm{Id}_Q$ by construction) and is orthogonal to $\ker h$. We then have

$$\tilde{\mathfrak{h}}(v,v) = \mathfrak{h}_F\left(h^*(hh^*)^{-1}v, h^*(hh^*)^{-1}v\right) = \mathfrak{h}_Q\left((hh^*)^{-1}v, v\right) = \frac{1}{\det(hh^*)}\mathfrak{h}_Q(Hv, v),$$

where $H \in \mathrm{End}(Q)$ is the endomorphism of Q whose matrix is the transposed comatrix of hh^*. Since we then have $\det \tilde{\mathfrak{h}} = \frac{1}{\det(hh^*)} \det \mathfrak{h}_Q$, in order for the hypothesis on curvature of Theorem 1.7 to be satisfied, we replace the metric $\mathrm{e}^{-\phi}$ for E with the metric

$$\mathrm{e}^{-\tilde{\phi}} := \frac{1}{\det(hh^*)^{\alpha q}} \, \mathrm{e}^{-\phi},$$

so that

$$\begin{aligned} \sqrt{-1}\,\partial\bar{\partial}\,\tilde{\phi} &= \sqrt{-1}\,\partial\bar{\partial}\,\phi + \alpha q \sqrt{-1}\,\partial\bar{\partial}\log\det(hh^*) \\ &\geq \alpha q \Theta(\mathfrak{h}_Q) + \alpha q \sqrt{-1}\,\partial\bar{\partial}\log\det(hh^*) = \alpha q \Theta(\tilde{\mathfrak{h}}). \end{aligned}$$

Theorem 1.7 is then obtained applying Theorem 1.9 to $\tilde{\mathfrak{h}}$ and $\mathrm{e}^{-\tilde{\phi}}$. □

Chapter 2

Berndtsson's positivity theorems

This Chapter is devoted to the two theorems of B. Berndtsson measuring the variation of Hilbert spaces of holomorphic sections of line bundles over families of complete Kähler manifolds (Theorem 1). The conclusion of both theorems is that if the line bundle is positive then the Hilbert spaces of sections form a vector bundle and the metric induced by the L^2 inner product has Nakano-positive curvature.

The first theorem, explained in Section 2.1, deals with families of Hilbert spaces for a single line bundle over a single Stein manifold, so that the only thing that is allowed to vary is the metric for the line bundle. In spite of the simplicity of the setting Berndtsson's first theorem has several applications, among which are new and very different proofs of L^2 extension and division (see Section 3.1 and Chapter 4), and a proof of the openness conjecture (Section 3.2).

The second theorem, recalled in Section 2.2, deals instead with families of compact Kähler manifolds. The applications of this theorem point to different directions than the first theorem, for instance giving a partial result towards P. Griffiths's conjecture on the existence of positively curved metrics on ample vector bundles.

2.1 Berndtsson's theorem for trivial fibrations

Let X be a Stein manifold and Ω a domain in $\mathbb{C}^m$. Suppose X is relatively compact in some ambient Stein manifold $\tilde{X}$, and let $L \to \tilde{X}$ be a holomorphic line bundle. Let $\tilde{p} : \tilde{X} \times \Omega \to \tilde{X}$ be the projection on the first factor and $\mathrm{e}^{-\tilde{\phi}}$ be a Hermitian metric for $\tilde{p}^*L \to \tilde{X} \times \Omega$. We write $p := \tilde{p}|_{X\times\Omega}$ and $\mathrm{e}^{-\phi} := \mathrm{e}^{-\tilde{\phi}}|_{X\times\Omega}$. For each $\tau \in \Omega$, define the Hilbert space

$$\mathcal{H}_\tau := \left\{ f \in H^0(X, L \otimes K_X) \,\middle|\, \|f\|_\tau^2 := \int_X |f|^2 \,\mathrm{e}^{-\phi_\tau} < +\infty \right\},$$

where $\mathrm{e}^{-\phi_\tau}$ is the restriction of $e^{-\phi}$ to $p^*L|_{X\times\{\tau\}}$.

Since the metric $\mathrm{e}^{-\phi}$ extends smoothly to $\tilde{X} \times \Omega$, the subspaces $\mathcal{H}_\tau \subseteq H^0(X, L \otimes K_X)$ are independent of τ, but have norms that vary with τ. If we fix $o \in \Omega$ and define $\tilde{f}(x,t) := f(x)$ for all $f \in \mathcal{H}_o$ then the map $\mathcal{H}_o \times \Omega \ni (f,\tau) \mapsto \tilde{f} \in \mathcal{H}$ is a bijection, which we declare to be holomorphic. Thus $\mathcal{H} \to \Omega$ has the structure of a trivial holomorphic vector bundle, with a smooth non-trivial metric defined by the inner products on the fibers $\mathcal{H}_\tau$, for $\tau \in \Omega$.

Theorem 2.1 (**Berndtsson's theorem for trivial fibrations** [4, Theorem 1.1])**.** *If* $\mathrm{e}^{-\phi}$ *has* (*strictly*) *positive curvature, then the curvature of the* L^2 *Hermitian metric induced on* $\mathcal{H} \to \Omega$ *by* $\mathrm{e}^{-\phi}$ *is* (*strictly*) *positive in the sense of Nakano.*

In particular $\mathcal{H}$ is Griffiths-(semi)positive, and thus we get the following convexity corollary.

Corollary 2.2. *Assume that* $\mathrm{e}^{-\phi}$ *is a possibly singular metric with* (*strictly*) *positive curvature, and let* $\xi \not\equiv 0$ *be a holomorphic section of* $\mathcal{H}^* \to \Omega$*. Then the function*

$$\Omega \ni \tau \longmapsto \log\|\xi_\tau\|_{\tau,*}^2$$

is plurisubharmonic. In particular, $\tau \mapsto \log\|\xi\|^2_{\tau,*}$ *is plurisubharmonic for any fixed non-zero* $\xi \in H^0(X, L \otimes K_X)^*$ *with finite* L^2*-norm.*

Proof. If the metric $e^{-\phi}$ is smooth, the statement follows directly from Theorem 2.1 and the characterization of Griffiths-(semi)negativity of Proposition 1.2.

Assume then that $e^{-\phi}$ is singular and fix a section ξ of $\mathcal{H}^* \to X$. Let $e^{-\phi_k}$ be a sequence of positive smooth Hermitian metrics for $\tilde{p}^*L \to \tilde{X} \times \Omega$ increasing to $e^{-\phi}$ (we can find such a sequence by regularizing the potentials ϕ and possibly compensating the loss of positivity by adding a small multiple of a positive strictly plurisubharmonic function, which exists because $\tilde{X}$ is Stein). Denote by $\|\cdot\|_{k,\tau}$ the norm induced on $\mathcal{H}_\tau$ by $e^{-\phi_k}$. Since these metrics are smooth up to the boundary, for each $k \in \mathbb{N}$ the function

$$\Omega \ni \tau \longmapsto \log\|\xi_\tau\|^2_{k,\tau,*}$$

is subharmonic. Moreover, since $e^{-\phi_k}$ increases to $e^{-\phi}$, for each $\tau \in \Omega$ the induced norms

$$\|f\|^2_{k,\tau} = \int_X |f|^2 e^{-\phi_{k,\tau}}$$

increase to $\|f\|^2_\tau$, and thus the subharmonic functions

$$\Omega \ni \tau \longmapsto \log\|\xi_\tau\|^2_{k,\tau,*}$$

decrease to the function

$$\Omega \ni \tau \longmapsto \log\|\xi_\tau\|^2_{\tau,*},$$

which is then subharmonic. □

The proof of Theorem 2.1 is contained in the next two sections, and follows Berndtsson's original exposition [4].

2.1.1 Proof of Theorem 2.1 in the strictly positive case

We first prove Theorem 2.1 assuming that the curvature of $\mathrm{e}^{-\phi}$ is strictly positive along fibers, i.e. $\sqrt{-1}\,\partial\bar{\partial}_X\,\phi_\tau > 0$ for all $\tau \in \Omega$.

Let $\mathcal{L} \to \Omega$ be the bundle whose fiber over $\tau \in \Omega$ are the L^2-integrable sections of L with respect to the metric $\mathrm{e}^{-\phi_\tau}$:

$$\mathcal{L}_\tau := \left\{ f \in \Gamma(X, L \otimes K_X) \,\middle|\, \int_X |f|^2\,\mathrm{e}^{-\phi_\tau} < +\infty \right\},$$

so that $\mathcal{H}$ is a trivial subbundle of the trivial bundle $\mathcal{L}$, with metric induced by the non-trivial L^2 metric of $\mathcal{L}$. Fix a chart $U \subseteq \Omega$ and local coordinates $t_1, \dots, t_m$ centered at $\tau \in U$. The Chern connection of $\mathcal{L}$ is then $\nabla^{\mathcal{L}1,0}_{\frac{\partial}{\partial t_j}} = \partial_{t_j} - \phi_j$, where ϕ_j is the operator of multiplication by the smooth function $\partial_{t_j}\phi$. Indeed, for any two local sections u, v of $\mathcal{L}$ such that $\partial_{t_j}u$ and $\partial_{t_j}v$ are also sections of $\mathcal{L}$, one has

$$\begin{aligned}
\int_X (\nabla^{\mathcal{L}1,0}_{\frac{\partial}{\partial t_j}} u)\bar{v}\,\mathrm{e}^{-\phi} &= \partial_{t_j}\int_X u\bar{v}\,\mathrm{e}^{-\phi} - \int_X u\overline{\partial_{\bar{t}_j} v} \\
&= \int_X (\partial_{t_j}u)\bar{v}\,\mathrm{e}^{-\phi} + \int_X u\overline{\partial_{t_j} v}\,\mathrm{e}^{-\phi} + \int_X u\bar{v}(\partial_{t_j}\phi)\,\mathrm{e}^{-\phi} - \int_X u\overline{\partial_{\bar{t}_j} v} \\
&= \int_X \left(\partial_{t_j}u - \phi_j u\right)\bar{v}\,\mathrm{e}^{-\phi}\,.
\end{aligned}$$

Consequently, the curvature of $\mathcal{L}$ is a densely defined operator given by $\Theta^{\mathcal{L}}_{j\bar{k}} = \phi_{j\bar{k}}$, so that $\Theta^{\mathcal{L}}$ is the operator of multiplication by the Hessian of ϕ with respect to the base variables $t_1, \dots, t_m$.

We now want to compute the curvature $\Theta^{\mathcal{H}}$ of the subbundle $\mathcal{H}$ from the curvature $\Theta^{\mathcal{L}}$ of $\mathcal{L}$. Fix smooth sections $u_1, \dots, u_m$ of $\mathcal{H}$: these are smooth sections of L on the total space of the fibration, and holomorphic along the fibers. Since the space of sections u of $\mathcal{H}$ represented by holomorphic sections of $\tilde{p}^*L \to \tilde{X} \times \Omega$ is dense in the space of holomorphic sections of $\mathcal{H}$, we can assume that $\partial_{t_k}u_j$ are also sections of $\mathcal{H}$ for all j, k. To check the Nakano-(semi)positivity of the L^2 metric induced on $\mathcal{H}$, from Griffiths's formula (Proposition 1.3) we need to estimate

$$\begin{aligned}\sum_{j,k=1}^m \int_X \Theta^{\mathcal{H}}_{j\bar{k}} u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau} &= \sum_{j,k=1}^m \int_X \Theta^{\mathcal{L}}_{j\bar{k}} u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau} - \int_X \left| \pi_\tau^\perp \sum_{j=1}^m \nabla^{\mathcal{L}1,0}_{t_j} u_j \right|^2 \mathrm{e}^{-\phi_\tau} \\ &= \sum_{j,k=1}^m \int_X \phi_{j\bar{k}} u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau} - \int_X \left| \pi_\tau^\perp \sum_{j=1}^m \phi_j u_j \right|^2 \mathrm{e}^{-\phi_\tau},\end{aligned} \tag{2.1}$$

where in this special setting $\pi : \mathcal{L} \to \mathcal{H}$ is the (fiberwise) Bergman projection, and therefore $\pi_\tau^\perp \partial_{t_j} u_j = 0$. Set $w := \pi_\tau^\perp(\sum_{j=1}^m \phi_j u_j)$, then

$$\bar{\partial}_X \, w = \bar{\partial}_X \sum_{j=1}^m \phi_j u_j = \sum_{j=1}^m (\bar{\partial}_X \, \phi_j) u_j,$$

where $\bar{\partial}_X$ denotes the $\bar{\partial}$ operator on X only. Note that $\bar{\partial}_X \, u_j = 0$ since u_j is holomorphic along the fibers. Since by definition w is orthogonal to the holomorphic sections along the fibers, it follows that w is the L^2-minimal solution to the $\bar{\partial}$ equation along the fiber. Since we are assuming that $\sqrt{-1}\,\partial\bar{\partial}\,\phi_\tau > 0$, we can estimate the second term in (2.1) by Hörmander–Skoda Theorem 1.4:

$$\int_X \left| \pi^\perp \sum_{j=1}^m \phi_j u_j \right|^2 \mathrm{e}^{-\phi_\tau} \leq \int_X \left| \sum_{j=1}^m (\bar{\partial}_X \, \phi_j) u_j \right|^2_{\sqrt{-1}\,\partial\bar{\partial}_X \, \phi_\tau} \mathrm{e}^{-\Phi_\tau} .$$

Choosing local coordinates $z_1, \ldots, z_n$ for X this becomes

$$\int_X \left| \pi^\perp \sum_{j=1}^m \phi_j u_j \right|^2 \mathrm{e}^{-\phi_\tau} \leq \int_X \sum_{\lambda,\mu=1}^n \phi_\tau^{\lambda\bar{\mu}} \left(\sum_{j=1}^m \phi_{j\lambda} u_j \right) \overline{\left(\sum_{k=1}^m \phi_{k\mu} u_k \right)} \mathrm{e}^{-\phi_\tau},$$

where $\phi_{j\lambda} = \frac{\partial^2 \phi}{\partial t_j \partial z_\lambda}$ and $\phi_\tau^{\lambda\bar{\mu}}$ is the $(\lambda, \bar{\mu})$ entry of the inverse matrix of the complex Hessian of ϕ_τ. This and (2.1) give

$$\sum_{j,k=1}^m \int_X \Theta^{\mathcal{H}}_{j\bar{k}} u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau} \geq \int_X \sum_{j,k=1}^m \left(\phi_{j\bar{k}} - \sum_{\lambda,\mu=1}^n \phi^{\lambda\bar{\mu}} \phi_{j\lambda} \overline{\phi_{k\mu}} \right) u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau} .$$

To conclude, notice that

$$\left(\phi_{j\bar{k}} - \sum_{\lambda,\mu=1}^{n} \phi^{\lambda\bar{\mu}} \phi_{j\lambda} \overline{\phi_{k\mu}}\right)_{j,\bar{k}}$$

is the Schur complement of the invertible block $\sqrt{-1}\,\partial\bar{\partial}_X\,\phi$ (the complex Hessian of ϕ along the fiber) in the (semi)positive definite Hermitian matrix $\sqrt{-1}\,\partial\bar{\partial}\,\phi$ (the complex Hessian of ϕ on the total space $X \times \Omega$), and thus it is (semi)positive definite by the following lemma.

Lemma 2.3. *Let $M = \left[\begin{smallmatrix} A & B \\ B^\dagger & C \end{smallmatrix}\right]$ be a Hermitian symmetric matrix (i.e. $M = M^\dagger$). Assume that M is (semi)positive definite and that A is invertible. Then the Schur complement $C - B^\dagger A^{-1} B$ of the block C is also (semi)positive definite.*

Proof. Since

$$\begin{bmatrix} I & 0 \\ -B^\dagger A^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix} \begin{bmatrix} I & 0 \\ -B^\dagger A^{-1} & I \end{bmatrix}^\dagger = \begin{bmatrix} A & 0 \\ 0 & C - B^\dagger A^{-1} B \end{bmatrix},$$

if M is (semi)positive definite, then $C - B^\dagger A^{-1} B$ is (semi)positive definite. □

2.1.2 The semipositive case

Assume now that $\sqrt{-1}\,\partial\bar{\partial}\,\phi \geq 0$. Since the ambient manifold $\tilde{X}$ is Stein, there is a strictly plurisubharmonic smooth function $\eta : \tilde{X} \to [0, \infty)$, so that the metrics $\mathrm{e}^{-\phi-\varepsilon\eta}$ have strictly positive curvature along the fibers of $X \times \Omega \to \Omega$ for all $\varepsilon > 0$. Then

$$\begin{aligned} \sum_{j,k=1}^{m} \int_X \Theta^{\mathcal{H}}_{\varepsilon,j\bar{k}} u_j \bar{u}_k \, \mathrm{e}^{-\phi_\tau-\varepsilon\eta} &= \sum_{j,k=1}^{m} \int_X (\phi_{j\bar{k}} + \varepsilon\eta_{j\bar{k}}) u_j \bar{u}_k \, \mathrm{e}^{-\phi_\tau-\varepsilon\eta} \\ &\quad - \int_X \left| \pi^\perp_{\varepsilon,\tau} \sum_{j=1}^{m} (\phi_j + \varepsilon\eta_j) u_j \right|^2 \mathrm{e}^{-\phi_\tau-\varepsilon\eta}, \end{aligned} \tag{2.2}$$

and we know from Subsection 2.1.1 that this quantity is non-negative, since $e^{-\phi-\varepsilon\eta}$ is strictly positively curved on the fibers. To conclude the proof of Theorem 2.1, we then need to show that

$$\lim_{\varepsilon\to 0}\sum_{j,k=1}^{m}\int_X \Theta^{\mathcal{H}}_{\varepsilon,j\bar{k}} u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau-\varepsilon\eta} = \sum_{j,k=1}^{m}\int_X \Theta^{\mathcal{H}}_{j\bar{k}} u_j \bar{u}_k \,\mathrm{e}^{-\phi_\tau} .$$

Because of (2.2), it suffices to show that

$$\lim_{\varepsilon\to 0}\int_X |\pi^{\perp}_{\varepsilon,\tau} u|^2 \,\mathrm{e}^{-\phi_\tau-\varepsilon\eta} = \int_X |\pi^{\perp}_{\tau} u|^2 \,\mathrm{e}^{-\phi_\tau} \tag{2.3}$$

for any section u of $L \to X$ that is L^2-integrable with respect to $\mathrm{e}^{-\phi-\varepsilon\eta}$ (which is the same as being L^2-integrable with respect to the metric $\mathrm{e}^{-\phi}$).

First recall that $\pi^{\perp}_{\tau} u$ is the section of $L \to X$ that is nearest to u in the norm induced by $\mathrm{e}^{-\phi}$ and such that

$$\int_X (\pi^{\perp}_{\tau} u)\bar{h}\,\mathrm{e}^{-\phi} = 0$$

for all L^2-integrable holomorphic sections $h \in H^0(X, L)$. Similarly, $\pi^{\perp}_{\varepsilon,\tau} u$ is the section of $L \to X$ that is nearest to u in the norm induced by $\mathrm{e}^{-\phi-\varepsilon\tau}$ and such that

$$\int_X (\mathrm{e}^{-\varepsilon\eta}\,\pi^{\perp}_{\varepsilon,\tau} u)\bar{h}\,\mathrm{e}^{-\phi} = \int_X (\pi^{\perp}_{\varepsilon,\tau} u)\bar{h}\,\mathrm{e}^{-\phi-\varepsilon\eta} = 0 \tag{2.4}$$

for all L^2-integrable holomorphic sections $h \in H^0(X, L)$. Therefore

$$\begin{aligned}\left\|\pi^{\perp}_{\tau} u - u\right\| &\le \left\|\mathrm{e}^{-\varepsilon\eta}\,\pi^{\perp}_{\varepsilon,\tau} u - u\right\| \\ &\le \left\|\mathrm{e}^{-\varepsilon\eta}(\pi^{\perp}_{\varepsilon,\tau} u - u)\right\| + \left\|(1-\mathrm{e}^{-\varepsilon\eta})u\right\| \\ &\le \left\|\pi^{\perp}_{\varepsilon,\tau} u - u\right\| + (\mathrm{e}^{\varepsilon M} - 1)\|u\| ,\end{aligned}$$

where $\|\cdot\|$ denotes the L^2-norm with respect to $\mathrm{e}^{-\phi}$ and $M = \sup_X \eta$. Similarly

$$\left\|\pi_{\varepsilon,\tau}^{\perp} u - u\right\|_{\varepsilon} \leq \mathrm{e}^{\varepsilon M}\left\|\pi_{\tau}^{\perp} u - u\right\|_{\varepsilon} + (\mathrm{e}^{\varepsilon M} - 1)\|u\|_{\varepsilon},$$

where $\|\cdot\|_{\varepsilon}$ denotes the L^2-norm with respect to $\mathrm{e}^{-\phi-\varepsilon\eta}$. Since

$$\left\|\pi_{\varepsilon,\tau}^{\perp} u - u\right\| \leq \mathrm{e}^{\varepsilon M}\left\|\pi_{\varepsilon,\tau}^{\perp} u - u\right\|_{\varepsilon} \quad \text{and} \quad \|u\|_{\varepsilon} \leq \|u\|,$$

it follows that

$$\left\|\pi_{\tau}^{\perp} u - u\right\| - (\mathrm{e}^{\varepsilon M} - 1)\|u\| \leq \left\|\pi_{\varepsilon,\tau}^{\perp} u - u\right\| \leq \mathrm{e}^{2\varepsilon M}\left\|\pi_{\tau}^{\perp} u - u\right\| + \mathrm{e}^{\varepsilon M}(\mathrm{e}^{\varepsilon M} - 1)\|u\|,$$

so that

$$\lim_{\varepsilon \to 0}\left\|\pi_{\varepsilon,\tau}^{\perp} u - u\right\| = \left\|\pi_{\tau}^{\perp} u - u\right\|. \tag{2.5}$$

Since

$$\pi_{\varepsilon,\tau}^{\perp} u = \mathrm{e}^{-\varepsilon\eta}\, \pi_{\varepsilon,\tau}^{\perp} u + (1 - \mathrm{e}^{-\varepsilon\eta})\pi_{\varepsilon,\tau}^{\perp} u$$

and

$$\left\|\pi_{\varepsilon,\tau}^{\perp}\right\|^2 \leq \mathrm{e}^{\varepsilon M}\left\|\pi_{\varepsilon,\tau}^{\perp} u\right\|_{\varepsilon}^2 \leq \mathrm{e}^{\varepsilon M}\|u\|_{\varepsilon}^2 \leq \mathrm{e}^{\varepsilon M}\|u\|^2,$$

it follows that $\pi_{\varepsilon,\tau}^{\perp} u$ converges to some section v which is orthogonal to holomorphic sections with respect to the norm induced by $\mathrm{e}^{-\phi}$ (recall that by (2.4) the term $\mathrm{e}^{-\varepsilon\eta}\, \pi_{\varepsilon,\tau}^{\perp} u$ is orthogonal to the L^2-integrable holomorphic sections with respect to the inner product induced by $\mathrm{e}^{-\phi}$). By (2.5) we then conclude that $v = \pi_{\tau}^{\perp} u$, and thus (2.3) follows by Lebesgue's dominated convergence theorem, proving Theorem 2.1 in the semipositive case.

2.1.3 Relation between the sharp L^2 extension theorem and Berndtsson's first theorem

As we will see in Section 3.1, Berndtsson's first theorem (or more precisely Corollary 2.2) implies a version of the L^2 extension theorem with sharp constants. The L^2 extension theorem with sharp constants (Theorem 1.5) in turn implies a stronger version of Corollary 2.2 in which the product manifold $X \times \Omega$ is replaced by a more general Stein manifold and the family of Hilbert spaces $\mathcal{H} \to \Omega$ does not need to be a vector bundle. We will follow the presentation of Section 11.4 in [57].

Let X be a Kähler manifold of complex dimension $m+n$, and let $T : X \to \Omega$ be a holomorphic submersion to a domain Ω in $\mathbb{C}^m$. For every $\tau \in \Omega$, we will denote by $X_\tau := T^{-1}(\tau)$ the fiber of T over τ. Let $L \to X$ be a holomorphic line bundle with Hermitian metric $\mathrm{e}^{-\varphi}$, and as before, for each $\tau \in \Omega$, define the Hilbert space

$$\mathcal{H}_\tau := \left\{ f \in H^0(X_\tau, L|_{X_\tau} \otimes K_{X_\tau}) \,\middle|\, \|f\|_\tau^2 := \int_{X_\tau} |f|^2 \mathrm{e}^{-\phi_\tau} < +\infty \right\}, \tag{2.6}$$

where $\mathrm{e}^{-\phi_\tau}$ is the restriction of $e^{-\phi}$ to $p^*L|_{X_\tau}$.

The family of Hilbert spaces $\mathcal{H} \to \Omega$ is in general not locally trivial, but we can still define sections as follows.

Definition 2.4. A section $\mathfrak{f}$ of $\mathcal{H} \to \Omega$ is a section $F_\mathfrak{f}$ of $L \otimes K_{X/\Omega} \to X$ such that $\mathfrak{f}_\tau := F_\mathfrak{f}|_{X_\tau} \in \mathcal{H}_\tau$ for all $\tau \in \Omega$ (here $K_{X/\Omega}$ denotes the relative canonical bundle). The section $\mathfrak{f}$ is said to be holomorphic if $F_\mathfrak{f}$ is holomorphic.

Similarly, we can define a notion of dual fibration and of section of the dual fibration.

Definition 2.5. The dual of $\mathcal{H} \to \Omega$ is the fibration $\mathcal{H}^* \to \Omega$ where $\mathcal{H}^*_\tau$ is the dual Hilbert space of $\mathcal{H}_\tau$, with the usual dual norm

$$\|\xi\|_{\tau,*} := \sup_{f \in \mathcal{H}_\tau \setminus \{0\}} \frac{|\langle \xi, f \rangle|}{\|\xi\|_\tau}.$$

A section of $\mathcal{H}^* \to \Omega$ is a map $\xi : \mathcal{H} \to \mathbb{C}$ such that $\xi_\tau := \xi|_{X_\tau} \in \mathcal{H}_\tau^*$ for all $\tau \in \Omega$. The section ξ of $\mathcal{H}^* \to \Omega$ is said to be holomorphic if the function

$$\Omega \ni \tau \longmapsto \langle \xi_\tau, F_{\mathfrak{f}} \rangle \in \mathbb{C}$$

is holomorphic for every holomorphic section $\mathfrak{f}$ of $\mathcal{H} \to \Omega$.

To prove (the generalization of) Corollary 2.2, it suffices to assume that Ω is the unit disk $\mathbb{D} \subset \mathbb{C}$.

Theorem 2.6. *Let X be a Stein manifold, and let $T : X \to \mathbb{D}$ be a holomorphic submersion. Let $L \to X$ be a holomorphic line bundle with Hermitian metric $\mathrm{e}^{-\varphi}$, and let $\mathcal{H} \to \mathbb{D}$ be the Hilbert space fibration defined by* (2.6)*. If $\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq 0$, then the function*

$$\mathbb{D} \ni \tau \longmapsto \log\|\xi_\tau\|^2_{\tau,*}$$

is subharmonic for all holomorphic sections ξ of $\mathcal{H}^* \to \mathbb{D}$.

Proof. The dual norm explicitly writes as

$$\|\xi_\tau\|^2_{\tau,*} = \sup_{f \in \mathcal{H}_\tau} \frac{|\langle \xi_\tau, f \rangle|^2}{\int_{X_\tau} |f|^2 \,\mathrm{e}^{-\varphi}}.$$

We want to check that $\tau \mapsto \log\|\xi_\tau\|^2_{\tau,*}$ satisfies the sub-mean-value inequality. Fix $\tau_0 \in \mathbb{D}$ and $\varepsilon > 0$ so that

$$D_\varepsilon(\tau_0) := \left\{ \tau \in \mathbb{C} \;\middle|\; |\tau - \tau_0| < \varepsilon \right\} \subset\subset \mathbb{D}.$$

Set $X(\varepsilon) := T^{-1}(D_\varepsilon(\tau_0))$ and $\tilde{T} := \frac{T-\tau_0}{\varepsilon} \in \mathcal{O}(X)$. Let $f_0 \in \mathcal{H}_{\tau_0}$ be the section realizing the supremum in $\|\xi_{\tau_0}\|_{\tau_0,*}$. By Theorem 1.5 there is a minimal-norm holomorphic section F_0 of $L|_{X(\varepsilon)} \to X(\varepsilon)$ such that

$$F_0|_{X_{\tau_0}} = \varepsilon f_0 \wedge \mathrm{d}\tilde{T} = f_0 \wedge \mathrm{d}T \quad \text{and} \quad \int_{X(\varepsilon)} |F_0|^2 \,\mathrm{e}^{-\varphi} \leq \pi\varepsilon^2 \int_{X_{\tau_0}} |f_0|^2 \,\mathrm{e}^{-\varphi}$$

(here the line bundle L_Z is trivial, and so we choose $\mathrm{e}^{-\lambda} = 1$ and we can send $\delta \to +\infty$). Let f_τ be defined by $F_0|_{X_\tau} = f_\tau \wedge \mathrm{d}T$ (so that f is a holomorphic section of $L \otimes K_{X/\mathbb{D}}$ over $X(\varepsilon)$). Notice that f_τ might not be integrable for all $\tau \in D_\varepsilon(\tau_0)$, but it must be integrable for all τ outside a measure zero subset of $D_\varepsilon(\tau_0)$, because F_0 has finite L^2 norm. Hence

$$\begin{aligned}
\log\|\xi_{\tau_0}\|^2_{\tau_0,*} &= \log|\langle \xi_{\tau_0}, f_0\rangle|^2 - \log \int_{X_{\tau_0}} |f_0|^2 \,\mathrm{e}^{-\varphi} \\
&\leq \log|\langle \xi_{\tau_0}, f_0\rangle|^2 - \log\left(\frac{1}{\pi\varepsilon^2}\int_{X(\varepsilon)} |F_0|^2\,\mathrm{e}^{-\varphi}\right) \\
&= \log|\langle \xi_{\tau_0}, f_0\rangle|^2 - \log\left(\frac{1}{\pi\varepsilon^2}\int_{\tau\in D_\varepsilon(\tau_0)} \left(\int_{X_\tau} |f_\tau|^2\,\mathrm{e}^{-\varphi}\right)\mathrm{d}T\wedge \mathrm{d}\bar{T}\right) \\
&\leq \log|\langle \xi_{\tau_0}, f_0\rangle|^2 - \frac{1}{\pi\varepsilon^2}\int_{\tau\in D_\varepsilon(\tau_0)} \log\left(\int_{X_\tau} |f_\tau|^2\,\mathrm{e}^{-\varphi}\right)\mathrm{d}T\wedge \mathrm{d}\bar{T} \\
&\leq \frac{1}{\pi\varepsilon^2}\int_{\tau\in D_\varepsilon(\tau_0)} \left(\log|\langle \xi_\tau, f_\tau\rangle|^2 - \log\int_{X_\tau} |f_\tau|^2\,\mathrm{e}^{-\varphi}\right)\mathrm{d}T\wedge \mathrm{d}\bar{T} \\
&\leq \frac{1}{\mathrm{Vol}_{\mathrm{d}T\wedge\mathrm{d}\bar{T}}(D_\varepsilon(\tau_0))}\int_{\tau\in D_\varepsilon(\tau_0)} \log\|\xi_\tau\|^2_{\tau,*}\,\mathrm{d}T\wedge\mathrm{d}\bar{T},
\end{aligned}$$

where the second inequality follows from the concavity of the logarithm, the third from the holomorphicity of $\tau \mapsto \langle \xi_\tau, f_\tau\rangle$, and the last from the definition of the dual norm. This proves Theorem 2.6. $\square$

2.2 Berndtsson's theorem for proper fibrations

Theorem 2.1 has an analogue for holomorphic fibrations with compact fibers. Consider a smooth holomorphic submersion $p : X \to Y$ of complex manifolds X and Y of dimensions $n+m$ and m, respectively, and assume that the fibers $X_t := p^{-1}(t)$ are compact. Let $L \to X$ a holomorphic line bundle with a smooth metric $\mathrm{e}^{-\phi}$ of non-negative curvature, and consider the family of vector spaces $\mathcal{H}_t := H^0(X_t, L|_{X_t} \otimes K_{X_t})$. Since the differential of p is surjective at all points of X, the canonical bundle of the fibers is identified with the restriction of the canonical bundle of the total space: $K_{X_t} \simeq K_X|_{X_t}$. Since L has a metric of semipositive curvature, by this identification and by a variant of the Ohsawa–Takegoshi

Extension Theorem [4, Theorem 8.1] it follows that any global holomorphic section of $L \otimes K_{X_t}$ over X_t extends to a holomorphic section of $L \otimes K_{X_s}$ over X_s with s close to t. Starting from a basis of $\mathcal{H}_t$ we therefore get a local holomorphic frame for $\mathcal{H}$, so that $\mathcal{H}$ has a natural holomorphic vector bundle structure.

Since the fibers are compact any element $\sigma \in \mathcal{H}_t$ is automatically L^2, and we thus obtain a metric for $\mathcal{H} \to Y$ by setting $\|\sigma\|_t^2 := \int_{X_t} |\sigma|^2 \, \mathrm{e}^{-\phi}$.

Theorem 2.7 (**Berndtsson's theorem for proper fibrations** [4, Theorem 1.2]). *If the total space X is Kähler and $\mathrm{e}^{-\phi}$ has (strictly) positive curvature then the curvature of the L^2 Hermitian metric induced on $\mathcal{H} \to Y$ by $\mathrm{e}^{-\phi}$ is (strictly) positive in the sense of Nakano.*

Berndtsson's proof of Theorem 2.7 is quite different from the proof of Theorem 2.1 since he had not "been able to find a natural complex structure on the space of all (not necessarily holomorphic) $(n, 0)$-forms, extending the complex structure on" $\mathcal{H}$. Instead, he computes "directly the Chern connection of the bundle E itself, and compute[s] the curvature from there, much as one proves Griffiths's formula". Recent work of D. Varolin [58] and P. Upadrashta [55] develop a theory of families of Hilbert spaces that might not form a vector bundle but nonetheless behave like one, and for which one has a reasonable notion of curvature. By the study of such objects, called Berndtsson–Lempert–Szőke fields, one obtains a proof of Theorem 2.7 that is basically identical to the proof of Theorem 2.1 [58], as well an analogue of Theorem 2.7 when the fibers are smoothly bounded pseudoconvex domains [55].

We end this Section by a nice application of Theorem 2.7 to bundles of projective spaces. Let V be a holomorphic vector bundle of rank $r < +\infty$ over a complex manifold Y, and consider the bundle $P(V^*) \to Y$ whose fiber at each point $t \in Y$ is the space of lines in V_t^*. A famous conjecture of Griffiths [25] is that, if the hyperplane line bundle $\mathcal{O}(1) \to P(V^*)$ has a smooth Hermitian metric of positive curvature (i.e. V is Hartshorne-ample [28]), then V has a Hermitian metric with Griffiths-positive curvature.

Denote by $S^m(V)$ the m^{th} symmetric power of V. Applying Theorem 2.7 to the fibration $X := P(V^*) \to Y$ and the line bundles $\mathcal{O}(l) \to P(V^*)$, one obtains the following.

Theorem 2.8 ([4, Theorem 1.3])**.** *Let V be a holomorphic vector bundle of finite rank over a complex Kähler manifold. If V is ample in the sense of Hartshorne then for any $m \in \mathbb{N}$ the bundle $S^m(V) \otimes \det V$ has a Hermitian metric with strictly positive curvature in the sense of Nakano.*

In particular, $V \otimes \det V$ has a metric with Nakano-positive curvature, providing partial evidence towards Griffiths conjecture. Indeed, it is a theorem of Demailly and Skoda that, if h is a metric of Griffiths-positive curvature, then $h \otimes \det h$ is a metric of Nakano-positive curvature [14], and in fact $S^m(V) \otimes \det V$ also has a Nakano-positive metric and a dual-Nakano-positive metric [35].

Chapter 3

A degeneration proof of the L^2 extension theorem and of the openness conjecture

Berndtsson's theorem for trivial fibrations (Theorem 2.1) can be used to provide new proofs of variants of the classical L^2-theorems we saw in Chapter 1. We present in Section 3.1 the Berndtsson–Lempert proof of the L^2 extension theorem, and in Section 3.2 a degeneration-based proof due to Berndtsson of the openness conjecture.

3.1 The L^2 extension theorem, revisited

In this Section we will prove the following slight generalization of Berndtsson and Lempert L^2-extension result [7]. We will follow [1].

Theorem 3.1 (**L^2 extension**). *Let X be a Stein manifold and $Z \subset X$ an analytic hypersurface. Let $L_Z \to X$ be the holomorphic line bundle associated to Z, with $T \in H^0(X, L_Z)$ such that $Z = (T = 0)$ and $\mathrm{d}T|_Z$ generically non-zero. Assume moreover that*

L_Z carries a (singular) Hermitian metric $\mathrm{e}^{-\lambda}$ such that $\mathrm{e}^{-\lambda}|_Z \not\equiv +\infty$ and

$$\sup_X |T|^2\,\mathrm{e}^{-\lambda} \leq 1.$$

Let $L \to X$ be a holomorphic line bundle with (singular) Hermitian metric $\mathrm{e}^{-\varphi}$ such that

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq -\sqrt{-1}\,\partial\bar{\partial}\,\lambda \quad \textit{and} \quad \sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \delta\,\sqrt{-1}\,\partial\bar{\partial}\,\lambda$$

for some $\delta > 0$. Then for any holomorphic section $f \in H^0(Z, L|_Z \otimes K_Z)$ such that

$$\|f\|_Z^2 := \int_Z |f|^2\,\mathrm{e}^{-\varphi} < +\infty$$

there is a holomorphic section $F \in H^0(X, L \otimes L_Z \otimes K_X)$ such that $F|_Z = f \wedge \mathrm{d}T$ and

$$\|F\|_X^2 := \int_X |F|^2\,\mathrm{e}^{-\lambda-\varphi} \leq \pi\left(1+\frac{1}{\delta}\right)\int_Z |f|^2\,\mathrm{e}^{-\varphi} = \pi\left(1+\frac{1}{\delta}\right)\|f\|_Z^2\,.$$

We emphasize that the sole difference between this last statement and the full L^2 extension theorem (Theorem 1.5) is that here we require

$$\begin{cases} \sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq -\sqrt{-1}\,\partial\bar{\partial}\,\lambda \\ \sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \delta\,\sqrt{-1}\,\partial\bar{\partial}\,\lambda \end{cases} \tag{3.1}$$

instead of the weaker

$$\begin{cases} \sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq 0 \\ \sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \delta\,\sqrt{-1}\,\partial\bar{\partial}\,\lambda \end{cases}.$$

In particular, the two conditions are clearly the same when $\sqrt{-1}\,\partial\bar{\partial}\,\lambda \geq 0$, and in this case Theorem 3.1 recovers the hyperplane case of [7, Theorem 3.8].

Following [1], we present here a slightly simplified variant of the proof given by Berndtsson and Lempert. As in [7], the main idea is that near Z any extension is nearly optimal and easy to estimate. By introducing a positively-curved family of metrics that "collapses" X to Z, Berndtsson's first theorem (or, more precisely, Corollary 2.2) implies that one can control the norm of the optimal extension on X by the norm of the optimal extension to the small neighborhood of Z.

3.1.1 Preliminary reductions

Since the constants of Theorem 3.1 are universal, we can make some standard reductions. First, as the singularities of Z are of codimension at least one in Z, they are contained in a hypersurface H of X not containing Z. Then we can reduce to the case of smooth Z (and $\mathrm{d}T|_Z \not\equiv 0$) by solving the problem for $Z \setminus H \subset X \setminus H$ and then extending the solution to X. Indeed, $X \setminus H$ is again a Stein manifold and if we assume that Theorem 3.1 holds for Z smooth then we obtain $\tilde{f} \in H^0(X \setminus H, L \otimes L_Z \otimes K_X)$ such that $\tilde{F}|_{Z\setminus H} = f \wedge \mathrm{d}T$ and

$$\int_{X\setminus H} |\tilde{F}|^2 \, \mathrm{e}^{-\lambda-\varphi} \leq \pi \left(1 + \frac{1}{\delta}\right) \int_{Z\setminus H} |f|^2 \, \mathrm{e}^{-\varphi} = \pi \left(1 + \frac{1}{\delta}\right) \|f\|_Z^2 < +\infty.$$

Hence, by Riemann's removable singularities theorem, $\tilde{F}$ extends to $F \in H^0(X, L \otimes L_Z \otimes K_X)$. As H has measure zero,

$$\int_{X\setminus H} |\tilde{F}|^2 \, \mathrm{e}^{-\lambda-\varphi} = \int_X |\tilde{F}|^2 \, \mathrm{e}^{-\lambda-\varphi} \leq \pi \left(1 + \frac{1}{\delta}\right) \|f\|_Z^2$$

and, because $F|_Z$ and $f \wedge \mathrm{d}T$ coincide on the open set $Z \setminus H$, we have $F|_Z = f \wedge \mathrm{d}T$ everywhere on Z, solving the division problem.

Remark 3.2. The same argument proves Theorem 3.1 when X is essentially Stein and $Z \subseteq X$ is an essentially Stein hypersurface, given that it has been proved for Stein manifolds. Recall that a manifold X is *essentially Stein* if there is an analytic set D such that $X \setminus D$ is Stein. For instance, projective manifolds are essentially Stein.

We can also assume that X is a relatively compact domain in some larger Stein manifold and that the metrics involved are smooth. Moreover, perhaps after shrinking X further, we can assume that Z meets the boundary of X transversely and that the section $f \in H^0(Z, L \otimes K_Z)$ to be extended is holomorphic up to the boundary of Z. If the result is proved under these assumptions, then the universality of the bounds yields the general case by standard weak-$*$ compactness theorems, Lebesgue-type limit theorems and approximation results for singular Hermitian metrics on Stein manifolds (see the first paragraph in Section 3 of [7]).

3.1.2 Dual formulation of the extension problem

Fix a section $f \in H^0(Z, L|_Z \otimes K_Z)$ to be extended. We can assume that f is holomorphic up to the boundary of Z by the remarks just made in Subsection 3.1.1.

Proposition 3.3. *There exists $F \in H^0(X, L \otimes L_Z \otimes K_X)$ such that $F|_Z = f \wedge \mathrm{d}T$ and $\|F\|_X^2 < +\infty$.*

Proof. Since X is a relatively compact domain in a Stein manifold and the metrics are smooth, any extension F in the ambient manifold will restrict to an extension on X with finite L^2-norm. Hence it suffices to show that for a Stein manifold X there is an extension, without any requirement on its L^2 norm.

Let $\mathcal{I}_Z \subseteq \mathcal{O}_X$ be the coherent ideal sheaf of germs of holomorphic functions vanishing at the points of Z. Twisting the short exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

by the line bundle $L \otimes L_Z \otimes K_X$ and using the adjunction formula we obtain the short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{I}_Z(L \otimes L_Z \otimes K_X) \longrightarrow \mathcal{O}_X(L \otimes L_Z \otimes K_X) \longrightarrow \mathcal{O}_Z(L|_Z \otimes K_Z) \longrightarrow 0.$$

From the induced long exact sequence in cohomology we then get

$$H^0(X, L \otimes L_Z \otimes K_X) \longrightarrow H^0(Z, L|_Z \otimes K_Z) \longrightarrow H^1(X, \mathfrak{I}_Z(L \otimes L_Z \otimes K_X)) = 0,$$

where the equality is a consequence of Cartan's Theorem B. Therefore

$$H^0(X, L \otimes L_Z \otimes K_X) \longrightarrow H^0(Z, L|_Z \otimes K_Z)$$

is surjective, meaning that for any $f \in H^0(Z, L|_Z \otimes K_Z)$ we can find $F \in H^0(X, L \otimes L_Z \otimes K_X)$ such that $F|_Z = f \wedge \mathrm{d}T$, as wanted. □

Since there is an extension f with finite L^2-norm, there is a (unique) extension $\tilde{F}$ with minimal L^2-norm. To prove Theorem 3.1 we shall estimate $\left\|\tilde{F}\right\|_X$.

Lemma 3.4. *Let $F \in H^0(X, L \otimes L_Z \otimes K_X)$ be any extension with finite L^2-norm. Then the extension $\tilde{F}$ with minimal L^2-norm has norm*

$$\left\|\tilde{F}\right\|_X^2 = \sup_{g \in C_c^\infty(Z, L|_Z \otimes K_Z)} \frac{|\xi_g(F)|^2}{\left\|\xi_g\right\|_*^2},$$

where $\|\cdot\|_*$ is the norm for the dual Hilbert space $(L^2 \cap H^0(X, L \otimes L_Z \otimes K_X))^*$ and

$$\xi_g(s) := (\sigma, g)_Z = \int_Z \sigma \bar{g}\, \mathrm{e}^{-\varphi},$$

with $s|_Z = \sigma \wedge \mathrm{d}T$.

Proof. We start by proving that

$$\left\|\tilde{F}\right\|_X^2 = \sup_{\xi \in \mathrm{Ann}(\mathfrak{I}_Z)} \frac{|\xi(F)|^2}{\|\xi\|_*^2},$$

where $\mathfrak{I}_Z$ is the subset of L^2-integrable holomorphic sections of $L \otimes L_Z \otimes K_X \to X$ that vanish along Z, and $\mathrm{Ann}(\mathfrak{I}_Z)$ is the subspace of all functionals on $(L^2 \cap H^0(X, L \otimes L_Z \otimes K_X))$ that vanish on $\mathfrak{I}_Z$. Notice that $\xi(F)$ is independent of the choice of the extension F.

We claim that $\tilde{F} \perp \mathfrak{I}_Z$. Indeed, if $k \in \mathfrak{I}_Z$ then $\tilde{F} + \varepsilon k$ is an extension of f for all $\varepsilon \in \mathbb{C}$, and since $\tilde{F}$ is the extension with minimal L^2 norm the function

$$\mathbb{C} \ni \varepsilon \longrightarrow \left\| \tilde{F} + \varepsilon k \right\|_X^2 = \left\| \tilde{F} \right\|_X^2 + 2\,\mathbb{R}\mathrm{e}[(\tilde{F}, k)_X \varepsilon] + O(|\varepsilon|^2)$$

has a minimum at $\varepsilon = 0$, so that $(\tilde{F}, k)_X = 0$.

Observe now that $\xi_g(k) = 0$ for all $k \in \mathfrak{I}_Z$ and that conversely, if $s|_Z = \sigma \wedge \mathrm{d}T$ and $\xi_g(s) = 0$ for all f then $\sigma = 0$. Therefore $\{\xi_g \mid g \in C_c^\infty(Z, L|_Z \otimes K_Z)\}$ is dense in $\mathrm{Ann}(\mathfrak{I}_Z)$ and we may thus restrict to the elements ξ_g when computing the supremum. □

By Lemma 3.4

$$\begin{aligned}
\left\| \tilde{F} \right\|_X^2 &= \sup_{g \in C_c^\infty(Z, L|_Z \otimes K_Z)} \frac{|\xi_g(F)|^2}{\left\| \xi_g \right\|_*^2} \\
&= \sup_{g \in C_c^\infty(Z, L|_Z \otimes K_Z)} \frac{|(f, \mathcal{P}g)_Z|^2}{\left\| \xi_g \right\|_*^2} \\
&\leq \|f\|_Z^2 \sup_{g \in C_c^\infty(Z, L|_Z \otimes K_Z)} \frac{\|\mathcal{P}g\|_Z^2}{\left\| \xi_g \right\|_*^2},
\end{aligned}$$

where

$$\mathcal{P} : L^2(Z, L|_Z \otimes K_Z) \longrightarrow L^2 \cap H^0(Z, L|_Z \otimes K_Z)$$

denotes the Bergman projection. Therefore, to prove Theorem 3.1 it suffices to prove that

$$\|\mathcal{P}g\|_Z^2 \leq \pi \left(1 + \frac{1}{\delta}\right) \left\| \xi_g \right\|_*^2 \tag{3.2}$$

for all $g \in C_c^\infty(Z, L|_Z \otimes K_Z)$.

3.1.3 A calculus lemma

We now establish the following lemma, which plays a key role in the proofs of both Theorem 3.1 and Theorem 2. The result is a slight modification of Lemma 3.4 in [7].

Lemma 3.5. *Let $\nu : (-\infty, 0] \to \mathbb{R}_+$ be an absolutely continuous increasing function such that*

$$\lim_{t\to-\infty} \mathrm{e}^{-Bt}\,\nu(t) = A < +\infty$$

for some $B > 0$. Then, for all $p > B$,

$$\lim_{t\to-\infty} \mathrm{e}^{-Bt} \int_t^0 \mathrm{e}^{-p(s-t)}\,\mathrm{d}\nu(s) = \frac{AB}{p-B}.$$

Remark 3.6. In contrast to Lemma 3.4 in [7], we do not require ν to be bounded above by $A\,\mathrm{e}^{Bt}$ for *all* $t < 0$. This weakened hypothesis allows us to obtain the more precise estimates needed for Theorem 3.1 and Theorem 4.1.

Proof. Integrating by parts one gets

$$\begin{aligned}
\mathrm{e}^{-Bt} \int_t^0 \mathrm{e}^{-p(s-t)}\,\mathrm{d}\nu(s) &= \mathrm{e}^{(p-B)t} \int_t^0 \mathrm{e}^{-ps}\,\mathrm{d}\nu(s) \\
&= \mathrm{e}^{(p-B)t} \left[\nu(0) - \mathrm{e}^{-pt}\,\nu(t) + p \int_t^0 \mathrm{e}^{-ps}\,\nu(s)\,\mathrm{d}s\right].
\end{aligned}$$

By the assumptions we have

$$\lim_{t\to-\infty} \mathrm{e}^{(p-B)t} \Big(\nu(0) - \mathrm{e}^{-pt}\,\nu(t)\Big) = -A.$$

Moreover, for any $\varepsilon > 0$ there is $t_\varepsilon < 0$ such that $(A-\varepsilon)\,\mathrm{e}^{Bt} \le \nu(t) \le (A+\varepsilon)\,\mathrm{e}^{Bt}$ for all $t \le t_\varepsilon$. Then

$$\begin{aligned}
\int_t^0 \mathrm{e}^{-ps}\,\nu(s)\,\mathrm{d}s &\le \int_{t_\varepsilon}^0 \mathrm{e}^{-ps}\,\nu(s)\,\mathrm{d}s + (A+\varepsilon) \int_t^{t_\varepsilon} \mathrm{e}^{-(p-B)s}\,\mathrm{d}s \\
&\le C_\varepsilon + \frac{A+\varepsilon}{p-B} \Big(\mathrm{e}^{-(p-B)t} - \mathrm{e}^{-(p-B)t_\varepsilon}\Big) = C'_\varepsilon + \frac{A+\varepsilon}{p-B}\,\mathrm{e}^{-(p-B)t}
\end{aligned}$$

and similarly

$$\int_t^0 \mathrm{e}^{-ps}\,\nu(s)\,\mathrm{d}s \ge C''_\varepsilon + \frac{A-\varepsilon}{p-B}\,\mathrm{e}^{-(p-B)t},$$

so that

$$\frac{A-\varepsilon}{p-B} \le \lim_{t\to-\infty} \mathrm{e}^{(p-B)t} \int_t^0 \mathrm{e}^{-ps}\, \nu(s)\, \mathrm{d}s \le \frac{A+\varepsilon}{p-B}.$$

Since this holds for all $\varepsilon > 0$ we conclude that

$$\lim_{t\to-\infty} \mathrm{e}^{(p-B)t}\, p \int_t^0 \mathrm{e}^{-ps}\, \nu(s)\, \mathrm{d}s = \frac{Ap}{p-B}$$

and then

$$\lim_{t\to-\infty} \mathrm{e}^{-Bt} \int_t^0 \mathrm{e}^{-p(s-t)}\, \mathrm{d}\nu(s) = -A + \frac{Ap}{p-B} = \frac{AB}{p-B},$$

as wanted. □

3.1.4 The family of metrics

We now define a family of metrics for $L \otimes L_Z \to X$ parametrized by

$$\tau \in \mathbb{L} := \{z \in \mathbb{C} \mid \mathbb{R}\mathrm{e}\, z < 0\}$$

by introducing a weight χ_τ that "collapses" X onto Z:

$$\chi_\tau := \max(\log |T|^2 - \lambda - \mathbb{R}\mathrm{e}\, \tau, 0).$$

Since this function only depends on $\mathbb{R}\mathrm{e}\, \tau =: t$, in the following we will write $\chi_\tau = \chi_t$. Notice that $\chi_0 = 0$ since $|T|^2 \mathrm{e}^{-\lambda} \le 1$, and that $\chi_t \sim -t$ when $t \to -\infty$ at all points not in Z.

We also obtain a corresponding family of metrics $\mathfrak{h}_\tau$ for $L \otimes L_Z \to X$ by setting

$$\mathfrak{h}_\tau := \mathrm{e}^{-\mathbb{R}\mathrm{e}\, \tau}\, \mathrm{e}^{-\varphi-\lambda}\, \mathrm{e}^{-(1+\delta)\chi_\tau},$$

and a family of norms for sections $s \in H^0(X, L \otimes L_Z \otimes K_X)$ given by

$$\|s\|_\tau^2 := \mathrm{e}^{-\mathbb{R}\mathrm{e}\,\tau} \int_X |s|^2 \,\mathrm{e}^{-\varphi-\lambda}\,\mathrm{e}^{-(1+\delta)\chi_\tau} \,.$$

We will denote by $\|\cdot\|_{\tau,*}$ the induced dual norms on linear functionals on $L^2 \cap H^0(X, L \otimes L_Z \otimes K_X)$, and interpret $\mathfrak{h}_\tau$ as a metric $\mathfrak{h}$ for $\mathrm{pr}_X^*(L \otimes L_Z) \to X \times \mathbb{L}$. Notice that $\mathfrak{h}$ has non-negative curvature by the hypotheses of Theorem 3.1.

Remark 3.7. Compared to the metrics used in [7], the multiplicative constant in front of χ_t is the *fixed* value $1 + \delta$ and thus we will not be able to send it to infinity. Rather than being a problem, this feature and Lemma 3.5 are what keep the curvature of $\mathfrak{h}$ under control without assuming that $\sqrt{-1}\,\partial\bar{\partial}\,\lambda \geq 0$ (see also [40, 41] for similar observations).

As for $\chi_\tau = \chi_t$, since the metrics and norms only depend on $t = \mathbb{R}\mathrm{e}\,\tau$, with a slight abuse of notation we will often write $\mathfrak{h}_t$ and $\|\cdot\|_t$ instead of $\mathfrak{h}_\tau$ and $\|\cdot\|_\tau$.

We now look at the extrema of the family. Clearly one has $\|s\|_0^2 = \|s\|_X^2$. To study the other extremum $t \to -\infty$, fix $t < 0$ and consider the set A_t of points in X for which the maximum in χ_t is attained by 0:

$$A_t := \left\{ x \in X \,\middle|\, |T(x)|^2 \,\mathrm{e}^{-\lambda} \leq \mathrm{e}^t \right\}.$$

We then write

$$\|s\|_t^2 = \mathrm{e}^{-t} \int_{A_t} |s|^2 \,\mathrm{e}^{-\varphi-\lambda} + \mathrm{e}^{-t} \int_{X \setminus A_t} |s|^2 \,\mathrm{e}^{-\varphi-\lambda} \left(\frac{\mathrm{e}^t}{|T|^2 \,\mathrm{e}^{-\lambda}} \right)^{1+\delta}$$

and proceed to estimate the two summands as $t \to -\infty$.

Notice first that the set A_t collapses to Z as $t \to -\infty$. More precisely, A_t asymptotically resembles a tube about Z whose radius-squared around each $z \in Z$ is asymptotic to $\frac{\mathrm{e}^t}{|\,\mathrm{d}T(z)|^2 \,\mathrm{e}^{-\lambda}}$. Write $s|_Z =: \sigma \wedge \mathrm{d}T$. Then

$$\lim_{t \to -\infty} \mathrm{e}^{-t} \int_{A_t} |s|^2 \,\mathrm{e}^{-\varphi-\lambda} = \pi \int_Z |\sigma|^2 \,\mathrm{e}^{-\varphi} \,. \tag{3.3}$$

The second integral can be rewritten as

$$\mathrm{e}^{-t}\int_t^0 \mathrm{e}^{-(1+\delta)(\tilde{t}-t)}\,\mathrm{d}\nu_s(\tilde{t}),\quad \text{with } \nu_s(t) := \int_{A_t} |s|^2\,\mathrm{e}^{-\varphi-\lambda}\,.$$

The function ν_s is clearly positive increasing and satisfies $\lim_{t\to-\infty}\mathrm{e}^{-t}\,\nu_s(t) = \pi\int_Z |\sigma|^2\,\mathrm{e}^{-\varphi}$. Moreover, it is absolutely continuous by the Fundamental Theorem of Calculus for Lebesgue integrals (see for instance Theorem 3.35 in [24]). Then Lemma 3.5 gives

$$\lim_{t\to-\infty}\mathrm{e}^{-t}\int_t^0 \mathrm{e}^{-(1+\delta)(\tilde{t}-t)}\,\mathrm{d}\nu_s(\tilde{t}) = \frac{\pi}{\delta}\int_Z |\sigma|^2\,\mathrm{e}^{-\varphi}\,.$$

All in all one gets

$$\lim_{t\to-\infty}\|s\|_t^2 = \pi\left(1+\frac{1}{\delta}\right)\int_Z |\sigma|^2\,\mathrm{e}^{-\varphi}\,. \tag{3.4}$$

3.1.5 Monotonicity of the family of dual norms and end of the proof

Now that we have a metric $\mathfrak{h}$ for $\mathrm{pr}_X^*(L\otimes L_Z)\to X\times\mathbb{L}$ with non-negative curvature, Berndtsson's Theorem gives the required estimate.

Lemma 3.8. *The function* $(-\infty,0]\ni t\mapsto \log\left\|\xi_g\right\|$ *is non-decreasing. In particular,*

$$\left\|\xi_g\right\|_*^2 = \left\|\xi_g\right\|_{0,*}^2 \geq \left\|\xi_g\right\|_{t,*}^2 \quad \textit{for all } t\leq 0$$

Proof. We first claim that $\sup_{\tau\in\mathbb{L}}\left\|\xi_g\right\|_{\tau,*}^2 = \sup_{t\leq 0}\left\|\xi_g\right\|_{t,*}^2 < +\infty$. In fact, it suffices to check that $\left\|\xi_g\right\|_{t,*}^2$ is uniformly bounded for all t sufficiently negative. Let

$$C_g := \int_Z |g|^2\,\mathrm{e}^{-\varphi} < +\infty.$$

Recall that g has compact support in Z. Then

$$\left\|\xi_g\right\|_{t,*}^2 = \sup_{\|s\|_t=1} \left|\int_Z \sigma \bar{g}\, \mathrm{e}^{-\varphi}\right|^2 \leq C_g \sup_{\|s\|_t=1} \int_Z |\sigma|^2\, \mathrm{e}^{-\varphi}.$$

By (3.3) one gets

$$\int_Z |\sigma|^2\, \mathrm{e}^{-\varphi} \leq \frac{2}{\pi}\, \mathrm{e}^{-t} \int_{A_t} |s|^2\, \mathrm{e}^{-\varphi-\lambda} \leq \frac{2}{\pi}\, \mathrm{e}^{-t} \int_X |s|^2\, \mathrm{e}^{-\varphi-\lambda}\, \mathrm{e}^{-(1+\delta)\chi_t},$$

so that $\left\|\xi_g\right\|_{t,*}^2 \leq \frac{2C_g}{\pi}$.

Next consider the trivial fibration $X \times \mathbb{L} \to \mathbb{L}$ and the line bundle $\mathrm{pr}_X^*(L \otimes L_Z) \to X \times \mathbb{L}$ with metric $\mathfrak{h}$. As already noted, $\mathfrak{h}$ has non-negative curvature by the hypotheses of Theorem 3.1. Then Corollary 2.2 to Berndtsson's Theorem 2.1 implies that $\mathbb{L} \ni \tau \mapsto \log\left\|\xi_g\right\|_{\tau,*}$ is subharmonic. Since $\left\|\xi_g\right\|_{\tau,*}$ only depends on $t = \mathbb{R}\mathrm{e}\,\tau$, it follows that $t \mapsto \log\left\|\xi_g\right\|_{t,*}$ is convex on $(-\infty, 0)$. If this map decreased anywhere on $(-\infty, 0)$, then by convexity we would have $\lim_{t\to-\infty} \log\left\|\xi_g\right\|_{t,*} = +\infty$, contradicting the uniform boundedness of $\log\left\|\xi_g\right\|_{t,*}$. □

To conclude the proof of Theorem 3.1, let $s \in H^0(X, L \otimes L_Z \otimes K_X)$ be any finite-norm extension of $\mathcal{P}g$ (so that $s|_Z = \mathcal{P}g \wedge \mathrm{d}T$). Then by (3.4)

$$\lim_{t\to-\infty} \left\|\xi_g\right\|_{t,*}^2 \geq \lim_{t\to-\infty} \frac{1}{\|s\|_t^2} \left|\int_Z |\mathcal{P}g|^2\, \mathrm{e}^{-\varphi}\right|^2 = \lim_{t\to-\infty} \frac{\|\mathcal{P}g\|_Z^4}{\|s\|_t^2} \geq \frac{\delta}{\pi(1+\delta)} \|\mathcal{P}g\|_Z^2,$$

so that by Lemma 3.8 one has

$$\|\mathcal{P}g\|_Z^2 \leq \pi\left(1 + \frac{1}{\delta}\right) \lim_{t\to-\infty} \left\|\xi_g\right\|_{t,*}^2 \leq \pi\left(1 + \frac{1}{\delta}\right) \left\|\xi_g\right\|_*^2,$$

i.e. we have proved (3.2) and hence Theorem 3.1.

3.1.6 Remarks on extension in higher codimension

When the subvariety Z has codimension k higher than 1, the adjoint formulation does not fit the extension problem as well as in the hypersurface case of Theorem 3.1.

A special context in which formulating extension in terms of canonical sections makes sense is the setting of the Ohsawa–Takegoshi–Manivel Theorem [36, 19]. Assume that Z is cut out by a holomorphic section T of some holomorphic vector bundle $E \to X$ of rank k and that T is generically transverse to the zero section of E. This means that we know a priori that the normal bundle of Z_{reg} in X extends to the vector bundle $E \to X$, and thus we have the adjunction formula $(K_X \otimes \det E)|_Z = K_Z$. Assume also that $\sup_X h(T, \bar{T}) \leq 1$ for some metric h for $E \to X$.

Then everything goes through in the same way as Theorem 3.1, up to replacing L_Z with $\det E$ and adapting the curvature assumptions. Explicitly, assume that

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \sqrt{-1}\,\partial\bar{\partial}\log\det h$$

and

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \sqrt{-1}\,\partial\bar{\partial}\log\det h - (k+\delta)\,\sqrt{-1}\,\partial\bar{\partial}\log h(T, \bar{T}).$$

Then for every holomorphic section $f \in H^0(Z, L|_Z \otimes K_Z)$ such that

$$\int_Z |f|^2\, \mathrm{e}^{-\varphi} < +\infty$$

there is a holomorphic section $F \in H^0(X, L \otimes \det E \otimes K_X)$ such that $F|_Z = f \wedge \det(\mathrm{d}T)$ and

$$\int_X |F|^2\, \mathrm{e}^{-\varphi + \log\det h} \leq \sigma_k \left(1 + \frac{k}{\delta}\right) \int_Z |f|^2\, \mathrm{e}^{-\varphi},$$

where $\sigma_k := \pi^k/k!$ is the volume of the unit ball in real dimension $2k$. Here the weight is the function $\chi_\tau := \max\left(\log h(T, \bar{T}) - \mathbb{R}\mathrm{e}\,\tau, 0\right)$ and the family of metrics is

$$\mathrm{e}^{-k\,\mathbb{R}\mathrm{e}\,\tau}\,\mathrm{e}^{-\varphi + \log\det h}\,\mathrm{e}^{-(k+\delta)\chi_\tau}\,.$$

Notice that this recovers Theorem 3.1 with $E = L_Z$ and $h = \mathrm{e}^{-\lambda}$.

Unfortunately in general we cannot assume that Z is cut out by a section of some vector bundle. In such case no clear analogue of adjunction is available and thus the non-adjoint formulation is preferable.

Theorem 3.9 (non-adjoint L^2 extension). *Let X be a Stein manifold with Kähler form ω and let $\rho : X \to [-\infty, 0]$ be such that*

$$\log \operatorname{dist}_Z^2 - \beta \leq \rho \leq \log \operatorname{dist}_Z^2 + \alpha$$

for some smooth function β on X and some constant α (so that $Z = \{\rho = -\infty\}$). Fix $\delta > 0$, let $L \to X$ be a holomorphic line bundle with metric $\mathrm{e}^{-\varphi}$, and assume that

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi + \mathrm{Ric}_\omega \geq 0, \quad \sqrt{-1}\,\partial\bar{\partial}\,\varphi + \mathrm{Ric}_\omega + (k+\delta)\sqrt{-1}\,\partial\bar{\partial}\,\rho \geq 0.$$

Then for every holomorphic section $f \in H^0(Z, L|_Z)$ such that

$$\|f\|_Z^2 := \int_Z |f|^2 \,\mathrm{e}^{-\varphi + k\beta}\,\mathrm{d}V_Z < +\infty$$

there is a holomorphic section $F \in H^0(X, L)$ such that $F|_Z = f$ and

$$\|F\|_X^2 := \int_Z |F|^2 \,\mathrm{e}^{-\varphi}\,\mathrm{d}V_X \leq \sigma_k \left(1 + \frac{k}{\delta}\right) \|f\|_Z^2 \,.$$

By taking X to be a pseudoconvex domain $D \subset \mathbb{C}^n$, $\rho = G - \psi$, $\varphi = \phi - k\psi$ and $\beta = B + \psi$ one obtains Theorem 3.8 in [7] (without the assumption that G and ψ are plurisubharmonic).

The argument is the same as the one for Theorem 3.1, except that the functionals ξ_g are now

$$\xi_g(s) := \int_Z s\bar{g}\,\mathrm{e}^{-\varphi + k\beta}\,\mathrm{d}V_Z$$

and the weights are $\chi_\tau := \max(\rho - \mathbb{R}\mathrm{e}\,\tau, 0)$. Then the family of norms becomes

$$\|s\|_t^2 := \mathrm{e}^{-kt} \int_X |s|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{e}^{-(k+\delta)\chi_t} \, \mathrm{d}V_X.$$

Clearly $\|s\|_0^2 = \|s\|_X^2$. For the other end of the family, let $A_t := \{\varphi < t\}$, then

$$\|s\|_t^2 = \mathrm{e}^{-kt} \int_{A_t} |s|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{d}V_X + \mathrm{e}^{-kt} \int_{X \setminus A_t} |s|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{e}^{-(k+\delta)(\rho - t)} \, \mathrm{d}V_X.$$

Since the set A_t is asymptotic to a tube around Z whose radius-squared is bounded above by $\mathrm{e}^{t+\beta}$, it follows that

$$\lim_{t \to -\infty} \mathrm{e}^{-kt} \int_{A_t} |s|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{d}V_X \leq \sigma_k \int_Z |s|^2 \, \mathrm{e}^{-\varphi + k\beta} \, \mathrm{d}V_Z$$

and, by Lemma 3.5,

$$\lim_{t \to -\infty} \mathrm{e}^{-kt} \int_{X \setminus A_t} |s|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{e}^{-(k+\delta)(\rho - t)} \, \mathrm{d}V_X \leq \frac{k\sigma_k}{\delta} \int_Z |s|^2 \, \mathrm{e}^{-\varphi + k\beta} \, \mathrm{d}V_Z.$$

All in all

$$\lim_{t \to -\infty} \mathrm{e}^{-kt} \int_X |s|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{e}^{-(k+\delta)\chi_t} \, \mathrm{d}V_X \leq \sigma_k \left(1 + \frac{k}{\delta}\right) \int_Z |s|^2 \, \mathrm{e}^{-\varphi + k\beta} \, \mathrm{d}V_Z,$$

which is what is needed to prove Theorem 3.9.

3.2 The openness conjecture

Let X be a complex manifold of dimension n and let φ be a plurisubharmonic function on X. Introduced by A. Nadel in [38], the multiplier ideal sheaf of φ is defined to be the subsheaf $\mathcal{I}(\varphi) \subseteq \mathcal{O}_X$ of germs of holomorphic functions $f \in \mathcal{O}_x$ such that $|f|^2 \, \mathrm{e}^{-\varphi}$ is locally integrable near $x \in X$ (see also [34]). Define $\mathcal{I}_+(\varphi) := \bigcup_{\varepsilon > 0} \mathcal{I}((1+\varepsilon)\varphi)$.

In [20], Demailly conjectured that $\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi)$ for any plurisubharmonic function φ on X. In 2015, Q. Guan and X. Zhou proved Demailly's strong openness conjecture [26] by using a version of the Ohsawa–Takegoshi L^2 extension theorem for subvarieties. Shortly after, P. H. Hiep obtained the same result using the standard Ohsawa–Takegoshi Theorem and induction on dimension [29]. Recently X. Wang and T. T. H. Nguyen have been able to use Berndtsson's first theorem to prove the full openness conjecture by using Lemma 3.11 below to give a bound on the maximal local integrability exponent [41]. Here we present an earlier argument of Berndtsson [5, 6] that proves the weaker Demailly–Kollár openness conjecture [21] using Berndtsson's first theorem on direct images (Theorem 2.1).

Theorem 3.10. *Let φ be a plurisubharmonic function in the unit ball $B \subset \mathbb{C}^n$ with $\varphi \leq 0$. Assume that*

$$\int_B \mathrm{e}^{-\varphi}\,\mathrm{d}V < +\infty$$

then there is $p > 1$ such that

$$\int_{B/2} \mathrm{e}^{-p\varphi}\,\mathrm{d}V < +\infty.$$

Moreover, p can be taken so that

$$p \geq 1 + \frac{\delta_n}{\int_B \mathrm{e}^{-\varphi}\,\mathrm{d}V},$$

where δ_n depends only on the dimension n.

Proof. Let $A^2(B, \mu)$ be the space of holomorphic functions on the unit ball B that are L^2-integrable with respect to the measure μ. For any $t \geq 0$ define $\varphi_t := \max(\varphi + t, 0)$. Notice that $\varphi_0 = 0$ and $0 \leq \varphi_t \leq t$. Fix $q > 1$ and define a new norm on $A^2(B, \mathrm{d}V)$ by setting

$$\|h\|_t^2 := \int_B |h|^2\,\mathrm{e}^{-q\varphi_t}\,\mathrm{d}V$$

(here $\mathrm{d}V$ denotes the standard Lebesgue measure). For $p < q$ we have

$$\|h\|_t^2 \leq \int_B |h|^2 \, \mathrm{e}^{-p\varphi_t} \, \mathrm{d}V \leq \mathrm{e}^{-pt} \int_B |h|^2 \, \mathrm{e}^{-p\varphi} \, \mathrm{d}V,$$

so that $\|h\|_t^2$ decreases like e^{-pt} when $h \in A^2(B, \mathrm{e}^{-p\varphi} \, \mathrm{d}V)$.

Lemma 3.11. *For any $0 < p < q$ we have*

$$\int_B |h|^2 \, \mathrm{e}^{-p\varphi} = p\left(1 - \frac{p}{q}\right) \int_0^{+\infty} \mathrm{e}^{pt} \|h\|_t^2 \, \mathrm{d}t + \left(1 - \frac{p}{q}\right) \|h\|_0^2$$

for all $h \in H^2(B, \mathrm{d}V)$.

Proof. We first compute that for any $x < 0$ we have

$$\begin{aligned}\int_0^{+\infty} \mathrm{e}^{pt} \, \mathrm{e}^{-q\max(x+t,0)} \, \mathrm{d}t &= \int_0^{-x} \mathrm{e}^{pt} \, \mathrm{d}t + \mathrm{e}^{-qx} \int_{-x}^{+\infty} \mathrm{e}^{(p-q)t} \, \mathrm{d}t \\ &= \frac{\mathrm{e}^{-px} - 1}{p} - \mathrm{e}^{-qx} \frac{\mathrm{e}^{-(p-q)x}}{p-q} = \frac{q}{p(q-p)} \mathrm{e}^{-px} - \frac{1}{p},\end{aligned}$$

so that

$$\mathrm{e}^{-px} = p\left(1 - \frac{p}{q}\right) \int_0^{+\infty} \mathrm{e}^{pt} \, \mathrm{e}^{-q\max(x+t,0)} \, \mathrm{d}t + \left(1 - \frac{p}{q}\right).$$

Then

$$\int_B |h|^2 \, \mathrm{e}^{-p\varphi} = p\left(1 - \frac{p}{q}\right) \int_0^{+\infty} \mathrm{e}^{pt} \|h\|_t^2 \, \mathrm{d}t + \left(1 - \frac{p}{q}\right) \|h\|_0^2,$$

as claimed. □

According to Lemma 3.11, to prove Theorem 3.10 we therefore need to estimate $\int_0^{+\infty} \mathrm{e}^{pt} \|h\|_t^2 \, \mathrm{d}t$, which is the motivation for the next statement.

Lemma 3.12. *Let H_0 be a separable Hilbert space equipped with a family of equivalent Hilbert norms $\|\cdot\|_t$, where $t \geq 0$. Assume this family defines a positively-curved Hermitian metric on the trivial bundle $H_0 \times \Omega \to \Omega$, where $\Omega := \{\tau \in \mathbb{C} \mid t := \mathbb{R}\mathrm{e}\, \tau \geq 0\}$ is the right* *half-plane. Let H be the subspace of elements $h \in H_0$ such that*

$$\|h\|^2 := \int_0^{+\infty} \mathrm{e}^t \|h\|_t \, \mathrm{d}t < +\infty.$$

Then for any $h \in H_0$, $\varepsilon > 0$, and $t > \varepsilon^{-1}$ there is an element $h_t \in H_0$ such that

$$\|h - h_t\|_0^2 \leq 2\varepsilon \|h\| \tag{3.5}$$

and

$$\|h_t\|_t^2 \leq \mathrm{e}^{-(1+\varepsilon)t} \|h\|_0^2 . \tag{3.6}$$

Proof. Since the norms are equivalent, for every $t \geq 0$ there is a bounded linear operator T_t on H_0 such that $\langle u, v\rangle = \langle T_t u, v\rangle_0$. By the spectral theorem (see for instance Theorem VII.3 and its immediate corollary in [48]) there is a finite measure space (X, μ), bounded functions $\lambda_t : X \to \mathbb{R}$, and a unitary transformation $U : H_0 \to L^2(X, \mathrm{d}\mu)$ such that

$$\|h\|_0 = \int_X |Uh|^2 \, \mathrm{d}\mu$$

and

$$\|h\|_t = \int_X |Uh|^2 \, \mathrm{e}^{-t\lambda_t} \, \mathrm{d}\mu$$

for all $t \geq 0$. Fix $\varepsilon > 0$. We define

$$h_t := U^{-1}(\{\chi_{\lambda_t > 1\}+\varepsilon} \, Uh),$$

where $\chi_{\{\lambda_t > 1+\varepsilon\}}$ is the characteristic function of the set $\{\lambda_t > 1 + \varepsilon\} \subset X$. Then

$$\|h_t\|_t^2 = \int_{\lambda_t > 1+\varepsilon} |Uh|^2 \, \mathrm{e}^{-t\lambda_t} \, \mathrm{d}\mu \leq \mathrm{e}^{-(1+\varepsilon)t} \int_X |Uh|^2 \, \mathrm{d}\mu = \mathrm{e}^{-(1+\varepsilon)t} \|h\|_0^2 ,$$

satisfying (3.6). To prove (3.5), for each fixed $t \geq 0$ we compare $\|\cdot\|_t$ with the flat metric $\|\cdot\|_{s,t}$, defined for $0 \leq s \leq t$ by

$$\|h\|_{s,t}^2 := \int_X |Uh|^2 \, \mathrm{e}^{-s\lambda_t} \, \mathrm{d}\mu.$$

Notice that $\|\cdot\|_{0,t} = \|\cdot\|_0$ and $\|\cdot\|_{t,t} = \|\cdot\|_t$, and moreover $\|\cdot\|_{s,t}$ is flat in the sense that any $h \in H_0$ (thought as a vector in the fiber $H_0 \times \{t_0\}$ of the trivial bundle $H_0 \times \Omega \to \Omega$) can be extended as $\tilde{h}_\zeta := U^{-1}(\mathrm{e}^{(\zeta - t_0)\lambda_t/2}\, Uh)$ so that $\left\|\tilde{h}_\zeta\right\|_{\mathbb{R}\mathrm{e}\,\zeta,t}$ is constant and $\tilde{h}_{t_0} = h$.

We now claim that, since $\|\cdot\|_{s,t}$ induces a flat metric and coincides with $\|\cdot\|_s$ for $s = 0$ and $s = t$, and since $\|\cdot\|_t$ induces a positively curved metric

$$\|\cdot\|_s \geq \|\cdot\|_{s,t} \quad \text{for all } 0 \leq s \leq t. \tag{3.7}$$

This follows by the convexity of the dual norms: $\|\cdot\|_{*,\mathbb{R}\mathrm{e}\,\sigma}$ is negatively curved and coincides with the flat norm $\|\cdot\|_{*,\mathbb{R}\mathrm{e}\,\sigma,t}$ when $\sigma = 0$ and $\mathbb{R}\mathrm{e}\,\sigma = t$, so the convex function $s \mapsto \|\cdot\|_{*,s}$ sits below the function $s \mapsto \|\cdot\|_{*,s,t}$ for all $0 \leq s = \mathbb{R}\mathrm{e}\,\sigma \leq t$. Hence $\|\cdot\|_s \geq \|\cdot\|_{s,t}$ for all $0 \leq s \leq t$, as wanted.

Going back to (3.5), since $h - h_t$ and h_t are orthogonal for the scalar product defined by $\|\cdot\|_{s,t}$ and by (3.7) we have

$$\int_0^t \mathrm{e}^s \|h - h_t\|_{s,t}^2 \, \mathrm{d}s \leq \int_0^t \mathrm{e}^s \|h\|_{s,t}^2 \, \mathrm{d}s \leq \int_0^t \mathrm{e}^s \|h\|_s^2 \, \mathrm{d}s.$$

By definition $h - h_t = U^{-1}(\chi_{\{\lambda_t < 1+\varepsilon\}} Uh)$, so

$$\|h - h_t\|_{s,t}^2 = \int_{\lambda_t < 1+\varepsilon} |Uh|^2 \, \mathrm{e}^{-s\lambda_t} \, \mathrm{d}\mu \geq \mathrm{e}^{-s(1+\varepsilon)} \|h\|_0^2 \,,$$

and thus

$$\int_0^t \mathrm{e}^s \|h - h_t\|_{s,t}^2 \, \mathrm{d}s \geq \|h - h_s\|_0^2 \int_0^t \mathrm{e}^{-s\varepsilon} \, \mathrm{d}s = \frac{1 - \mathrm{e}^{-\varepsilon t}}{\varepsilon} \|h - h_s\|_0^2 \,.$$

Restricting to $t \geq \varepsilon^{-1}$ and putting everything together, we obtain

$$\|h - h_s\|_0^2 \leq 2\varepsilon \int_0^t \mathrm{e}^s \|h\|_s^2 \, \mathrm{d}s \leq 2\varepsilon \|h\|^2 ,$$

proving (3.5) and Lemma 3.12. □

With our choice of $\|\cdot\|_t$, by Lemma 3.11 for $p = 1$ we have

$$\|h\|^2 = \int_0^{+\infty} \mathrm{e}^t \|h\|_t \, \mathrm{d}t = \frac{q}{q-1} \int_B |h|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{d}V - \|h\|_0^2 ,$$

and in particular

$$\|1\|^2 = \frac{q}{q-1} \int_B \mathrm{e}^{-\varphi} \, \mathrm{d}V - \mathrm{Vol}(B) < +\infty$$

(recall that $\varphi_0 = 0$). Since φ is plurisubharmonic, by Berndtsson's theorem for trivial fibrations (Theorem 2.1) the family of norms $\|\cdot\|_t$ defines a positively-curved Hermitian metric on the trivial bundle $A^2(B, \mathrm{d}V) \times \Omega \to \Omega$. Then we can apply Lemma 3.12 to $h = 1$: for any $\varepsilon > 0$ and $t > \varepsilon^{-1}$ we get a holomorphic function h_t on the unit ball B such that

$$\int_B |h_t - 1|^2 \, \mathrm{d}V \leq 2\varepsilon \left(\frac{q}{q-1} \int_B \mathrm{e}^{-\varphi} \, \mathrm{d}V - \mathrm{Vol}(B) \right)$$

and

$$\int_B |h_t|^2 \, \mathrm{e}^{-q\varphi_t} \, \mathrm{d}V \leq \mathrm{Vol}(B) \, \mathrm{e}^{-(1+\varepsilon)t} .$$

Observe now that by the sub-mean-value property there is a constant $\delta_n > 0$, depending only on the dimension of B, such that $\sup_{B/2} |g| \leq \frac{1}{2}$ whenever g is holomorphic in B and $\int_B |g|^2 \, \mathrm{d}V \leq \delta_n$. Therefore, choosing ε small enough, we can assume that $\sup_{B/2} |h_t - 1| \leq \frac{1}{2}$, and thus $|h_t| \geq \frac{1}{2}$ on $B/2$. Hence

$$\int_{B/2} \mathrm{e}^{-q\varphi_t} \, \mathrm{d}V \leq 4 \int_B |h_t|^2 \, \mathrm{e}^{-q\varphi_t} \, \mathrm{d}V = 4\|h_t\|_t^2 \leq 4 \, \mathrm{Vol}(B) \, \mathrm{e}^{-(1+\varepsilon)t} .$$

By taking for instance $p \leq 1 + \frac{\varepsilon}{2}$, we have

$$\mathrm{e}^{pt} \int_{B/2} \mathrm{e}^{-q\varphi_t} \,\mathrm{d}V \leq 4 \operatorname{Vol}(B) \, \mathrm{e}^{pt-(1+\varepsilon)t} \leq 4 \operatorname{Vol}(B) \, \mathrm{e}^{-\frac{\varepsilon}{2}t},$$

so that

$$\int_0^{+\infty} \mathrm{e}^{pt} \int_{B/2} \mathrm{e}^{-q\varphi_t} \,\mathrm{d}t \leq \int_0^{\frac{1}{\varepsilon}} \mathrm{e}^{pt} \int_{B/2} \mathrm{e}^{-q\varphi_t} \,\mathrm{d}t + 4 \operatorname{Vol}(B) \int_{\frac{1}{\varepsilon}}^{+\infty} \mathrm{e}^{-\frac{\varepsilon}{2}t} \,\mathrm{d}t =: C < +\infty.$$

Applying Lemma 3.11 with B replaced by $B/2$, $h = 1$, and $p \leq \min\left(q, 1 + \frac{\varepsilon}{2}\right)$ we then obtain

$$\int_{B/2} \mathrm{e}^{-p\varphi} \leq \left(1 - \frac{p}{q}\right) (pC + \operatorname{Vol}(B)) < +\infty,$$

proving Theorem 3.10. □

Chapter 4

A degeneration proof of a Skoda-type L^2 division theorem

In this Chapter we present the degeneration-based proof of Theorem 2. For the reader's convenience, we state the following slightly more concretely formulated version of Theorem 2.

Theorem 4.1 (L^2 **division**). *Let X be a Stein manifold and let $E, G \to X$ be holomorphic line bundles with (singular) Hermitian metrics $\mathrm{e}^{-\varphi}$ and $\mathrm{e}^{-\psi}$, respectively. Fix $h = (h_1, \ldots, h_r) \in H^0(X, (E^* \otimes G)^{\oplus r})$ and $1 < \alpha < \frac{r+1}{r-1}$. Assume that*

$$\sqrt{-1}\,\partial\bar{\partial}\,\varphi \geq \frac{\alpha(r-1)}{\alpha(r-1)+1}\,\sqrt{-1}\,\partial\bar{\partial}\,\psi.$$

Then for any holomorphic section $g \in H^0(X, G \otimes K_X)$ such that

$$\|g\|^2_G := \int_X \frac{|g|^2\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}} < +\infty$$

there is a holomorphic section $f = (f_1, \ldots, f_r) \in H^0(X, E^{\oplus r} \otimes K_X)$ such that

$$g = h \dot{\otimes} f := h_1 \otimes f_1 + \cdots + h_r \otimes f_r$$

and

$$\|f\|^2_{E^{\oplus r}} := \int_X \frac{|f|^2 \,\mathrm{e}^{-\varphi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}}$$
$$\leq r\frac{\alpha}{\alpha-1}\int_X \frac{|g|^2\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}} = r\frac{\alpha}{\alpha-1}\|g\|^2_G\,.$$

The main idea is to look at all possible linear combinations $v_1 \otimes f_1(x) + \cdots + v_r \otimes f_r(x)$. Then one constructs a positively curved family of metrics that at one extreme "localizes" the problem at the point of interest $v = h(x)$ and at the other extreme retrieves the usual L^2-norm for f. Near $h(x)$ the optimal solution to the division problem is "trivial"; for instance, if $h(x) = (h_1(x), 0, \ldots, 0)$, one takes $f(x) = (g(x)h_1(x)^{-1}, 0, \ldots, 0)$. The positivity of the direct image bundle [4] then implies that one can control $\|f\|_{E^{\oplus r}}$ by the norm of the trivial solution near $h(x)$. The proof follows the author's work in [1].

4.1 Preliminary reductions

No base locus. We may assume that the sections $h_1, \ldots, h_r$ have no common zeros. Indeed, let D be the zero-set of h_r. Then $X \setminus D$ is again Stein and $h|_{X\setminus D}$ has no zeros. Assuming that Theorem 4.1 holds if $\{h_1 = \cdots = h_r = 0\} = \varnothing$, we obtain $\tilde{f} \in H^0(X \setminus D, (E^{\oplus r} \otimes K_X)|_{X\setminus D})$ such that

$$g|_{X\setminus D} = h|_{X\setminus D} \dot{\otimes} \tilde{f}$$

and

$$\int_{X\setminus D} \frac{|\tilde{f}|^2\,\mathrm{e}^{-\varphi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \leq r\frac{\alpha}{\alpha-1}\|g\|^2_G < +\infty.$$

As h is bounded on any bounded chart $U \subset\subset X$,

$$\int_{U\setminus D} |\tilde{f}|^2\,\mathrm{e}^{-\varphi} \leq C\int_{U\setminus D} \frac{|\tilde{f}|^2\,\mathrm{e}^{-\varphi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \leq Cr\frac{\alpha}{\alpha-1}\|g\|^2_G < +\infty,$$

where $C > 0$ depends on U, h and $\alpha(r-1)$. Hence, by Riemann's Removable Singularities Theorem, $\tilde{f}$ extends to $f \in H^0(X, E^{\oplus r} \otimes K_X)$. As D has measure 0,

$$\int_X \frac{|f|^2 \,\mathrm{e}^{-\varphi}}{(|h|^2 \,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} = \int_{X\setminus D} \frac{|\tilde{f}|^2 \,\mathrm{e}^{-\varphi}}{(|h|^2 \,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \leq r\frac{\alpha}{\alpha-1}\|g\|_G^2 < +\infty,$$

and because $h \dot{\otimes} f$ and g coincide on the open set $X \setminus D$ we have $h \dot{\otimes} f = g$ everywhere on X, solving the division problem.

As in Remark 3.2, the same argument proves Theorem 4.1 when X is essentially Stein, given that it has been proved for Stein manifolds.

X **bounded pseudoconvex.** As in the proof of the L^2 extension theorem, we can reduce X to a relatively compact domain in some larger Stein manifold. We say that sections extend up to the boundary of X if they extend to a neighborhood of X in the ambient Stein manifold. We can also assume that ω and E, G extend to the ambient manifold and that the metrics $\mathrm{e}^{-\varphi}$ and $\mathrm{e}^{-\psi}$ are smooth. If the result is proved under these assumptions then the universality of the bounds yields the general case by standard weak-$*$ compactness theorems, Lebesgue-type limit theorems and approximation results for singular Hermitian metrics on Stein manifolds.

4.2 Dual formulation of the division problem

Fix a section $g \in H^0(X, G \otimes K_X)$ to be divided. We may assume, possibly after shrinking X, that g is holomorphic up to the boundary of X. Let $\gamma : E^{\oplus r} \otimes K_X \to G \otimes K_X$ be defined by

$$\gamma(e_1, \ldots, e_r) := h_1 \otimes e_1 + \cdots + h_r \otimes e_r.$$

Proposition 4.2. *There exists* $f = (f_1, \dots, f_r) \in H^0(X, E^{\oplus r} \otimes K_X)$ *such that*

$$g = h_1 \otimes f_1 + \cdots + h_r \otimes f_r = h \dot{\otimes} f$$

and

$$\|f\|^2_{E^{\oplus r}} < +\infty.$$

Proof. Since X is a relatively compact domain in a Stein manifold, any solution f in the ambient manifold will restrict to a solution on X with finite L^2-norm. Hence, it suffices to show that for a Stein manifold X there is a not-necessarily-L^2 solution of the division problem.

As the $h_1, \dots, h_r$ have no common zeros, the map γ is a surjective morphism of vector bundles and thus we have the short exact sequence of vector bundles

$$0 \longrightarrow \ker\gamma \longrightarrow E^{\oplus r} \otimes K_X \longrightarrow G \otimes K_X \longrightarrow 0.$$

The induced sequence in cohomology then yields

$$0 \to H^0(X, \ker\gamma) \to H^0(X, E^{\oplus r} \otimes K_X) \to H^0(X, G \otimes K_X) \to H^1(X, \ker\gamma) = 0,$$

where equality is a consequence of Cartan's Theorem B. Hence, the map induced by γ in cohomology is surjective, meaning that for any $g \in H^0(X, G \otimes K_X)$ we can find

$$f = (f_1, \dots, f_r) \in H^0(X, E^{\oplus r} \otimes K_X)$$

such that

$$g = \gamma \circ f = h_1 \otimes f_1 + \cdots + h_r \otimes f_r,$$

proving the statement. □

Since there is a solution f of the division problem with finite L^2-norm, there is a (unique) solution $\tilde{f}$ with minimal L^2-norm. To prove Theorem 4.1 it then suffices to estimate $\left\|\tilde{f}\right\|_{E^{\oplus r}}$.

Lemma 4.3. *Let $f \in H^0(X, E^{\oplus r} \otimes K_X)$ be any solution to the division problem with finite L^2-norm. Then the solution $\tilde{f}$ with minimal L^2-norm has norm*

$$\left\|\tilde{f}\right\|^2_{E^{\oplus r}} = \sup_{\xi \in \operatorname{Ann} H^0(X, \ker\gamma)} \frac{|\xi(f)|^2}{\|\xi\|^2_*},$$

where $\|\cdot\|_*$ is the norm for the dual Hilbert space $H^0(X, E^{\oplus r} \otimes K_X)^*$ and $\operatorname{Ann} H^0(X, \ker\gamma)$ is the annihilator of $H^0(X, \ker\gamma)$, i.e. the set of all linear functionals on $H^0(X, E^{\oplus r} \otimes K_X)$ that vanish on $H^0(X, \ker\gamma)$. Moreover, one can restrict the supremum to functionals $\xi_\eta \in H^0(X, E^{\oplus r} \otimes K_X)^*$ of the form

$$\xi_\eta(f) := (\gamma \circ f, \eta)_G = \int_X \frac{(h \dot{\otimes} f)\bar{\eta}\, \mathrm{e}^{-\psi}}{(|h|^2 \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}},$$

for $\eta \in C^\infty_c(X, G \otimes K_X)$ (smooth compactly supported sections of $G \otimes K_X$).

Proof. Note first that the supremum is independent of the choice of the arbitrary L^2 solution f. Indeed, if $\xi \in \operatorname{Ann} H^0(X, \ker\gamma)$ and $\gamma \circ f = \gamma \circ f' = g$, then by linearity $f - f' \in H^0(X, \ker\gamma)$, so that $\xi(f) = \xi(f')$.

Next, we claim that $\tilde{f} \perp H^0(X, \ker\gamma)$. Indeed, if $k \in H^0(X, \ker\gamma)$, then $\gamma \circ (\tilde{f} + \varepsilon k) = g$ for all $\varepsilon \in \mathbb{C}$. As $\tilde{f}$ is the minimal norm solution,

$$\mathbb{C} \ni \varepsilon \longmapsto \left\|\tilde{f} + \varepsilon k\right\|^2_{E^{\oplus r}} = \left\|\tilde{f}\right\|^2_{E^{\oplus r}} + 2\,\mathbb{R}\mathrm{e}[(\tilde{f}, k)_{E^{\oplus r}}\varepsilon] + O(|\varepsilon|^2)$$

has minimum at $\varepsilon = 0$ (here $(\cdot,\cdot)_{E^{\oplus r}}$ denotes the L^2 inner product on $E^{\oplus r} \otimes K_X$). Hence $(\tilde{f}, k)_{E^{\oplus r}} = 0$.

Finally, notice that if $k \in H^0(X, \ker\gamma)$ then

$$\xi_\eta(k) = (\gamma \circ k, \eta)_G = 0,$$

i.e. $\xi_\eta \in \operatorname{Ann} H^0(X, \ker\gamma)$. Conversely, if

$$0 = \xi_\eta(f) = (\gamma \circ f, \eta)_G$$

for all $\eta \in C_c^\infty(X, G \otimes K_X)$, then $\gamma \circ f = 0$. Hence

$$\left\{\xi_\eta \mid \eta \in C_c^\infty(X, G \otimes K_X)\right\} \subseteq \operatorname{Ann} H^0(X, \ker\gamma)$$

is dense, so we may restrict to elements ξ_η when computing the supremum. $\square$

By Lemma 4.3

$$\begin{aligned} \left\|\tilde{f}\right\|_{E^{\oplus r}}^2 &= \sup_{\eta \in C_c^\infty(X, G\otimes K_X)} \frac{|(\gamma \circ f, \eta)|^2}{\left\|\xi_\eta\right\|_*^2} \\ &= \sup_{\eta \in C_c^\infty(X, G\otimes K_X)} \frac{|(g, \mathcal{P}\eta)|^2}{\left\|\xi_\eta\right\|_*^2} \leq \|g\|_G^2 \sup_{\eta \in C_c^\infty(X, G\otimes K_X)} \frac{\|\mathcal{P}\eta\|_G^2}{\left\|\xi_\eta\right\|_*^2}, \end{aligned} \tag{4.1}$$

where

$$\mathcal{P} : L^2(X, G \otimes K_X) \longrightarrow H^0(X, G \otimes K_X) \cap L^2(X, G \otimes K_X)$$

denotes the Bergman projection. Therefore, to prove Theorem 4.1 it suffices to prove that

$$\|\mathcal{P}\eta\|_G^2 \leq r\frac{\alpha}{\alpha - 1}\left\|\xi_\eta\right\|_*^2$$

for all $\eta \in C_c^\infty(X, G \otimes K_X)$.

4.3 The degenerating family of norms

Instead of working directly on the vector bundle $E^{\oplus r} \otimes K_X \to X$, we lift everything to the line bundle

$$L := \operatorname{pr}_X^*(E \otimes K_X) \otimes \operatorname{pr}_{\mathbb{P}_{r-1}}^* \mathcal{O}_{\mathbb{P}_{r-1}}(1) \longrightarrow X \times \mathbb{P}_{r-1},$$

where $\mathcal{O}_{\mathbb{P}_{r-1}}(1)$ is the hyperplane bundle of $\mathbb{P}_{r-1}$ and pr_X, $\mathrm{pr}_{\mathbb{P}_{r-1}}$ are the projections of $X \times \mathbb{P}_{r-1}$ on X, $\mathbb{P}_{r-1}$ respectively. Explicitly, fix once for all coordinates $v_1, \ldots, v_r$ for $\mathbb{C}^r$ (descending to the homogeneous coordinates $[v_1 : \cdots : v_r]$ for $\mathbb{P}_{r-1}$) and declare the lift of a section $s \in H^0(X, E^{\oplus r} \otimes K_X)$ to be the section $\hat{s} \in H^0(X \times \mathbb{P}_{r-1}, L)$ defined by

$$\hat{s}(x, [v]) := v^* \cdot s(x) = v_1^* s_1(x) + \cdots + v_r^* s_r(x) \in H^0(X \times \mathbb{P}_{r-1}, L), \tag{4.2}$$

where $v_1^*, \ldots, v_r^*$ are the dual coordinates of $v_1, \ldots, v_r$ of $\mathbb{C}^r$.

Notice that the lift is a bijective map, since all sections of L are of the form (4.2). We can then lift the functionals $\xi_\eta \in H^0(X, E^{\oplus r} \otimes K_X)^*$ of Lemma 4.3 to functionals $\hat{\xi}_\eta \in H^0(X \times \mathbb{P}_{r-1}, L)^*$ defined as $\hat{\xi}_\eta(\hat{s}) := \xi_\eta(s)$ for all $\hat{s} \in H^0(X \times \mathbb{P}_{r-1}, L)$.

Remark 4.4. One can interpret the lifted section $\hat{s}$ by thinking of the projective space $\mathbb{P}_{r-1}$ as parametrizing all possible choices of linear combinations (up to scaling). Hence, the value of the section $\hat{s}$ at $(x, [v])$ can be thought of (tautologically) as the linear combination parametrized by v of the entries of the vector $s(x)$. What follows is essentially a procedure to "single out" the linear combination given by $[h(x)]$ (the equivalence class of $h(x)$ in $\mathbb{P}_{r-1}$).

Next, we define a family of metrics for $L \to X \times \mathbb{P}_{r-1}$, parametrized by

$$\tau \in \mathbb{L} := \{z \in \mathbb{C} \mid \operatorname{\mathbb{R}e} z < 0\}.$$

Toward this end, let

$$\chi_\tau(x, v) := \max\left(\log(|v|^2 |h(x)|^2 - |v \cdot h(x)|^2)\, \mathrm{e}^{-\psi+\varphi} - \operatorname{\mathbb{R}e} \tau,\ \log |v|^2 |h(x)|^2\, \mathrm{e}^{-\psi+\varphi}\right)$$

and, for $\sigma \in L_{(x,[v])}$, set

$$\mathfrak{h}_\tau(\sigma, \bar{\sigma})_{(x,[v])} := \frac{r!}{\pi^{r-1}}\, \mathrm{e}^{-(r-1)\operatorname{\mathbb{R}e}\tau}\, \frac{|\sigma|^2\, \mathrm{e}^{-\varphi}}{|v|^2} \left(|v|^2\, \mathrm{e}^{-\chi_\tau}\right)^{\alpha(r-1)}.$$

Note that whether the maximum defining χ_τ is attained by the first or the second entry is independent of the choice of the representative v of $[v] \in \mathbb{P}_{r-1}$, and that the weight $|v|^2 \,\mathrm{e}^{-\chi_\tau}$ is a well-defined function on $X \times \mathbb{P}_{r-1}$. Notice also that χ_τ depends only on $t := \mathbb{R}\mathrm{e}\,\tau$ (as does $\mathfrak{h}_\tau$), so in the following we will write χ_t (and $\mathfrak{h}_t$) instead.

Remark 4.5. As we shall soon see, the choice of χ_t is motivated as follows. For $t = 0$, the maximum is realized by the second entry, so that

$$(|v|^2 \,\mathrm{e}^{-\chi_t})^{\alpha(r-1)} = \frac{1}{(|h(x)|^2 \,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}},$$

providing the weighting by the norm of h in $\|\cdot\|_{E^{\oplus r}}$. On the other hand, as $t \to -\infty$, the function

$$|v|^2 \,\mathrm{e}^{-\chi_t} = \frac{1}{(|h(x)|^2 \,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \min\left(\frac{\mathrm{e}^t}{1 - \frac{|v \cdot h(x)|^2}{|v|^2 |h(x)|^2}}, 1 \right)$$

gets very small at all $v \in \mathbb{P}_{r-1}$ that are not "sufficiently aligned" with $h(x)$. The precise sense of this statement will be more evident in Section 4.4, but for the moment note that $1 - \frac{|v \cdot h(x)|^2}{|v|^2 |h(x)|^2}$ constitutes a measurement of the angle between v and $h(x)$.

The family of metrics $\mathfrak{h}_\tau$ induces the family of L^2-norms

$$\|\sigma\|_\tau^2 := \frac{r!}{\pi^{r-1}} \,\mathrm{e}^{-(r-1)\,\mathbb{R}\mathrm{e}\,\tau} \int_{X \times \mathbb{P}_{r-1}} \frac{|\sigma|^2 \,\mathrm{e}^{-\varphi}}{|v|^2} \left(|v|^2 \,\mathrm{e}^{-\chi_\tau} \right)^{\alpha(r-1)} \wedge \mathrm{d}V_{\mathrm{FS}},$$

where $\mathrm{d}V_{\mathrm{FS}}$ is the (fixed) Fubini–Study volume form of $\mathbb{P}_{r-1}$.

We interpret the family $\mathfrak{h}_\tau$ as a metric $\mathfrak{h}$ for the pull-back of L on $X \times \mathbb{P}_{r-1} \times \mathbb{L}$, and we claim that sum of the curvature of $\mathfrak{h}$ and the Ricci curvature is non-negative. To start, $\mathrm{e}^{-\varphi-\alpha(r-1)\chi}$ is non-negatively curved: $\mathrm{e}^{-\varphi-\alpha(r-1)(-\psi+\varphi)}$ contributes semipositively by the hypothesis on curvature of Theorem 4.1, and by Lagrange's identity we have

$$\log\left(|v|^2 |h(x)|^2 - |v \cdot h(x)|^2 \right) = \log \sum_{1 \le i < j \le r} |h_i(x) v_j - h_j(x) v_i|^2,$$

which is (locally) plurisubharmonic, being the logarithm of a sum of norms squared of (locally) holomorphic functions (likewise for the right-hand side of the maximum). This leaves us to check that the negativity coming from the factor $(|v|^2)^{\alpha(r-1)-1}$ is compensated for by the Ricci curvature coming from the Fubini–Study volume form: in fact, in local coordinates we can write $\frac{1}{|v|^2}|v|^{2\alpha(r-1)}\,\mathrm{d}V_{\mathrm{FS}}$ as $\frac{\mathrm{d}V(z)}{(1+|z|^2)^{r+1-\alpha(r-1)}}$ (where $\mathrm{d}V(z)$ is the standard Euclidean volume form), which is positively curved for $\alpha < \frac{r+1}{r-1}$.

4.4 Extrema of the family of norms

We now investigate the behavior of the family of norms $\|\cdot\|_t$ at $t = 0$ and as $t \to -\infty$.

At the $t = 0$ extreme we have $\chi_0(x, v) = \log\left(|v|^2|h(x)|^2\,\mathrm{e}^{-\psi+\varphi}\right)$, so that

$$\begin{aligned}\|\hat{s}\|_0^2 &= \frac{r!}{\pi^{r-1}}\int_{X\times\mathbb{P}_{r-1}}\frac{|v\cdot s|^2\,\mathrm{e}^{-\varphi}}{|v|^2}\wedge\frac{\mathrm{d}V_{\mathrm{FS}}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}}\\ &= \int_X\frac{|s|^2\,\mathrm{e}^{-\varphi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} = \|s\|_{E^{\oplus r}}^2\,,\end{aligned} \tag{4.3}$$

which recovers the norm-squared of s before the lifting. Consequently, for $t = 0$, lifting functionals also preserves norms:

$$\left\|\hat{\xi}_\eta\right\|_{0,*} = \sup_{\hat{s}\in H^0(X\times\mathbb{P}_{r-1},L)}\frac{|\hat{\xi}_\eta(\hat{s})|}{\|\hat{s}\|_0} = \sup_{s\in H^0(X,E^{\oplus r}\otimes K_X)}\frac{|\xi_\eta(s)|}{\|s\|_{E^{\oplus r}}} = \left\|\xi_\eta\right\|_*.$$

To study the other extreme of the family, i.e. $t \to -\infty$, fix $x \in X$ and let $A_{t,x}$ be the set of $v \in \mathbb{P}_{r-1}$ such that the maximum in χ_t is achieved by the second entry, i.e.

$$A_{t,x} = \left\{v\in\mathbb{P}_{r-1}\;\middle|\;1-\frac{|v\cdot h(x)|^2}{|v|^2|h(x)|^2} < \mathrm{e}^t\right\}.$$

By choosing homogeneous coordinates so that v_1 is parallel to $h(x)$ (which is not 0 since we are assuming that the h_i's have no common zeros), and by choosing local coordinates so that $v_1 = 1$, one sees that $A_{t,x}$ is a ball of real dimension $2r-2$, centered at $[h(x)]$ (the origin, in local coordinates) and of radius $\sqrt{\frac{\mathrm{e}^t}{1-\mathrm{e}^t}} \underset{t\to-\infty}{\sim} \mathrm{e}^{t/2}$.

We can then split $\|\hat{s}\|_t^2$ into two summands:

$$\|\hat{s}\|_t^2 = \int_X \mathbf{I}_{t,s}(x) + \int_X \mathbf{II}_{t,s}(x),$$

where

$$\mathbf{I}_{t,s}(x) := \frac{r!}{\pi^{r-1}} \frac{\mathrm{e}^{-(r-1)t}}{(|h|^2 \, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \int_{A_{t,x}} \frac{|v \cdot s(x)|^2 \, \mathrm{e}^{-\varphi}}{|v|^2} \wedge \mathrm{d}V_{\mathrm{FS}}$$

and

$$\mathbf{II}_{t,s}(x) := \frac{r!}{\pi^{r-1}} \frac{\mathrm{e}^{-(r-1)t}}{(|h|^2 \, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \int\limits_{\mathbb{P}_{r-1} \setminus A_{t,x}} \frac{|v \cdot s(x)|^2 \, \mathrm{e}^{-\varphi} \, \mathrm{e}^{\alpha(r-1)t}}{|v|^2 \left(1 - \frac{|v \cdot h|^2}{|v|^2 |h|^2}\right)^{\alpha(r-1)}} \wedge \mathrm{d}V_{\mathrm{FS}}\,.$$

For the first term we get

$$\lim_{t \to -\infty} \mathbf{I}_{t,s}(x) = r \frac{|h \mathbin{\dot{\otimes}} s|^2 \, \mathrm{e}^{-\psi}}{(|h|^2 \, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}}, \tag{4.4}$$

since asymptotically

$$\int_{A_{t,x}} \frac{|v \cdot s(x)|^2 \, \mathrm{e}^{-\varphi}}{|v|^2} \wedge \mathrm{d}V_{\mathrm{FS}} \underset{t \to -\infty}{\sim} \frac{\pi^{r-1}}{(r-1)!} \, \mathrm{e}^{(r-1)t} \, \frac{|h \mathbin{\dot{\otimes}} s|^2 \, \mathrm{e}^{-\varphi}}{|h|^2}.$$

For the second

$$\mathbf{II}_{t,s}(x) = \mathrm{e}^{-(r-1)t} \int_t^0 \mathrm{e}^{-\alpha(r-1)(\tilde{t}-t)} \, \mathrm{d}\nu_{x,s}(\tilde{t}),$$

with

$$\nu_{x,s}(t) := \frac{r!}{\pi^{r-1}} \frac{1}{(|h|^2 \, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)}} \int_{A_{t,x}} \frac{|v \cdot s|^2 \, \mathrm{e}^{-\varphi}}{|v|^2} \wedge \mathrm{d}V_{\mathrm{FS}} = \mathrm{e}^{(r-1)t} \, \mathbf{I}_{t,s}(x).$$

The function $\nu_{x,s}$ is increasing and positive, and is absolutely continuous by the Fundamental Theorem of Calculus for Lebesgue integrals [24, Theorem 3.35]. Moreover, by (4.4),

$$\lim_{t\to-\infty} \mathrm{e}^{-(r-1)t}\, \nu_{x,s}(t) = r\frac{|h \dot{\otimes} s|^2\, \mathrm{e}^{-\psi}}{(|h|^2\, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}}.$$

Hence, by Lemma 3.5,

$$\lim_{t\to-\infty} \mathbf{II}_{t,s}(x) = r\frac{|h \dot{\otimes} s|^2\, \mathrm{e}^{-\psi}}{(|h|^2\, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}}\frac{r-1}{\alpha(r-1)-(r-1)}. \tag{4.5}$$

All together

$$\lim_{t\to-\infty} \|\hat{s}\|_t^2 = r\frac{\alpha}{\alpha-1}\int_X \frac{|h \dot{\otimes} s|^2\, \mathrm{e}^{-\psi}}{(|h|^2\, \mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}} = r\frac{\alpha}{\alpha-1}\left\|h \dot{\otimes} s\right\|_G^2, \tag{4.6}$$

retrieving (a multiple of) the norm-squared of the image $h \dot{\otimes} s$ of s.

4.5 Monotonicity of the family of dual norms and end of the proof

Now that we have a metric $\mathfrak{h}$ for (the pull-back of) L on $X \times \mathbb{P}_{r-1} \times \mathbb{L}$ with positive enough curvature, Berndtsson's Theorem 2.1 gives the core step of the argument. Fix $\eta \in C_c^\infty(X, G \otimes K_X)$.

Lemma 4.6. *The function*

$$\begin{array}{ccc} (-\infty, 0] & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \log\left\|\hat{\xi}_\eta\right\|_{t,*}^2 \end{array}$$

is non-decreasing. In particular,

$$\left\|\xi_\eta\right\|_*^2 = \left\|\hat{\xi}_\eta\right\|_{0,*}^2 \geq \left\|\hat{\xi}_\eta\right\|_{t,*}^2 \quad \textit{for all } t \leq 0.$$

Proof. **Step 1.** We first prove that $\sup_{\tau\in\mathbb{L}}\left\|\hat{\xi}_\eta\right\|^2_{\tau,*} < +\infty$. Once more, $\left\|\hat{\xi}_\eta\right\|^2_{\tau,*}$ only depends on $\operatorname{Re}\tau =: t$, so it suffices to prove that $\left\|\hat{\xi}_\eta\right\|^2_{t,*}$ is uniformly bounded for all t sufficiently negative. Let

$$C_\eta := \int_X \frac{|\eta|^2 \,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}} < +\infty,$$

then

$$\left\|\hat{\xi}_\eta\right\|^2_{t,*} = \sup_{\|\hat{s}\|^2_t=1}\left|\int_X \frac{(h\dot{\otimes} s)\bar{\eta}\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}}\right|^2 \leq C_\eta \sup_{\|\hat{s}\|^2_t=1}\int_X \frac{|h\dot{\otimes} s|^2\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}}.$$

By (4.4), if t is sufficiently negative,

$$\frac{|h\dot{\otimes} s|^2\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}} \leq \frac{2}{r}\,\mathbf{I}_{t,s}(x)$$

in the sense of top forms, so that

$$\left\|\hat{\xi}_\eta\right\|^2_{t,*} \leq \frac{2C_\eta}{r}\sup_{\|\hat{s}\|^2_t=1}\int_X \mathbf{I}_{t,s}(x) \leq \frac{2C_\eta}{r} < +\infty,$$

as wanted.

Step 2. Consider now the trivial fibration $(X\times\mathbb{P}_{r-1})\times\mathbb{L} \xrightarrow{\mathrm{pr}_{\mathbb{L}}} \mathbb{L}$. We have already checked at the end of Section 4.3 that the curvature of $\mathfrak{h}$, seen as a metric for $\mathrm{pr}^*_{X\times\mathbb{P}_{r-1}} L \to (X\times\mathbb{P}_{r-1})\times\mathbb{L}$, plus the Ricci curvature coming from the Fubini–Study volume form is non-negative on the total space $X\times\mathbb{P}_{r-1}\times\mathbb{L}$. Then Corollary 2.2 implies that $\tau\mapsto\log\left\|\hat{\xi}_\eta\right\|^2_{\tau,*}$ is subharmonic in $\mathbb{L}$. Since $\left\|\hat{\xi}_\eta\right\|_{\tau,*}$ only depends on $t=\operatorname{Re}\tau$, it follows that $t\mapsto\log\left\|\hat{\xi}_\eta\right\|^2_{t,*}$ is convex on $(-\infty,0)$. If this map decreases anywhere on $(-\infty,0)$, then by convexity we would have $\lim_{t\to-\infty}\log\left\|\hat{\xi}_\eta\right\|^2_{t,*} = +\infty$, contradicting the uniform boundedness of $\left\|\hat{\xi}_\eta\right\|^2_{t,*}$ obtained in Step 1. Hence the statement follows. $\square$

Let now $s \in H^0(X,(E\otimes K_X)^{\oplus r})$ be any solution of $h\,\dot{\otimes}\,s = \mathcal{P}\eta$ (such s exists with bounded L^2-norm by the same argument of Proposition 4.2). Then by (4.6) and Lemma 4.6 we have

$$\begin{aligned}\left\|\xi_\eta\right\|_*^2 = \left\|\hat{\xi}_\eta\right\|_{0,*}^2 &\geq \lim_{t\to-\infty}\left\|\hat{\xi}_\eta\right\|_{t,*}^2\\ &\geq \lim_{t\to-\infty}\frac{1}{\|\hat{s}\|_t^2}\left|\int_X \frac{(h\,\dot{\otimes}\,s)\overline{\mathcal{P}\eta}\,\mathrm{e}^{-\psi}}{(|h|^2\,\mathrm{e}^{-\psi+\varphi})^{\alpha(r-1)+1}}\right|^2\\ &= \lim_{t\to-\infty}\frac{\|\mathcal{P}\eta\|_G^4}{\|\hat{s}\|_t^2} = \frac{\alpha-1}{\alpha r}\|\mathcal{P}\eta\|_G^2\end{aligned}$$

for all $\eta \in C_c^\infty(X, G\otimes K_X)$. Hence, by (4.1), we conclude that the minimal-norm solution $\tilde{f}$ to the division problem $h\,\dot{\otimes}\,f = g$ has norm

$$\left\|\tilde{f}\right\|_{E^{\oplus r}}^2 \leq \|g\|_G^2 \sup_{\eta\in C_c^0(X,G\otimes K_X)} \frac{\|\mathcal{P}\eta\|_G^2}{\left\|\xi_\eta\right\|_*^2} \leq r\frac{\alpha}{\alpha-1}\|g\|_G^2\,,$$

proving Theorem 4.1.

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