Tutorial – The Constraint Satisfaction Problem Dichotomy Theorem. Lecture 2

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Example of a pp-interpretation

Recall $\mathbf{K}_3 = (\{0,1,2\}, \neq)$ and $\mathbf{M}_{3SAT} = (\{0,1\}, R_{3SAT})$ where

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So $(D,S) \cong \mathbf{K}_3$, which proves $\mathbf{K}_3 \overset{pp}{\hookrightarrow} \mathbf{M}_{3SAT}$.

Summary of Lecture 1

$\mathrm{CSP}_p(\mathbf{M})$: decision problem about satisfiability of $\wedge$at-fmlas/$\mathbf{M}$.

CSP Dichotomy Theorem of Bulatov and Zhuk (2017, 2020):

$$\mathbf{M}_{3SAT} \overset{pp}{\not\to} \mathbf{M} \implies \mathrm{CSP}_p(\mathbf{M}) \text{ is in P.}$$

Algebraic perspective

- $\mathbf{M} \mapsto$ idempotent polymorphism algebra $\mathbb{M}$.
- Connections between {pp-definable relations over $\mathbf{M}$} and $\mathrm{HSP}(\mathbb{M})$.

Positive characterization of $\mathbf{M}_{3SAT} \overset{pp}{\not\to} \mathbf{M}$ (Theorem 1):

"$\mathbb{M}$ has a Taylor operation"

Plan for today

Intro to solving $\mathrm{CSP}_p(\mathbf{M})$ when $\mathbb{M}$ has a Taylor operation

1. Preliminary remarks
   - $\wedge$at-fmlas as multi-sorted structures
   - Preprocessing – enforcing local consistency and irreducibility.
   - Generalized $\wedge$at-fmlas.
2. A "crazy" reduction strategy
3. The module-free case
4. Zhuk's extension/refinement to the general (Taylor) case

Part 1 – Preliminary remarks

Simplifying assumptions

Fix **M** (finite structure).

Fix φ ($\wedge$at-fmla/**M**), say $$\varphi = \bigwedge_{i=1}^{N} \alpha_i \quad (\alpha_i \text{ atomic}).$$

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Atomic subformulas $\alpha_1, \ldots, \alpha_N$ now are called the **constraints** (of φ).

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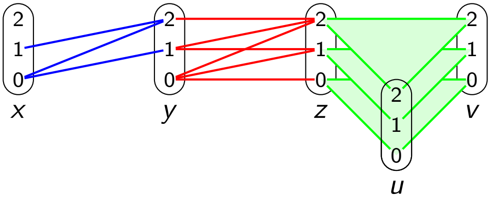
$$\varphi \;=\; (x < y) \;\wedge\; (y \leq z) \;\wedge\; E(z,u,v).$$

Construct 

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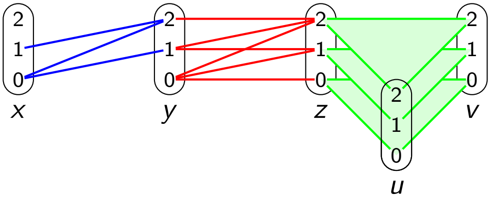
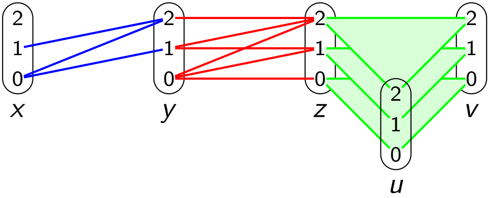
Construct  $=: \Phi$

Formally: the **microstructure (multi-)hypergraph** of a $\wedge$at-fmla φ over $\mathbf{M}$ is the multi-sorted structure Φ whose:

- Sorts are indexed by the variables occurring in φ.
- Domain of each sort is M.
- Each constraint $R(x_{i_1},\ldots,x_{i_k})$ of φ gives a relation $R_{x_{i_1},\ldots,x_{i_k}}$ of Φ, which is just R interpreted as having type $(x_{i_1},\ldots,x_{i_k})$.

Solution to φ = choice of one value in each domain of Φ collectively satisfying every relation of Φ.

Preprocessing: 1-consistency



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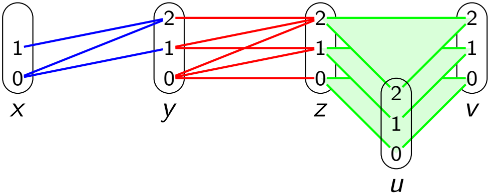

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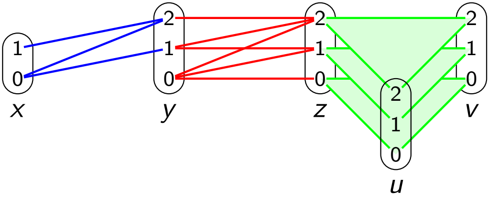

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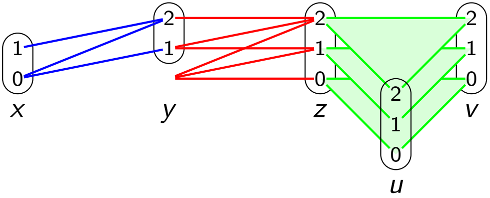

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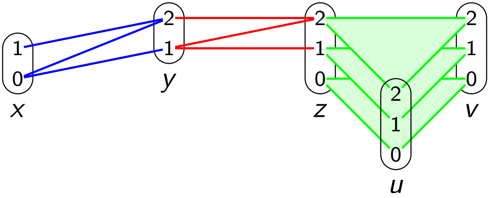

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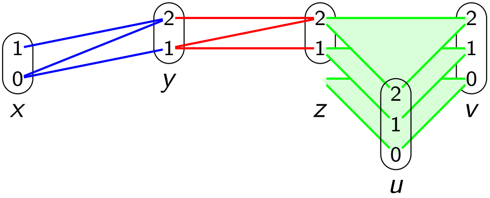

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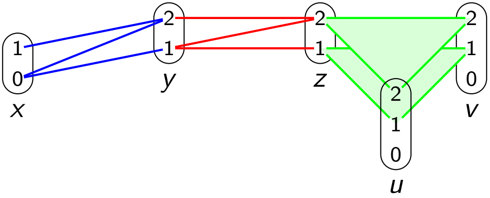

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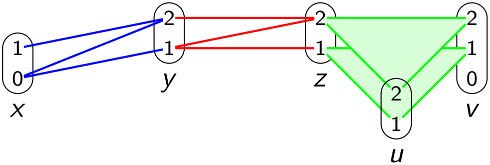

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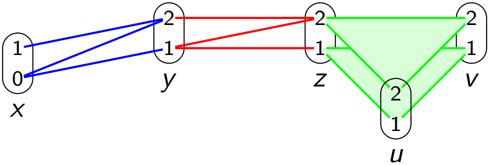

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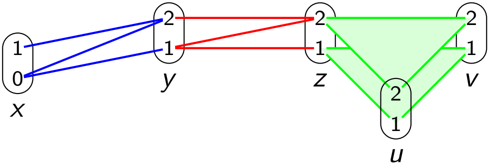

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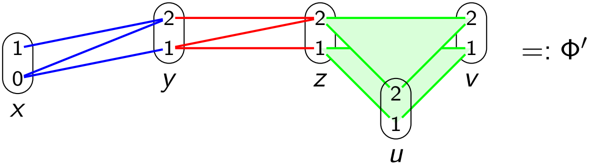

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- Φ' has the same solutions as Φ (and φ).
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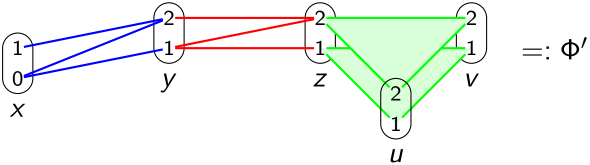

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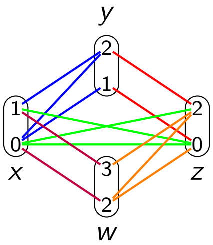
- Φ' can be found efficiently (polynomial time in the size of φ).
- Each domain of Φ' is pp-definable in Φ (hence also in **M**).

Cycle consistency



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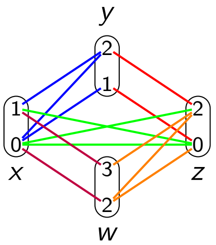

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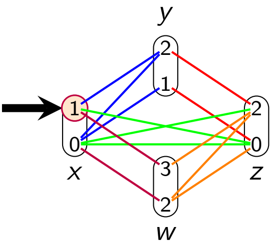

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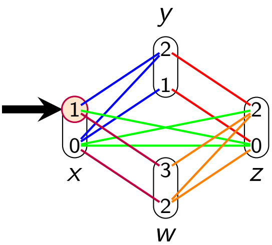

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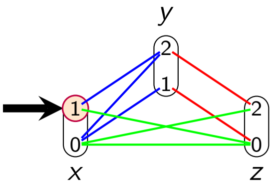

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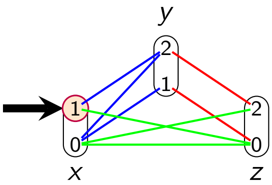

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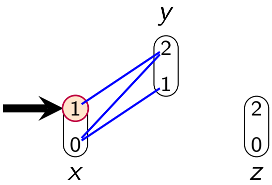

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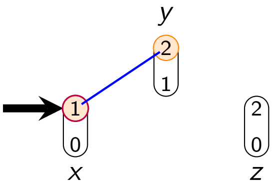

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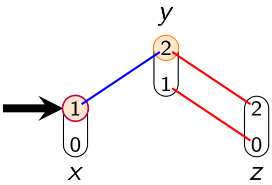

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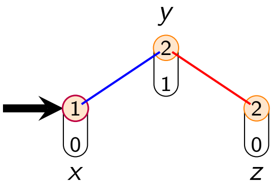

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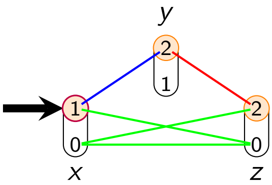

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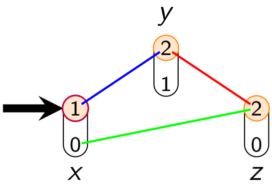

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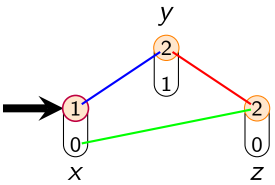

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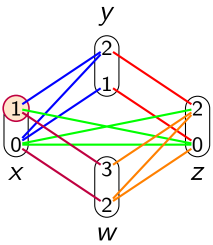

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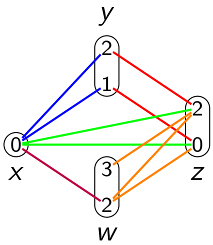

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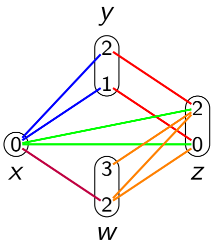

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"Cycle consistency" = 1-consistency + enforcing this cycle condition.

Irreducibility; generalized fmlas

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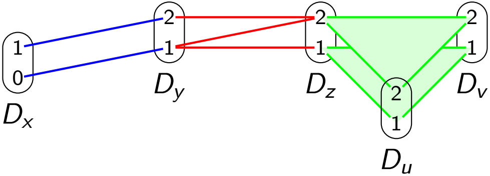
Enforcing any/all of these conditions (1-consistency, cycle-consistency, irreducibility):

- Does not change the set of solutions.
- Might lead to empty domain(s) $\rightsquigarrow$ "proof of inconsistency."
- Else, leads to a **generalized** $\wedge$at-**formula** or **Gen**$\wedge$at-**fmla**:

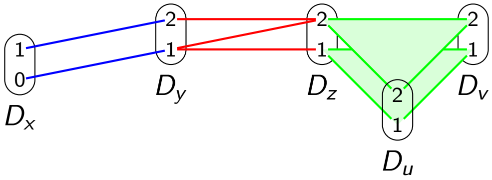
  Substructure of a microstructure hypergraph of a $\wedge$at-fmla, where the domains are pp-definable subsets of $\mathbf{M}$.

Going forward, I focus entirely on Gen∧at-fmlas (usually 1-consistent).

$\Phi =$ 

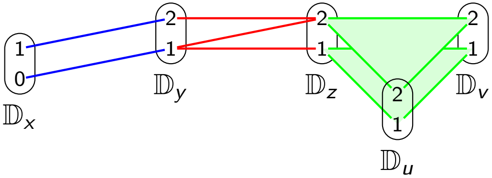
Domains $D_x, D_y, \ldots$ and relations $R \subseteq D_x \times D_y$ etc. are pp-definable in $\mathbf{M}$.

Going forward, I focus entirely on Gen∧at-fmlas (usually 1-consistent).

$\Phi =$ 

Domains $D_x, D_y, \ldots$ and relations $R \subseteq D_x \times D_y$ etc. are pp-definable in **M**.

Flipping to the algebraic perspective: domains are **subalgebras** $\mathbb{D}_x \leq \mathbb{M}$, and relations are **subalgebras** $\mathbb{R} \leq \mathbb{D}_x \times \mathbb{D}_y$.

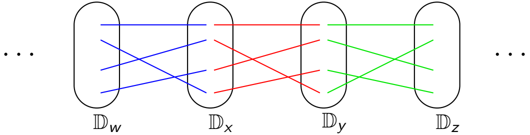
$\Phi =$ 

Call Φ a Gen∧at-fmla $\boxed{\text{over } \mathbb{M}}$.

Part 2 – Reduction Strategy

Crazy Idea

You are given a Gen$\wedge$at-fmla $\Phi/\mathbb{M}$. (Assume cycle-consistent, irreducible, ...)

$\Phi =$ 

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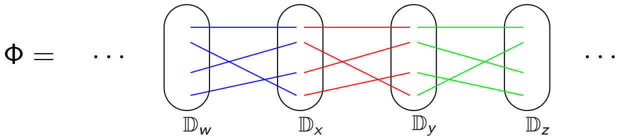

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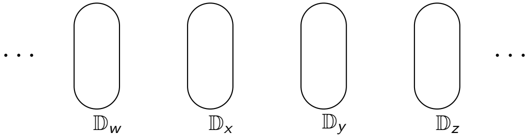

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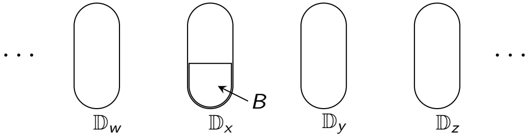

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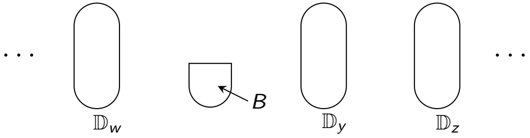

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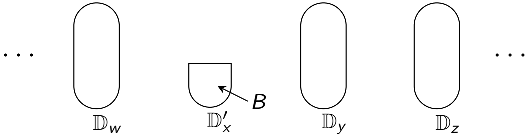

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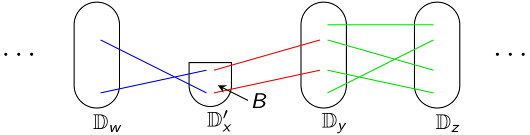
$\Phi' =$ 

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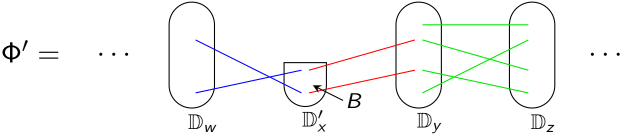

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What could possibly go wrong?

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I.e., so that

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(Without knowing the relations of Φ?)

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There is a precedent (Barto, Kozik): "Yes" in the *module-free* case.

Part 3 – The module-free case

"Module-free case" – refers to finite structures **M** for which

- $\mathbf{M}_{3SAT} \overset{pp}{\not\hookrightarrow} \mathbf{M}$, and
- HSP($\mathbb{M}$) contains no (idempotent reduct of a) module.

Equivalent relational characterization:

$$(\mathbb{Z}_p^n, \text{“}x - y + z - w = 0\text{”}) \overset{pp}{\not\hookrightarrow} \mathbf{M} \text{ for all primes } p \text{ and all } n \geq 1.$$

Barto & Kozik (2009) proved CSP Dichotomy for the module-free case, by showing that the "crazy idea" strategy can be implemented.

What "special" subuniverses did they choose?

WARNING:

The Surgeon General has determined

that listening to rest of this lecture

may cause nausea and/or headaches

Absorbing subuniverses

Definition (Barto, Kozik)

Let $\mathbb{A}$ be a finite idempotent algebra and $B \leq \mathbb{A}$.

Say that B is a **2-absorbing subuniverse** of $\mathbb{A}$, and write $B \lhd_2 \mathbb{A}$, if there exists a binary (term) operation $t(x, y)$ of $\mathbb{A}$ such that

$$t(A, B) \subseteq B \quad \text{and} \quad t(B, A) \subseteq B.$$

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Similarly for n-**absorbing subuniverse** and $B \lhd_n \mathbb{A}$.

Examples

1. $\mathbb{A} = (A, *, \ldots)$ with $0 \in A$ such that $0 * x = x * 0 = 0 \quad \forall x \in A$.

$$\{0\} \lhd_2 \mathbb{A} \quad \text{witnessed by} \quad x * y.$$

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2. $\mathbb{A} = (\{0,1\}, \mathrm{maj}(x,y,z))$.

   $\mathbb{A}$ has no proper 2-absorbing subuniverse.

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3. $\mathbb{A} = (\{0,1,2\}, \cdot\,)$ where $\cdot$ is the "rock-paper-scissors" operation.

$\cdot$	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2

   $\mathbb{A}$ has no proper n-absorbing subuniverse (for any n).

$$B \lhd_2 \mathbb{A} \implies B \lhd_3 \mathbb{A} \implies B \lhd_4 \mathbb{A} \implies \cdots$$

B is an **absorbing subuniverse** (written $B \lhd \mathbb{A}$) if it is an n-absorbing subuniverse for some n.

A good lecture would spend ≥ 10 minutes talking about interesting formal properties of $\lhd$.

Here are two:

1. $\lhd$ propagates within pp-definitions (e.g., over Gen$\wedge$at-fmlas).
2. Suppose $\mathbb{A} =$ the idempotent polymorphism algebra of $\mathbf{M}$ and $B \lhd_n \mathbb{A}$.

   Then $\forall m \geq n$, $\forall$ pp-formula $\varphi(x_1, \ldots, x_m)/\mathbf{M}$, if for every i there exists a solution to φ in $\mathbf{M}$ passing through B in all but coordinate i, then there exists a solution to φ in B^m.

PC algebras

$\mathbb{A}$ is **polynomially complete** (PC) if every operation $f : A^n \to A$ can be realized as a term operation of $\mathbb{A}$ with parameters:

$$f(x_1, \ldots, x_n) = t(x_1, \ldots, x_n, a_1, \ldots, a_k).$$

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2. $\mathbb{A} = (\{0, 1, 2\}, \cdot\,)$ where $\cdot$ is the "rock-paper-scissors" operation.

   Every subset of A is a subuniverse. No proper subset is an absorbing subuniverse.

Reduction strategy (module-free case)

Theorem 2 (Kozik 2016, improving Barto-Kozik 2009)

Suppose

- $\mathbb{M}$ is finite, idempotent, and has a Taylor operation.
- HSP($\mathbb{M}$) is module-free. (i.e., congruence meet-semidistributive, i.e., omits $\mathbf{1}, \mathbf{2}$)
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1. If $\mathbb{D}_x$ is a domain and $B \lhd \mathbb{D}_x$, then B "works" for the red. strategy:

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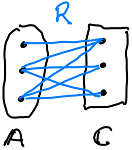
2. If no $\mathbb{D}_y$ has a proper absorbing subuniverse, then for every $\mathbb{D}_x$ with $|D_x| > 1$ and for every maximal congruence θ of $\mathbb{D}_x$,
   (a) (Zhuk) $\mathbb{D}_x/\theta$ is PC, and
   (b) Every θ-class works for the reduction strategy.

Part 4 – Zhuk's extension/refinement

Left centers, Zhuk centers

$\mathbb{A}, \mathbb{C}$ idempotent algebras

Suppose $R \leq_{sd} \mathbb{A} \times \mathbb{C}$. sd = "subdirect," i.e., $\mathrm{proj}_1(R) = A$ and $\mathrm{proj}_2(R) = C$



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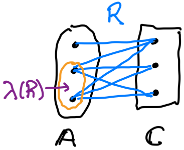
The **left center** of R is

$$\lambda(R) := \{a \in A : \{a\} \times C \subseteq R\}.$$



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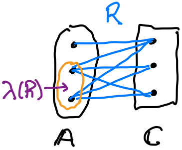

Definition

Suppose $\mathbb{A}$ is finite and idempotent, and $B \leq \mathbb{A}$.

Say B is a **Zhuk center** of $\mathbb{A}$, and write $B \leq_{ZC} \mathbb{A}$, if

$B = \lambda(R)$ for some $R \leq_{sd} \mathbb{A} \times \mathbb{C}$, where $\mathbb{C}$ is finite, idempotent,

and $\boxed{\mathbb{C} \text{ has no proper 2-absorbing subuniverse}}$.

B is a **Zhuk center** of $\mathbb{A} \iff B = \lambda(R)$ for some $R \leq_{sd} \mathbb{A} \times \mathbb{C}$, where $\mathbb{C}$ has no proper 2-absorbing subuniverse.

Example. $\mathbb{A} = (\{0, 1\}, \text{maj})$

$B = \{0\}$.

B is a **Zhuk center** of $\mathbb{A} \iff B = \lambda(R)$ for some $R \leq_{sd} \mathbb{A} \times \mathbb{C}$, where $\mathbb{C}$ has no proper 2-absorbing subuniverse.

Example. $\mathbb{A} = (\{0,1\}, \text{maj})$

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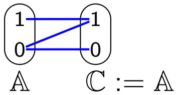
$\text{maj}(x,y,z)$ is monotone, so $\leq$ is a subuniverse of $\mathbb{A}^2$:



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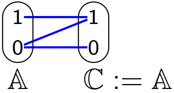

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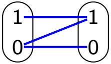

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The left center of $\leq$ is

$$\lambda(\leq) = \{0\}.$$

$\therefore$ $\{0\}$ is a Zhuk center of $\mathbb{A}$.

A good lecture would spend ≥ 10 minutes talking about interesting formal properties of $\leq_{ZC}$.

Here are two (from Zhuk).

1. If $\mathbb{A}$ has a Taylor operation, then

$$B \leq_{ZC} \mathbb{A} \implies B \lhd_3 \mathbb{A}.$$

2. $\leq_{ZC}$ propagates within pp-definitions.

Unlike absorbing subuniverses, Zhuk centers are fragile under adding extra operations to $\mathbb{A}$.

Theorem 2 (Kozik 2016, improving Barto-Kozik 2009)

Suppose

- $\mathbb{M}$ is finite, idempotent, and has a Taylor operation.
- HSP($\mathbb{M}$) is module-free.
- Φ is a Gen$\wedge$at-fmla over $\mathbb{M}$, and is cycle-consistent.

Then:

1. If $\mathbb{D}_x$ is a domain and $B \lhd \mathbb{D}_x$, then B "works" for the red. strategy:

$$\Phi \text{ has a solution } \Longrightarrow \Phi \text{ has a solution passing through } B.$$

2. If no $\mathbb{D}_y$ has a proper absorbing subuniverse, then for every $\mathbb{D}_x$ with $|D_x| > 1$ and for every maximal congruence θ of $\mathbb{D}_x$,
   (a) $\mathbb{D}_x/\theta$ is PC, and
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Theorem 3 ~~(Kozik 2016, improving Barto-Kozik 2009)~~ Zhuk 2017/20

Suppose

- $\mathbb{M}$ is finite, idempotent, and has a Taylor operation.
- ~~HSP($\mathbb{M}$) is module-free.~~
- Φ is a Gen$\wedge$at-fmla over $\mathbb{M}$, and is cycle-consistent and irreducible.

Then:

$\overbrace{B \lhd_2 \mathbb{D}_x \text{ or } B <_{ZC} \mathbb{D}_x}$

1. If $\mathbb{D}_x$ is a domain and ~~$B \lhd \mathbb{D}_x$~~, then B "works" for the red. strategy:

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2-absorbing subuniverse or Zhuk center

2. If no $\mathbb{D}_y$ has a proper ~~absorbing subuniverse~~, then for every $\mathbb{D}_x$ with $|D_x| > 1$ and for every maximal congruence θ of $\mathbb{D}_x$,
   (a) $\mathbb{D}_x/\theta$ is PC or a simple module, and
   (b) If PC, then every θ-class works for the reduction strategy.