Tutorial – The Constraint Satisfaction Problem Dichotomy Theorem. Lecture 3

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Waterloo (Canada)

Assoc. Sym. Logic meeting – Ames, IA
17 May 2024

Summary of Lecture 2

"$\mathrm{CSP}_p(\mathbf{M})$ when $\mathbb{M}$ has a Taylor operation"

Representation of $\wedge$at-fmla/$\mathbf{M}$ by its microstructure hypergraph Φ.

- Preprocessing: 1-consistency, cycle-consistency, irreducibility.
- Generalized $\wedge$at-fmlas/$\mathbb{M}$.

"Crazy" reduction strategy (for solving satisfiability of Gen$\wedge$at-fmlas)

Previous success in the module-free case (Theorem 2, Kozik)

Theorem 3 (Zhuk): success "up to modules" in the general (Taylor) case:

1. 2-absorbing subuniverses
2. Zhuk centers (special kind of 3-absorbing)
3. PC congruence classes (when (1) and (2) not available)

Plan for today

"One aspect of Zhuk's proof: linear equations"

1. Examples
2. Definitions: SIs, similarity
3. Definitions: Linear constraints
   - Rectangular relations, linear relations
   - Adjacency, components
4. Weakening inconsistent formulas
   - Critical relations, crucial weakenings
   - Expanded covers
   - A "crucial" theorem of Zhuk
5. Postscript (time permitting)
   - Proof sketch of Theorem 3 (2-absorbing case)
   - Lies that I have told

Part 1 – Examples

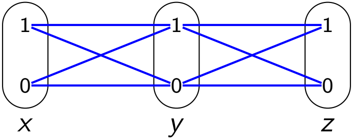
Linear equations manifest in Gen$\wedge$at-fmlas in two ways:

- Explicitly – via a single constraint.
- Implicitly – via a combination of constraints (= pp-definition).

Example 1 (explicit).

$$\mathbb{Z}_2 := (\{0,1\}, x+y+z), \quad R \leq_{sd} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$R := \{(a,b,c) \in (Z_2)^3 \,:\, a+b+c=0\}.$$



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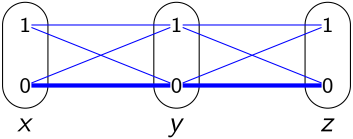

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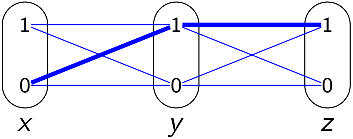

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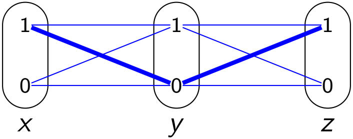

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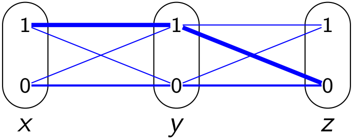

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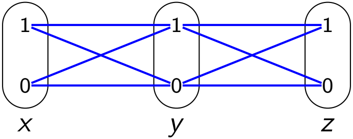

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Example 2 (explicit). $\mathbb{Q} = (Q_8,\ xy^{-1}z)$ where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

Define $S \leq_{sd} \mathbb{Q} \times \mathbb{Q} \times \mathbb{Z}_2$ by

$$S = \{(a, b, c) \in Q_8 \times Q_8 \times Z_2 \,:\, ab^{-1} = (-1)^c\}.$$

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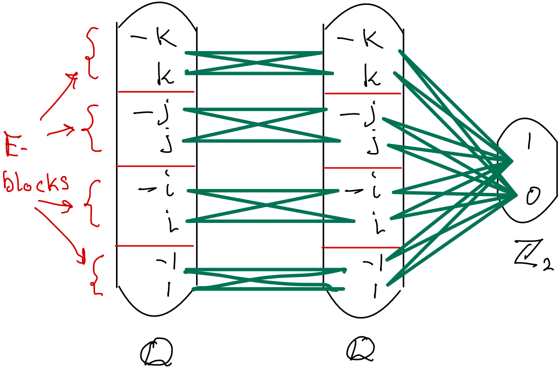

The point here is that:

- We have a minimal congruence E of $\mathbb{Q}$, whose classes are $\cong \mathbb{Z}_2$.
- S imposes a linear equation on each "branch" through E-classes.

Example 3 (implicit). Recall $R \leq_{sd} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ from Example 1.

Consider R as an algebra $\mathbb{R}$.

Define
$$\begin{aligned} \text{proj}_i : \mathbb{R} \to \mathbb{Z}_2 \quad & \text{for } i = 1, 2, 3 \\ S_i := \text{graph}(\text{proj}_i) & \leq_{sd} \mathbb{R} \times \mathbb{Z}_2. \end{aligned}$$

Let Φ be the Gen$\wedge$at-fmla

$$S_1(u, x_1) \;\&\; S_2(u, x_2) \;\&\; S_3(u, x_3)$$

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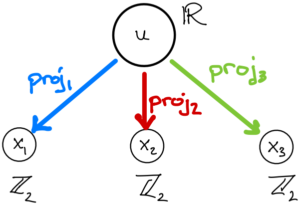
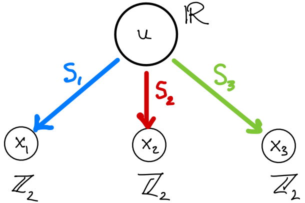
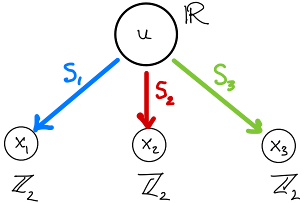


This gadget implicitly defines the linear relation $R(x_1, x_2, x_3)$, yet each individual constraint relation is "simple."

$\Phi =$



- Each constraint "is simply" a homomorphism from $\mathbb{R}$ to $\mathbb{Z}_2$
- . . . and so in essence "is" nothing more than an isomorphism between two copies of $\mathbb{Z}_2$ ($\mathbb{R}/E_i$ and $\mathbb{Z}_2$ where where $E_i = \ker(S_i)$) . . .
- . . . yet the implied constraint $\exists u\, \Phi$ on x_1, x_2, x_3 is linear. This can be explained by lattice-theoretic relationships between E_1, E_2, E_3 in the congruence lattice of $\mathbb{R}$.

$\Phi =$ 

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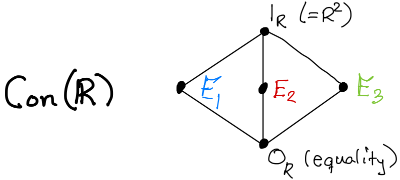

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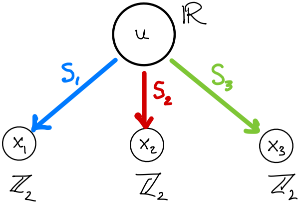

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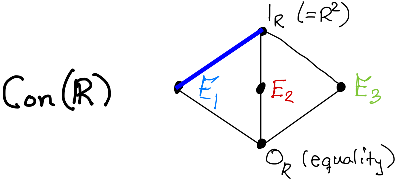

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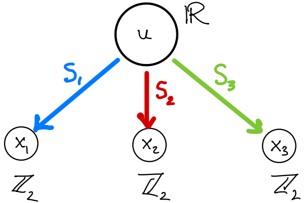

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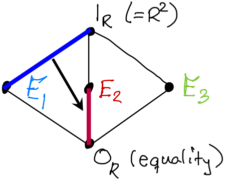
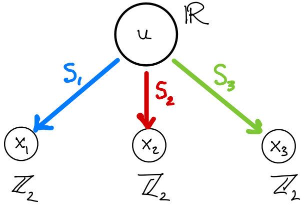
Con($\mathbb{R}$)



$\Phi =$ 

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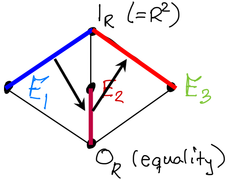
Con(ℝ) 

Fuzzification

Both explicit and implicit linear constraints can be "fuzzified."

Example 4 (explicit, fuzzified). Let $R' \leq_{sd} \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ be

$$R' = \{(a, b, c) \in Z_6 \times Z_6 \times Z_6 \ : \ a + b + c \equiv 0 \pmod 2\}.$$

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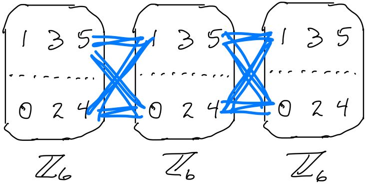
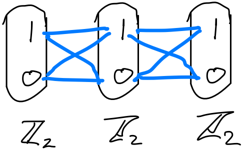


Example 4 is the "pullback" of Example 1 via the obvious homomorphism $\mathbb{Z}_6 \to \mathbb{Z}_2$.



Part 2 – Definitions: SIs, similarity

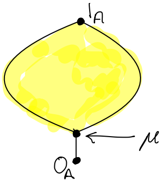
In the examples, the "essential" algebras were

$$\mathbb{Z}_2 = (Z_2,\ x+y+z) \quad \text{and} \quad \mathbb{Q} = (Q_8, xy^{-1}z).$$

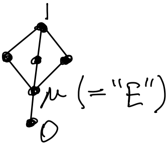
Both are examples of "SI algebras with abelian monolith."

An algebra $\mathbb{A}$ is **subdirectly irreducible (SI)** if $|A| > 1$ and $\mathbb{A}$ has a unique smallest nontrivial congruence μ (called the **monolith**).

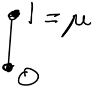
Con($\mathbb{A}$)

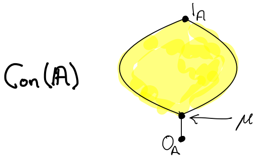


Examples: $\text{Con}(\mathbb{Q}) =$



$\text{Con}(\mathbb{Z}_2) =$



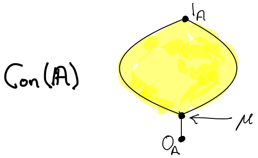


The monolith μ is abelian if . . . blah blah

Necessary condition (assuming $\mathbb{A}$ is idempotent and has a Taylor op):

- Each μ-class C is a module[1] (as $\mathbb{C} \leq \mathbb{A}$). Moreover, the underlying group of C is elementary p-abelian (same prime p for each class).

[1] I.e., there exists a unital R-module structure on C with respect to which the term operations of $\mathbb{A}$ are exactly the R-linear operations $\sum_1^n r_i x_i$ satisfying $\sum_i r_i = 1$.



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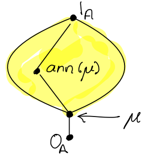
We'll also need to know about the **annihilator** of μ. It is blah blah. . .

- . . . a congruence $\mathrm{ann}(\mu) \geq \mu$.
- . . . akin to the "centralizer of an abelian normal subgroup."



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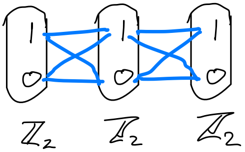
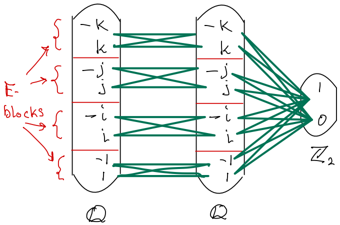
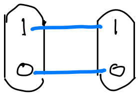
In both explicit Examples 1 and 2, the linear equations were supported on "branches through μ-classes."





This is also true of the binary constraints S_i in the implicit Example 3: after defuzzification, they were



Similarity

Not all pairs of SIs with abelian monolith can "jointly participate" in a linear constraint.

For example:

1. The sizes of the monolith classes for $\mathbb{A}$ and $\mathbb{B}$ must be powers of the same prime p.

The "can-jointly-participate" relation was previously worked out in the "classical" (congruence modular) case.

Theorem 4 (Freese 1983)

Blah blah blah. (He worked it out in the "congruence modular" case.)

Much of Freese's theorem extends to the Taylor case (next slide).

Theorem 5 (Zhuk $+ 3\varepsilon$. Bulatov proves something similar)

Suppose $\mathbb{M}$ is finite and HSP($\mathbb{M}$) is Taylor.

There exists an equivalence relation $\sim$ on the class $\mathcal{K}$ of finite SIs in HSP($\mathbb{M}$) with abelian monolith, defined by blah blah blah and satisfying:

1. If $\mathbb{A} \sim \mathbb{B}$, then $\mathbb{A}/\operatorname{ann}(\mu_{\mathbb{A}}) \cong \mathbb{B}/\operatorname{ann}(\mu_{\mathbb{B}})$.
2. (**Coordinatization**) Each $\sim$-class contains a privileged member...

(See slide 15 of my Siena 2019 lecture)

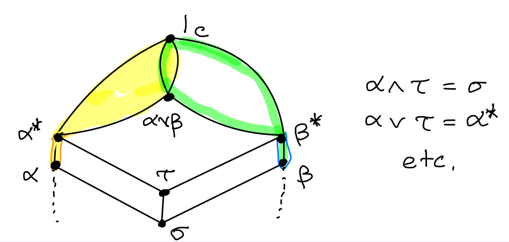
Special case: If $\mathbb{A} \in \mathcal{K}$ satisfies $\operatorname{ann}(\mu) = 1_A$, then

- There exists a **simple module** $\mathbb{U} \sim \mathbb{A}$, say with $|U| = p^k$, a surjective homomorphism $h : \mu \to \mathbb{U}$, and an element $u \in U$ such that $h^{-1}(u) = \{(a, a) : a \in A\} = 0_A$.
- For each μ-class C and element $c \in C$, the map $h(-, c)|_C$ is an embedding $(C, x-y+z) \hookrightarrow (U, x-y+z)$. Thus $|C| = p^\ell$ $(\ell \leq k)$.

(Theorem 5 continued)

3. (**Internal witnesses**) Suppose $\mathbb{A}, \mathbb{B} \in \mathcal{K}$. **If** there exist $\mathbb{C} \in \mathsf{HSP}(\mathbb{M})$ and $\alpha, \beta, \sigma, \tau \in \mathrm{Con}(\mathbb{C})$ such that $\mathbb{A} \cong \mathbb{C}/\alpha$, $\mathbb{B} \cong \mathbb{C}/\beta$, $\sigma \prec \tau$, and

Con($\mathbb{C}$):



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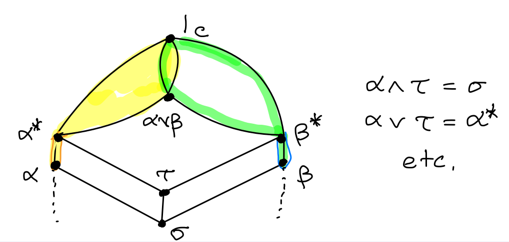

then $\mathbb{A} \sim \mathbb{B}$. (Say that $(\alpha, \beta, \sigma, \tau)$ **witness** $\mathbb{A} \sim \mathbb{B}$ **in** $\mathbb{C}$.)

(The converse is also true.)

Part 3 – Definitions: Linear constraints

Suppose $R \leq_{sd} \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$.

R is **completely functional** if "any $n-1$ coordinates determine the last."

(I.e., for every $i = 1, \ldots, n$ and all $\mathbf{a}, \mathbf{b} \in R$, if $a_j = b_j$ for all $j \neq i$, then $a_i = b_i$.

Example: $R \leq_{sd} \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n}$ given by

$$R = \{(a_1, \ldots, a_n) \,:\, a_1 + \cdots + a_n = 0\}.$$

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Example: Let $\mathbb{A}$ be **any** algebra.

$$\text{Let } R = \{(a, a) \,:\, a \in A\} \quad (= 0_A).$$

Call this example **stupid**.

$R \leq_{sd} \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$.

R is **rectangular**[1] if it is the "fuzzification" (pullback to $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$) of some completely functional

$$\overline{R} \leq_{sd} (\mathbb{D}_1/\delta_1) \times \cdots \times (\mathbb{D}_n/\delta_n)$$

for some (necessarily unique) $\delta_i \in \mathrm{Con}(\mathbb{D}_i)$.

(I will call $\mathbb{D}_1/\delta_1, \ldots, \mathbb{D}_n/\delta_n$ the **reduced domains** of R.)

[1] Or has the $(1, n-1)$-**parallelogram property**.

$R \leq_{sd} \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$.

R is **rectangular**[1] if it is the "fuzzification" (pullback to $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$) of some completely functional

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(I will call $\mathbb{D}_1/\delta_1, \ldots, \mathbb{D}_n/\delta_n$ the **reduced domains** of R.)

Example: $R \leq_{sd} \underbrace{\mathbb{Z}_6 \times \cdots \times \mathbb{Z}_6}_{n}$ given by

$$R = \{(a_1, \ldots, a_n) : a_1 + \cdots + a_n \equiv 0 \pmod 2\}.$$

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(Tentative definition) (Invented for this lecture)

A subdirect relation $R \leq_{sd} \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ is **linear** if:

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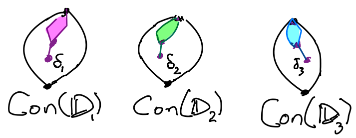

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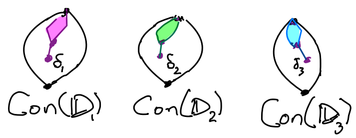

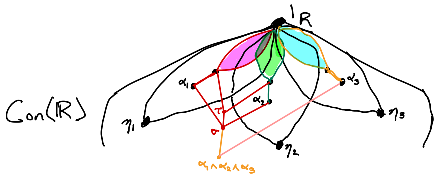

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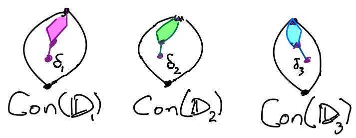
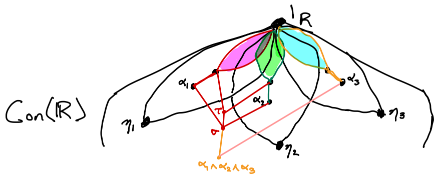


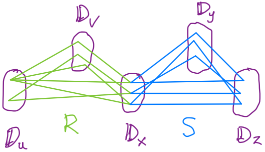


Then $\mathbb{A}_1 \sim \mathbb{A}_2$ is witnessed <u>in $\mathbb{R}$</u> by α_1, α_2, and some congruences $\sigma \prec \tau$ with $\alpha_1 \wedge \cdots \wedge \alpha_n \leq \sigma$. (And similarly for all $\mathbb{A}_i \sim \mathbb{A}_j$.)

Adjacency

Suppose Φ is a Gen$\wedge$at-fmla.

Supppose $R(u, v, x)$ and $S(x, y, z)$ are two <u>linear</u> constraints sharing x.



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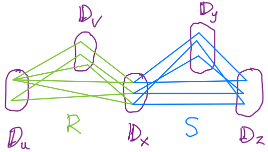
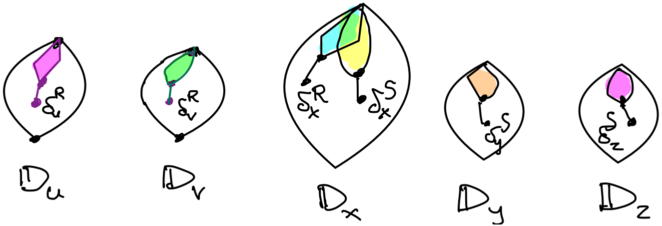


Congruence lattices of the domains



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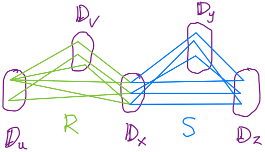

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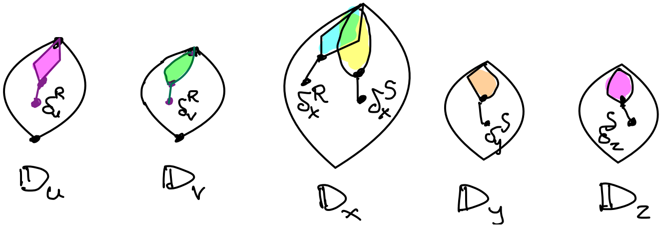

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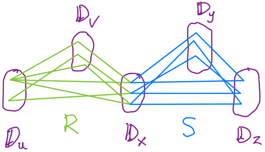

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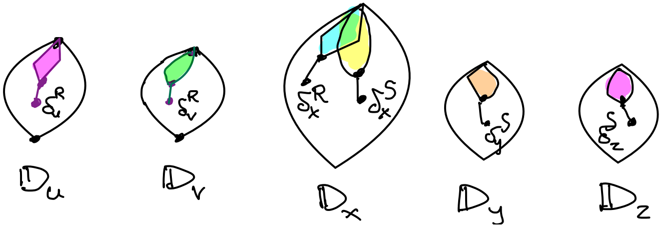

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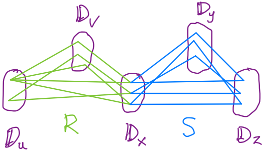

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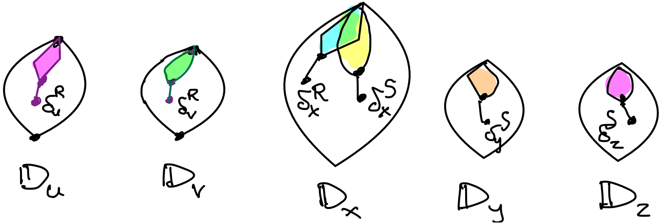

Say that $R(u,v,x)$ and $S(x,y,z)$ are **adjacent at** x if $\mathbb{D}_x/\delta_x^R \sim \mathbb{D}_x/\delta_x^S$ witnessed <u>in $\mathbb{D}_x$</u> by δ_x^R, δ_x^S and some $\sigma \prec \tau$ in $\mathrm{Con}(\mathbb{D}_x)$.

Components

Let Φ be a Gen$\wedge$at-fmla$/\mathbb{M}$. $\mathbb{M}$ has a Taylor op

A **(linear) component** of Φ is a set Ω of linear constraints in Φ which is connected by the adjacency relation.

Heuristic: if Φ is cycle-consistent, then components "coherently encode" systems of linear equations.

Part 3 – Weakening inconsistent formulas

Suppose a given Gen$\wedge$at-fmla Φ is inconsistent (has no solutions).

Do: for each constraint $R(x_1, \ldots, x_n)$ in Φ:

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  - replace $R(\mathbf{x})$ with $S(\mathbf{x})$ & $T(\mathbf{x})$.

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- Else (R is $\cap$-**irreducible**): if R does not depend on coordinate i,
  - let $R' = \mathrm{proj}_{[n] \smallsetminus \{i\}}(R)$ and replace $R(\mathbf{x})$ with $R'(\mathbf{x} \smallsetminus x_i)$.

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- Else (R is **critical**): let R^* be the unique smallest subuniverse of $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ properly containing R.

  If "Φ with $R(\mathbf{x})$ replaced by $R^*(\mathbf{x})$" is still inconsistent,
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Repeat.

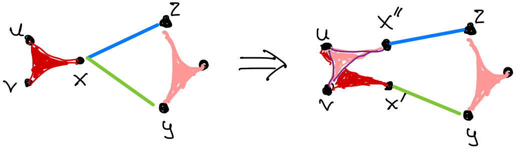
The final Gen$\wedge$at-fmla Ψ is called a **crucial weakening** of Φ. It is still inconsistent, every relation is critical, and replacing any relation R by R^* makes Ψ consistent.

Expanded covering

Assume Φ inconsistent, Ψ a crucial weakening of Φ.

An **expanded covering** of Ψ is blah blah blah...

Basically, you are allowed to "create multiple copies of variables, and of constraints"



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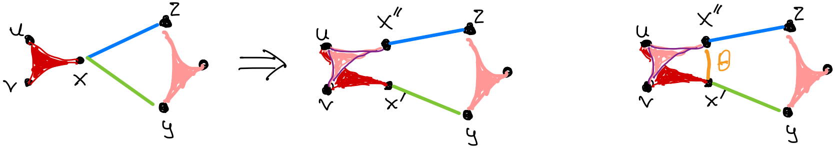

and optionally, add binary reflexive constraints (e.g., congruences) between multiple copies of the same variable.

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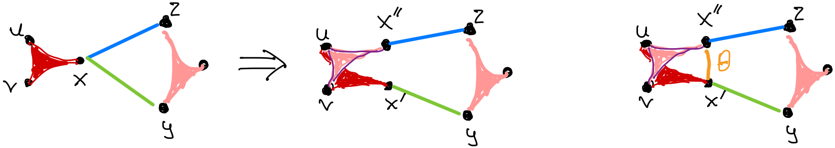

and optionally, add binary reflexive constraints (e.g., congruences) between multiple copies of the same variable.

If equality relations were added between all copies of the same variable, the expanded covering would be "equivalent" to the original Ψ.

<u>Not</u> adding all equalities creates something formally "weaker" than Ψ.

Theorem 6 (Zhuk)

Assume

- $\mathbb{M}$ is finite, idempotent, and has a Taylor operation.
- Φ is a Gen$\wedge$at-fmla over $\mathbb{M}$.
- Φ is cycle-consistent, irreducible, and inconsistent.
- Ψ is a crucial weakening of Φ.

Then

1. Every constraint relation of Ψ is critical and **rectangular**.
2. Hence (extending Kearnes, Szendrei 2012) every constraint relation of Ψ is either linear or stupid.
3. There exists an expanded covering Ψ' of Ψ which is still inconsistent (hence crucial), and which has a component Ω of Ψ' such that
   - **(Full annihilators)** If $\mathbb{A}$ is an SI of some constraint of Ω, with abelian monolith μ, then $\text{ann}(\mu) = 1_A$.
   - **(Implicit reduction)** The solution set of Ω is **not** subdirect.

Postscript

Zhuk's Theorem 3

Theorem 3 – part (1), 2-absorbing case

Suppose

- $\mathbb{M}$ is finite, idempotent, and has a Taylor operation.
- Φ is a Gen$\wedge$at-fmla over $\mathbb{M}$.
- Φ is cycle-consistent and irreducible.
- $\mathbb{D}_x$ is a domain of Φ.
- B is a proper 2-absorbing subuniverse of $\mathbb{D}_x$.

Then reducing $\mathbb{D}_x$ to B "works" for the reduction strategy:

$$\Phi \text{ has a solution} \implies \Phi \text{ has a solution passing through } B.$$

Proof sketch.

Can assume that every constraint relation of Φ is critical.

Assume that Φ has no solution passing through B.

$B \lhd_2 \mathbb{D}_x$. Fix $t(x,y)$ witnessing this. Notation: $B \lhd_2^t \mathbb{D}_x$

Exercize: every $\mathbb{D}_y$ has a unique $\subseteq$-minimal 2-absorbing subuniverse witnessed by t. Call it $D_y^{(1)}$. In particular, $D_x^{(1)} \subseteq B$

Let $D^{(1)} = (D_y^{(1)} : \mathbb{D}_y \text{ is a domain of } \Phi)$. "Minimal 2-absorbing reduction"

Say that an assignment to the domains $\boxed{\text{passes through } D^{(1)}}$ if its value at every variable y is in $D_y^{(1)}$.

By assumption, Φ has no solution passing through $D^{(1)}$.

Φ has no solution passing through $D^{(1)}$.

Weaken Φ to get Ψ which is "crucial for having no solutions passing through $D^{(1)}$."

Zhuk proves an extension of his Theorem 6 for this type of "crucial relative to $D^{(1)}$ formula."

- Get an expanded covering Ψ' of Ψ, still crucial relative to having no solution passing through $D^{(1)}$ (if y was split into $y', y'', \ldots$, this means Ψ' has no solutions whose values at $y', y'', \ldots$ all lie in $D_y^{1)}$), and a component Ω of Ψ' with full annihilators (and non-subdirect solution set, but we don't need this).

Pick a constraint, say $R(x, y, z)$, in Ω.

Let $\mathbb{A} = \mathbb{D}_x/\delta_x^R$ be the SI with abelian monolith corresponding to R at x.

- For simplicity, assume $\delta_x^R = 0$, so $\mathbb{A} = \mathbb{D}_x$.
- Let μ be its monolith.

Let $R^*(x, y, z)$ be $\exists x'[R(x', y, z) \;\&\; x' \overset{\mu}{\equiv} x]$.

$$R^*(x,y,z) \;=\; \exists x'[R(x',y,z) \;\&\; x' \overset{\mu}{\equiv} x].$$

$R(x,y,z)$ is a constraint in $\Omega \subseteq \Psi'$, $R^*(x,y,z)$ is strictly weaker, and Ψ' is "crucial for not having solutions in $D^{(1)}$." Thus if we were to replace $R(x,y,z)$ with $R^*(x,y,z)$ in Ψ', the resulting weakening will has a solution in $D^{(1)}$.

Also recall (Theorem 5) that there exists a simple module $\mathbb{U} \sim \mathbb{A}$ and a surjective homomorphism $h : \mu \to \mathbb{U}$ such that $h^{-1}(u) = 0_{D_x}$ ("equality") for some $u \in U$.

Now assume that Φ has a solution. Then Ψ' also has a solution.

Leveraging:

- Ψ' has a solution, but no solution through $D^{(1)}$,
- Ψ' with $R(x,y,z)$ relaxed to $R^*(x,y,z)$ has a solution through $D^{(1)}$,
- $h^{-1}(U) = \mu$ but $h^{-1}(u) =$ equality,

Zhuk obtains a pp-formula with one free variable of sort $\mathbb{U}$, whose quantified variables range over the sorts of Ψ', such that

- The pp-formula defines some subset $V \subseteq U$ (with $u \in V$), but
- When the domains of the quantified variables are restricted to $D^{(1)}$, the formula defines a proper subset $W \subset V$ (with $u \notin W$).

By the relation-algebra correspondence, $\mathbb{V}$ is a sub<u>algebra</u> of $\mathbb{U}$ (so is also a module) and W is a proper sub<u>universe</u> of $\mathbb{V}$.

Recall that $D_y^{(1)} \lhd_2^t \mathbb{D}_y$ for all y. "2-absorbing witnessed by $t(x, y)$"

Fact: $\lhd_2^t$ propagates under pp-definitions.

Hence $W \lhd_2^t \mathbb{V}$.

But a module has no proper 2-absorbing subuniverse (exercize).

Contradiction! $\square$

Lies I have told

1. Much of my terminology is nonstandard.

2. Theorem 2 clearly follows from Kozik's proof, but is not explicitly in his paper; Theorem 5 is within 3ε of "morally" being in Zhuk's paper; Theorem 6 doesn't appear in Zhuk's paper, but a more complicated variant occurs there.

3. Zhuk only proved Theorems 3, 5 and 6 for "weak" Taylor algebras $\mathbb{M}$:

   > Those having a weak near unanimity operation $w(x_1, \ldots, x_n)$ which generates all other operations via composition.

   (This was sufficient to prove the CSP Dichotomy Theorem.) Only Theorem 5 needs work to extend to arbitrary Taylor algebras, and I did that.

4. The proof sketch of Theorem 3 (in the Postscript) used Theorem 6.

   Zhuk's proof of Theorem 6 used Theorem 3!

   - Zhuk proved both by a very complicated simultaneous induction.

Annotated Bibliography

Manuel Bodirsky, *The complexity of infinite-domain constraint satisfaction*, Lect. Notes Log., **52**, Cambridge U. Press, 2021.

- Chapters 1-3 give excellent an introduction to CSP, pp-definability, and the algebraic perspective.

Libor Barto, Andrei Krokhin, R.W., "Polymorphisms and how to use them," in *The constraint satisfaction problem: complexity and approximability*, 1-44, Dagstuhl Follow-Ups, **7**, 2017.

- Survey paper on polymorphisms and CSP (pre-Bulatov/Zhuk's solution).

Marcin Kozik, "Weak consistency notions for all the CSPs of bounded width," LICS 2016. Full paper at arXiv:1605.00565.

- Full proof of Theorem 2 (success of "crazy strategy" in "module-free" case).

Dmitriy Zhuk, "A proof of the CSP Dichotomy Conjecture," J. ACM **67** (2020), 78 pp.

- Zhuk's full proof. Complete, but not for the faint of heart.

Andrei Bulatov, “A dichotomy theory for nonuniform CSPs simplified,” arXiv:2007.09099 (2020).

- A good starting point for those wanting to understand Bulatov’s algorithm and proof. Full details are spread across several papers on arXiv. Also not for the faint of heart.

Libor Barto, Zarathustra Brady, Andrei Bulatov, Marcin Kozik, Dmitriy Zhuk, “Minimal Taylor algebras as a common framework for the three algebraic approaches to the CSP,” LICS 2021. Full paper at arXiv:2104.11808.

- A very interesting first work aiming to harmonize Bulatov’s and Zhuk’s “reduction” strategies in their algorithms.

Andrei Bulatov, “A dichotomy theory for nonuniform CSPs simplified,” arXiv:2007.09099 (2020).

- A good starting point for those wanting to understand Bulatov's algorithm and proof. Full details are spread across several papers on arXiv. Also not for the faint of heart.

Libor Barto, Zarathustra Brady, Andrei Bulatov, Marcin Kozik, Dmitriy Zhuk, “Minimal Taylor algebras as a common framework for the three algebraic approaches to the CSP,” LICS 2021. Full paper at arXiv:2104.11808.

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Thank you!