Abelian congruences in locally finite Taylor varieties

Tutorial – Lecture 3

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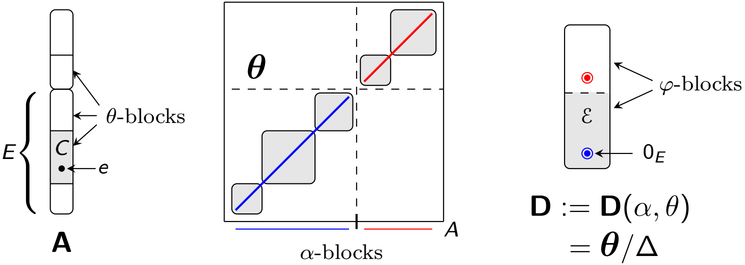
BLAST 2025 – Boulder, USA
22 May 2025

Recap

Suppose $\mathbf{A} \in \mathcal{V}$ with a WDT, $\theta \leq \alpha$ in Con $\mathbf{A}$, $[\alpha, \theta] = 0$, and $\Delta := \Delta_{\theta,\alpha}$ the smallest witness.

Let $\mathbf{D} = \mathbf{D}(\alpha, \theta) := \boldsymbol{\theta}/\Delta$ and $\varphi := \overline{\alpha}/\Delta$.



θ-blocks and φ-blocks support abelian groups: Grp(θ, e), Grp$(\varphi, 0_E)$.

If E is an α-block and $\mathcal{E}$ is the corresponding φ-block, then we have embeddings $\chi_e : \text{Grp}(\theta, e) \longrightarrow \text{Grp}(\varphi, 0_E)$ from each θ-block in E.

Recap (continued)

When **A** is finite and $0 \prec \theta$, $\exists$ a finite field $\mathbb{F}$ such that:

1. Each θ-block naturally inherits the structure of a vector space over $\mathbb{F}$.
2. For each $\alpha \geq \theta$ with $[\alpha, \theta] = 0$:
   (a) Each φ-block of the difference algebra $\mathbf{D}(\alpha, \theta)$ also inherits the structure of a vector space over $\mathbb{F}$.
   (b) Each $\chi_e : \mathrm{Grp}(\theta, e) \hookrightarrow \mathrm{Grp}(\varphi, 0_E)$ is an $\mathbb{F}$-vector space embedding.
3. $\mathbb{F}$ and its actions on θ-blocks can be defined (up to isomorphism) from $(\mathbf{D}(\alpha, \theta), \varphi, T_\alpha)$, for any $\alpha \geq \theta$ satisfying $[\alpha, \theta] = 0$.
4. $\mathbb{F}$ is determined by the <u>isomorphism type</u> of $(\mathbf{D}(\alpha, \theta), \varphi, T_\alpha)$.

Problem from 2nd lecture:

Why didn't I just let $\alpha = \theta$ and use $\mathbf{D}(\theta,\theta)$ and $\mathbb{F} = \mathbb{F}_{(\mathbf{D}(\theta,\theta),\varphi,T_\theta)}$?

Answer: It will be more useful to use the largest α satisfying $[\alpha,\theta] = 0$.

- Called the **annihilator** of θ; denoted $(0:\theta)$.
- Cute Fact:

$$\Big(0 \prec \theta \ \text{ and } \ \alpha = (0:\theta)\Big) \implies \mathbf{D}(\alpha,\theta) \text{ is subdirectly irreducible (SI)}$$

(with monolith φ)

Consequence of the Cute Fact

Suppose $\mathbf{A}, \mathbf{A}'$ are finite algebras in a WDT variety, with

$\theta \in \operatorname{Con} \mathbf{A}$ and $\theta' \in \operatorname{Con} \mathbf{A}'$, both minimal and abelian.

$\alpha = (0 : \theta)$ and $\alpha' = (0 : \theta')$.

$\mathbf{D} = \mathbf{D}(\alpha, \theta)$ with monolith φ, $\mathbf{D}' = \mathbf{D}(\alpha', \theta')$ with monolith φ'.

$\mathbb{F} =$ the finite field for θ, and $\mathbb{F}' =$ the finite field for θ'.

Lemma 8

In this situation: if $\mathbf{D}(\alpha, \theta) \cong \mathbf{D}(\alpha', \theta')$, then $\mathbb{F} \cong \mathbb{F}'$.

Proof sketch:

$$\mathbf{D} \cong \mathbf{D}' \;\Rightarrow\; \underbrace{(\mathbf{D}, \varphi) \cong (\mathbf{D}', \varphi')}_{\text{because SI}} \;\Rightarrow\; \underbrace{(\mathbf{D}, \varphi, T_\alpha) \cong (\mathbf{D}', \varphi', T_{\alpha'})}_{\text{transversal subuniverses are conjugate}} \;\Rightarrow\; \mathbb{F} \cong \mathbb{F}'.$$

Similarity

Definition (Freese 1983)

Suppose $\mathbf{A}, \mathbf{A}'$ are SI algebras in a congruence modular variety, with abelian monoliths μ, μ' respectively.

Say that $\mathbf{A}$ and $\mathbf{A}'$ are *similar* and write $\mathbf{A} \sim \mathbf{A}'$ if (some complicated shit).

Freese also proved:

$$\text{(same complicated shit)} \iff \mathbf{D}(\alpha, \mu) \cong \mathbf{D}(\alpha', \mu') \ \text{ where } \alpha = (0 : \mu) \text{ and } \alpha' = (0 : \mu').$$

Use this to define $\sim$ between SI's with abelian monoliths in WDT varieties.

Corollary 3

If $\mathbf{A}, \mathbf{A}'$ are finite SI algebras in a WDT variety, with abelian monoliths and respective fields $\mathbb{F}, \mathbb{F}'$, then $\mathbf{A} \sim \mathbf{A}' \implies \mathbb{F} \cong \mathbb{F}'$.

(Deep breath)

Critical, completely functional relations

Definition

Suppose $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are finite algebras in a common signature.

Let $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$.

R is **completely functional** if each coordinate is determined by (i.e., is a function of) the remaining coordinates:

$$\forall i \in [n], \qquad \left.\begin{array}{l} (\mathbf{r},\ a,\ \mathbf{s}) \in R \\ (\mathbf{r},\ \underset{\substack{\uparrow \\ i}}{b},\ \mathbf{s}) \in R \end{array}\right\} \implies a = b.$$

$n = 2$: R completely functional $\iff$ R is the graph of some $\mathbf{A}_1 \cong \mathbf{A}_2$.

Boring

Examples with $n = 3$.

1. $\{(a, a, a) : a \in A\} \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$. Also boring.

2. Let $\mathbf{A} = (A, +, 0)$ be a finite abelian group,

   $R_+ := \{(a_1, a_2, a_3) \in A^3 \,:\, a_1 + a_2 + a_3 = 0\} \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$.

   Not boring!

Definition (Kearnes & Szendrei 2012; Zhuk "Key relations" 2017)

Suppose $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. R is **critical** if

1. R is meet-irreducible in the lattice $\mathrm{Sub}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.
2. R has no dummy coordinates. (*) In particular, $|A_i| > 1$ for each i.

Examples:

1. $\{(a, a, a) \,:\, a \in A\} \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ — Not critical
2. $R_+ \leq_{sd} \mathbf{A} \times \mathbf{A} \times \mathbf{A}$ — Critical when $\mathbf{A}$ is cyclic of prime power order.

(*) $R \neq A_1 \times \mathrm{proj}_{\{2,\ldots,n\}}(R)$, and similarly for the other coordinates.

Rectangular relations

Another kind of relation:

rectangular (a.k.a. "having the $(1, n-1)$-parallelogram property").

I won't define them. But let

$$\begin{aligned} CCF &= \{\text{all critical, completely functional relations}\} \\ CR &= \{\text{all critical, rectangular relations}\} \\ PCCF &= \{\text{all pullbacks of critical, completely functional relations} \\ &\qquad \text{on quotients of the factor algebras } \mathbf{A}_1, \ldots, \mathbf{A}_n\}. \end{aligned}$$

Then

$$CCF \subseteq CR \subseteq PCCF.$$

Understanding CR (in a locally finite variety) is sometimes important.

Understanding CCF is typically enough.

A worthy goal:

To describe critical rectangular relations in locally finite WDT varieties*.

We focus on critical, completely functional relations of arity ≥ 3 in locally finite WDT varieties.

Building on

- Kearnes & Szendrei (2012) in CM case.
- Zhuk "CSP" (2017).

(*) = locally finite Taylor varieties.

Theorem 4

Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be finite algebras in a locally finite WDT variety, $n \geq 3$.

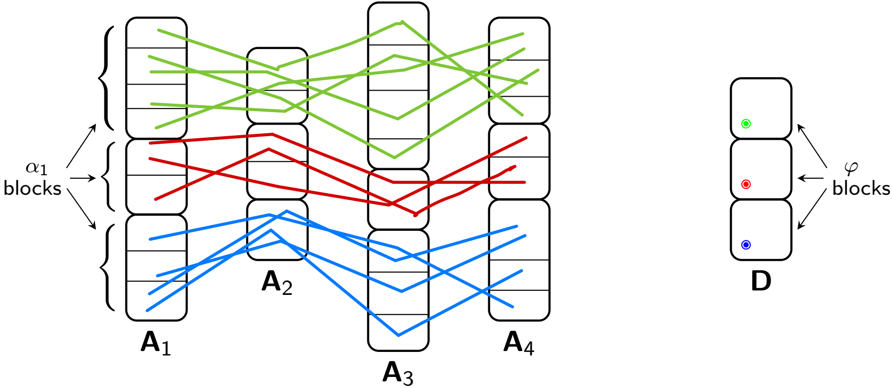
Suppose $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ is completely functional and critical.

1. Each $\mathbf{A}_i$ is subdirectly irreducible, say with monolith μ_i.
2. Each μ_i is abelian. Let $\alpha_i := (0 : \mu_i)$ for each $i \in [n]$.
3. $\mathbf{A}_1 \sim \mathbf{A}_2 \sim \cdots \sim \mathbf{A}_n$, meaning
$$\mathbf{D}(\alpha_1, \mu_1) \cong \mathbf{D}(\alpha_2, \mu_2) \cong \cdots \cong \mathbf{D}(\alpha_n, \mu_n).$$
4. R is "annihilator-coherent," meaning:
$$\mathbf{a}, \mathbf{b} \in R \implies \left(\forall i, j \ \ (a_i, b_i) \in \alpha_i \iff (a_j, b_j) \in \alpha_j\right).$$

(1) is due to Kearnes & Szendrei (and does not need "WDT" or $n \geq 3$).

(2)–(4) were proved by Kearnes & Szendrei in CM case; I extend to WDT.

Theorem 4 picture:



A completely functional, critical $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_4$

Device for proving (3), (4)

To show e.g.

$$\mathbf{a}, \mathbf{b} \in R \implies \Big((a_1, b_1) \in \alpha_1 \iff (a_2, b_2) \in \alpha_2 \Big)$$

$$\text{and} \quad \mathbf{A}_1 \sim \mathbf{A}_2$$

define $\mathbf{C} := \mathrm{proj}_{1,2}(R) \leq_{sd} \mathbf{A}_1 \times \mathbf{A}_2$ and

$$\tau := \left\{ \begin{array}{ccc} a & \overset{C}{\text{——}} & b \\ {\scriptstyle \mu_1}\Big| & & \Big|{\scriptstyle \mu_2} \\ a' & \underset{C}{\text{——}} & b' \end{array} \; : \; \exists\, \mathbf{x} \text{ with } (a, b, \mathbf{x}), (a', b', \mathbf{x}) \in R \right\}$$

"Zhuk Bridge"

As a binary relation on rows, $\tau \in \mathrm{Con}\, \mathbf{C}$.

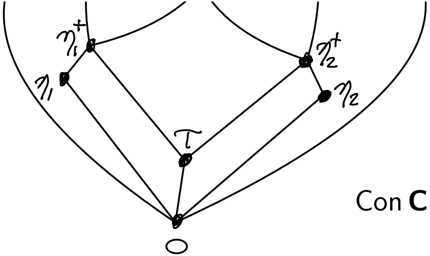
As a binary relation on columns, $\tau \leq_{sd} \boldsymbol{\mu}_1 \times \boldsymbol{\mu}_2$.

Using τ on rows:

$\eta_i = \ker(\mathrm{proj}_i) \in \mathrm{Con}\,\mathbf{C}$

$\eta_i^+/\eta_i \searrow \tau/0$ for $i = 1, 2$.

So τ is abelian.



In WDT varieties, perspective abelian intervals have = relative annihilators.

So $(\eta_1 : \eta_1^+) = (0 : \tau) = (\eta_2 : \delta_2^+)$. Proves (α_1, α_2)-coherence of R.

Using τ on columns: the rule

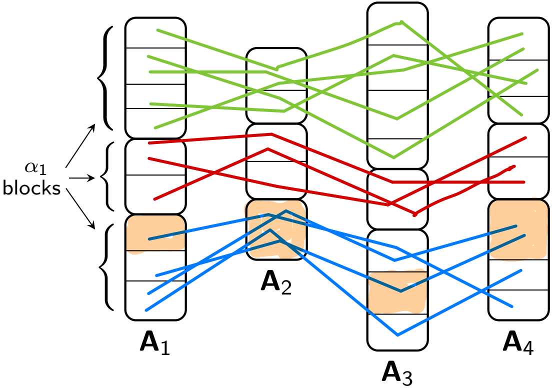
$$(a, a')/\Delta_1 \longmapsto (b', b)/\Delta_2 \qquad \text{for all} \qquad \begin{array}{ccc} a & \overset{c}{\rule{2em}{0.4pt}} & b \\ \mu_1 \Big| & & \Big| \mu_2 \\ a' & \underset{c}{\rule{2em}{0.4pt}} & b' \end{array} \in \tau$$

determines a well-defined isomorphism $\lambda_{12} : \mathbf{D}(\alpha_1, \mu_1) \cong \mathbf{D}(\alpha_2, \mu_2)$.

Proves $\mathbf{A}_1 \sim \mathbf{A}_2$.

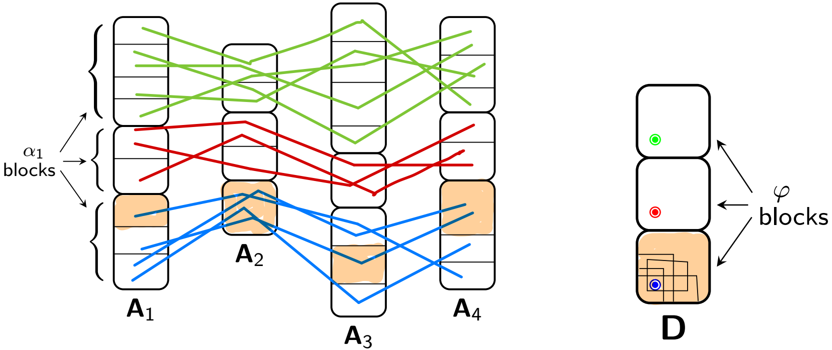
Strands

Suppose $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ as in Theorem 4.



A **strand** of R is a product $C = C_1 \times \cdots \times C_n$ where each C_i is a μ_i-block and $R \cap C \neq \varnothing$.

Fix a strand $C = C_1 \times \cdots \times C_n$.



Let **D** be the common (up to $\cong$) difference algebra for $\mathbf{A}_1, \ldots, \mathbf{A}_n$.

Let $\mathbb{F}$ be the common finite field.

Let $\mathcal{E}$ be the φ-block corresponding to the α_i-blocks containing C.

Embed $C_1, \ldots, C_n$ in $\mathcal{E}$ via $\chi_{e_1}, \ldots, \chi_{e_n}$ composed with $\cong$'s.

Intuition: $R \cap C$ "should" be the pullback to C of the solution set in $\mathcal{E}^n$ of a single $\mathbb{F}$-linear equation.

I can't prove this intuition.

But I can prove it "locally" (using ideas from Zhuk "Key relations" 2017).

Let R^* be the unique upper cover of R in $\mathrm{Sub}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.

Let $C = C_1 \times \cdots \times C_n$ be a strand of R.

Define an edge relation E on $R^* \cap C$ by

$$\mathbf{a} \; E \; \mathbf{b} \iff \exists\, i \in [n] \text{ with } a_j = b_j \text{ for all } j \neq i.$$

Definition

An R-**block** (in C) is a connected component of the graph $(R^* \cap C, E)$.

I <u>can</u> verify the intuition on each R-block in C.

Blocks have a lot of structure.

Theorem 5

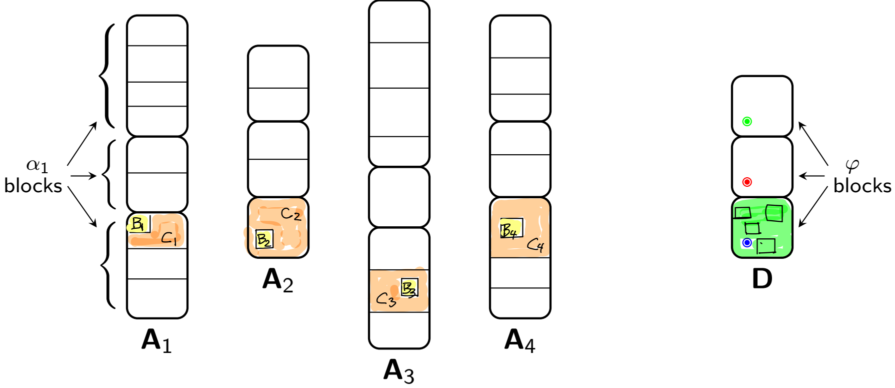
Let $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ as in Theorem 4.

Let $C = C_1 \times \cdots \times C_n$ be a strand of R.

Let B be an R-block in C.

1. $B \cap R \neq \varnothing$. (In fact, every $\mathbf{a} \in B \setminus R$ is "essentially" in $B \cap R$.)
2. $B = B_1 \times \cdots \times B_n$.
3. $|B_1| = \cdots = |B_n|$.
4. Each B_i is a coset of a subgroup of $\mathrm{Grp}(\mu_i, e_i)$ ($e_i \in C_i$).
5. $\forall i \in [n]$, $R \cap B$ "is" the graph of a function $\prod_{j \neq i} B_j \to B_i$.

Kearnes & Szendrei proved (1).



Theorem 6

(With everything as before): If $B = B_1 \times \cdots \times B_n$ is an R-block in the strand $C = C_1 \times \cdots \times C_n$, and each C_i is embedding in the appropriate φ-block of $\mathbf{D}$, then $R \cap B$ is the pullback to B of the solution set of a single $\mathbb{F}$-linear equation

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = u$$

in $\mathcal{E}^n$.

(Congruence modular case)

Lemma 9

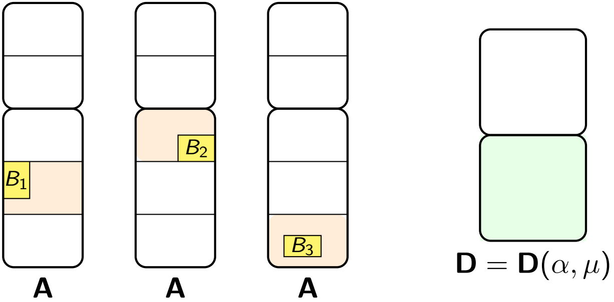
(With everything as before): If $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are in a congruence modular variety, then every strand of R is contained in R^* and is an R-block.

So the intuition holds in CM varieties.

Question: where do $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ in Theorem 6 come from?

Consider the special case where $n = 3$ and

- $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}_3 =: \mathbf{A}$. (monolith μ, annihilator α)
- $(a_1, a_2, a_3) \in R \implies a_1 \overset{\alpha}{\equiv} a_2 \overset{\alpha}{\equiv} a_3$.



Recall that R induces isomorphisms $\lambda_{ij} : \mathbf{D}(\alpha_i, \mu_i) \cong \mathbf{D}(\alpha_j, \mu_j)$.

I.e., $\lambda_{ij} \in \mathrm{Aut}(\mathbf{D})$. Can check that $\lambda_{ij}(\varphi) \subseteq \varphi$ and $\lambda_{ij}(0_E) = 0_E \ \forall E$.

The linear equations used to define $R \cap B$, for each R-block B, use the same coefficients $\lambda_1, \ldots, \lambda_n$. Only the right-hand side constants vary.

Question 1: what (if anything) can be said about the right-hand side constants, for R-blocks within a fixed strand C?

If $B = B_1 \times \cdots \times B_n$ is an R-block, then $B_1, \ldots, B_n$ are cosets of subgroups of their respective μ_i-blocks.

Question 2: Are $B_1, \ldots, B_n$ cosets of $\mathbb{F}$-subspaces of their respective μ_i-blocks?

Thank you!

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