$\aleph_0$-CATEGORICAL STRONGLY MINIMAL COMPACT COMPLEX MANIFOLDS

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ABSTRACT. *Essential* $\aleph_0$-categoricity; i.e., $\aleph_0$-categoricity in some full countable language, is shown to be a robust notion for strongly minimal compact complex manifolds. Characterisations of triviality and essential $\aleph_0$-categoricity are given in terms of complex-analytic automorphisms, in the simply connected case, and correspondences in general. As a consequence it is pointed out that an example of McMullen yields a strongly minimal compact Kähler manifold with trivial geometry but which is not $\aleph_0$-categorical, giving a counterexample to a conjecture of the second author and Tom Scanlon.

1. INTRODUCTION AND PRELIMINARIES

This paper is concerned with a model-theoretic study of compact complex manifolds (*ccm*'s) X which have "little structure", in the sense of there being "few" subvarieties of X^n for all n. Among the motivations for writing the current paper is to point out the existence of a strongly minimal compact Kähler manifold with trivial geometry in the model-theoretic sense, but which is not $\aleph_0$-categorical. This provides a counterexample to a conjecture of the second author and Tom Scanlon (analogous to a similar conjecture for strongly minimal differential algebraic varieties) that appears in [9]. However, a closer look reveals that the notion of $\aleph_0$-categoricity itself is not so clear-cut for *ccm*'s, and so a large part of the current paper is dedicated to showing that at least for strongly minimal *ccm*'s, $\aleph_0$-categoricity is a robust notion, and in fact can be characterized by the existence of only finitely many *correspondences*: proper complex-analytic subsets of $X \times X$ that project onto X in each co-ordinate.

A compact complex manifold X can be considered as a first-order structure $\mathcal{A}(X)$ by adjoining predicates for all (closed) complex-analytic subsets of X^n for all n. The first-order theory of the corresponding structure is very tractable from the model-theoretic point of view; it has finite Morley rank. There is a considerable (geometric) model-theoretic machinery around first-order theories of finite Morley rank. In so far as compact complex manifolds X are concerned, the relevance of this model-theoretic machinery is inversely proportional to the extent to which X is an algebraic variety. Loosely speaking, the strongly minimal compact complex manifolds, which are exactly the irreducible *ccm*'s with no proper infinite complex-analytic subsets, are the building blocks of arbitrary *ccm*'s. There is a

rudimentary classification of strongly minimal structures M according to the behaviour of algebraic closure in a saturated elementary extension; (a) nonmodular, (b) modular nontrivial, and (c) trivial. When $M = \mathcal{A}(X)$, this essentially corresponds to (a) X is an algebraic curve, (b) X is a nonalgebraic simple complex torus, and (c) X has algebraic and Kummer dimension zero, or equivalently X admits no positive-dimensional compact complex-analytic family of correspondences. The identification of strongly minimal ccm's X of type (c) would seem to be a central problem in bimeromorphic geometry. A further distinction within type (c) is between $\aleph_0$-categorical and non $\aleph_0$-categorical. But the notion is problematic. If M is a structure for a *countable* language then M (or rather the first-order theory of M) is said to be $\aleph_0$-*categorical* if $\mathrm{Th}(M)$ has a unique countable model, equivalently if for each n, there are only finitely many $\emptyset$-definable subsets of M^n. However the underlying first-order language of the structure $\mathcal{A}(X)$ is on the face of it *uncountable*, as for example each point of X is named by a predicate. A possible definition of $\aleph_0$-categoricity of X is that there is *some* full countable language $\mathcal{L}$ for X (see Definition 1.1 below) such that $(X, \mathcal{L})$ is $\aleph_0$-categorical. Now X need not have a full countable language, for example if X is a Hopf surface. On the other hand, if X does have a full countable language (which is the case when X is of Kähler type), then as the first author points out in [3], there is a "canonical" choice for such a language, the so-called Douady language. We prove (Theorem 3.15) that for X strongly minimal X is $\aleph_0$-categorical in some full countable language iff X is $\aleph_0$-categorical in the Douady language iff X has only finitely many correspondences. In section 2 we go through the special case when X is simply connected, where the arguments are easier and where the third condition becomes $\mathrm{Aut}(X)$ is finite. McMullen's example of a general $K3$ surface X with $\mathrm{Aut}(X) = \mathbb{Z}$, provides then a trivial strongly minimal Kähler manifold which is not $\aleph_0$-categorical.

There are several overviews of the model theory of compact complex manifolds for a general audience, such as [4] and [6], which we point the reader towards. A starting point for this theory is the fact that compact complex manifolds are complete ω_1-compact Zariski structures in the sense of Zilber (see §3.4.2 of [11]) and this substantially informs the approach taken here. While we will restrict ourselves to manifolds for our main results, smoothness is not essential to the basic model-theoretic development and it is convenient to sometimes work more generally with arbitrary reduced and irreducible compact complex-analytic spaces, that is, *compact complex varieties*. Douady spaces and full countable languages play an important role in the current paper, and the reader is referred to [3] for a comprehensive treatment. An introduction to key notions of model theory, especially in regard to applications, appears in [8] which we again recommend for the non expert.

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In the remainder of this section we briefly review essential saturation for compact complex varieties and introduce "essential $\aleph_0$-categoricity". But first, as strong minimality and triviality are central to the paper we give brief accounts. When we speak of a *definable set* in a structure M we mean a set (typically a subset of some M^n) definable in M possibly with parameters from M. A one-sorted structure M (in a possibly uncountable language L) is said to be *strongly minimal* if for any elementary extension M' of M every definable subset of M' is finite or cofinite. If M is strongly minimal and M' a saturated elementary extension of M, algebraic closure yields an infinite-dimensional pregeometry or matroid on M'. The structure

M is said to have *trivial geometry* if this pregeometry on M' is trivial, in the sense that for any subset A of M', $\mathrm{acl}(A) \cap M = \cup_{a \in A} \mathrm{acl}(a) \cap M$.

If X is a compact complex variety, then by $\mathcal{A}(X)$ we mean the structure which has X as its universe and a predicate for each complex-analytic subset of each finite cartesian power of X. We say that X *is strongly minimal* or that X *has trivial geometry*, if it is true of the structure $\mathcal{A}(X)$. It follows from quantifier elimination and ω_1-compactness that X is strongly minimal just if X has no positive-dimensional proper complex-analytic subsets. We denote by $\mathcal{A}$ the many-sorted structure where there is a sort for each compact complex variety and a predicate for each complex-analytic subset of each finite cartesian product of sorts. We typically work in the many-sorted structure $\mathcal{A}$. But note that by definability of types, a subset of X^n is definable in $\mathcal{A}(X)$ if and only if it is definable in $\mathcal{A}$.

Definition 1.1 (cf. [3]). Suppose X is a compact complex variety. A *full countable language for X* is a countable (one-sorted, relational) language $\mathcal{L}$ and an $\mathcal{L}$-structure on X such that

    (1) for all $n < \omega$, a subset of X^n is definable in $\mathcal{A}(X)$ if and only if it is definable (with parameters) in $(X, \mathcal{L})$.

We will say that $\mathcal{L}$ is *analytic* if in addition

    (2) whenever σ is an automorphism of $(X, \mathcal{L})$ and $A \subseteq X^n$ is a complex-analytic subset, then $\sigma(A)$ is complex-analytic.

We also say that X is *essentially saturated* if it has some full countable language.

Example 1.2 (The Douady Language). Suppose X is an essentially saturated compact complex variety. In [3] it is shown that for all $n > 0$, every irreducible complex-analytic subset of X^n lives in a compact component of the Douady space $D(X^n)$. (In fact this characterises essential saturation.) By a *prime* component of $D(X^n)$ we will mean any irreducible component of $D(X^n)_{\mathrm{red}}$ in which an irreducible complex-analytic subset of X^n lives. So there are countably many prime components, each is a compact complex variety, and every irreducible complex-analytic subset of X^n is in a prime component. Let $Z(X^n) \subseteq D(X^n) \times X^n$ be the universal family of complex-analytic subspaces of X^n. Consider the reduct $\mathcal{A}_X$ of $\mathcal{A}$, where there is a predicate for the restriction of $Z(X^n) \to D(X^n)$ to each prime component, as $n > 0$ varies. Then, by quantifier elimination, a subset of a cartesian power of X is definable in $\mathcal{A}_X$ if and only if it is definable in $\mathcal{A}(X)$. Now let $\mathcal{L}_{\mathrm{Douady}}$ be the language where there is a predicate for each subset of X^n that is 0-definable in $\mathcal{A}_X$, for each $n > 0$. By definability of types this is a full countable language for X, and every automorphism of $(X, \mathcal{L}_{\mathrm{Douady}})$ extends to an automorphism of $\mathcal{A}_X$. Since automorphisms of $\mathcal{A}_X$ preserve complex-analyticity, $\mathcal{L}_{\mathrm{Douady}}$ is a full countable analytic language for X. We call it the *Douady language* of X. Note also that $(X, \mathcal{L}_{\mathrm{Douady}})$ admits quantifier elimination.

Throughout, by acl we mean the algebraic closure in "eq".

Lemma 1.3. *Suppose X is a compact complex variety. If $\mathcal{L}$ is a full countable analytic language for X and $C \subseteq X^n$ is F-definable in $(X, \mathcal{L})$, then C is of the form $\bigcup_{i=1}^{\ell} A_i \setminus B_i$ where each A_i is an irreducible $\mathrm{acl}(F)$-definable complex-analytic subset of X^n and B_i is a proper $\mathrm{acl}(F)$-definable complex-analytic subset of B_i.*

Proof. Let G be the group of automorphism of $(X, \mathcal{L})$, in the model-theoretic sense, that fixes F point-wise. Suppose D is any F-definable set. Since $\mathcal{L}$ is analytic, every member of G will permute the collection of complex-analytic subsets of X^n that contain D. Hence the closure of D, $\bar{D}$, which we know is definable in $(X, \mathcal{L})$ by fullness, is fixed set-wise by every automorphism in G. By saturation, it follows that $\bar{D}$ is F-definable in $(X, \mathcal{L})$.

Now, by quantifier elimination in $\mathcal{A}(X)$ we can write C irredundantly as $C = \bigcup_{i=1}^{\ell} C_i$ where $C_i = A_i \setminus B_i$ with A_i irreducible complex-analytic and B_i a proper complex-analytic subset of A_i. Moreover this decomposition is unique up to a permutation of $\{C_1, \dots, C_\ell\}$. Since $\mathcal{L}$ is analytic and C is F-definable, it follows that every member of G permutes $\{C_1, \dots, C_\ell\}$. Hence, by saturation, each C_i is $\mathrm{acl}(F)$-definable in $(X, \mathcal{L})$. By the discussion in the first paragraph, applied to $D = C_i$ and the parameter set $\mathrm{acl}(F)$, $A_i = \bar{C}_i$ is also $\mathrm{acl}(F)$-definable. Hence, so is $B_i = A_i \setminus C_i$. $\square$

Definition 1.4 (Essential $\aleph_0$-categoricity)**.** A compact complex variety X is *essentially $\aleph_0$-categorical* if there exists a full countable language $\mathcal{L}$ for X such that $(X, \mathcal{L})$ is $\aleph_0$-categorical.

This paper is primarily concerned with essentially $\aleph_0$-categorical strongly minimal manifolds. If X is such, and $\mathcal{L}$ is a full countable language witnessing $\aleph_0$-categoricity, then by Zilber's theorem algebraic closure in $(X, \mathcal{L})$, and hence also in $\mathrm{Th}\big(\mathcal{A}(X)\big)$, is a modular geometry. But modular strongly minimal manifolds are characterised in Proposition 5.1 of [10], they are either of trivial geometry or are simple complex tori. Since the latter are not essentially $\aleph_0$-categorical (by what we know about $\aleph_0$-categorical groups, for example), we obtain:

Fact 1.5. *Essentially $\aleph_0$-categorical strongly minimal compact complex varieties are necessarily of trivial geometry.*

The following is a useful characterisation of triviality.

Lemma 1.6. *Suppose X is a strongly minimal compact complex variety. The following are equivalent:*

    (i) *X has trivial geometry,*
    (ii) *there is no infinite definable family of irreducible complex-analytic subsets of X^2 projecting onto X in each co-ordinate.*

Proof. This is well-known and comes easily out of the definitions, but we give a sketch of the proof anyway, using freely model-theoretic language. Let us first assume that there is some infinite definable family of irreducible complex-analytic subsets of X^2 projecting onto X in each coordinate. Namely there is some definable subset W of X^k (some k), and some definable subset Z of $W \times X^2$ such that for each $w \in W$ the fibre $Z_w \subset X^2$ is an irreducible complex-analytic set that projects onto X in each coordinate, and $\{Z_w : w \in W\}$ is infinite. As we may exclude X^2 itself from being among the fibres, we may assume (by strong minimality of X) that each Z_w is generically finite-to-one over X in each co-ordinate. In particular each Z_w is itself a strongly minimal compact complex variety. Moreover, shrinking W if necessary, we may also assume that the complex-analytic set $\overline{W}$, the closure of W in X^k, is irreducible.

Now we pass to a saturated elementary extension $\mathcal{A}'$ of $\mathcal{A}$. Let $(c, a, b) \in Z(\mathcal{A}')$ be a generic point of $\overline{Z}$ in $\mathcal{A}'$. In particular, c is generic in $\overline{W}$. As $Z(\mathcal{A}')_c$ is

generically finite-to-one over $X(\mathcal{A}')$ in each co-ordinate, a and b are interalgebraic over c. On the other hand, as acl-dim$(Z) >$ acl-dim(W), $a, b \notin$ acl(c).

We claim that $b \notin$ acl(a). To prove this, consider $d \models$ stp(c/a) independent of c over a. Since $a \notin$ acl(c) and X is of rank 1, c is independent of a over the empty set. Hence d is independent of c over the empty set, and is thus generic in $\overline{W}$ over c. Thus, if $Z(\mathcal{A}')_c = Z(\mathcal{A}')_d$, then as this is a closed $\{c\}$-definable condition on d (and remembering that $\overline{W}$ is irreducible) we would have that all the fibres Z_w are equal, contradicting our assumption that the family is infinite. Hence, $Z(\mathcal{A}')_c \neq Z(\mathcal{A}')_d$ and so $Z(\mathcal{A}')_c \cap Z(\mathcal{A}')_d$ is finite. As $a \notin$ acl(c, d), it follows that $(a, b) \notin Z(\mathcal{A}')_c \cap Z(\mathcal{A}')_d$. So tp$(d/ab) \neq$ tp(c/ab), which implies that $b \notin$ acl(a).

So $b \in$ acl$(c, a) \setminus \big($acl$(c) \cup$ acl$(a)\big)$, showing nontriviality of X.

Conversely assume X is nontrivial. Let $a, b \in X(\mathcal{A}')$ and $c \in X(\mathcal{A}')^k$ be such that $b \in$ acl$(c, a) \setminus \big($acl$(c) \cup$ acl$(a)\big)$. Extending c we may further assume that tp(ab/c) is stationary. Let $Z \subset X^k \times X^2$ be the *locus* of (c, a, b); that is, Z is the smallest complex-analytic subset of $X^k \times X^2$ such that $Z(\mathcal{A}')$ contains (c, a, b). Then for some definable $W \subseteq X^k$ with $c \in W(\mathcal{A}')$, each fibre of X over W is an irreducible complex-analytic subset of X^2 that projects onto X in each co-ordinate. (Note that irreduciblity is definable in parameters, see for example page 30 of [1].) If E is the definable equivalence relation on W where wEw' if and only if $Z_w = Z'_w$, then $Z(\mathcal{A}')_c$ is defined over c/E and so $b \in$ acl$(c/E, a)$. Since $b \notin$ acl(a), it follows that $\{Z_w : w \in W\}$ must be infinite. $\qquad\square$

2. A warm-up: the simply connected case

Lemma 2.1. *Suppose X is a simply connected strongly minimal compact complex manifold of dimension greater than one.*

  (a) *The only irreducible complex-analytic subsets of X^2 are points, X^2 itself, vertical and horizontal "slices" $\{a\} \times X$ and $X \times \{a\}$ where $a \in X$, and graphs of automorphisms.*

  (b) *Suppose moreover that X has trivial geometry, and $A \subseteq X^n$ is an irreducible complex-analytic subset. Then, after some permutation of the co-ordinates, there exists $0 \leq r \leq n$ such that A is defined by equations $\phi_1, \ldots, \phi_{n-r}$ where ϕ_j is either of the form '$x_{r+j} = \sigma(x_i)$' for some $\sigma \in$ Aut X and $i \leq r$, or of the form '$x_{r+j} = b$' for some $b \in X$.*

Proof. To prove part (a) we first recall the following fact:

Fact 2.2. *Suppose X is a strongly minimal compact complex manifold of dimension greater than one, $A \subseteq X^n$ is an irreducible complex-analytic subset such that one of the co-ordinate projections $\pi : A \to X$ is surjective and finite-to-one. If $\rho : A' \to A$ is a normalisation of A, then $\pi \circ \rho : A' \to X$ is an unramified covering.*

Indeed, arguments for this fact can be found in Lemma 7 of [7], Proposition 2.12 of [9], and Lemma 4.2 of [5]. Here is a sketch: The branch locus of $\pi \circ \rho : A' \to X$, where the morphism is not locally a biholomorphism, is a proper complex-analytic subset of A'. Since $\pi \circ \rho$ is finite-to-one and A' is irreducible, the image of the branch locus cannot be all of X, and so by strong minimality it must be a finite subset of X. Hence the branch locus itself is finite. But the smoothness of X and the normality of A' imply that the branch locus is either of codimension one or empty (by the purity of branch theorem). It follows that $\pi \circ \rho$ is everywhere locally biholomorphic, as desired.

Now, if $A \subseteq X^2$ is an irreducible complex-analytic set that is neither all of X^2 nor a "slice", then by strong minimality each co-ordinate projection $\pi : A \to X$ is finite-to-one onto X. Applying Fact 2.2 to this situation, and remembering that X is simply connected, we get that $\pi \circ \rho$, and hence $\pi : A \to X$ itself, is a biholomorphism. This proves part (a).

We prove part (b) by induction on n. The case of $n = 1$ is by strong minimality, and the case of $n = 2$ is part (a). For the induction step, suppose $n > 2$ and let $a = (a_1, \dots, a_n) \in X(\mathcal{A}')$ be a generic point of A in a sufficiently saturated elementary extension $\mathcal{A}'$ of $\mathcal{A}$. After permuting co-ordinates we may assume that $\{a_1, \dots, a_r\}$ is an acl-basis for $\{a_1, \dots, a_n\}$. If $r = n$ then $A = X^n$ and we are done. If $r = 0$ then A is a point and we are done. Suppose $0 < r < n$ and consider $a_{r+1} \in \operatorname{acl}(a_1, \dots, a_r)$. By triviality, $a_{r+1} \in \operatorname{acl}(a_i)$ for some $i \leq r$. Let $S = \operatorname{loc}(a_i, a_{r+1}) \subseteq X^2$ be the locus of (a_i, a_{r+1}). Then $S \to X$, under the first co-ordinate projection is a generically finite-to-one map. By part (a), S must either be of the form $X \times \{b\}$ or S is the graph of some $\sigma \in \operatorname{Aut} X$. In the first case $a_{r+1} = b$ is a standard point in X. But as a was generic in A and A was irreducible, this means that, after a co-ordinate permutation, $A = B \times \{b\}$ for some $B \subseteq X^{n-1}$ irreducible and complex-analytic. The desired description of A then follows by applying the induction hypothesis to B. So we may assume that S is the graph of some $\sigma \in \operatorname{Aut} X$. Again by genericity of a and irreducibility of A, the $(r+1)$st co-ordinate of every element of A is obtained by applying σ to the ith co-ordinate. That is, after a co-ordinate permutation, we get $A \subseteq f(B)$ where $B \subseteq X^{n-1}$ is the projection of A to the first $n-1$ co-ordinates and $f(x_1, \dots, x_{n-1}) := (x_1, \dots, x_{n-1}, \sigma(x_i))$. As $f(B)$ is irreducible and $\dim A \geq \dim B = \dim f(B) \geq \dim A$, we get that $A = f(B)$. The desired description of A then follows from the description of B given by the induction hypothesis. $\qquad\square$

Lemma 2.1(b) implies in particular that if X is simply connected strongly minimal trivial, then any irreducible complex-analytic set $A \subseteq X^n$, for $n \geq 2$, is completely determined by its co-ordinate projections to X^2.

Proposition 2.3. *Suppose X is a simply connected strongly minimal compact complex manifold. Then the following are equivalent:*

   (i) $\operatorname{Aut} X$ *is countable.*
   (ii) X *is essentially saturated and has trivial geometry.*

Moreover, in this case, the language $\mathcal{L}_{\mathrm{Aut}}$ consisting of a predicate symbol for the graph of each automorphism of X is a full countable analytic language for X.

Proof. Suppose $\operatorname{Aut} X$ is countable. Since by Lemma 2.1 (a) the only irreducible complex-analytic subsets of X^2 that project onto each co-ordinate are graphs of automorphisms, there can be no infinite definable family of such. Hence by condition (ii) of Lemma 1.6 X must have trivial geometry. It follows immediately from Lemma 2.1 (b) now that the language of automorphisms of X is a full countable analytic language for X. In particular, X is essentially saturated.

For the converse, suppose $\operatorname{Aut} X$ is uncountable and X is essentially saturated. Then, by the existence of a full countable language, there must exist an infinite definable family of automorphisms of X, which contradicts triviality. $\qquad\square$

Proposition 2.4. *Suppose X is a simply connected strongly minimal compact complex manifold with $\operatorname{Aut} X$ countable.*

(a) $(X, \mathcal{L}_{\mathrm{Douady}}) = (X, \mathcal{L}_{\mathrm{Aut}})$ *in the sense that every basic relation of one is 0-definable in the other, and vice versa.*

(b) *In* $(X, \mathcal{L}_{\mathrm{Aut}})$, *and hence also in* $(X, \mathcal{L}_{\mathrm{Douady}})$ *by part (a),*

$$\mathrm{acl}(\emptyset) \cap X = \bigcup_{\mathrm{id} \neq \sigma \in \mathrm{Aut}\, X} \mathrm{Fix}(\sigma).$$

Proof. Since $\mathrm{Aut}\, X$ is discrete, the graph of each automorphism is *isolated in* X^2 in the sense that it lives in a zero-dimensional prime component of $D(X^2)$. Hence the graph of each automorphism is a basic relation of $(X, \mathcal{L}_{\mathrm{Douady}})$. In order to prove part (a) it therefore suffices to show that every basic relation in $(X, \mathcal{L}_{\mathrm{Douady}})$ is 0-definable in $(X, \mathcal{L}_{\mathrm{Aut}})$. To that end, fix $n > 0$ and consider $A \subset X^n$ an irreducible complex-analytic subset. We know by essential saturation that A lives in a compact prime component of $D(X^n)$. We wish to describe this component.

Claim 2.5. *The prime component C of $D(X^n)$ in which A lives is (up to biholomorphism) of the form $C = X^m$ for some $0 \leq m \leq n$, and the universal family restricted to C, $Z := Z(X^n)|_C \subseteq C \times X^n \subseteq X^{m+n}$ is defined by equations of the form $\sigma(x_i) = x_j$ where $\sigma \in \mathrm{Aut}\, X$.*

Proof of Claim 2.5. We prove this by induction on n. For $n = 1$, A is either X itself in which case $C = X^0$ and $Z = X$, or A is a point in which case $C = X$ and Z is the diagonal in X^2.

Suppose $n > 1$. Assume moreover that there exists a co-ordinate projection $\pi : X^n \to X$ such that $\pi(A)$ is a point. Then after a possible permutation of co-ordinates we have that $A = A' \times \{a\}$ for some irreducible complex-analytic $A' \subseteq X^{n-1}$ and $a \in X$. Now A' lives in some prime component C' of $D(X^{n-1})$ with $Z' \subseteq C' \times X^{n-1}$ the restriction of the universal family to C'. As $A = A' \times \{a\}$ the family $Z \to C$ is obtained from $Z' \to C'$ by base change with respect to $C' \times X \to C'$. That is, $C = C' \times X$ and

$$Z = \{(c, x_0, x_0, x_1, \ldots, x_{n-1}) : c \in C', x_0 \in X, (c, x_1, \ldots, x_{n-1}) \in Z'\}.$$

Applying the induction hypothesis to C' and Z' we see that the claim is true of C and Z also.

We may therefore assume that no co-ordinate projection of X is a point. By Lemma 2.1(a), for all co-ordinate projections $\pi : X^n \to X^2$, $\pi(A)$ is the graph of an automorphism. One consequence of this is that A itself is defined by equations of the form $\sigma(x_i) = x_j$ where $\sigma \in \mathrm{Aut}\, X$; indeed, looking at the defining formulas for A given in Lemma 2.1(b) we see that these are the only possibilities. On the other hand, by the discreteness of $\mathrm{Aut}\, X$, we also get that each $\pi(A)$ lives in a zero-dimensional prime component of $D(X^2)$. It follows by triviality that A lives in a zero-dimensional prime component of $D(X^n)$ – this is exactly Proposition 3.4 of [5]. Hence $C = X^0$ and $Z = A$. $\square$

It follows from the claim that $\mathcal{A}_X$, the reduct of $\mathcal{A}$ where only the components of the universal families $Z(X^n)$ are named, is in fact one-sorted (the sort being X itself) and that the basic relations in $\mathcal{A}_X$ are 0-definable in $\mathcal{L}_{\mathrm{Aut}}$. By definition the same then holds for $\mathcal{L}_{\mathrm{Douady}}$. This proves part (a).

To prove part (b), first note that the right-to-left containment is clear since if $\sigma \neq \mathrm{id}$ then $\mathrm{Fix}(\sigma)$ is finite as it is a proper closed subset of the strongly minimal X. The left-to-right containment on the other hand is true in any faithful group

action: working in $(X, \mathcal{L}_{\text{Aut}})$ we let $F = \bigcup\limits_{\text{id} \neq \sigma \in \text{Aut } X} \text{Fix}(\sigma)$ and show that there is a unique 1-type in $X \setminus F$ over F. Note that each member of Aut X fixes F setwise: if $f \in \text{Fix}(\sigma)$ then $\tau(f) \in \text{Fix}(\tau \sigma \tau^{-1})$. Hence Aut X acts on $X \setminus F$, and this action is clearly free. It follows that X is the disjoint union of Aut X orbits and F, and each such orbit is being acted on regularly by $\text{Aut}(X)$. It is then clear that for any $a, b \in X \setminus F$ there is an automorphism of the *structure* $(X, \mathcal{L}_{\text{Aut}})$ which fixes F pointwise and takes a to b. So all elements of $X \setminus F$ have the same $\mathcal{L}_{\text{Aut}}$-type over the empty set. $\qquad \square$

Now we investigate $\aleph_0$-categoricity for simply connected strongly minimal compact complex manifolds.

Theorem 2.6. *Suppose X is a simply connected strongly minimal compact complex manifold. Then the following are equivalent.*

  (i) *Aut X is finite.*
  (ii) *X is essentially saturated and $(X, \mathcal{L}_{\text{Aut}})$ is $\aleph_0$-categorical.*
  (iii) *X is essentially saturated and $(X, \mathcal{L}_{\text{Douady}})$ is $\aleph_0$-categorical.*
  (iv) *X is essentially $\aleph_0$-categorical.*

Proof. (i) $\implies$ (ii). Assume that Aut X is finite. By Proposition 2.3 we know that X is essentially saturated and trivial and that $\mathcal{L}_{\text{Aut}}$ is a full countable analytic language for X. We need to check that for each $n > 0$, there are only finitely many 0-definable subsets of X^n in $(X, \mathcal{L}_{\text{Aut}})$. By Lemma 1.3, since we are working in an analytic language, it suffices to show that there are only finitely many irreducible complex-analytic subsets of X^n that are $\text{acl}(\emptyset)$-definable in $(X, \mathcal{L}_{\text{Aut}})$.

For $n = 1$ we need to count the number of $\text{acl}(\emptyset)$-definable points in X. Since Aut X is finite and the set of fixed points of each nontrivial member of Aut X is finite, Proposition 2.4(b) tells us that $\text{acl}(\emptyset) \cap X$ is finite. The $n = 2$ case is taken care of by the description of the irreducible complex-analytic subsets of X^2 given by Lemma 2.1(a): the only possibilities are X^2, points, slices, or graphs of automorphisms. The first and last of these only contribute finitely many. In the case of points or slices, note that the singletons involved, by $\text{acl}(\emptyset)$-definability and saturation, must be in $\text{acl}(\emptyset) \cap X$ – and hence these also only contribute finitely many possibilities for irreducible complex-analytic subsets of X^2.

The $n = 2$ case now implies the general case: If $n > 2$ and $A \subseteq X^n$ is irreducible complex-analytic $\text{acl}(\emptyset)$-definable, then each of the projections of A to X^2 are also irreducible complex-analytic $\text{acl}(\emptyset)$-definable. But by triviality A is determined by its projections to X^2 – see Lemma 2.1(b). Hence there are only finitely many possibilities for A.

(ii) $\implies$ (iii). Because of $\aleph_0$-categoricity X must have trivial geometry (see Fact 1.5). The implication now follows immediately from Proposition 2.4(a).

(iii) $\implies$ (iv). Clear.

(iv) $\implies$ (i). Suppose $\mathcal{L}$ is a full countable language for X such that $(X, \mathcal{L})$ is $\aleph_0$-categorical. Then $(X, \mathcal{L})$ must have trivial geometry. It follows that every automorphism of X is $\text{acl}(\emptyset)$-definable in $(X, \mathcal{L})$. By $\aleph_0$-categoricity, there can be only finitely many such. $\qquad \square$

This, together with an example of McMullen discussed below, resolves in the negative a conjecture of the second author and Thomas Scanlon from [9]:

Corollary 2.7. *There exist trivial strongly minimal compact Kähler manifolds which are not ℵ$_0$-categorical in any full countable language.*

Proof. This comes from the study of generic analytic K3 surfaces due to Gross, McMullen and Oguiso, though we were informed by the survey article [2]. An *analytic K3 surface* is a smooth simply connected compact surface X with trivial canonical bundle. They are Kähler manifolds (and hence essentially saturated). A K3 surface is *generic* if it has trivial Picard group (these are in fact dense in the moduli space of K3 surfaces). Generic K3 surfaces are strongly minimal (as any curve on X would give rise to an effective divisor and hence a nontrivial line bundle). Oguiso has shown that a generic K3 surface either has trivial automorphism group or $\operatorname{Aut} X = \mathbb{Z}$. In particular, by Proposition 2.3, all generic K3 surfaces have trivial geometry. McMullen produced examples with $\operatorname{Aut} X = \mathbb{Z}$, which by Theorem 2.6 cannot be ℵ$_0$-categorical in any full countable language. □

3. The general case

Fix a strongly minimal compact complex manifold X. Some of what we did in the previous section goes through without the assumption of simply connectedness if we replace automorphisms by finite-to-finite correspondences, but there are additional complications. By a *finite-to-finite correspondence on X* we mean an irreducible complex-analytic subset $S \subset X^2$ such that both co-ordinate projections are surjective finite-to-one maps. By strong minimality this is equivalent to saying that S is a proper irreducible complex-analytic subset of X^2 that projects onto X in each co-ordinate. We denote the set of all finite-to-finite correspondences by $\operatorname{Corr} X$. Unlike in the simply connected case, triviality will not completely reduce the study of irreducible complex-analytic subsets of X^n to members of $\operatorname{Corr} X$. The main problem is that intersections of pull-backs of correspondences may not be irreducible, and so their irreducible components need to be taken into account.

Definition 3.1. By a *generalised correspondence* on X we mean an irreducible $\dim(X)$-dimensional complex-analytic subset of X^n that projects onto X in each co-ordinate.

Lemma 3.2. *Suppose $A \subseteq X^n$ is a generalised correspondence. For each $i = 2, \ldots n$, let $\pi_i : X^n \to X^2$ be the co-ordinate projection $(x_1, \ldots, x_n) \mapsto (x_1, x_i)$. Then $S_i := \pi_i(A) \in \operatorname{Corr} X$ for all $i = 1, \ldots, \ell$ and A is an irreducible component of $\bigcap_{i=2}^n \pi_i^{-1}(S_i)$.*

Proof. Clearly each $S_i \subseteq X^2$ is proper (it is of dimension $\dim X$), irreducible, complex-analytic and projects onto X in each co-ordinate. So $S_i \in \operatorname{Corr} X$.

Note that $B := \bigcap_{i=2}^n \pi_i^{-1}(S_i)$ is of acl-dimension at most 1 as every co-ordinate is algebraic over the first co-ordinate. Hence $\dim B \leq \dim X$. But $A \subseteq B$, so that $\dim B = \dim X$, and A must be an irreducible component of B. □

We have the following analogue of Lemma 2.1.

Lemma 3.3.   (a) *The only irreducible complex-analytic subsets of X^2 are points, X^2 itself, vertical and horizontal "slices" $\{a\} \times X$ and $X \times \{a\}$ where $a \in X$, and finite-to-finite correspondences.*
  (b) *Suppose moreover that X has trivial geometry. If $A \subseteq X^n$ is an irreducible complex-analytic subset, then, after some permutation of the co-ordinates, A is a product of generalised correspondences and singletons.*

Proof. Suppose $A \subseteq X^2$ is an irreducible complex-analytic subset. Note that by irreducibility the co-ordinate projections of A are either all of X or points. If both projections are points then A is a point. If one projection is a point and the other is all of X then A is of the form $\{a\} \times X$ and $X \times \{a\}$. If both projections are onto X than either $A = X^2$ or $A \in \operatorname{Corr} X$.

For part (b) the $n = 1$ case is clear, and the $n = 2$ case is part (a). For $n > 2$, let $a = (a_1, \ldots, a_n) \in X(\mathcal{A}')$ be a generic point of A in a sufficiently saturated elementary extension $\mathcal{A}'$ of $\mathcal{A}$. Note that if some $a_i \in \operatorname{acl}(\emptyset)$ then a_i is a standard point of X and so after permuting co-ordinates A is of the form $A' \times \{a_i\}$, and we are done by induction. We may therefore assume that all $a_i \notin \operatorname{acl}(\emptyset)$. Let $\{b_1, \ldots, b_r\}$ be an acl-basis for $\{a_1, \ldots, a_n\}$. By triviality, and the fact that no co-ordinate is in $\operatorname{acl}(\emptyset)$, each a_j is in $\operatorname{acl}(b_i)$ for a unique b_i. Hence, after permuting the co-ordinates, we can write $(a_1, \ldots, a_n) = (\bar{b}_1, \ldots, \bar{b}_r)$ where $\bar{b}_i = (b_i = b_{i,1}, \ldots, b_{i,k_i})$ and $b_{i,j} \in \operatorname{acl}(b_{i,1})$, for all $i \leq r$ and $j \leq k_i$. For each $i \leq r$ let $A_i = \operatorname{loc}(\bar{b}_i)$. Then $A_i \subseteq X^{k_i}$ is irreducible complex-analytic; it is of dimension $\dim X$ since $\bar{b}_i$ is of acl-dimension 1, and it projects onto X in each co-ordinate since every co-ordinate of $\bar{b}_i$ is not in $\operatorname{acl}(\emptyset)$. That is, each A_i is a generalised correspondence. Since $\{b_{1,1}, \ldots, b_{r,1}\}$ is acl-independent, we have $A = A_1 \times \cdots \times A_r$, as desired. $\qquad \square$

Definition 3.4. The *language of generalised correspondences* for X, $\mathcal{L}_{\mathrm{GC}}$, is the language where there is a predicate for each generalised correspondence on X.

It follows from Lemma 3.2 that if $\operatorname{Corr} X$ is countable then so is $\mathcal{L}_{\mathrm{GC}}$.

The arguments for Proposition 2.3 and Proposition 2.4(a) now generalise to:

Proposition 3.5. *The following are equivalent:*

  (i) $\operatorname{Corr} X$ *is countable.*
  (ii) X *is essentially saturated and has trivial geometry.*

In this case $\mathcal{L}_{\mathrm{GC}}$ is a full countable analytic language for X, and $(X, \mathcal{L}_{\mathrm{Douady}}) = (X, \mathcal{L}_{\mathrm{GC}})$ in the sense that every basic relation of one is 0-definable in the other, and vice versa.

Proof. The equivalence of (i) and (ii) is almost exactly as in Proposition 2.3. The countability of $\operatorname{Corr} X$ implies the countability of $\mathcal{L}_{\mathrm{GC}}$ by Lemma 3.2. It also implies that there is no infinite definable family of finite-to-finite correspondences. Since finite-to-finite correspondences are the only irreducible complex-analytic subsets of X^2 that project onto both co-ordinates (Lemma 3.3(a)), we see by condition (ii) of Lemma 1.6 that the geometry on X must be trivial. Hence every irreducible complex-analytic subset is up to a co-ordinate permutation a product of generalised correspondences and singletons (Lemma 3.3(b)). From this it follows that $\mathcal{L}_{\mathrm{GC}}$ is a full countable analytic language for X. For the converse, note that if X were essentially saturated and $\operatorname{Corr} X$ were uncountable then there would be an infinite definable family of finite-to-finite correspondences, contradicting triviality.

Now suppose that the equivalent conditions (i) and (ii) are satisfied. We want to show that $(X, \mathcal{L}_{\mathrm{Douady}}) = (X, \mathcal{L}_{\mathrm{GC}})$. Following the argument for automorphisms, in order to show that every generalised correspondence is 0-definable in $(X, \mathcal{L}_{\mathrm{Douady}})$ we prove that they are isolated in the sense that each one lives in a zero-dimensional prime component of the Douady space. Suppose $A \subseteq X^n$ is a generalised correspondence on X living in the irreducible prime component C of the Douady space of X^n. There exists a proper complex-analytic subset $E \subset C$, such that distinct points in

$C \setminus E$ correspond to distinct irreducible $\dim(X)$-dimensional complex-analytic subsets of X^n. In Lemma 2.3 of [5] it is pointed out that projecting onto a singleton in some co-ordinate is a property that is preserved in irreducible components of $D(X^n)$. Since A does not project onto a singleton in any co-ordinate, this must also be true of each of the complex-analytic subsets of X^n given by points in C. By strong minimality it follows that each of the irreducible complex-analytic sets corresponding to points of $C \setminus E$ project onto X in every co-ordinate. That is, distinct points of $C \setminus E$ give rise to distinct generalised correspondences on X. As there are only countably many generalised correspondences, C must be zero-dimensional, as desired.

Finally, still assuming countability of $\operatorname{Corr} X$, we need to prove that every basic relation in $(X, \mathcal{L}_{\mathrm{Douady}})$ is 0-definable in $(X, \mathcal{L}_{\mathrm{GC}})$. Here the argument is exactly as in Proposition 2.4(a) once we replace Claim 2.5 by: *For every irreducible complex-analytic $A \subseteq X^n$, the prime component C of $D(X^n)$ in which A lives is (up to biholomorphism) of the form $C = X^m$ for some $0 \leq m \leq n$, and the universal family restricted to C, $Z := Z(X^n)|_C \subseteq C \times X^n \subseteq X^{m+n}$ is defined by a conjunction of atomic $\mathcal{L}_{\mathrm{GC}}$-formulas.* The claim is also proved by induction on n, with $n = 1$ being clear. For $n > 1$, if some co-ordinate projection of A to X is a singleton then one reduces to the induction hypothesis exactly as in the proof of Claim 2.5. Hence, by Lemma 3.3(a), we may assume that every co-ordinate projection of A to X^2 is a finite-to-finite correspondence on X. We have already seen that the elements of $\operatorname{Corr} X$ are all isolated in X^2, so all the projections of A to X^2 are isolated. It follows by triviality, using Proposition 3.4 of [5], that A must be isolated in X^n. Hence $C = X^0$ and $Z = A$. So it remains to observe that in this case, A itself is defined by a conjunction of atomic $\mathcal{L}_{\mathrm{GC}}$-formulas. But that is just what Lemma 3.3(b) says, given that no co-ordinate projection of A is a singleton. $\qquad \square$

We now need to analyse the structure $(X, \mathcal{L}_{\mathrm{GC}})$. In particular we need to describe the algebraic closure of the empty set. Unlike in the simply connected case, we are not just working with a pure group action. However, as it turns out, we are not so very far away from that situation.

Definition 3.6. Given $S, T \in \operatorname{Corr} X$ and $x \in X$ we set

$$T \circ S := \big\{ (a, b) \in X^2 : \text{ for some } c \in X, (a, c) \in S \text{ and } (c, b) \in T \big\},$$
$$S^{-1} := \big\{ (a, b) \in X^2 : (b, a) \in S \big\},$$
$$\Delta := \big\{ (a, a) : a \in X \big\}$$
$$\operatorname{orbit}(x) := \big\{ a \in X : (x, a) \in U \text{ for some } U \in \operatorname{Corr} X \big\},$$
$$X_{00} := \big\{ a \in X : (a, a) \in U \text{ for some } U \in \operatorname{Corr} X \text{ with } U \neq \Delta \big\},$$
$$X_0 := \bigcup_{a \in X_{00}} \operatorname{orbit}(a).$$

Note that in the simply connected case when the only finite-to-finite correspondences are the graphs of automorphisms, $T \circ S$ is the graph of the composition of the automorphisms, S^{-1} is the graph of the inverse automorphism, Δ is the graph of the identity automorphism, $\operatorname{orbit}(x)$ is the orbit of x under the action of $\operatorname{Aut} X$ on X, and $X_0 = X_{00}$ is the set of fixed points of X under this action.

Proposition 3.7. *The relation $y \in \operatorname{orbit}(x)$ is an equivalence relation on X.*

Proof. Reflexivity is by the fact that $\Delta \in \operatorname{Corr} X$ and symmetry is by the fact that if $S \in \operatorname{Corr} X$ then $S^{-1} \in \operatorname{Corr} X$. Finally, while it is not necessarily the case

that $T \circ S \in \operatorname{Corr} X$ whenever $S, T \in \operatorname{Corr} X$, the following lemma shows that the irreducible components of $T \circ S$ are – and that suffices for transitivity. $\qquad \square$

Lemma 3.8. *Suppose $n > 1$ and $A_1, A_2 \subseteq X^n$ are generalised correspondences such that $\pi(A_1) = \pi(A_2)$, where $\pi : X^n \to X^{n-1}$ is the projection onto the last $n-1$ co-ordinates. Let*

$$B := \left\{ (a_1, a_2) \in X^2 : \text{ for some } z \in X^{n-1}, \ (a_1, z) \in A_1 \text{ and } (a_2, z) \in A_2 \right\}$$

Then every irreducible component of B is in $\operatorname{Corr} X$. Moreover, unless $A_1 = A_2$, none of these components is Δ.

In particular, if $S, T \in \operatorname{Corr} X$, then every irreducible component of $T \circ S$ is in $\operatorname{Corr} X$, and unless $S = T^{-1}$ none of these components is Δ.

Proof. The "in particular" clause follows by letting $n = 2$, $A_1 = S$, and $A_2 = T^{-1}$.

For the "moreover" clause, observe that if $\Delta \subseteq B$, then $A_1 \cap A_2$ projects onto X in the first co-ordinate, so that $\dim(A_1 \cap A_2) = \dim X$, and hence $A_1 = A_2$.

Now let us prove that main statement. It is not hard to see that $B \subseteq X^2$ is a complex-analytic set that projects onto X in each co-ordinate in a finite-to-one manner. It follows from Lemma 3.3(a) that each irreducible component of B must be either a singleton or a finite-to-finite correspondence. We need to rule out the possibility of zero-dimensional components.

Let $A = \pi(A_1) = \pi(A_2) \subseteq X^{n-1}$, and let $\rho : A' \to A$ be a normalisation of A. Note that A is again a generalised correspondence, and in particular projects onto X in a finite-to-one manner in each co-ordinate. Then, by Fact 2.2, composing ρ with any co-ordinate projection shows that A' is a unramified cover of X. In particular, A' is smooth and of dimension $\dim X$.

Let $p_1 : X \times A' \times X \to X^n$ be the holomorphic surjection $(x_1, z, x_2) \mapsto (x_1, \rho(z))$, let $p_2 : X \times A' \times X \to X^n$ be $(x_1, z, x_2) \mapsto (x_2, \rho(z))$ and let $q : X \times A' \times X \to X^2$ be $(x_1, z, x_2) \mapsto (x_1, x_2)$. Set $W := p_1^{-1}(A_1) \cap p_2^{-1}(A_2) \subseteq X \times A' \times X$. By the surjectivity of ρ, $B = q(W)$. By the smoothness of $X \times A' \times X$ we know that each irreducible component of W is of dimension at least $2 \dim X + 2 \dim X - 3 \dim X = \dim X$. But since $q|_W$ is also finite-to-one, every irreducible component of B is at least of dimension $\dim X$, as desired. $\qquad \square$

Note that the intersection of distinct generalised correspondences, $A_1, A_2 \subseteq X^n$, must be finite. Indeed, as the first co-ordinate projection of A_1 is finite-to-one onto X, if $A_1 \cap A_2$ were infinite, then its projection, being complex-analytic, would be all of X by strong minimality. Hence $\dim(A_1 \cap A_2) \geq \dim X$. Irreducibility would then imply that $A_1 = A_2$. So such intersections will always land in the algebraic closure of the empty set in $(X, \mathcal{L}_{\mathrm{GC}})$. Hence the following lemma is a necessary part of showing that that algebraic closure is X_0.

Lemma 3.9. *Suppose $A_1, A_2 \subseteq X^n$ are distinct generalised correspondences. Then $A_1 \cap A_2 \subseteq X_0^n$.*

Proof. By induction on $n > 0$. The case of $n = 1$ is vacuous since then $A_i = X$ for $i = 1, 2$. Suppose $n > 1$. Let $\pi : X^n \to X^{n-1}$ be some co-ordinate projection. Note that $\pi(A_1)$ and $\pi(A_2)$ are generalised correspondences, so that by induction if $\pi(A_1) \neq \pi(A_2)$ then $\pi(A_1) \cap \pi(A_2) \subseteq X_0^{n-1}$. If this were the case for all co-ordinate projections to X^{n-1}, then we would be done. So we may assume that for some π, $\pi(A_1) = \pi(A_2)$. After permuting co-ordinates we may in addition

assume that π is the projection onto the last $n-1$ co-ordinates. We are thus in a situation to which Lemma 3.8 applies. Now let $(a, z) \in A_1 \cap A_2$, where $a \in X$ and $z = (z_2, \ldots, z_n) \in X^{n-1}$. Then (a, a) is in the set B of Lemma 3.8, and so by that lemma, $(a, a) \in S$ for some $\Delta \neq S \in \operatorname{Corr} X$. Hence $a \in X_{00}$. By Lemma 3.2, for all $i = 2, \ldots, n$, $(a, z_i) \in T$ for some $T \in \operatorname{Corr} X$. It follows that each $z_i \in \operatorname{orbit}(a)$, and so $z_i \in X_0$, as desired. $\qquad \square$

In order capture algebraic closure in $\mathcal{L}_{\mathrm{GC}}$ we need to also consider the following exceptional points: For each $S \in \operatorname{Corr} X$ the second co-ordinate projection, say $\pi : S \to X$, is finite-to-one and is thus generically m-to-one. Let E_S be the finite subset of X over which the fibres of π are of cardinality different from m. Let
$$E := \bigcup_{S \in \operatorname{Corr} X} E_S.$$
The following is the key step in proving that any two elements outside $X_0 \cup E$ have the same $\mathcal{L}_{\mathrm{GC}}$-type.

Proposition 3.10. *Suppose* $\operatorname{Corr} X$ *is countable. For all* $a, b \in X \setminus (X_0 \cup E)$, *there is a bijection* $\phi : \operatorname{orbit}(a) \to \operatorname{orbit}(b)$ *such that* $\phi(a) = b$, *and for any generalised correspondence* $A \subseteq X^n$ *and* $x_1, \ldots, x_n \in \operatorname{orbit}(a)$, $(x_1, \ldots, x_n) \in A$ *if and only if* $\big(\phi(x_1), \ldots, \phi(x_n)\big) \in A$.

Proof. Let $(S_i : i < \omega)$ be an enumeration of $\operatorname{Corr} X \setminus \{\Delta\}$.

Let $\mathcal{A}'$ be a sufficiently saturated elementary extension of $\mathcal{A}$, and denote by $X(\mathcal{A}')$ the interpretation of X in the extension. Fix a generic point of X in $\mathcal{A}'$ – that is a point $c \in X(\mathcal{A}') \setminus X$. Let

$$\operatorname{orbit}(c) := \{x \in X(\mathcal{A}') : (x, c) \in S(\mathcal{A}') \text{ for some } S \in \operatorname{Corr} X\}$$

We will show that there is a bijection $\phi : \operatorname{orbit}(a) \to \operatorname{orbit}(c)$ such that $\phi(a) = c$, and for any generalised correspondence $A \subseteq X^n$ and $x_1, \ldots, x_n \in \operatorname{orbit}(a)$, $(x_1, \ldots, x_n) \in A$ if and only if $\big(\phi(x_1), \ldots, \phi(x_n)\big) \in A(\mathcal{A}')$. Applying this to b also, will prove the proposition.

Let $(c_j : 1 \le j < \omega)$ be an enumeration of $\bigcup_{i < \omega} S_i(\mathcal{A}')_c$, where $S_i(\mathcal{A}')_c := \{x \in X(\mathcal{A}') : (x, c) \in S_i(\mathcal{A}')\}$. Moreover, assume the enumeration is coherent in the sense that for some increasing sequence $\ell_0 < \ell_1 < \cdots$, $(c_1, \ldots, c_{\ell_r})$ is an enumeration of $\bigcup_{i \le r} S_i(\mathcal{A}')_c$. For each $r < \omega$ let $A_r := \operatorname{loc}(c, c_1, \ldots, c_{\ell_r}) \subseteq X^{\ell_r + 1}$.

Claim 3.11. *Each* A_r *is a generalised correspondence.*

Proof of 3.11. That A_r is irreducible and complex-analytic is by definition. Note that for each j, $c \in \operatorname{acl}(c_j)$ since $(c_j, c) \in S(\mathcal{A}')$ for some $S \in \operatorname{Corr} X$. Hence the genericity of c implies the genericity of each c_j. It follows that the co-ordinate projections of A_r are all onto X. Finally, it is $\dim(X)$-dimensional because each $c_j \in \operatorname{acl}(c)$ and so the first co-ordinate projection is generically finite-to-one. $\qquad \square$

The coherence of the enumeration yields a direct system of surjective maps $X \leftarrow A_0 \leftarrow A_1 \leftarrow \cdots$ given by the natural initial segment co-ordinate projections. We can therefore find $(a_j : 1 \le j < \omega)$ such that for all r, $(a, a_1, \ldots, a_{\ell_r}) \in A_r$.

Claim 3.12. $(a_j : 1 \le j < \omega)$ *enumerates* $\operatorname{orbit}(a) \setminus \{a\}$.

Proof of 3.12. Note that $\mathrm{orbit}(a) \setminus \{a\}$ is the increasing union of the sets $\bigcup_{i=0}^{r}(S_i)_a$, as r goes to infinity. (Note that a is not in any of the latter as $a \notin X_0$.) Hence it suffices to show, fixing r, that $(a_1, \ldots, a_{\ell_r})$ enumerates $\bigcup_{i=0}^{r}(S_i)_a$.

By construction, for each $j \leq \ell_r$, $(c_j, c) \in S_i(\mathcal{A}')$ for some $i \leq r$. By genericity of c, it follows that the projection to the $(1+j, 1)$-co-ordinate of $A_r = \mathrm{loc}(c, c_1, \ldots, c_{\ell_r})$ has infinite intersection with S_i. But as A_r is a generalised correspondence, this projection is itself a finite-to-finite correspondence (see Lemma 3.2), so that it must equal S_i. So $(a_j, a) \in S_i$. It suffices to show therefore that ℓ_r is the cardinality of $\bigcup_{i=0}^{r}(S_i)_a$. By construction ℓ_r is the cardinality of $\bigcup_{i=0}^{r} S_i(\mathcal{A}')_c$. But note that for $i \neq j$, $S_i \cap S_j$ is finite and so $S_i(\mathcal{A}')_c \cap S_j(\mathcal{A}')_c = \emptyset$. Hence,

$$
\begin{aligned}
\ell_r &= \Big| \bigcup_{i=0}^{r} S_i(\mathcal{A}')_c \Big| \\
&= \sum_{i=0}^{r} |S_i(\mathcal{A}')_c| \\
&= \sum_{i=0}^{r} m_i
\end{aligned}
$$

where m_i is the cardinality of the general fibres of the second co-ordinate projection on S_i. On the other hand, for $i \neq j$, $S_i \cap S_j \subseteq X_0^2$ by Lemma 3.9, and so as $a \notin X_0$, $(S_i)_a \cap (S_j)_a = \emptyset$ also. Hence

$$
\Big| \bigcup_{i=0}^{r}(S_i)_a \Big| = \sum_{i=0}^{r} |(S_i)_a| = \sum_{i=0}^{r} m_i
$$

since $a \notin E$ implies that $(S_i)_a$ has the typical size m_i. Hence $\ell_r = \Big| \bigcup_{i=0}^{r}(S_i)_a \Big|$. $\quad\square$

Since $(c_0 := c, c_j : 1 \leq j < \omega)$ enumerates $\mathrm{orbit}(c)$ by construction, and now we know that $(a_0 := a, a_j : 1 \leq j < \omega)$ enumerates $\mathrm{orbit}(a)$, all that remains to be proved is that if $A \subseteq X^n$ is any generalised correspondence, and $i_1, \ldots, i_n < \omega$ are arbitrary, then $(a_{i_1}, \ldots, a_{i_n}) \in A$ if and only if $(c_{i_1}, \ldots, c_{i_n}) \in A(\mathcal{A}')$. Fix r so that $i_1, \ldots, i_n$ are all $\leq \ell_r$. For the right-to-left direction, note that $(c_{i_1}, \ldots, c_{i_n}) \in A(\mathcal{A}')$ implies that the $(i_1, \ldots, i_n)$-co-ordinate projection of A_r has infinite intersection with A, and so as both are generalised correspondences, must equal A. Hence, $(a_{i_1}, \ldots, a_{i_n}) \in A$. Conversely, suppose $(c_{i_1}, \ldots, c_{i_n}) \notin A(\mathcal{A}')$, $(a_{i_1}, \ldots, a_{i_n}) \in A$, and seek a contradiction. Then the $(i_1, \ldots, i_n)$-co-ordinate projection of A_r, say $B \subseteq X^n$, is a generalised correspondence, different from A. By Lemma 3.9, $A \cap B \subseteq X_0^n$, so that $a_{i_1} \in X_0$. Hence $a \in \mathrm{orbit}(a_{i_1}) \subseteq X_0$. This contradiction proves that $(a_{i_1}, \ldots, a_{i_n}) \notin A$, and thus completes the proof of Proposition 3.10. $\quad\square$

Putting 3.7 and 3.10 together, we obtain the desired characterisation of algebraic closure in $(X, \mathcal{L}_{\mathrm{GC}})$.

Proposition 3.13. *If $\mathrm{Corr}\, X$ is countable, then $\mathrm{acl}(\emptyset) \cap X = X_0 \cup E$ in $(X, \mathcal{L}_{\mathrm{GC}})$.*

Proof. For the right-to-left containment, suppose $a \in X_0$. Then by definition $a \in$ orbit(b) for some $b \in X_{00}$. But then $(b, b) \in S \cap \Delta$ for some $\Delta \neq S \in \text{Corr } X$. As this intersection must be finite, $b \in \text{acl}(\emptyset)$. Now $a \in \text{orbit}(b)$ implies that $a \in \text{acl}(b) \subseteq \text{acl}(\emptyset)$, as desired. On the other hand, if $a \in E$ then $a \in E_S$ for some $S \in \text{Corr } X$, and E_S, the set of points over which the fibre of the second co-ordinate projection on S is not the generic value, is a finite 0-definable set in $(X, \mathcal{L}_{\text{GC}})$. Hence $X_0 \cup E$ is contained in $\text{acl}(\emptyset)$.

To prove the left-to-right direction we take $a, b \in X \setminus (X_0 \cup E)$ and show that they have the same type in $(X, \mathcal{L}_{\text{GC}})$. We do this by exhibiting an automorphism of $(X, \mathcal{L}_{\text{GC}})$ taking a to b. By Proposition 3.7, X is partitioned into disjoint orbits. Let $\phi : \text{orbit}(a) \to \text{orbit}(b)$ be the bijection given by Proposition 3.10. We define $\sigma : X \to X$ to be the permutation that is the identity on every orbit except orbit(a) and orbit(b), $\sigma|_{\text{orbit}(a)} = \phi$, and $\sigma|_{\text{orbit}(b)} = \phi^{-1}$. To show that σ is an $\mathcal{L}_{\text{GC}}$-automorphism we need to show that it preserves all the generalised correspondences. Let $A \subseteq X^n$ be a generalised correspondence, and $x = (x_1, \ldots, x_n) \in A$. Then all the x_i's are in the same orbit. If that orbit is orbit(a) or orbit(b) then Proposition 3.10 implies that $\sigma(x) \in A$. If not, then $\sigma(x) = x \in A$. $\qquad \square$

Now the characterisation of essential $\aleph_0$-categoricity goes through:

Theorem 3.14. *The following are equivalent.*

(i) $\text{Corr } X$ *is finite.*
(ii) X *is essentially saturated and* $(X, \mathcal{L}_{\text{GC}})$ *is* $\aleph_0$*-categorical.*
(iii) X *is essentially saturated and* $(X, \mathcal{L}_{\text{Douady}})$ *is* $\aleph_0$*-categorical.*
(iv) X *is essentially* $\aleph_0$*-categorical.*

Proof. The proof is analogous to the simply connected case (Theorem 2.6).

(i) $\Longrightarrow$ (ii). Assume $\text{Corr } X$ is finite. First observe that $\text{acl}(\emptyset) \cap X$ is finite. Indeed, by Proposition 3.13, $\text{acl}(\emptyset) \cap X = X_0 \cup E$. Since $\text{Corr } X$ is finite and the intersection of distinct finite-to-finite correspondences are finite, X_{00} is finite. But every orbit is also finite. Hence X_0 is finite. On the other hand as each E_S is finite and $\text{Corr } X$ is finite, E is also finite.

Also, by Lemma 3.2 there are only finitely many generalised correspondences in X^n, for each $n > 0$.

Now, if $A \subseteq X^n$ is irreducible complex-analytic, then (after a permutation of co-ordinates) A is a product of generalised correspondences and singletons – this is Lemma 3.3(b). By saturation of $(X, \mathcal{L}_{\text{GC}})$ – which holds because of Proposition 3.5 – one can use automorphisms to show that if A is $\text{acl}(\emptyset)$-definable in $(X, \mathcal{L}_{\text{GC}})$, then the singletons that appear must come from $\text{acl}(\emptyset) \cap X$. Hence there are only finitely many $\text{acl}(\emptyset)$-definable irreducible complex-analytic subsets of X^n in $(X, \mathcal{L}_{\text{GC}})$, for all $n > 0$. Exactly as in Theorem 2.6, this implies $\aleph_0$-categoricity.

(ii) $\Longrightarrow$ (iii). This is because $\aleph_0$-categoricity implies triviality, and then we know that $(X, \mathcal{L}_{\text{Douady}})$ is inter-0-definable with $(X, \mathcal{L}_{\text{GC}})$ by Proposition 3.5.

(iii) $\Longrightarrow$ (iv). Clear.

(iv) $\Longrightarrow$ (i). Let $\mathcal{L}$ be a full countable language for X such that $(X, \mathcal{L})$ is $\aleph_0$-categorical. Again we conclude that X has trivial geometry and hence every finite-to-finite correspondence is $\text{acl}(\emptyset)$-definable in $(X, \mathcal{L})$. So, by $\aleph_0$-categoricity, $\text{Corr } X$ is finite. $\qquad \square$

In conclusion let us discuss some possible extensions and generalisations. First of all, we expect the equivalence of (iii) and (iv) – that is, the robustness of $\aleph_0$-categoricity – to hold for arbitrary compact complex varieties, and not just for smooth strongly minimal ones. In fact, the right setting in which to investigate $\aleph_0$-categoricity in bimeromorphic geometry seems to be what we might call *meromorphic varieties*: Zariski open subsets of compact complex varieties. On the other hand, it also makes sense to leave the complex-analytic context altogether and ask whether (i), (ii), and (iv) are equivalent for strongly minimal *complete pre-smooth ω_1-compact Zariski structures* in the sense of Zilber (see [11]). In fact, much of what we have done here works in that setting; the only part of our argument that does not immediately extend is Lemma 3.8 where we made essential use of the existence of normalisations and the purity of branch theorem.

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