A MODEL THEORY FOR MEROMORPHIC VECTOR FIELDS

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Dedicated to Boris Zilber on the occasion of his 75th birthday

Abstract. Motivated by the study of meromorphic vector fields, a model theory of "compact complex manifolds equipped with a generic derivation" is here proposed. This is made precise by the notion of a *differential* CCM-*structure*. A first-order axiomatisation of existentially closed differential CCM-structures is given. The resulting theory, DCCM, is a common expansion of the theories of differentially closed fields and compact complex manifolds. A study of the basic model theory of DCCM is initiated, including proofs of completeness, quantifier elimination, elimination of imaginaries, and total transcendentality. The finite-dimensional types in DCCM are shown to be precisely the generic types of meromorphic vector fields.

1. Introduction

The model-theoretic approach to systems of (ordinary) algebraic differential equations is via the first-order theory of differentially closed fields in characteristic zero (DCF_0). Such systems of equations, at least in the autonomous case when the equations have constant parameters, can be presented geometrically as algebraic vector fields; namely a projective algebraic variety X equipped with a rational section $v : X \to TX$ to the tangent space. In fact, the finite-dimensional fragment of DCF_0 essentially coincides with the birational geometry of algebraic vector fields. (See, for example, [13] for an exposition of DCF_0 from this point of view.) Here, I am interested in generalising this model-theoretic framework to *meromorphic* vector fields; namely when X is a compact complex-analytic space that is not necessarily algebraic and v is a meromorphic section to the holomorphic tangent bundle. While DCF_0 is built on the theory of algebraically closed fields (ACF_0), the new theory I am seeking should be built on a first-order theory of compact complex manifolds.

About thirty years ago, in [18], as part of the development of the notion of "Zariski-type structure", Zilber proposed a model theory for compact complex manifolds. Unlike ACF_0 and DCF_0, the first-order theory proposed by Zilber for compact complex manifolds was not given by an explicit axiomatisation, nor as the model companion of a natural class of algebraic structures, but rather as theories of particular structures: a compact complex manifold M is viewed as a first-order structure in the language where there is a predicate for each closed complex-analytic subset of each finite cartesian power of M. Zilber showed that the theory of any such structure shares many properties with its algebraic predecessors: in particular, they admit quantifier elimination and are of finite Morley rank (bounded by the dimension of M). Later, in the work of Hrushovski [5] and Pillay [15], for example,

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it became common to consider all compact complex manifolds, indeed all (reduced) compact complex-analytic spaces, at once, in a multisorted structure whose theory now goes by the name CCM. Like differentially closed fields, CCM is a proper expansion of ACF_0. Also like DCF_0, much of the richness of geometric stability theory absent in ACF_0 is present in CCM. For example, all cases of the Zilber trichotomy appear.

In this paper, I will present a common expansion of CCM and DCF_0, which I will call DCCM. It will turn out (in Section 8, below) that the finite-dimensional fragment of DCCM will capture, precisely, the bimeromorphic geometry of meromorphic vector fields. As such, it achieves the goal set out in this introduction.

The theory DCCM arises by considering *differential* CCM-*structures*, essentially by adding a "derivation" to the definable closure of a generic point of a sort, say X, in CCM. This can be made sense of because the elements of the definable closure of a generic point of X can be viewed as meromorphic maps from X to other sorts, and hence can be differentiated. See Sections 2 and 3 for a detailed explanation.

The specific goals of this paper are:

(1) to show that the (universal) theory of differential CCM-structures admits a model companion, which is DCCM, by giving a geometric first-order axiomatisation of the existentially closed models (Theorem 5.5);
(2) to show that DCCM is complete, admits quantifier elimination (Proposition 6.3) and elimination of imagainaries (Theorem 7.6), and to give a geometric characterisation of definable and algebraic closure (Proposition 6.5);
(3) to show that DCCM is totally transcendental (Theorem 7.5), and to give geometric characterisations of nonforking independence (Corollary 7.4); and,
(4) to establish the correspondence between finite-dimensional types (over the empty set) in DCCM and meromorphic vector fields (Theorem 8.3).

The proofs proceed largely by finding geometric analogues for the algebraic arguments already familiar from DCF_0.

The next step in the study of DCCM, not attempted here, would be to establish the canonical base property for finite dimensional types following the strategy of [16] in the case of DCF_0. This will involve developing a theory of jet spaces in DCCM, see for example [1] where this was done for compact complex manifolds with a generic automorphism (CCMA). In any case, once the canonical base property is established, a concrete manifestation of the Zilber dichotomy for finite dimensional minimal types in DCCM will follow. It would then be reasonable to expect that many of the recent applications of model theory to algebraic vector fields, as carried out in [4] and [6] for example, would extend to meromorphic vector fields.

The process of adding an automorphism to any given first-order theory of interest, and then seeking a model companion, is well-studied (see [2]). Here we have "added a derivation" instead. Clearly, this does not make sense for an arbitrary theory. But following the ideas presented here, it may be worth investigating a robust general setting where adding a derivation does make sense. A likely candidate might be that of Zariski-type structures in Zilber's sense; one that expands ACF_0 and admits a functor that extends to all sorts the tangent space construction on algebraic varieties.

2. Meromorphic varieties and their tangent spaces

In this section I want to slightly loosen the usual formalism for doing the model theory of compact complex-analytic spaces, so as to work directly in the "compactifiable" rather than compact setting.

For the fundamental notions from complex-analytic geometry we suggest [3]. Given a reduced compact complex-analytic space X, by the *Zariski topology* on X we will mean the (noetherian) topology of closed complex-analytic subsets of X. This will not conflict with the usual meaning of the Zariski topology in the case that X is a projective complex-algebraic variety, because in that case the complex-analytic and complex-algebraic sets agree (Chow's Theorem).

Definition 2.1. By a *meromorphic variety* we will mean a pair $(X, \overline{X})$ where X is a Zariski dense and open subset of a reduced compact complex-analytic space $\overline{X}$. Note that X inherits from $\overline{X}$ the structure of a reduced complex-analytic space in its own right, which may admit other compactifications.[1] We will usually abbreviate our notation by referring to X as the meromorphic variety, but it is important to keep in mind that we view X as embedded in a fixed given compactification $\overline{X}$.

Cartesian products of meromorphic varieties, $X \times Y$, are viewed as meromorphic varieties with the compactification $\overline{X \times Y} = \overline{X} \times \overline{Y}$.

By the *Zariski topology* on X we mean the topology induced by the Zariski topology on $\overline{X}$. Note that this is a coarser topology than that of the closed complex-analytic subsets of X; such a set is Zariski closed in X if and only if its (euclidean) closure in $\overline{X}$ is Zariski closed.

By a *definable holomorphic map* $f : X \to Y$ of meromorphic varieties we mean a holomorphic map that extends to a meromorphic map $\overline{f} : \overline{X} \to \overline{Y}$. Equivalently, the graph of f is Zariski closed in $X \times Y$. More generally, a *definable meromorphic map* $f : X \to Y$ is a meromorphic map that extends to a meromorphic map from $\overline{X}$ to $\overline{Y}$. Such a map is *dominant* if its image is Zariski dense in Y.

Meromorphic varieties, as I have defined them here, are intended to extend the notion of quasi-projective variety from the complex-algebraic to the complex-analytic setting. Indeed, the quasi-projective varieties are precisely the meromorphic varieties X where $\overline{X}$ is projective algebraic. Note that while every regular or rational function on a quasi-projective variety X extends to a rational function on the projective closure $\overline{X}$, the same is not true of holomorphic and meromorphic functions on meromorphic varieties, and this is why we restrict our attention to *definable* holomorphic and meromorphic maps, namely the ones that do so extend.

Remark 2.2. When $X = \overline{X}$ is compact, every holomorphic (respectively meromorphic) map to a meromorphic variety, $f : X \to Y$, is definable holomorphic (respectively definable meromorphic). This is because, by the Proper Mapping Theorem, the image of f in Y will be Zariski closed, and hence we can take $\overline{f}$ to be f itself, viewed as a map from X to $\overline{Y}$.

The usual model-theoretic set-up is to consider the first-order theory of the multi-sorted structure $\mathcal{A}$ where there is a sort for each reduced and irreducible compact complex-analytic space, and a predicate for each Zariski closed subset of each finite cartesian product of sorts. See, for example, the surveys [11] and [14].

[1]Thanks to the anonymous referee for pointing out that different compactifications of X need not be bimeromorphically equivalent.

Every meromorphic variety, in the above sense, is 0-definable in $\mathcal{A}$, as is every Zariski closed subset of every finite cartesian product of meromorphic varieties. It therefore does no harm to work instead with the expansion of $\mathcal{A}$ to the multi-sorted structure $\mathcal{M}$ where there is a sort for each irreducible meromorphic variety, and a predicate for each Zariski closed subset of each finite cartesian product of sorts. So we have added some sorts and some predicates, but they were all already 0-definable in the original structure. I will denote by L the language of $\mathcal{M}$, and by CCM the first-order L-theory of $\mathcal{M}$. It admits quantifier elimination and elimination of imaginaries, and, sort by sort, is of finite Morley rank.

Every quasi-projective complex-algebraic variety V, given with an embedding in a projective compactification $\overline{V}$, is a meromorphic variety, and the algebraic and analytic Zariski topologies on V agree. In particular, definable holomorphic maps in this case are just regular morphisms, and definable meromorphic maps are rational. In this way, algebraic geometry lives as a pure reduct of CCM.

Our main use of the flexibility that $\mathcal{M}$ affords is that the collection of sorts is closed under taking tangent spaces. Recall that the *tangent space* of a complex-analytic space X is the linear fibre space $\pi : TX \to X$ associated to the sheaf of differentials, $\underline{\Omega}^1_X$, on X. So TX is a complex-analytic space and $\pi : TX \to X$ is a surjective holomorphic map whose fibres are uniformly equipped with the structure of a complex vector space, in the sense that there are holomorphic maps for addition, $+ : TX \times_X TX \to TX$, scalar multiplication $\lambda : \mathbb{C} \times TX \to TX$, and zero section $z : X \to TX$, all over X, satisfying the vector space axioms. For any point $p \in X$, the *tangent space to* X *at* p is the fibre of $\pi : TX \to X$ above p, denoted by T_pX, and it is canonically isomorphic as a complex vector space to $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}_{X,p}/\mathfrak{m}^2_{X,p}, \mathbb{C})$, where $\mathfrak{m}_{X,p}$ is the maximal ideal of the local ring of X at p.

We claim that when X is a meromorphic variety so is TX, and that $\pi, +, \lambda, z$ are all definable holomorphic maps. Let us first consider the case when $X = \overline{X}$ is already compact. We are looking for a natural compactification of TX. In fact, there is a canonical way to do this for any linear fibre space $\mathcal{L}(\mathcal{F}) \to X$ associated to a coherent analytic sheaf $\mathcal{F}$ on X; it is just the relativisation of the usual embedding of $\mathbb{C}^n$ in the projectivisation of $\mathbb{C}^{n+1}$. One considers the coherent analytic sheaf $\mathcal{F} \times \mathcal{O}_X$ of rank one greater than $\mathcal{F}$, and then the associated projective linear space $\mathbb{P}(\mathcal{F} \times \mathcal{O}_X) \to X$. See [3, Section 1.9] for details. Then $\mathcal{L}(\mathcal{F})$ embeds in $\mathbb{P}(\mathcal{F} \times \mathcal{O}_X)$ over X as a Zariski open set in such a way that the linear structure (namely, $\pi, +, \lambda, z$) extends meromorphically to the projective linear space. Applying this to $\mathcal{F} = \underline{\Omega}^1_X$ gives $TX \to X$ the meromorphic structure we are looking for, namely $\overline{TX} := \mathbb{P}(\underline{\Omega}^1_X \times \mathcal{O}_X)$. Now, if we consider a general meromorphic variety X embedded in $\overline{X}$, then the linear space $TX \to X$ is just the restriction to X of $T\overline{X} \to \overline{X}$, hence $\overline{T\overline{X}}$ will serve as a compactification for TX, to which the linear structure extends meromorphically.

Remark 2.3. While we have not been assuming here that X is smooth, it is the case that the tangent space is better behaved and more familiar under that assumption. As we are only interested in the bimeromorphic structure, we can achieve smoothness by replacing X with its nonsingular locus. Note that the set of nonsingular points of X is of the form $X \cap U$ where U is the (Zariski dense and open) set of nonsingular points of the compactification $\overline{X}$. It follows that the nonsingular locus of X is again a meromorphic variety given with the same compactification $\overline{X}$.

Recall that the tangent space construction is functorial: For each meromorphic (respectively, holomorphic) map $g : X \to Y$ between complex-analytic spaces there is a meromorphic (respectively, holomorphic) map $dg : TX \to TY$, such that

$$\begin{tikzcd} TX \arrow[r, "dg"] \arrow[d, "\pi_X"'] & TY \arrow[d, "\pi_Y"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

commutes, and we have the functoriality property $d(g \circ h) = (dg) \circ (dh)$. If X and Y are meromorphic varieties, and $g : X \to Y$ is definable meromorphic (respectively, holomorphic), then so is $dg : TX \to TY$. That is, if g extends to a meromorphic map $\overline{X} \to \overline{Y}$ the dg extends to a meromorphic map $\overline{TX} \to \overline{TY}$.

3. The differential structure

By a CCM-*structure* I will mean a definably closed subset of a model of CCM. In other words, a model of $\mathrm{CCM}_\forall$. The goal of this section is to describe what we might consider a "derivation" on a CCM-structure. But first, let us recall what CCM-structures themselves look like.

Since we are in a relational language in which all elements of $\mathcal{M}$ are named, a model of $\mathrm{CCM}_\forall$ is simply a subset A of an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{M} \subseteq A$. As we are in a multi-sorted setting, this is meant relative to every sort: so $S(\mathcal{M}) \subseteq S(A)$ for all sorts S of L. But we will mostly be interested in finitely generated definably closed substructures, so where $A = \mathrm{dcl}(a)$ for some $a \in X(\mathcal{N})$ and some irreducible meromorphic variety X. Replacing X by the locus of a, we may assume that a is a generic point of X in the sense that it is not contained in $Y(\mathcal{N})$ for any proper Zariski closed subset $Y \subsetneq X$. In that case we can identify A with the set of all definable meromorphic maps $g : X \to S$ as S ranges over all other sorts. Indeed the identification is given by $g \mapsto g(a) \in S(A)$, noting that every point of $S(A)$ arises this way as $A = \mathrm{dcl}(a)$, and that if two definable meromorphic maps agree on a then they agree on X by genericity.

It is worth comparing to the algebraic case, so when X happens to be a quasi-projective complex-algebraic variety. In that case one only needs to consider the single target sort $S = \mathbb{P}$, the projective line. Indeed, in that case, $\mathrm{dcl}(a) = \mathbb{C}(X)$ is just the field of rational functions. For nonalgebraic meromorphic varieties, if we only considered $S = \mathbb{P}$ we would obtain the *meromorphic function field* of X, and not necesarilly the full definable closure of a generic point. Indeed, on some compact complex-analytic spaces, namely those of *algebraic dimension* 0, there are no nonconstant meromorphic functions, but many nonconstant meromorphic maps to other sorts.

The differential structure I want to consider is motivated by the study of the following natural objects in bimeromorphic geometry:

Definition 3.1. By a *meromorphic vector field* we will mean an irreducible meromorphic variety X equipped with a definable meromorphic section $v : X \to TX$ to the tangent space of X.

Remark 3.2. When $X = \overline{X}$ is compact, "definable" is redundant and a meromorphic vector field is simply a meromorphic section to the tangent space – see Remark 2.2.

So this notion does generalise what I called a meromorphic vector field in the Introduction. However, as we are only interested in the bimeromorphic geometry, it is not much of a generalisation: we can always pass from (X, v) to $(\overline{X}, \overline{v})$.

Of course, every meromorphic variety equipped with its zero section is a meromorphic vector field, which we call the *trivial vector field*.

Every *(rational) algebraic vector field*, by which we mean an irreducible quasi-projective complex-algebraic variety equipped with a rational section to the tangent space, is a meromorphic vector field. Indeed, these are the only meromorphic vector fields on algebraic varieties. In particular, as all compact complex-analytic spaces of dimension 1 are projective algebraic curves, every 1-dimensional meromorphic vector field is algebraic.

We already get nonalgebraic examples in dimension 2. As is pointed out in [17, Example 2], for instance, all elliptic surfaces admit interesting meromorphic vector fields. Since there are nonalgebraic elliptic surfaces (every compact complex surface of algebraic dimension 1 is such), this is a class of nontrivial meromorphic vector fields that are not algebraic. These examples also show that meromorphic vector fields can be ubiquitous in situations where no holomorphic ones exist.

But there are also nonalgebraic holomorphic vector fields. Suppose $X = \overline{X}$ is compact and $G = \mathrm{Aut}_0(X)$ is the connected component of the automorphism group of X. Then G is a complex Lie group whose Lie algebra consists precisely of the holomorphic vector fields on X, see [9, III.1]. It follows that if $X = \overline{X}$ is nonalgebraic and $\mathrm{Aut}_0(X)$ is positive dimensional, then X admits many nontrivial and nonalgebraic holomorphic (and hence meromorphic) vector fields. So, for example, if X is any complex torus, then $X = \mathrm{Aut}_0(X)$ acting by translation, and hence each point of the Lie algebra of X gives rise to an (invariant) holomorphic vector field on X.

Finally, it is worth noting, and was pointed out to me by the anonymous referee, that, unlike in the algebraic case, there are compact complex manifolds that admit no nontrivial meromorphic vector fields. For example, suppose X is a generic K3 surface. If X did admit a meromorphic vector field, v, then, as X has no proper infinite closed analytic subsets, the indeterminacy locus of v would be finite, and so, by Hartogs' theorem, v would extend to a holomorphic vector field on X. But K3 surfaces do not admit any nontrivial global holomorphic vector fields.

Suppose (X, v) is a meromorphic vector field, $\mathcal{N} \succeq \mathcal{M}$ is an elementary extension, and $a \in X(\mathcal{N})$ is a generic point of X. What structure does v induce on $A := \mathrm{dcl}(a)$? Well, for any definable meromorphic $g : X \to S$, we have the definable meromorphic map $\nabla_v(g) := dg \circ v : X \to TS$. Viewing $g \in S(A)$ we have defined a function $\nabla_v : S(A) \to TS(A)$, for all sorts S. Here are two salient properties of this function that are easily verified using the functoriality of the tangent space construction:

- $\pi \circ \nabla_v(g) = g$ where $\pi : TS \to S$ is the projection.
- $df \circ \nabla_v(g) = \nabla_v(f \circ g)$ for any definable meromorphic $f : S \to T$.

We are thus lead to consider the following notion:

Definition 3.3. Let $L_\nabla = L \cup \{\nabla\}$ where $\nabla = (\nabla_S : S \text{ sort of } L)$ and ∇_S is a function symbol from the sort S to the sort TS. Let $\mathrm{CCM}_{\forall,\nabla}$ denote the universal L_∇-theory which is obtained by adding to $\mathrm{CCM}_\forall$ the following axioms:

(1) For each sort S, $\nabla_S : S \to TS$ is a section to $\pi : TS \to S$.

(2) For each definable meromorphic map $f : S_1 \to S_2$ between sorts, the following diagram commutes:

$$\begin{tikzcd} TS_1 \arrow[r, "df"] & TS_2 \\ S_1 \arrow[u, "\nabla_{S_1}"] \arrow[r, "f"] & S_2 \arrow[u, "\nabla_{S_2}"'] \end{tikzcd}$$

Remembering that f and df are not function symbols in the language but rather their graphs are predicates, what we mean by this is the axiom

$$\forall xy\big((x,y) \in \Gamma(f) \implies (\nabla_{S_1} x, \nabla_{S_2} y) \in \Gamma(df)\big).$$

We will usually drop the subscript and write ∇ for ∇_S whenever it is clear from context which sort we are working in.

One consequence of Axiom (2) that gets used often without mention is that $\nabla(a_1, a_2) = (\nabla a_1, \nabla a_2)$ under the identification $T(S_1 \times S_2) = TS_1 \times TS_2$.

We can always extend uniquely to the definable closure:

Proposition 3.4. *Suppose $A \subseteq \mathcal{N} \models \mathrm{CCM}$ and $(A, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$. Then there is a unique extension of ∇ to $\mathrm{dcl}(A)$ making it a model of $\mathrm{CCM}_{\forall,\nabla}$.*

Proof. Let $B := \mathrm{dcl}(A)$. Given a sort S we need to define ∇ on $S(B)$. Fix $b \in S(B)$ and let $X := \mathrm{loc}(b) \subseteq S$ so that $b \in X(B)$ is generic. Since $b \in \mathrm{dcl}(A)$, there exists some other irreducible meromorphic variety Y admitting a dominant definable meromorphic map $f : Y \to X$, and a generic point $a \in Y(A)$, such that $b = f(a)$. Now, $df : TY \to TX$ and $\nabla(a) \in TY(A)$. Define $\nabla(b) := df(\nabla a)$. Indeed, this is forced upon us by Axiom (2) of Definition 3.3, and hence takes care of the uniqueness part of the statement.

We have to check that it is well-defined. Suppose we have another $f' : Y' \to X$ and $a' \in Y'(A)$ generic such that $b = f'(a')$ as well. Let $Z = \mathrm{loc}(a, a') \subseteq Y \times Y'$ and consider $\bar{f} := (f, f') : Z \to X^2$. Since $\bar{f}$ takes a generic point of Z to the diagonal $D \subseteq X^2$ we have that $\bar{f}(Z) \subseteq D$. Hence $d\bar{f} : TZ \to T(X^2)$ lands in TD which is the diagonal in $T(X^2) = (TX)^2$. Since $d\bar{f}(\nabla(a, a')) = (df(\nabla a), df'(\nabla a'))$ by functoriality, this means that $df(\nabla a) = df'(\nabla a')$, as desired.

Next, observe that ∇ so defined is a function from $S(B)$ to $TS(B)$, and is a section to $\pi : TS \to S$. That is, (B, ∇) does satisfy Axiom (1) of Definition 3.3. Taking $f = \mathrm{id}$ in the above construction, we see also that $(A, \nabla) \subseteq (B, \nabla)$.

It remains to verify Axiom (2). That is, given $g : S_1 \to S_2$ a definable meromorphic map between sorts, and $b_i \in S_i(B)$ with $g(b_1) = b_2$, we need to show that $dg(\nabla b_1) = \nabla b_2$. Note that by concatenating – namely working in cartesian products – we can arrange things so that b_1 and b_2 are defined over the same tuple from A. That is, there is a sort S with $a \in S(A)$ such that $b_1 = f_1(a)$ and $b_2 = f_2(a)$ where $f_i : S \to S_i$ are definable meromorphic maps. Taking Zariski loci we may assume that a is generic in S and that each b_i is generic in S_i. Hence

$$\begin{aligned} dg(\nabla b_1) &= dg(df_1(\nabla a)) && \text{by how } \nabla \text{ is defined on } B \\ &= d(gf_1)(\nabla a)) && \text{by functoriality} \\ &= df_2(\nabla a)) && \text{as } gf_1 = f_2 \text{, as that is the case on the generic } a \\ &= \nabla b_2 && \text{by how } \nabla \text{ is defined on } B, \end{aligned}$$

as desired. $\square$

Definition 3.5. A *differential* CCM-*structure* is a model $(A, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$ such that $A = \mathrm{dcl}(A)$.

As a consequence of Proposition 3.4, when working with models of $\mathrm{CCM}_{\forall,\nabla}$ there is little loss of generality in assuming that we have a differential CCM-structure, namely that the underlying set is definably closed in CCM.

It is worth observing that standard points are always constant:

Lemma 3.6. *Suppose (A, ∇) is a differential* CCM-*structure and S is a sort. If $p \in S(\mathcal{M})$ then $\nabla(p) = 0 \in T_pX$.*

Proof. Note that $X := \{p\}$ is itself an irreducible meromorphic variety, and we can consider the containment as a definable holomorphic map $f : X \to S$. Now $TX = \{(p, 0)\}$, and hence $\nabla_X = 0$. But, by Axiom (2) of Definition 3.3, this forces $\nabla_S(p) = df(\nabla_X(p)) = 0$ as $df_p : T_pX \to T_{f(p)}S$ is a linear map. □

In the finitely dcl-generated case we recover precisely the meromorphic vector fields that motivated Definition 3.3:

Proposition 3.7. *Suppose X is an irreducible meromorphic variety, a is a generic point of X in some elementary extension, and $A = \mathrm{dcl}(a)$. Then the differential* CCM-*structures on A are precisely the ∇_v induced by meromorphic vector fields $v : X \to TX$.*

Proof. We have already seen that $(A, \nabla_v) \models \mathrm{CCM}_{\forall,\nabla}$ if (X, v) is a meromorphic vector field. For the converse, suppose $(A, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$. Note that $a \in X(A)$ and $\nabla(a) \in TX(A)$. As definable meromorphic maps, $a \in X(A)$ is the identitiy map on X and $\nabla(a) \in TX(A)$ is some $v : X \to TX$. Axiom (1) ensures that v is a section to $\pi : TX \to X$, and hence a meromorphic vector field on X. It remains to verify that $\nabla = \nabla_v$. Let $g(a) \in S(A)$ where $g : X \to S$ is a definable meromorphic map and S is a sort. Then

$$\nabla_v(g(a)) = dg \circ v(a) = dg \circ \nabla(a) = \nabla(g(a))$$

where the final equality is by Axiom (2). □

Note that Proposition 3.7 extends to meromorphic varieties the (well known) correspondence, in the case when X is quasi-projective algebraic, between $\mathbb{C}$-linear derivations on $\mathbb{C}(X)$ and rational vector fields on X.

So the study of meromorphic vector fields amounts to the study of (finitely generated) differential CCM-structures. In the usual model-theoretic way, we will eventually look for a model companion; a theory that axiomatises the existentially closed differential CCM-structures.

We conclude this section by extending the notion of differential CCM-structure to a setting where ∇ is allowed to take values in an extension. This will be useful in what follows.

Definition 3.8. Suppose $\mathcal{N} \models \mathrm{CCM}$ and $A \subseteq \mathcal{N}$ is a definably closed set. By an *$\mathcal{N}$-valued differential* CCM-*structure* on A we mean a map $\nabla : S(A) \to TS(\mathcal{N})$, for every sort S, such that ∇ is a section to $\pi : TS \to S$, and such that $df(\nabla a) = \nabla(f(a))$ for all $a \in S(A)$ and all definable meromorphic maps f.

4. Prolongations

In this section we construct a version of the tangent space that is twisted by a differential structure. Since differential structure only has content in proper elementary extensions of $\mathcal{M}$, this will necessarily be about "meromorphic varieties over parameters" in arbitrary models of CCM, which we begin by reviewing.

Fix a model $\mathcal{N} \models \mathrm{CCM}$. Given an irreducible meromorphic variety X, we view it as a sort of L consider its $\mathcal{N}$-points $X(\mathcal{N})$. Let us recall the Zariski topology on $X(\mathcal{N})$ with parameters from $\mathcal{N}$, sometimes referred to as the *nonstandard* Zariski topology to emphasise that we are not necessarily in the prime model $\mathcal{M}$. See [10, Section 2] for a more detailed discussion. Every Zariski closed subset $Y \subseteq X(\mathcal{M})$ is named as a predicate in L and so we can consider $Y(\mathcal{N})$. These are the 0-definable Zariski closed subsets of $X(\mathcal{N})$. More generally, given a set of parameter $A \subseteq \mathcal{N}$, a Zariski closed subset $Y \subseteq X(\mathcal{N})$ over A is a subset of the form $Y = Z_a$ where $a \in S(A)$ is a generic point of another sort S and $Z \subseteq S \times X$ is a (0-definable) Zariski closed subset that projects dominantly on S. In diagrams:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\rho"'] & S \times X \\ S & \end{tikzcd}$$

That is, $Y = Z_a$ arises as the generic member of a 0-definable family of Zariski closed subsets of X. This forms a noetherian topology on $X(\mathcal{N})$. If Y is A-irreducible then we can take Z to be irreducible, and if Y is absolutely irreducible the we can take Z so that $\rho : Z \to S$ is a fibre space, meaning its general fibres in the standard model are irreducible.

The general standard fibres of $\rho : Z \to S$ will be of constant dimension when Z is irreducible, giving rise to a notion of dimension for irreducible Zariski closed subsets of $X(\mathcal{N})$, which we denote by $\dim Y$.

The tangent space construction extends to nonstandard Zariski closed sets. Fix $Y = Z_a$ as above. Then the tangent spaces of the fibres of ρ in the standard model vary uniformly: Consider the following diagram:

$$\begin{tikzcd} S \times X & Z \arrow[l, hook'] \arrow[d, "\rho"'] & TZ \arrow[l] \arrow[r, hook] \arrow[d, "d\rho"] & TS \times TX \\ & S & TS \arrow[l] & \end{tikzcd}$$

and let $z : S \to TS$ be the zero section. For any $p \in S(\mathcal{M})$, the fibre $(TZ)_{z(p)} \subseteq TX$ of $d\rho$ above $z(p)$ is nothing other than $T(Z_p)$, the tangent space of $Z_p \subseteq X$. Hence we define the *tangent space of* $Y = Z_a$ in $\mathcal{N}$, denoted TY, to be $(TZ)_{z(a)}$.

Suppose, now, that $A = \mathrm{dcl}(A)$ and we have a $\mathcal{N}$-valued differential CCM-structure ∇ on A. Then, instead of considering the zero section, we can consider the differential section ∇. That is, since $\nabla(a) \in TS(\mathcal{N})$, we can consider the fibre $(TZ)_{\nabla(a)} \subseteq TX(\mathcal{N})$ of $d\rho$ over $\nabla(a)$. We define this to be the *prolongation space of* $Y = Z_a$, and denote it by τY. That is, $\tau Y := (TZ)_{\nabla(a)}$.

Lemma 4.1. *The above definition of τY depends only on Y and not on the presentation of Y as Z_a.*

Proof. Suppose Y also appears as Z'_b for some 0-definable Zariski closed $Z' \subseteq S' \times X$ with $b \in S'(A)$ generic. Replacing b with (a, b), we may assume that there is a dominant definable meromorphic map $f : S' \to S$ with $f(b) = a$, and that $Z' \subseteq Z \times_S S'$. Hence $f' := (f, \mathrm{id}_X) \restriction_{Z'} : Z' \to Z$ restricts to the identity on $Z'_b = Y = Z_a$. Moreover, we have

$$\begin{tikzcd} Z \arrow[d, "\rho"'] & Z' \arrow[l, "f'"'] \arrow[d, "\rho'"] \\ S & S' \arrow[l, "f"'] \end{tikzcd}$$

which yields

$$\begin{tikzcd} TZ \arrow[d, "d\rho"'] & TZ' \arrow[l, "df'"'] \arrow[d, "d\rho'"] \\ TS & TS' \arrow[l, "df"'] \end{tikzcd}$$

Since (A, ∇) is a differential CCM-structure, $df(\nabla b) = \nabla(a)$, so that

$$df'_{\nabla(b)} : (TZ')_{\nabla(b)} \to (TZ)_{\nabla(a)}$$

Since $df' = (df, \mathrm{id}_{TX}) \restriction_{TZ'}$, this shows that $(TZ')_{\nabla(b)} = (TZ)_{\nabla(a)}$, as desired. □

Remark 4.2. Given two such nonstandard Zariski closed sets Y_1, Y_2, there is a natural identification of $T(Y_1 \times Y_2)$ with $T(Y_1) \times T(Y_2)$ induced by the corresponding indentification for (standard) meromorphic varieties. Moreover, if (A, ∇) is an ($\mathcal{N}$-valued) differential CCM-structure over which Y_1, Y_2 are defined, then we also have an identification of $\tau(Y_1 \times Y_2)$ with $\tau(Y_1) \times \tau(Y_2)$.

We denote the restriction of $\pi : TX \to X$ to τY also as $\pi : \tau Y \to Y$, and it is canonically attached to the prolongation space. For any $b \in Y$, we denote the fibre by $\tau_b Y$, and call it the *prolongation space to Y at b*. Note that if Y is an a-definable Zariski closed subset of X then τY is a $\nabla(a)$-definable Zariski closed subset of TX and $\tau_b Y$ is a $\nabla(a)b$-definable Zariski closed subset of $T_b X$.

Lemma 4.3. *Suppose (B, ∇) is a $\mathcal{N}$-valued differential* CCM*-structure extending (A, ∇), and $b \in Y(B)$. Then $\nabla(b) \in \tau_b Y$.*

Proof. Since $b \in X(B)$ we must have $\nabla(b) \in T_b X$. So it remains to verify that $\nabla(b) \in \tau Y$. Write $Y = Z_a$ as above. Then $(\nabla(a), \nabla(b)) = \nabla(a, b) \in TZ(\mathcal{N})$. In particular, $\nabla(b) \in (TZ)_{\nabla(a)}$ which is τY by construction. □

Lemma 4.4. *If Y is A-irreducible and $b \in Y$ is generic over A then $\tau_b Y$ is absolutely irreducible and* $\dim(\tau_b Y) = \dim Y$*.*

Proof. Let a from A be such that $Y = Z_a$ with $Z = \mathrm{loc}(a, b) \subseteq S \times X$ as above. Because $\rho : Z \to S$ is dominant, $d\rho$ restricts to a surjective $\mathbb{C}$-linear map between the tangent spaces at standard general points. Hence, at the generic point in $\mathcal{N}$, we have that $d\rho_{(a,b)} : T_{(a,b)} Z \to T_a S$ is a surjective $\mathbb{C}(\mathcal{N})$-linear map, where $\mathbb{C}(\mathcal{N})$ is the interpretation in $\mathcal{N}$ of the complex field, itself an algebraically closed field extending $\mathbb{C}$. By definition, the tangent space $T_b Y$ is the kernel of $d\rho_{(a,b)}$ while the prolongation space $\tau_b Y$ is $d\rho^{-1}_{(a,b)}(\nabla a)$. So $\tau_b Y$ is a coset of $T_b Y$ in $T_{(a,b)} Z$. Absolute irreducibility of $\tau_b Y$ follows, and $\dim(\tau_b Y) = \dim(T_b Y)$. Finally, note that $\dim(T_b Y) = \dim Y$ because for standard general $(p, q) \in Z(\mathcal{M})$, the tangent space to Z_p at q is of dimension $\dim(Z_p)$. □

Finally, it is worth thinking about the case when Y is 0-definable. That is, using the above notation, when $a \in \mathcal{M}$. In that case, by Lemma 3.6, ∇ agrees with the zero section at a, and hence $\tau Y = TZ$ is just the tangent space of Y. That is, for 0-definable Zariski closed sets, the prolongation and tangent spaces agrees.

5. Differentially closed CCM-structures

We aim to prove that $\mathrm{CCM}_{\forall,\nabla}$ admits a model companion. We begin by exploring some properties of the existentially closed (e.c.) models. This amounts to proving extension lemmas. For example, Proposition 3.4, which says that every model of $\mathrm{CCM}_{\forall,\nabla}$ extends to the definable closure of the underlying model of $\mathrm{CCM}_{\forall}$, implies that if (A,∇) is an e.c. model of $\mathrm{CCM}_{\forall,\nabla}$ then it is a differential CCM-structure. Moreover, the e.c. models of $\mathrm{CCM}_{\forall,\nabla}$ are precisely the existentially closed differential CCM-structures. This justifies:

Definition 5.1. A *differentially closed* CCM-*structure* is an e.c. model of $\mathrm{CCM}_{\forall,\nabla}$.

Here is the main extension lemma:

Proposition 5.2. *Suppose* $\mathcal{N} \models \mathrm{CCM}$ *and* (A,∇) *is a* $\mathcal{N}$*-valued differential* CCM-*structure. Suppose X is an irreducible meromorphic variety, $b \in X(\mathcal{N})$, and $Y :=$* $\mathrm{loc}(b/A)$ *is the smallest A-definable Zariski closed subset of $X(\mathcal{N})$. For any $c \in \tau_b Y$ there is an extension of ∇ to a $\mathcal{N}$-valued differential* CCM-*structure on* $\mathrm{dcl}(Ab)$ *such that $\nabla(b) = c$.*

Proof. Let $D := \mathrm{dcl}(Ab)$. We follow the approach of Proposition 3.4. That is, given an element of D, say $d = f(a,b)$ where a is from A and $f : \mathrm{loc}(a,b) \to \mathrm{loc}(d)$ is a definable meromorphic map, we set $\nabla(d) := df(\nabla a, c)$. We have to verify that $(\nabla a, c) \in T_{(a,b)} \mathrm{loc}(a,b)$ for this to even make sense, that is to be able to apply df to $(\nabla a, c)$. Note, first of all, that since $Y = \mathrm{loc}(b/A) \subseteq \mathrm{loc}(b)$ we do have that $c \in \tau_b Y \subseteq T_b \mathrm{loc}(b)$. So $(\nabla a, c) \in T_a \mathrm{loc}(a) \times T_b \mathrm{loc}(b)$, that is, $(\nabla a, c)$ lies above (a,b), and it only remains to check that $(\nabla a, c) \in T \mathrm{loc}(a,b)$. Since $Y = \mathrm{loc}(b/A) \subseteq \mathrm{loc}(b/a)$, and the latter is the fibre of the co-ordinate projection $\mathrm{loc}(a,b) \to \mathrm{loc}(a)$ over a, we have that $\tau Y \subseteq \tau \mathrm{loc}(b/a)$, and the latter is by definition the fibre of $T \mathrm{loc}(a,b) \to T \mathrm{loc}(a)$ over $\nabla(a)$. Since $c \in \tau Y$, this tells us that $(\nabla a, c) \in T \mathrm{loc}(a,b)$, as desired.

Considering the case when $d = a$ and $f : \mathrm{loc}(a,b) \to \mathrm{loc}(a)$ is the co-ordinate projection, we see that this definition of ∇ on D extends the given ∇ on A. Considering the case when $d = b$ (so that a is the empty tuple and $f = \mathrm{id}$), we see that $\nabla(b) = c$, as desired.

While the proof of Proposition 3.4 was carried out in the context of models of $\mathrm{CCM}_{\forall,\nabla}$, it works equally well in the setting of $\mathcal{N}$-valued differential CCM-structures, showing that the way we have defined ∇ on D above yields, for any sort S, a well-defined map $\nabla : S(D) \to TS(\mathcal{N})$ that is a section to $TS \to S$, and such that $dg(\nabla d) = \nabla(g(d))$ for any definable meromorphic map g and tuple $d \in S(D)$. So (D,∇) is again a $\mathcal{N}$-valued differential CCM-structure. □

Corollary 5.3. *If* (A,∇) *is a differentially closed* CCM-*structure then* $A \models \mathrm{CCM}$.

Proof. We have that $A \subseteq \mathcal{N}$ for some $\mathcal{N} \models \mathrm{CCM}$. Let (B,∇) be a maximal $\mathcal{N}$-valued differential CCM-structure extending (A,∇). This exists as $\mathcal{N}$-valued differential CCM-structures are preserved under unions of chains, as can be easily verified from the definition.

We claim that $B = \mathcal{N}$. Given $b \in X(\mathcal{N})$ for some sort X, let $Y = \operatorname{loc}(b/B)$ and choose $c \in \tau_b Y$. By Proposition 5.2 we can extend ∇ to a $\mathcal{N}$-valued differential CCM-structure on $\operatorname{dcl}(Bb)$. By maximality, it follows that $b \in X(B)$ to start with. As X and b were arbitrary, this shows that $B = \mathcal{N}$.

We have that $(A, \nabla) \subseteq (\mathcal{N}, \nabla)$ is an extension of differential CCM-structures. By quantifier elimination, CCM has a universal-existential axiomatisation. Since (A, ∇) is existentially closed, the truth of such axioms in $(\mathcal{N}, \nabla)$ will imply their truth in (A, ∇). That is, $A \models \mathrm{CCM}$, as desired. $\square$

This is, of course, not enough. That is, not every differential CCM-structure on a model of CCM is differentially closed. For example, the standard model $\mathcal{M}$ admits the trivial differential structure $\nabla = 0$, but is not existentially closed as we can use Proposition 5.2 to produce nontrivial differential CCM-structures extensions.

The following property of differentially closed CCM-structures, which we refer to as the *geometric axiom*, can be read as saying that ∇ is a "generic" section to the tangent space:

Proposition 5.4. *If $(\mathcal{N}, \nabla)$ is a differentially closed* CCM*-structure then it satisfies the following condition:*

(GA) *Suppose S is a sort, $X \subseteq S$ is an $\mathcal{N}$-definable irreducible Zariski closed subset, $Y \subseteq \tau X$ is an $\mathcal{N}$-definable irreducible Zariski closed subset that projects dominantly onto X, and $Y_0 \subsetneq Y$ is a proper $\mathcal{N}$-definable Zariski closed subset. Then there exists $a \in X(\mathcal{N})$ such that $\nabla(a) \in Y \setminus Y_0$.*

Proof. We already know, by Corollary 5.3, that $\mathcal{N} \models \mathrm{CCM}$. Let $\mathcal{U} \succeq \mathcal{N}$ be a sufficiently saturated elementary extension, and let $c \in Y(\mathcal{U})$ be generic in Y over $\mathcal{N}$. In particular, $c \in Y \setminus Y_0$. By dominance, $b := \pi(c) \in X(\mathcal{U})$ is generic over $\mathcal{N}$. In particular, $\operatorname{loc}(b/\mathcal{N}) = X$ and $c \in \tau_b X(\mathcal{U})$. So, by Proposition 5.2, we can extend ∇ to a $\mathcal{U}$-valued differential CCM-structure on $\operatorname{dcl}(\mathcal{N}b)$ such that $\nabla(b) = c$. Then, as in the proof of Corollary 5.3, we can extend ∇ further to all of $\mathcal{U}$ so that $(\mathcal{U}, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$. Now, b witnesses that in $(\mathcal{U}, \nabla)$ there is a point of X that is sent by ∇ into $Y \setminus Y_0$. By existential closedness of $(\mathcal{N}, \nabla)$, there must exist $a \in X(\mathcal{N})$ such that $\nabla(a) \in Y \setminus Y_0$. $\square$

As the terminology already indicates, the geometric axiom characterises differentially closed CCM-structures:

Theorem 5.5. *A model $(\mathcal{N}, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$ is existentially closed if and only if $\mathcal{N} \models \mathrm{CCM}$ and condition* (GA) *of Proposition 5.4 holds.*

Proof. Corollary 5.3 and Proposition 5.4 gave the left-to-right direction. We therefore assume that $\mathcal{N} \models \mathrm{CCM}$ and $(\mathcal{N}, \nabla)$ satisfies (GA), and show that $(\mathcal{N}, \nabla)$ is existentially closed. Let S be a sort, x a variable belonging to S, and $\phi(x)$ a (finite) conjunction of L_∇-literals over $\mathcal{N}$ that is realised by $c \in S(A)$ in some extension $(A, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$ of $(\mathcal{N}, \nabla)$. We need to show that $\phi(x)$ has a realisation already in $(\mathcal{N}, \nabla)$. As in the proof of Corollary 5.3, we can extend (A, ∇) further to $(\mathcal{U}, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$ where $\mathcal{U} \models \mathrm{CCM}$.

Let d be the *order* of $\phi(x)$, that is, the largest positive integer such that $\nabla^d(x)$, namely ∇ iterated d-times and applied to x, appears in $\phi(x)$. We leave it to the reader to verify that $\phi(x)$ can then be rewritten as $\big(\nabla^d(x) \in U\big) \wedge \big(\nabla^d(x) \notin V\big)$ where U and V are $\mathcal{N}$-definable Zariski closed subsets of $T^d(S)$, the dth iterated tangent space of S.

Let $Y := \operatorname{loc}(\nabla^d c/\mathcal{N}) \subseteq T^d(S)(\mathcal{U})$. Since c realises $\phi(x)$, we must have that $\nabla^d(c) \in U \setminus V$ and so $Y \subseteq U$ and $Y \not\subseteq V$. In particular, $Y_0 := Y \cap V$ is a proper $\mathcal{N}$-definable Zariski closed subset of Y. We aim to find $a \in S(\mathcal{N})$ such that $\nabla^d(a) \in Y \setminus Y_0$; this will suffice as such an a would be a realisation of $\phi(x)$ in $(\mathcal{N}, \nabla)$.

Let $\overline{c} := \nabla^{d-1}(c)$ and $X := \operatorname{loc}(\overline{c}/\mathcal{N})$. Then $\nabla^d(c) = \nabla(\overline{c})$, so that Y is contained in τX and projects dominantly onto X. Hence, by (GA), there is an $\overline{a} \in X(\mathcal{N})$ such that $\nabla(\overline{a}) \in Y \setminus Y_0$. Consider the first co-ordinate projection $\pi : T^{d-1}(S) \to S$, and set $a := \pi(\overline{a}) \in S(\mathcal{N})$. It will suffice to show, therefore, that $\nabla^{d-1}(a) = \overline{a}$.

For each $\ell \geq 0$, let us denote by $\pi_\ell : T^{\ell+1}S \to T^\ell S$ the canonical projection. Moreover, for each $\ell = 0, \dots, d-1$, let $\overline{a}_\ell$ be the image of $\overline{a}$ in $T^\ell S$. So, in particular, $\overline{a}_0 = a$ and $\overline{a}_{d-1} = \overline{a}$. We claim that it suffices to show that

$$\overline{a}_{\ell+1} = \nabla(\overline{a}_\ell) \tag{5.1}$$

for all $\ell = 0, \dots, d-2$. Indeed, this would imply that $\overline{a} = \overline{a}_{d-1} = \nabla(\overline{a}_{d-2}) = \nabla^2(\overline{a}_{d-3}) = \cdots = \nabla^{d-1}(a)$, as desired. So let us fix $\ell = 0, \dots, d-2$ and show (5.1). The idea is to construe (5.1) as a Zariski closed condition on $\nabla(\overline{a})$. First of all, noting that $\pi_{\ell+1}(\nabla\overline{a}_{\ell+1}) = \overline{a}_{\ell+1}$ and $d\pi_\ell(\nabla\overline{a}_{\ell+1}) = \nabla(\pi_\ell\overline{a}_{\ell+1}) = \nabla(\overline{a}_\ell)$, we see that (5.1) is equivalent to

$$\pi_{\ell+1}(\nabla\overline{a}_{\ell+1}) = d\pi_\ell(\nabla\overline{a}_{\ell+1}). \tag{5.2}$$

Next, letting $\rho : T^{d-1}S \to T^{\ell+1}S$ be the projection, we have that $\rho(\overline{a}) = \overline{a}_{\ell+1}$, and hence $\nabla(\overline{a}_{\ell+1}) = \nabla(\rho\overline{a}) = d\rho(\nabla(\overline{a}))$. So (5.2) is equivalent to

$$\pi_{\ell+1} d\rho(\nabla\overline{a}) = d(\pi_\ell\rho)(\nabla\overline{a}). \tag{5.3}$$

This is a Zariski closed condition on $\nabla\overline{a}$, and as $\nabla\overline{a}$ is in $Y = \operatorname{loc}(\nabla\overline{c}/\mathcal{N})$, it suffices to verify that the identity holds of $\nabla\overline{c}$. But this follows from the fact that $\nabla\overline{c} = \nabla^d c$,

$$\begin{aligned}
\pi_{\ell+1} d\rho(\nabla\overline{c}) &= \pi_{\ell+1} d\rho(\nabla^d c) \\
&= \pi_{\ell+1}\nabla(\rho(\nabla^{d-1}c)) \\
&= \rho(\nabla^{d-1}c) \\
&= \nabla^{\ell+1}c \\
&= \nabla(\nabla^\ell c) \\
&= \nabla(\pi_\ell\rho(\nabla^{d-1}c)) \\
&= d(\pi_\ell\rho)(\nabla^d c)) \\
&= d(\pi_\ell\rho)(\nabla\overline{c})).
\end{aligned}$$

Hence (5.3) holds, as desired. □

That condition (GA) of Proposition 5.4 is first-order expressible follows from the fact that as X varies in an L-definable family, τX varies in an L_∇-definable family by construction (see Section 4), and that in CCM irreducibility and domination are definable in parameters (see [10, Section 2]). Theorem 5.5 thus gives us a model companion to $\mathrm{CCM}_{\forall,\nabla}$, namely the theory of differentially closed CCM-structures, which we denote DCCM.

6. Basic model theory of DCCM

From general model theory, we have that DCCM is model-complete. In this section we prove that $\mathrm{CCM}_{\forall,\nabla}$ has the amalgamation property, from which we can deduce that DCCM is complete and admits quantifier elimination. As a consequence we obtain a geometric description of algebraic and definable closure.

But first we need an extension lemma for algebraic closure, whereas we have only dealt with definable closure (in Proposition 3.4) so far.

Lemma 6.1. *Suppose (A, ∇) is a differential CCM-structure with $A \subseteq \mathcal{N} \models \mathrm{CCM}$, and $b \in \mathrm{acl}(A)$. Then there is a unique $\mathcal{N}$-valued differential CCM-structure on $\mathrm{dcl}(Ab)$ that extending ∇. Moreover, this extension is in fact $\mathrm{dcl}(Ab)$-valued.*

Proof. Let $Y = \mathrm{loc}(b/A)$ and $c \in \tau_b Y$. By Proposition 5.2 we can extend ∇ from A to a $\mathcal{N}$-valued differential CCM-structure on $\mathrm{dcl}(Ab)$ by sending $\nabla(b) := c$. We will show that $\tau_b Y = \{c\}$ is a singleton and hence $c \in \mathrm{dcl}(Ab)$, so that the above extension is in fact $\mathrm{dcl}(Ab)$-valued, and so $(\mathrm{dcl}(Ab), \nabla) \models \mathrm{CCM}_{\forall,\nabla}$. This will also show uniqueness as any extension of ∇ to $\mathrm{dcl}(Ab)$ would have to take b into $\tau_b Y = \{c\}$, by Lemma 4.3, and hence would agree with the one we just constructed.

Let $X := \mathrm{loc}(b)$ and write $Y = Z_a$ where $Z = \mathrm{loc}(a, b) \subseteq S \times X$ is a 0-definable irreducible Zariski closed set and S is a sort with $a \in S(A)$ generic. The fact that $b \in \mathrm{acl}(A)$ means that Y is finite, and hence the co-ordinate projection $\rho : Z \to S$ is generically finite-to-one. It follows that $d_p\rho : T_p Z \to T_{\rho(p)} S$ is an isomorphism for general $p \in Z(\mathcal{M})$. Hence $d_{(a,b)}\rho : T_{(a,b)} Z \to T_a S$ is a bijection. If $c, c' \in \tau_b Y$ then we know, by the proof of Proposition 5.2, that $(\nabla a, c), (\nabla a, c') \in T_{(a,b)} Z$. But as $d\rho$ takes both $(\nabla a, c)$ and $(\nabla a, c')$ to $\nabla(a) \in T_a S$ we must have $c = c'$. So $\tau_b Y = \{c\}$, as desired. $\square$

Next we prove independent amalgamation. We will use $\overset{\mathrm{CCM}}{\downarrow}$ to mean nonforking independence in CCM.

Lemma 6.2. *Suppose A, B_1, B_2 are definably closed subsets of $\mathcal{N} \models \mathrm{CCM}$, with $A \subseteq B_1 \cap B_2$ and $B_1 \underset{A}{\overset{\mathrm{CCM}}{\downarrow}} B_2$. Suppose ∇_i is a differential CCM-structure on B_i, for $i = 1, 2$, such that ∇_1 and ∇_2 agree on A. Then there is a common extension, ∇, of ∇_1 and ∇_2 to $B := \mathrm{dcl}(B_1 B_2)$ such that $(B, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$.*

Proof. Using Lemma 6.1 we can extend the differential CCM-structure on A, B_1, B_2 uniquely to their algebraic closures in $\mathcal{N}$. In particular, ∇_1 and ∇_2 will agree on $\mathrm{acl}(A)$. So we may as well assume that $A = \mathrm{acl}(A)$, and $B_i = \mathrm{acl}(B_i)$ for $i = 1, 2$. One consequence of A being acl-closed is that Zariski loci over A are absolutely irreducible, and hence independence over A has the following Zariski-topological characterisation:

$$b_1 \underset{A}{\overset{\mathrm{CCM}}{\downarrow}} b_2 \quad \text{if and only if} \quad \mathrm{loc}(b_1, b_2/A) = \mathrm{loc}(b_1/A) \times \mathrm{loc}(b_2/A).$$

See [10, Section 2] for details.

Every tuple from B is of the form $b = f(b_1, b_2)$ where each b_i is from B_i, and $f : \mathrm{loc}(b_1, b_2) \to \mathrm{loc}(b)$ is a definable meromorphic map. Our only choice is to define $\nabla(b) := df(\nabla_1 b_1, \nabla_2 b_2)$. But we need $(\nabla_1 b_1, \nabla_2 b_2) \in T_{(b_1,b_2)} \mathrm{loc}(b_1, b_2)$ for this to make sense. This is what we now check.

Let a be a tuple from A such that $\operatorname{loc}(b_1, b_2/A) = \operatorname{loc}(b_1, b_2/a)$. Let us denote by ∇ the common restriction of ∇_1 and ∇_2 to A. Taking prolongations with respect to the differential CCM-structure (A, ∇), and using that for $i = 1, 2$ we have $(A, \nabla) \subseteq (B_i, \nabla_i)$, we see that $\nabla_i(b_i) \in \tau_{b_i} \operatorname{loc}(b_i/a)$. Hence,

$$\begin{aligned}
(\nabla_1 b_1, \nabla_2 b_2) &\in \tau_{b_1} \operatorname{loc}(b_1/a) \times \tau_{b_2} \operatorname{loc}(b_2/a) \\
&= \tau_{(b_1,b_2)}\big(\operatorname{loc}(b_1/a) \times \operatorname{loc}(b_2/a)\big) \\
&= \tau_{(b_1,b_2)} \operatorname{loc}(b_1, b_2/a) \qquad \text{as } b_1 \underset{a}{\overset{\mathrm{CCM}}{\downarrow}} b_2 \\
&\subseteq T_{(b_1,b_2)} \operatorname{loc}(b_1, b_2)
\end{aligned}$$

as desired.

So it does make sense to set $\nabla(b) := df(\nabla_1 b_1, \nabla_2 b_2)$ for $b = f(b_1, b_2)$. The next step is to make sure this is well defined. What if we also have $b = f'(b_1', b_2')$? This is dealt with exactly as it was done in Proposition 3.4. Namely, let $Z := \operatorname{loc}(b_1, b_2, b_1', b_2')$ and consider $\bar{f} := (f, f') : Z \to \operatorname{loc}(b)^2$. Since $\bar{f}$ takes a generic point of Z to the diagonal we have that $d\bar{f} : TZ \to T(\operatorname{loc}(b)^2) = (T\operatorname{loc}(b))^2$ lands in the diagonal. Now, the argument in the previous paragraph, applied to $b_i b_i'$, shows, in particular, that $(\nabla_1(b_1 b_1'), \nabla_2(b_2 b_2')) \in T\operatorname{loc}(b_1 b_1', b_2 b_2')$. Hence, $(\nabla_1 b_1, \nabla_2 b_2, \nabla_1 b_1', \nabla_2 b_2') \in TZ$ and we get that $df(\nabla_1 b_1, \nabla_2 b_2) = df'(\nabla_1 b_1', \nabla_2 b_2')$.

We have defined ∇ on B in such a way that it is a section to $TS \to S$ for any sort S. It remains to check Axiom (2) of Definition 3.3. That is, suppose $g : S \to S'$ a definable meromorphic map between sorts, and $b \in S(B)$, $b' \in S'(B)$ with $g(b) = b'$. We need to show that $dg(\nabla b) = \nabla b'$. We may assume that there are b_1, b_2 from B_1, B_2, respectively, and definable meromorphic maps f, f' such that $b = f(b_1, b_2)$ and $b' = f'(b_1, b_2)$. It follows that $gf = f'$ on $\operatorname{loc}(b_1, b_2)$, and so we compute:

$$\begin{aligned}
dg(\nabla b) &= dg(df(\nabla_1 b_1, \nabla_2 b_2)) \quad \text{by definition of } \nabla(b) \\
&= d(gf)(\nabla_1 b_1, \nabla_2 b_2)) \quad \text{by functoriality} \\
&= df'(\nabla_1 b_1, \nabla_2 b_2)) \quad \text{as } gf = f' \\
&= \nabla b' \quad \text{by definition of } \nabla(b').
\end{aligned}$$

This completes the proof that $(B, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$. □

Proposition 6.3. $\mathrm{CCM}_{\forall,\nabla}$ *has the amalgamation property. In particular,* DCCM *admits quantifier elimination and is complete.*

Proof. Suppose $(B_i, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$, for $i = 1, 2$, with a common substructure (A, ∇). We seek a model $(B, \nabla) \models \mathrm{CCM}_{\forall,\nabla}$ into which (B_1, ∇) and (B_2, ∇) both embed over A. Let $\mathcal{U} \supseteq B_1$ be a sufficiently saturated model of CCM. By universality, there is an embedding of B_2 into $\mathcal{U}$ over A. Moreover, after taking nonforking extensions in CCM, we can find such an embedding such that the image of B_2 is independent from B_1 over A. We may as well assume, therefore, that $B_2 \subseteq \mathcal{U}$ already, and that $B_1 \underset{A}{\overset{\mathrm{CCM}}{\downarrow}} B_2$. Applying Lemma 6.2, we have a differential CCM-structure ∇ on $B := \operatorname{dcl}(B_1 B_2)$ that extends ∇ on both both B_1 and B_2.

Quantifier elimination now follows for DCCM, as a general consequence for a model companion of a universal theory with amalgamation.

Completeness also follows as we have a prime substructure: all differentially closed CCM-structures extend the standard model $\mathcal{M} \models \mathrm{CCM}$ equipped with the trivial differential structure (Lemma 3.6). $\square$

Remark 6.4. In the case of DCF_0 quantifier elimination implies that every definable set is a finite boolean combination of the closed sets of a certain noetherian topology; namely the Kolchin topology. There is a natural analogue of the Kolchin topology here, a *meromorphic Kolchin topology* on each sort, which one expects to also be noetherian. I leave this to the interested reader to pursue.

Next, we wish to characterise definable and algebraic closure in DCCM. First of all, given $(\mathcal{N}, \nabla) \models \mathrm{DCCM}$ and $A \subseteq \mathcal{N}$, let us denote by $\langle A \rangle$ the L_∇-structure generated by A. If A is already an L_∇-substructure and a is a tuple then we denote by $A\langle a \rangle$ the L_∇-structure generated by $A \cup \{a\}$. Note that

$$A\langle a \rangle = A \cup \{a, \nabla(a), \nabla^2(a), \dots\}.$$

Quantifier elimination tells us that $\mathrm{tp}(a/A) = \mathrm{tp}(a'/A)$ if and only if there is an L-isomorphism $\alpha : A\langle a \rangle \to A\langle a' \rangle$ that fixes A point-wise and sends $\nabla^n(a)$ to $\nabla^n(a')$ for all $n \geq 0$.

We have been using acl and dcl for algebraic and definable closure in the L-theory CCM. We will continue to do so, using acl_∇ and dcl_∇ for algebraic and definable closure in the L_∇-theory DCCM.

Proposition 6.5. *Suppose* $(\mathcal{N}, \nabla)$ *is a differentially closed* CCM*-structure and* $A \subseteq \mathcal{N}$. *Then* $\mathrm{dcl}_\nabla(A) = \mathrm{dcl}(\langle A \rangle)$ *and* $\mathrm{acl}_\nabla(A) = \mathrm{acl}(\langle A \rangle)$.

Proof. By Proposition 3.4, $\mathrm{dcl}(\langle A \rangle)$ is a differential CCM-substructure of $(\mathcal{N}, \nabla)$. Replacing A by $\mathrm{dcl}(\langle A \rangle)$, we may as well assume that A is a differential CCM-substructure to start with, and show that $\mathrm{dcl}_\nabla(A) = A$ and $\mathrm{acl}_\nabla(A) = \mathrm{acl}(A)$. The right-to-left containments are clear.

For the converses, let us first suppose that $b \notin \mathrm{acl}(A) =: B$. By Lemma 6.1, (B, ∇) is a differential CCM-substructure of $(\mathcal{N}, \nabla)$. We can find, in some elementary extension $\mathcal{U}$ of $\mathcal{N}$, a copy of $\mathcal{N}$ over B, say $\mathcal{N}'$, such that $\mathcal{N} \underset{B}{\overset{\mathrm{CCM}}{\downarrow}} \mathcal{N}'$. Let $\alpha : \mathcal{N} \to \mathcal{N}'$ be an L-isomorphism over B witnessing this, and consider $b' := \alpha(b')$. The fact that $b \underset{B}{\overset{\mathrm{CCM}}{\downarrow}} b'$ and that $b \notin B$ forces $b \neq b'$. On the other hand, setting $\nabla' := \alpha \nabla \alpha^{-1}$ we have that $(\mathcal{N}', \nabla') \models \mathrm{DCCM}$ and that $\alpha : (\mathcal{N}, \nabla) \to (\mathcal{N}', \nabla')$ is an L_∇-isomorphism over B. Now, we can find a common extension of ∇ and ∇' to $\mathrm{dcl}(\mathcal{N}\mathcal{N}')$ in $\mathcal{N}$ by Lemma 6.2 and then further to a model $(\mathcal{K}, \nabla) \models \mathrm{DCCM}$. So, in $(\mathcal{K}, \nabla)$ we have produced at least two distinct realisations, b and b', of $\mathrm{tp}(b/B)$. Repeating the process we can show that $\mathrm{tp}(b/B)$ has arbitrarily many realisations. That is, $b \notin \mathrm{acl}_\nabla(B) = \mathrm{acl}_\nabla(A)$, as desired.

Finally, suppose, toward a contradiction, that $b \in \mathrm{dcl}_\nabla(A) \setminus A$. This time we produce two distinct realisations of $\mathrm{tp}(b/A)$ for our contradiction. Since $\mathrm{dcl}_\nabla(A) \subseteq \mathrm{acl}_\nabla(A) = \mathrm{acl}(A)$, we have that $b \in \mathrm{acl}(A) \setminus A$. Hence $\mathrm{tp}_L(b/A)$ has a second realisation, $b' \in \mathrm{acl}(A)$ with $b' \neq b$. We thus have an L-isomorphism $\alpha : \mathrm{dcl}(Ab) \to \mathrm{dcl}(Ab')$ that fixes A point-wise and sends b to b'. But, by Lemma 6.1, $\mathrm{dcl}(Ab)$ and $\mathrm{dcl}(Ab')$ are differential CCM-substructures of $(\mathcal{N}, \nabla)$, and, as they each admit unique differential structures extending ∇ on A, we must have that α is an L_∇-isomorphism. By quantifier elimination, this means $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$. $\square$

7. Stability and elimination of imaginaries

We work now in a fixed sufficiently saturated model $(\mathcal{U}, \nabla) \models \mathrm{DCCM}$ and adopt the usual convention that all parameter sets are assumed to be of cardinality less than that of the saturation.

In order to prove that DCCM is a stable theory, and to capture the meaning of nonforking independence therein, we will follow an axiomatic approach. That is, we first introduce a natural notion of independence and then show that it has all the properties that characterise nonforking independence in stable theories.

Definition 7.1. Given sets A, B, C, we say that A *is independent of* B *over* C, denoted by $A \underset{C}{\downarrow} B$, to mean that $\mathrm{dcl}_\nabla(A) \overset{\mathrm{CCM}}{\underset{\mathrm{dcl}_\nabla(C)}{\downarrow}} \mathrm{dcl}_\nabla(B)$.

Note that we do not yet know that $\downarrow$ is nonforking independence, but we allow ourselves the notation as we will soon see that it is.

Let us first verify that $\downarrow$ is a notion of independence, in the sense introduced by Kim and Pillay [8]. First of all, it is clearly invariant under the action of automorphisms of $(\mathcal{U}, \nabla)$. Local character, finite character, symmetry, and transitivity all follow easily from the corresponding properties for $\overset{\mathrm{CCM}}{\downarrow}$.

Lemma 7.2 (Extension)**.** *Given $a, C \subseteq B$ there is $a' \models \mathrm{tp}(a/C)$ such that $a' \underset{C}{\downarrow} B$.*

Proof. We may assume that $C = \mathrm{dcl}_\nabla(C)$ and $B = \mathrm{dcl}_\nabla(B)$. By extension in CCM there is sequence $(a'_n : n \geq 0) \overset{\mathrm{CCM}}{\underset{C}{\downarrow}} B$ and an L-isomorphism

$$\alpha : C\langle a \rangle \to C \cup \{a'_n : n \geq 0\}$$

that fixes C point-wise and takes $\nabla^n(a)$ to a'_n for all $n \geq 0$. Extend α to an L-isomorphism

$$\alpha : A := \mathrm{dcl}(C\langle a \rangle) \to A' := \mathrm{dcl}(C \cup \{a'_n : n \geq 0\}).$$

Set $\nabla' := \alpha \nabla \alpha^{-1}$ on A' so that (A', ∇') is a differential CCM-structure isomorphic to (A, ∇). On the other hand, since $A' \overset{\mathrm{CCM}}{\underset{C}{\downarrow}} B$, Lemma 6.2 gives us a common extension of (A', ∇') and (B, ∇) to a model $(\mathcal{N}, \nabla') \models \mathrm{DCCM}$. By universality we have an embedding $\iota : (\mathcal{N}, \nabla') \to (\mathcal{U}, \nabla)$ over B. Then $\beta := \iota \circ \alpha : A \to \mathcal{U}$ is an L-isomorphism with its image that fixes C point-wise and takes $\nabla^n(a)$ to $\nabla^n(\beta(a))$ for all $n \geq 0$. Hence, by quantifier elimination, $a' := \beta(a) \models \mathrm{tp}(a/C)$. Now $A' \overset{\mathrm{CCM}}{\underset{C}{\downarrow}} B$ implies that $\iota(A') \overset{\mathrm{CCM}}{\underset{C}{\downarrow}} B$ as ι is over B. But $\iota(A') = \beta(A) = \mathrm{dcl}(C\langle a' \rangle)$. So $a' \underset{C}{\downarrow} B$, as desired. $\square$

Lemma 7.3 (Stationarity over algebraically closed sets)**.** *Suppose $C = \mathrm{acl}_\nabla(C) \subseteq B$, and a, a' are tuples. If $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$, and both a and a' are independent of B over C, then $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$.*

Proof. We may assume, without loss of generality, that $B = \mathrm{dcl}_\nabla(B)$. Since $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$, there is an L-isomorphism $\alpha : C\langle a \rangle \to C\langle a' \rangle$ that fixes C

pointwise and takes $\nabla^n(a)$ to $\nabla^n(a')$ for all $n \geq 0$. Since C is acl-closed, and the sequences $(\nabla^n a : n \geq 0)$ and $(\nabla^n a' : n \geq 0)$ are both CCM-independent from B over C, stationarity over algebraically closed sets in CCM implies that there is an L-isomorphism $\beta : B\langle a\rangle \to B\langle a'\rangle$ that fixes B pointwise and takes $\nabla^n(a)$ to $\nabla^n(a')$ for all $n \geq 0$. By quanitifier elimination, $\operatorname{tp}(a/B) = \operatorname{tp}(a'/B)$. $\square$

Corollary 7.4. DCCM *is a stable theory and* $\downarrow$ *is nonforking independence.*

Proof. This follows from the above observations by the characterisation of nonforking independence in simple (and hence stable) theories, see [8]. $\square$

We can also deduce stability by counting types. In fact, we get total transcendentality:

Theorem 7.5. DCCM *is λ-stable for every cardinal $\lambda \geq 2^{\aleph_0}$.*

Proof. We count types. Fix $\lambda \geq 2^{\aleph_0}$ and a subset $A \subseteq \mathcal{U}$ of cardinality at most λ. We show that there are at most λ-many complete types over A. We may assume that $A = \operatorname{dcl}_\nabla(A)$ is a differential CCM-substructure.

Suppose X is a sort, $a \in X(\mathcal{U})$, and consider $\operatorname{tp}(a/A)$. By quantifier elimination, it is determined by the sequence of types $\big(\operatorname{tp}_L(\nabla^n a/A) : n \geq 0\big)$ in CCM. Let

$$Z_n := \operatorname{loc}(\nabla^n a/A\nabla^{n-1}a).$$

I claim that there is some $N \geq 0$ such that $Z_{n+1} = \tau_{\nabla^n a}(Z_n)$, for all $n \geq N$. This will suffice, since then $\operatorname{tp}(a/A)$ is determined by the pair $\big(N, \operatorname{tp}_L(\nabla^N a/A)\big)$ of which there are at most λ-many possibilities by the λ-stability of CCM.

Note that $\nabla^{n+1}(a) \in \tau_{\nabla^n a}(Z_n)$, and so $Z_{n+1} \subseteq \tau_{\nabla^n a}(Z_n)$. But $\dim(\tau_{\nabla^n a}(Z_n)) = \dim(Z_n)$ by Lemma 4.4. So $\dim(Z_{n+1})$ is a nonincreasing function of n that must eventually stabilise. By irreducibility of $\tau_{\nabla^n a}(Z_n)$, this forces $Z_{n+1} = \tau_{\nabla^n a}(Z_n)$ for large enough n, as desired. $\square$

Theorem 7.6. DCCM *admits elimination of imaginaries.*

Proof. A general criterion for elimination of imaginaries in a stable theory is that finite sets have codes and global types have canonical bases, in the home sorts; see, for example, [7, Section 3]. That finite sets in DCCM have codes in the home sort follows from elimination of imaginaries in CCM, see [15] and [12, Appendix].

So, we fix a saturated $\mathcal{N} \preceq \mathcal{U}$ and a complete type $p = \operatorname{tp}(a/\mathcal{N})$, and show that p has a canonical base in $\mathcal{N}$. Let N be as in the proof of Theorem 7.5; that is,

$$\operatorname{loc}(\nabla^{n+1}a/\mathcal{N}\nabla^n a) = \tau_{\nabla^n a}\big(\operatorname{loc}(\nabla^n a/\mathcal{N}\nabla^{n-1}a)\big),$$

for all $n \geq N$. By elimination of imaginaries for CCM, there is a code c for $\operatorname{loc}(\nabla^N a/\mathcal{N})$ in $\mathcal{N}$. We claim that c is a canonical base for p.

Fix σ, an L_∇-automorphism of $\mathcal{N}$. We need to show that $p^\sigma = p$ if and only if $\sigma(c) = c$. One direction is clear: if $p^\sigma = p$ then $\operatorname{tp}_L(\nabla^N a/\mathcal{N})^\sigma = \operatorname{tp}_L(\nabla^N a/\mathcal{N})$ and hence $\operatorname{loc}(\nabla^N a/\mathcal{N})^\sigma = \operatorname{loc}(\nabla^N a/\mathcal{N})$, so that $\sigma(c) = c$.

For the converse, suppose $\sigma(c) = c$. Extend σ to an L_∇-automorphism $\hat\sigma$ of $\mathcal{U}$, and let $\hat a := \hat\sigma(a)$. We have that

$$\begin{array}{rcll} \operatorname{tp}_L(\nabla^N \hat a/\mathcal{N}) &=& \operatorname{tp}_L(\nabla^N a/\mathcal{N}) & \text{since } \sigma(c) = c \\ \operatorname{loc}(\nabla^{N+1}a/\mathcal{N}\nabla^N a) &=& \tau_{\nabla^N a}\big(\operatorname{loc}(\nabla^N a/\mathcal{N}\nabla^{N-1}a)\big) & \text{by choice of } N\text{, and} \\ \operatorname{loc}(\nabla^{N+1}\hat a/\mathcal{N}\nabla^N \hat a) &=& \tau_{\nabla^N \hat a}\big(\operatorname{loc}(\nabla^N \hat a/\mathcal{N}\nabla^{N-1}\hat a)\big) & \text{by applying } \hat\sigma. \end{array}$$

These imply that $\operatorname{tp}_L(\nabla^{N+1}\hat{a}/\mathcal{N}) = \operatorname{tp}_L(\nabla^{N+1}a/\mathcal{N})$. We can iterate to prove that $\operatorname{tp}_L(\nabla^n\hat{a}/\mathcal{N}) = \operatorname{tp}_L(\nabla^n a/\mathcal{N})$ for all $n \geq 0$. By quantifier elimination,

$$p^\sigma = \operatorname{tp}(\hat{a}/\mathcal{N}) = \operatorname{tp}(a/\mathcal{N}) = p,$$

as desired. □

8. Meromorphic vector fields and finite-dimensional types

We return in this final section to the motivating objects of interest: meromorphic vector fields. Our goal is to show that they are captured, up to bimeromorphic equivalence, in DCCM, by the "finite-dimensional" types.

We continue to work in a fixed sufficiently saturated model $(\mathcal{U}, \nabla) \models$ DCCM.

Definition 8.1. Suppose A is an L_∇-substructure and $p = \operatorname{tp}(b/A)$ is a complete type. By the *dimension of* p, denote by $\dim_\nabla(p)$ or $\dim_\nabla(b/A)$, we mean the sequence of nonnegative nondecreasing integers $\big(\dim(\operatorname{loc}(\nabla^n b/A)) : n \geq 0\big)$ ordered lexicographically. If $\dim_\nabla(p)$ is eventually constant then we say that p is *finite-dimensional* and we (re)use $\dim_\nabla(p)$ to denote that eventual finite number.

Note that the dimension depends only on the type p and not on the choice of realisation b. On the other hand, this dimension is not invariant under definable bijection – for example b and $\nabla(b)$ are interdefinable over the empty set but the dimension sequences are not always the same (one is a shift of the other). Nevertheless, whether or not a type is finite-dimensional, and the value of that finite dimension in the case that it is, is invariant under definable bijection.

Dimension witnesses forking:

Proposition 8.2. *Suppose a is a tuple and $C \subseteq B$ are L_∇-substructures. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $\dim_\nabla(a/B) = \dim_\nabla(a/C)$.*

Proof. We may assume that $B = \operatorname{dcl}_\nabla(B)$ and $C = \operatorname{dcl}_\nabla(C)$. We have, by Proposition 6.5, that $\operatorname{dcl}_\nabla(Ca) = \operatorname{dcl}(C\langle a\rangle)$. Hence $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$ is equivalent to $\nabla^n(a) \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{CCM}}_C B$ for all $n \geq 0$. But, as dimension witnesses forking in CCM, this is equivalent to $\dim(\operatorname{loc}(\nabla^n a/B)) = \dim(\operatorname{loc}(\nabla^n a/C))$ for all $n \geq 0$. □

It follows that if p is finite-dimensional then it is of finite U-rank, and in fact that $U(p) \leq \dim_\nabla(p)$. One can ask whether the same holds of Morley rank: is it the case that Morley rank is bounded by dimension? It is also natural to ask about the converse: if p is of finite rank (U-rank or Morley rank, it is the same thing) must it be of finite dimension? One expects affirmative answers to these questions, as is the case for DCF_0, but I do not pursue them here.

A natural source of finite-dimensional types over the empty set are meromorphic vector fields in the sense of Definition 3.1. Suppose (X, v) is such. Consider the type $p(x)$, over the empty, which says that $x \in X$ is generic and that $\nabla(x) = v(x)$. This is consistent by the geometric axiom of Proposition 5.4. Indeed, given any proper Zariski closed $X_0 \subseteq X$, apply (GA) to Y the Zariski closure of the image of v in $TX = \tau X$ and Y_0 the restriction of Y to X_0, yielding a $\mathcal{U}$-point $a \in X \setminus X_0$ with $\nabla(a) = v(a)$. Moreover, by quantifier elimination, this type is complete: the L-type of x is determined by x being generic in X, and $\nabla(x) = v(x)$ implies

$\nabla^n(x) = v_n(x)$ for appropriate definable meromorphic $v_n : X \to T^n X$, for all n. We call p the *generic type of* (X, v). Note that p is finite-dimensional; in fact, $\dim_\nabla(p) = \dim X$. Indeed, for all $n \geq 0$, we have that $\nabla^n(b) = v_n(b)$ and v_n is a definable meromorphic section to $T^n X \to X$, and hence, as b is generic in X, we get $\dim(\mathrm{loc}(\nabla^n b/A)) = \dim X$.

It turns out that all finite-dimensional types arise this way:

Theorem 8.3. *Every finite-dimensional type over the empty set in* DCCM *is, up to interdefinability, the generic type of a meromorphic vector field.*

Proof. Suppose $p = \mathrm{tp}(b)$ is finite dimensional. Let $d \geq 0$ be such that

$$\dim(\mathrm{loc}(\nabla^{d+1} b)) = \dim(\mathrm{loc}(\nabla^d b)).$$

Since the projection $\mathrm{loc}(\nabla^{d+1} b) \to \mathrm{loc}(\nabla^d b)$ is dominant, this means that it must be generically finite-to-one. Hence, setting $c := \nabla^{d+1}(b)$, we have that $c \in \mathrm{acl}(\nabla^d(b))$. As in the proof of Lemma 6.1, it follows that if $Y = \mathrm{loc}(c/\nabla^d(b))$ then $\tau_c Y$ is a singleton. Since Y is defined over $\nabla^d(b)$, the prolongation space τY is defined over $\nabla^{d+1}(b) = c$, and hence also $\tau_c Y$ is defined over c. By Lemma 4.3, $\nabla(c) \in \tau_c Y$. So $\nabla(c) \in \mathrm{dcl}(c)$. By quantifier elimination in CCM we can write $\nabla(c) = v(c)$ for some definable meromorphic map v. Then, setting $X := \mathrm{loc}(c)$, we have that $v : X \to TX$ is a section to the tangent space of X. That is, (X, v) is a meromorphic vector field and $q = \mathrm{tp}(c)$ is its generic type. Finally, observe that b and $c = \nabla^{d+1}(b)$ are interdefinable over the empty set, so that p and q are interdefinable types. $\square$

The upshot is that the finite-dimensional fragment of DCCM, over the empty set, captures precisely the bimeromorphic geometry of meromorphic vector fields.

Remark 8.4. We have restricted our attention in this discussion to the empty set for brevity; we could have worked more generally over arbitrary parameters A. The result would be that the finite-dimensional types over A are precisely, up to interdefinability, the generic types of *meromorphic D-varieties* over A. We leave it to the reader to both articulate precisely, and verify, this claim.

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